The homotopy theory of differentiable sheaves

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Abstract

Many important theorems in differential topology relate properties of manifolds to properties of their underlying homotopy types – defined e.g. by the total singular complex or the Čech nerve of a good open cover. Upon embedding the category of manifolds into the ∞-topos $\mathbf{Diff}^\infty$ of differentiable sheaves one gains a further notion of underlying homotopy type: the *shape* of the corresponding differentiable sheaf.

In a first series of results we prove using simple cofinality and descent arguments that the shape of any manifold coincides with many other notions of underlying homotopy types such as the ones mentioned above. Our techniques moreover allow for computations, such as the homotopy type of the Haefliiger stack.

This leads to more refined questions, such as what it means for a mapping differential sheaf to have the correct shape. To answer these we construct model structures as well as more general homotopical calculi on the ∞-category $\mathbf{Diff}^\infty$ (which restrict to its full subcategory of 0-truncated objects, $\mathbf{Diff}^\infty_{\leq 0}$) with shape equivalences as the weak equivalences. These tools are moreover developed in such a way so as to be highly customisable, with a view towards future applications, e.g. in geometric topology.

Finally, working with the ∞-topos $\mathbf{Diff}^0$ of sheaves on topological manifolds, we give new and conceptual proofs of some classical statements in algebraic topology. These include Dugger and Isaksen's hypercovering theorem, and the fact that the Quillen adjunction between simplicial sets and topological spaces is a Quillen equivalence.

Contents

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1 Introduction

1.1 Overview

Many important results about differentiable manifolds such as the classification of compact surfaces or the Poincaré-Hopf theorem express differential topological properties in terms of suitably defined underlying homotopy types of differentiable manifolds. Similarly, important invariants of differentiable manifolds such as their de Rham cohomology only depend on their underlying homotopy type. For a given differentiable manifold M the latter is commonly defined explicitly, e.g., one may take

1. its smooth total singular complex;
2. its underlying topological space;
3. the Čech complex of a good open cover of M; or,
4. a triangulation of M.

A further method for extracting homotopy types from manifolds may be obtained using shape theory. Any (∞-)topos $\mathcal{E}$ comes equipped with a colimit preserving functor $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$, which associates to any object X a pro-homotopy type called its *shape*, which parametrises covering spaces on X with fibres in homotopy types. For example, when $\mathcal{E}$ is the ∞-topos of sheaves on the ∞-category of schemes w.r.t. the étale topology, then the shape coincides with the étale homotopy type (see [Hoy18, §5]). For a simpler example, consider a small ∞-category A and the ∞-topos $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ of presheaves on A. The shape functor factors through $\mathcal{S}$ and is given by colim : $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$; in particular, the shape of any representable object is contractible. For a second small ∞-category B and a functor $u : A \to B$, we obtain a triple adjunction

$$\begin{tikzcd} \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \arrow[r, shift left=2, "u_!"] \arrow[r, shift right=2, "u_*"'] & \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S}) \arrow[l, "u^*"'] \end{tikzcd} \tag{1}$$

where $u_! : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S})$ preserves shapes, and $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \leftarrow \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S}) : u^*$ preserves shapes precisely when $u : A \to B$ is initial (a.k.a. cofinal, a.k.a. coinitial, ...).

Denote by $\mathbf{Diff}^r$ the ∞-topos of r-times diferentiable sheaves on the category of manifolds w.r.t. the usual Grothendieck topology. Using the technology of fractured ∞-toposes we are able to give a simple Galois theoretic proof that $\pi_!\mathbf{R}^d \simeq 1$ for all $d \in \mathbf{N}$. This is the starting point for showing that $\mathbf{Diff}^r$ has many of the pleasant properties of presheaf ∞-categories. For instance, as for presheaf categories, it follows that the shape functor factors through $\mathcal{S}$.

More importantly, we are able to make similar cofinality arguments as for presheaf ∞-toposes. For example, (any number of variants of) the functor $u : \Delta \to \mathbf{Diff}^r$ sending $[n]$ to the standard simplex can be regarded as initial in some sense, and one obtains an adjunction

$$u_! : \underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^r : u^* \tag{2}$$

in which both adjoints preserve shapes. Moreover, if $r \geq s$, then it is possible to obtain a triple adjunction

$$\begin{tikzcd} \mathbf{Diff}^r \arrow[r, shift left=2ex, "u_!"] \arrow[r, shift right=2ex, "u_*"'] & \mathbf{Diff}^s \arrow[l, "u^*" description] \end{tikzcd} \tag{3}$$

analogous to (1), where again $u_!$ and u^* preserve shapes. If $s = 0$, then $u_!$ sends any manifold to its underlying topological space.

Finally, taking the Čech nerve of any open cover $\mathcal{U}$ of M, we observe that

$$\pi_! M \simeq \pi_! \operatorname*{colim}_{[n]\in\Delta} (\underbrace{\mathcal{U} \times_M \cdots \times_M \mathcal{U}}_{n\times}) \simeq \operatorname*{colim}_{[n]\in\Delta} \pi_! (\underbrace{\mathcal{U} \times_M \cdots \times_M \mathcal{U}}_{n\times}) \tag{4}$$

by descent and the fact that $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ preserves colimits. Applying (2) to point 1. above, (3) to 2., and (4) to 3. & 4., we obtain the following theorem.

Theorem A. *The homotopy types described in points 1. - 4. above are all equivalent to the shape of M.* □

Let A, X be differentiable sheaves, we next consider the problem of studying the shape of the internal mapping sheaf $\underline{\mathbf{Diff}}^r(A, X)$. To illustrate why this might be useful, consider the analogous situation in the context of compactly generated topological spaces, where we assume that A is a CW-complex, and X is any compactly generated topological space. The internal mapping space $\underline{\mathbf{TSpc}}(A, X)$ (consisting of the set of continuous maps equipped with the compact-open topology) is then a model for the mapping homotopy type of the homotopy types modelled by A and X. In the differentiable setting, when A and X are manifolds, with A closed, then the set $\mathbf{Diff}^r(A, X)$ may be endowed with the structure of an infinite dimensional Fréchet manifold [GG73, Th. 1.11], and it is a folk theorem that its underlying homotopy type is again equivalent to $\mathcal{S}(\pi_! A, \pi_! X)$. By [Wal12, Lm A.1.7] the Fréchet manifold of smooth maps from M to N is canonically equivalent to $\underline{\mathbf{Diff}}^r(M, N)$. Also, the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with products, so that we obtain a comparison morphism $\pi_! \underline{\mathbf{Diff}}^r(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$. A differentiable sheaf A is then said to satisfy the *smooth Oka principle* if the comparison map $\pi_! \underline{\mathbf{Diff}}^r(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an isomorphism for all differentiable sheaves X (see [SS21]), and it is natural to ask for which differentiable sheaves the smooth Oka principle holds. We obtain the following generalisation of the main statement of [BEBP19].

Theorem B. *Any paracompact differentiable manifold locally modelled on Hilbert spaces, nuclear Fréchet spaces, or nuclear Silva spaces satisfies the smooth Oka principle.* □

The theory leading up to the proof of Theorem A above could be viewed as a study of the interaction of shapes with colimits—which is quite simple, because shape functors commute with colimits. The proof of Theorem B on the other hand boils down to showing that the shape functor $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ commutes with certain pullbacks—which is more difficult. Specifically one needs a method for identifying morphisms $X \to Y$ in $\mathbf{Diff}^r$ such that any pullback along $X \to Y$ commutes with $\pi_!$. It turns out that the ∞-toposes considered in this article are such that if they admit homotopical calculi (such as model structures) then $X \to Y$ has the desired property whenever it is a fibration in any of these calculi. Thus we are led to develop flexible tools for constructing such homotopical calculi, which we do using the theory of test categories.

1.2 Applications to geometric topology

Here we discuss some of the good properties of $\mathbf{Diff}^r_{\leq 0}$, the topos of set valued sheaves on manifolds, and illustrate how these might be relevant to problems in geometric topology, and in particular to the sheaf theoretic h-principle (see [Aya09], [RW11], [Dot14], [Kup19]).

Let $\mathbf{Emb}^{\infty}_n$ denote the topological category whose objects are the d-dimensional smooth manifolds, and where $\underline{\mathbf{Emb}}^{\infty}_d(M, N)$ is the set of smooth embeddings of M in N, equipped with, equivalently, the underlying topology of the Fréchet manifold $\mathbf{Emb}^{\infty}_d(M, N)$ or the C^{∞}-compact-open topology. Recall that a sheaf F on $\mathbf{Emb}^{\infty}_d$ valued in topological spaces is ***invariant*** if the map $\underline{\mathbf{Emb}}^{\infty}_d(M, N) \times F(M) \to F(N)$ is continuous.

Fixing a smooth manifold N, the following are examples of invariant sheaves:

1. The sheaf $\underline{\mathrm{Imm}}(\_, N)$ sending each manifold M to the space of immersions of M in N.
2. The sheaf $\underline{\mathrm{Subm}}(\_, N)$ sending each manifold M to the space of submersion of M to N.
3. The sheaf Conf of configuration spaces sending any manifold M to the space of finite subsets of M, topologised in such a way that points may "disappear off to infinity" when M is open (See [RW11, §3]).
4. The sheaf sending any manifold M to the space of symplectic forms on M (for n even).

An invariant sheaf F is ***microflexible*** ([RW11, Def. 5.1]) if for

(i) any polyhedron K,

(ii) any manifold M,

(iii) compact subsets $A \subseteq B \subseteq M$, and

(iv) subsets $U \subseteq V \subseteq M$ containing A and B, respectively,

the lifting problem

$$\begin{tikzcd} & \{0\} \times K \arrow[r] \arrow[dl] \arrow[d] & F(V) \arrow[d] \\ {[0,\varepsilon] \times K} \arrow[r] \arrow[urr, dashed] & {[0,1] \times K} \arrow[r] & F(U) \end{tikzcd} \tag{5}$$

admits a solution for some $0 < \varepsilon < 1$, possibly after passing to a smaller pair $U \subseteq V$ containing A and B, respectively. Of the above examples 1. and 3. are microflexible, while 2. is not.

For any invariant sheaf F and any manifold M one may construct the *scanning map* (see [Fra11, Lect. 17])

$$\mathrm{scan} : F(M) \to \Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big), \tag{6}$$

and F is said to *satisfy the h-principle on M* if the scanning map is an equivalence.

Theorem 1.2.1 ([Fra11, Lect. 20])**.** *Every microflexible invariant sheaf satisfies the h-principle on any open manifold.* $\square$

This is a very powerful theorem, as the study of $\Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big)$ is often easier than that of $F(M)$.

Example 1.2.2. For $F = \underline{\mathrm{Imm}}(\_, N)$ (as in 1. above), $\Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} F(\mathbf{R}^n) \to M\big)$ can with little effort be shown to be equivalent to the space of formal immersions of M into N, that is, the set of bundle maps

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

which are induce monomorphisms $T_m M \to T_{fm} N$ for all $m \in M$. The h-principle can thus be used to prove the famed Smale-Hirsch theorem (see [Sma59] & [Hir59] for details). ⌟

Theorem 1.2.1 may be viewed as a statement that any microflexible invariant sheaf $F : (\mathbf{Emb}_n^\infty)^{\mathrm{op}} \to \mathbf{TSpc}$ retains many of its exactness properties when composed with the functor $\mathbf{TSpc} \to \mathcal{S}$ sending any topological space to its (singular) homotopy type. The geometry of the constituent spaces of F is generally crucial for proving microflexibility; however

1. it is often difficult to construct suitable topologies on these spaces, and

2. many of these spaces admit a natural smooth structure.

In fact, the constituent spaces of F are oftentimes more naturally viewed as objects of $\mathbf{Diff}^\infty$ (as already observes in [GTMW09] and [Kup19]), so that one is lead to consider sheaves of the form $F : (\mathbf{Emb}_n^\infty)^{\mathrm{op}} \to \mathbf{Diff}^\infty$. At a first glance, it may look as if we are introducing a new complication by considering sheaves valued in an ∞-category rather than an ordinary category. However, in most cases, such as in the examples 1. - 4. considered above, we obtain sheaves valued in $\mathbf{Diff}^\infty_{\leq 0}$. The following theorem provides a first justification for replacing $\mathbf{TSpc}$ with $\mathbf{Diff}^\infty_{\leq 0}$.

Theorem C ([Cis03, §6.1])**.** *The topos $\mathbf{Diff}^\infty_{\leq 0}$ admits a model structure such that the restriction of the shape functor $\pi_! : \mathbf{Diff}^r_{\leq 0} \to \mathcal{S}$ exhibits $\mathbf{Diff}_{\leq 0}$ as a localisation along the weak equivalences.* $\square$

Thus many of the techniques developed here may be used without knowledge of ∞-categories. Moreover, $\mathbf{Diff}^\infty_{\leq 0}$ has excellent formal properties, which are directly relevant to the microflexibility condition.

Theorem D. *Filtered colimits in $\mathbf{Diff}^\infty_{\leq 0}$ are are homotopy colimits.* $\square$

Theorem E. *Closed manifolds are categorically compact in $\mathbf{Diff}^\infty_{\leq 0}$.* $\square$

To give a simple illustration of how these properties are relevant to the sheaf theoretic h-principle, consider how the lifting condition (5) may now be replaced with

$$\begin{tikzcd} & \{0\} \times K \arrow[r] \arrow[dl] \arrow[d] & \operatorname{colim}_{V \supseteq B} F(V) \arrow[d] \\ {[0,\varepsilon] \times K} \arrow[r] \arrow[urr, dashed] & {[0,1] \times K} \arrow[r] & \operatorname{colim}_{U \supseteq A} F(U) \end{tikzcd} \tag{7}$$

eliminating the necessity to gradually choose smaller and smaller open neighbourhoods $V \supseteq U$ of $B \supseteq A$. Indeed, this is close to how Gromov originally formulated the microflexibility condition (see [Gro86, §1.4.2]) but using quasi-topological spaces (introduced by Spanier; [Spa63]) with the intention of obataining the correct colimits (as explained in [Gro86, §1.4.1]). Unfortunately, both theorems D and E fail for quasi-topological spaces, as shown in Example 1.2.3 below.

A further use of the good formal properties of $\mathbf{Diff}^{\infty}_{\leq 0}$ is suggested by Ayala in [Aya09, p. 19]: The construction of the scanning map in (6) involves carefully choosing and then modifying an exponential map $\exp : TM \to M$ (see [RW11, §6]). In order to formulate an h-principle which works for any exponential function, Ayala constructs a variant of the scanning map given by

$$\mathrm{scan} : F(M) \to \Gamma\big(\mathrm{Fr}(TM) \times_{\mathrm{O}_n} \operatorname*{colim}_{\delta>0} F\big(\mathring{B}^n{}_\delta(0)\big)\big). \tag{8}$$

The colimit $\mathrm{colim}_{\delta>0} F\big(\mathring{B}^n{}_\delta(0)\big)$ is again taken in the category of quasi-topological spaces in [Aya09] with the expectation that it has the same homotopy type as $F(\mathbf{R}^n)$, but this once more fails by Example 1.2.3. Fortunately, by Theorem D the colimit does have the correct homotopy type when taken in $\mathbf{Diff}^{r}_{\leq 0}$. More generally, we believe that working with differentiable sheaves throughout in [Aya09] would fix all issues which arise from working with quasi-topological spaces.

Example 1.2.3. For each $\delta > 0$ the space $\mathrm{Conf}\big(\mathring{B}^n_\delta(0)\big)$ is weakly equivalent to S^n. In Ayala's variant of quasi-topological spaces (see [Aya09, Def. 2.7]) the colimit is equivalent to the Sierpinski space, which is contractible. In other variants of quasi-topological spaces (e.g., [SW57, §3], [Gro86, §1.4.1]) one still obtains a contractible two-point space. ⌟

Part I
Foundations

2 Shapes and cofinality

Here we briefly recall how any object in a topos admits a canonical underlying (pro-)homotopy type.

Definition 2.0.1. Let $\mathcal{X}$ be an ∞-topos, and denote by $\pi : \mathcal{X} \to \mathcal{S}$ the unique geometric morphisms to $\mathcal{S}$, then the ***shape*** of $\mathcal{X}$ is the pro-left adjoint of π^*, i.e. $\pi_*\pi^* : \mathcal{S} \to \mathcal{S}$, and is denoted by $\pi_!\mathcal{X}$. ⌟

The following offers a useful method for detecting when an ∞-topos has trivial shape.

Proposition 2.0.2. *Let $\mathcal{X}$ be hypercomplete ∞-topos, then the shape fo $\mathcal{X}$ is contractible iff the canonical map*

$$E \to H^0(\mathcal{X}, E)$$

is an equivalence for all sets E, and

$$H^i(\mathcal{X}, G) = 0$$

for all i and all G, where G is a group for $i = 1$, and an Abelian group for all $i \geq 2$.

Proof. For the if statement we want to prove that for any homotopy type K the unit map $K \to \pi_*\pi^*K$ is an equivalence. It is enough to show this for the special case when K is n-truncated for some $n \in \mathbf{N}$, because for general K we then have

$$\begin{aligned} K &= \lim_n (K_{\leq n}) \\ &\xrightarrow{\simeq} \lim_n \pi_*\pi^*(K_{\leq n}) \\ &= \lim_n \pi_*((\pi^*K)_{\leq n}) \\ &= \pi_*(\lim_n (\pi^*K)_{\leq n}) \\ &= \pi_*\pi^*K \end{aligned}$$

where the third isomorphism follows from [Lur09, 5.5.6.28], and the last isomorphism follows from the hypercompleteness assumption on $\mathcal{X}$.

We prove this special case via induction: The base case holds by assumption. Let $n > 0$, and assume the statement holds for all k-truncated objects, for $0 \leq k < n$. Let K be an n-truncated homotopy type, then we obtain the commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \pi_*\pi^*K \arrow[d] \\ K_{\leq n} \arrow[r] & \pi_*\pi^*(K_{\leq n}) \end{tikzcd}$$

in which the bottom arrow is an isomorphism by the induction hypothesis. To show that $K \to \pi_*\pi^*K$ is an equivalence it is thus enough to show that $L \to \pi_*\pi^*L$ is an equivalence for all fibres L of $X \to X_{\leq n}$. We check the equivalence on homotopy groups. For $0 \leq i < n$ we have $\pi_i L = 0 = H^{n-i}(\mathcal{X}, \pi_0\Omega L) = \pi_i\, \pi_*\pi^*L$ by assumption. For $i = n$ we have

$$\begin{aligned} \pi_n L &= \pi_0\, \Omega^n L \\ &\xrightarrow{\simeq} \pi_0\, \pi_*\pi^*\Omega^n L \\ &= \pi_0\, \Omega^n \pi_*\pi^* L \\ &= \pi_n\, \pi_*\pi^* L \end{aligned}$$

where the second isomorphism follows from the base case.

The only if statement is obvious. □

Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism, then the unit $\mathrm{id}_{\mathcal{X}} \to f_*f^*$ induces a natural transformation $(\pi_{\mathcal{Y}})_*(\pi_{\mathcal{Y}})^* \to (\pi_{\mathcal{X}})_*(\pi_{\mathcal{X}})^* = (\pi_{\mathcal{Y}})_* f_* f^* (\pi_{\mathcal{Y}})^*$, i.e., a morphism $\pi_!\mathcal{X} \to \pi_!\mathcal{Y}$ in $\mathrm{Pro}(\mathcal{S})$.

Definition 2.0.3. geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is a ***shape equivalence*** if the induced $\pi_!\mathcal{X} \to \pi_!\mathcal{Y}$ map is an isomorphism. A morphism $X \to Y$ in $\mathcal{X}$ is a *shape equivalence* if the geometric morphism $\mathcal{X}_{/X} \to \mathcal{X}_{/Y}$ is a shape equivalence. ⌟

Example 2.0.4. A functor $A \to B$ between small ∞-categories is a homotopical equivalence iff the induced geometric morphism $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S})$ is a shape equivalence. ⌟

Proposition 2.0.5. *Let $\mathcal{E}$ be an ∞-topos, then the left adjoint $\pi^* : \mathcal{S} \to \mathcal{E}$ of the final geometric morphism extends to the diagram*

$$\begin{tikzcd}
& \mathrm{Pro}(\mathcal{S}) \arrow[dr, shift left, "\pi^*"] \arrow[dr, phantom, "\perp"] & \\
\mathcal{S} \arrow[ur, hook] \arrow[rr] & & \mathcal{E}, \arrow[ul, shift left, "\pi_!"]
\end{tikzcd}$$

where $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$ sends any object $E \in \mathcal{E}$ to $\pi_! \mathcal{E}_{/E}$. □

Thus for any geometric morphism $\mathcal{E} \to \mathcal{F}$ we obtain the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, "(\pi_{\mathcal{E}})_!"'] & & \mathcal{F} \arrow[ll, "f^*"'] \arrow[dl, "(\pi_{\mathcal{F}})_!"] \\
& \mathrm{Pro}(\mathcal{S}) \arrow[u, phantom, "\Rightarrow"] &
\end{tikzcd} \tag{9}$$

and f^* is a shape equivalence iff $(\pi_{\mathcal{E}})_! \circ f^*(1) \to (\pi_{\mathcal{F}})_!$ is an equivalence.

Definition 2.0.6. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is ***essential*** if $\mathcal{E} \leftarrow \mathcal{F} : f^*$ admits a left adjoint, denoted by $f_! : \mathcal{E} \to \mathcal{F}$. ⌟

Proposition 2.0.7. *Let $f : \mathcal{E} \to \mathcal{F}$ be an essential geometric morphism, then $f_! : \mathcal{E} \to \mathcal{F}$ preserves shapes.* □

Definition 2.0.8. The geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is a ***local shape equivalence*** iff the natural transformation $(\pi_{\mathcal{E}})_! \circ f^* \Rightarrow (\pi_{\mathcal{F}})_!$ from diagram (9) is an equivalence. ⌟

Example 2.0.9. A functor $A \to B$ between small ∞-categories is initial iff the induced geometric morphism $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \underline{\mathrm{Hom}}(B^{\mathrm{op}}, \mathcal{S})$ is a local shape equivalence. ⌟

The following simple observation will turn out to be remarkably useful.

Proposition 2.0.10. *Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, and assume that $\mathcal{F}$ is generated under colimits by a small subcategory C, then f is a local shape equivalence, iff for each object F in C the natural morphism $\pi_! f^* F \to \pi_! F$ is an equivalence.* □

2.1 Locally contractible toposes

Definition 2.1.1. An ∞-topos $\mathcal{E}$ is called ***locally ∞-connected*** if the functor $\pi_! : \mathcal{E} \to \mathrm{Pro}(\mathcal{S})$ factors through the inclusion $\mathcal{S} \hookrightarrow \mathrm{Pro}(\mathcal{S})$, i.e., $\mathcal{E} \leftarrow \mathcal{S} : \pi^*$ admits a left adjoint, which we again denote by $\pi_!$. ⌟

Let $f : \mathcal{E} \to \mathcal{F}$ be a geometric morphism, then the diagram (9) restricts to the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, "(\pi_{\mathcal{E}})_!"'] & & \mathcal{F} \arrow[ll, "f^*"'] \arrow[dl, "(\pi_{\mathcal{F}})_!"] \\
& \mathcal{S}. \arrow[u, phantom, "\Rightarrow"] &
\end{tikzcd}$$

Proposition 2.1.2 ([Lur17, Prop. A.1.11])**.** *Let $\mathcal{E}$ be a locally ∞-connected topos, then the functor $\psi_! : \mathcal{E} \to \mathcal{S}_{/\pi_! \mathbf{1}}$ admits a fully faithful right adjoint.* □

Proposition 2.1.3. *Let $a : \mathcal{E} \hookrightarrow \mathcal{F}$ be a geometric embedding which is also a local shape equivalence, then $(\pi_{\mathcal{E}})_! = (\pi_{\mathcal{F}})_! \circ a_*$.*

Proof. By assumption $(\pi_{\mathcal{E}})_! \circ a^* = (\pi_{\mathcal{F}})_!$, so the corollary follows from precomposing with a_*. □

Definition 2.1.4. An n-topos $(1 \leq n \leq \infty)$ is ***locally contractible*** if it is generated under colimits by a *set* of objects of contractible shape. ⌟

Proposition 2.1.5. *Let $\mathcal{X}$ be a hypercomplete ∞-topos, then the following are equivalent:*

(I) *$\mathcal{X}$ is locally ∞-connected.*

(II) *$\mathcal{X}$ is locally contractible.*

Proof. The implication (I) $\implies$ (II) is the content of Proposition **??**. To show (II) $\implies$ (I), consider a small subcategory $C \subseteq \mathcal{X}$, generating $\mathcal{X}$ under colimits. For any object $c \in C$ and any morphism $1 \to c$ the apex of the pullback

$$\begin{tikzcd} c \times_{\pi^*\pi_! c} 1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ c \arrow[r] & \pi^*\pi_! c \end{tikzcd}$$

has contractible shape by [Lur17, Prop. A.1.9]. For all $c \in C$, choose an effective epimorphism $\coprod_{I_c} 1 \twoheadrightarrow \pi_! c$, then the morphisms $c \times_{\pi^*\pi_! c} \pi^* \left(\coprod_{I_c} 1\right) = \coprod_{I_c} c \times_{\pi^*\pi_! c} 1 \to c$ are effective epimorphisms, so that the proposition follows from [Lur18, Prop. 20.4.5.1]. □

Example 2.1.6. Let A be a small category, then $\widehat{A}$ is locally contractible. ⌟

Proposition 2.1.7. *Let $\mathcal{E}$ be an ∞-topos generated under colimits by a small subcategory C of objects with contractible shape, then the diagram*

$$\begin{tikzcd} \underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathcal{S}) \arrow[drr, "\mathrm{colim} = \pi_!"'] & & \mathcal{E} \arrow[ll, hook'] \arrow[d, "\pi_!"] \\ & & \mathcal{S} \end{tikzcd}$$

commutes.

Proof. Combine Propositions 2.0.10 and 2.1.3. □

Theorem 2.1.8. *Let $\mathcal{E}$ be an ∞-topos generated under colimits by a small subcategory C of objects with contractible shape, let A be a small ∞-category A, and let $u : A \to C$ be an initial functor, then*

$$\pi_! E = \mathrm{colim}\, \mathcal{E}(u, E)$$

for any object E in $\mathcal{E}$. □

In many cases of interest, the functor $\alpha : A \to C$ will be a bijection on objects. We will use the following two propositions repeatedly to check the conditions of the above proposition.

Proposition 2.1.9. *Let (A, I) and (B, J) be pairs consisting of small ordinary categories together with an interval, and let $u : A \to B$ be a functor carrying I to J (including the inclusions of the final object, which u must then preserve). Assume that*

(a) $\pi_! : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ *preserves finite products, and that*

(b) *every object in B is J-contractible*

then u is initial.

Proof. The functor u is initial iff for every object b in B the shape of u^*b is contractible (see [Cis19, Cor. 4.4.31]). Let $J \times b \to b$ be an J-contraction of b, then the unit morphisms produce a diagram

$$\begin{tikzcd}
u^*b \cong e \times u^*b \arrow[r] \arrow[d] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
I \times u^*b \arrow[r] & u^*u_!I \times u^*b \cong u^*(J \times b) \arrow[r] & u^*b, \\
u^*b \cong e \times u^*b \arrow[r] \arrow[u] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

showing that u^*b is I-contractible by (a). □

Proposition 2.1.10. *Let (B, J) be a pair consisting of a small ordinary category together with an interval. Let $u : \square \to B$ be a functor carrying the interval $\square^1$ to J (including the inclusions of the final object, which u must then preserve). If every object in B is J-contractible then u is initial.*

Proof. The functor u is initial iff for every object b in B the shape of u^*b is contractible (see [Cis19, Cor. 4.4.31]).
Claim: There exists natural morphism $X_1 \otimes X_2 \to X_1 \times X_2$.

Let $J \times b \to b$ be an J-contraction of b, then the unit morphisms produce a diagram

$$\begin{tikzcd}
u^*b \cong e \otimes u^*b \arrow[r] \arrow[d] & e \times u^*b \arrow[r] \arrow[d] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
\square^1 \otimes u^*b \arrow[r] & \square^1 \times u^*b \arrow[r] & u^*u_!\square^1 \times u^*b \cong u^*(J \times b) \arrow[r] & u^*b, \\
u^*b \cong e \otimes u^*b \arrow[r] \arrow[u] & e \times u^*b \arrow[r] \arrow[u] & u^*u_!e \times u^*b \cong u^*(e \times b) \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

showing that u^*b is $\square^1$-contractible because $\pi_!(X_1 \otimes X_2) \simeq \pi_!X_1 \times \pi_!X_2$ for all cubical sets X_1, X_2 (see [Cis06, Cor. 8.4.32]).

Proof of claim: We note that for any two cubical sets X_1, X_2 there are canonical morphisms $X_1 \otimes X_2 \to X_i$ $(i = 1, 2)$. To see this, note that for the for any $k_1, k_2 \in \mathbf{N}$ we have projection maps (in **Set**) $\square^{k_1} \otimes \square^{k_2} \cong \{0,1\}^{k_1} \times \{0,1\}^{k_2} \to \{0,1\}^{k_i}$ for $i = 1, 2$; the canonical morphisms $X_1 \otimes X_2 \to X_i$ $(i = 1, 2)$ are then obtained by extending by colimits, yielding the desired morphism. □

3 Fractured ∞-toposes

3.1 Basic definitions

The following definition is equivalent to Lurie's ([Lur18, Def. 20.1.2.1]), but highlights the salient properties necessary for us. Their equivalence is proved in the end of this section.

Definition 3.1.1. A ***fractured*** ∞***-topos*** is an adjunction

$$j_! : \mathcal{E}^{\mathrm{corp}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E} : j^*$$

between ∞-toposes $\mathcal{E}^{\mathrm{corp}}$ and $\mathcal{E}$ satisfying properties (a) -(d) below:

(a) The topos $\mathcal{E}$ is generated under colimits by the objects in the image of $j_!$.

(b) For every object U in $\mathcal{E}^{\mathrm{corp}}$, the the left adjoint in

$$(j_!)_{/U} : \mathcal{E}^{\mathrm{corp}}_{/U} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{E}_{/U} : (j^*)_{/U}$$

is fully faithful.

(c) The functor $\mathcal{E}^{\mathrm{corp}} \leftarrow \mathcal{E} : j^*$ preserves colimits (and thus admits a right adjoint by the adjoint functor theorem),

(d) For any pullback square

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

in which $U \to V$ and V' are in the image of $j_!$, the map $U' \to V'$ is in the image of $j_!$.

⌟

The axioms imply that $j_!$ is faithful, and $\mathcal{E}^{\mathrm{corp}}$ will often be identified with its image under $j_!$. Paradigmatic examples are given by sheaves on some class of nice gemoetric objects, and $\mathcal{E}^{\mathrm{corp}}$ then consists of an appropriate notion Deligne-Mumford stacks and étale morphisms between them. In particular, the ∞-topos $\mathbf{Diff}^r$ of r-times differentiable sheaves may be endowed with the structure of a fractured ∞-topos. Manifolds will then be the most important examples of Deligne-Mumford stacks, and local homeomorphism are the étale morphism between them.

Definition 3.1.2. A morphism $U \to X$ in a fractured topos $\mathcal{E}$ is called ***admissible*** if for every pullback diagram

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

in which X' is in $\mathcal{E}^{\mathrm{corp}}$, the morphism $U' \to X'$ is in $\mathcal{E}^{\mathrm{corp}}$. ⌟

These should be though of as étale morphisms between not necessarily corporeal objects in $\mathcal{E}$. The admissible morphisms are local: [Lur18, Cor. 20.3.2.8] Under mild conditions the structure of a fractured ∞-topos may be recovered from its class of admissible morphisms (see [Lur18, Rmk. 20.3.4.6]). From this perspective, (d) can then be thought of as consisting of two parts:

1. Admissible morphisms are closed under pullbacks.

2. If $U \to V$ is an admissible morphism, and V is corporeal, then U is corporeal.

We now work towards Theorem 3.1.7, which will allow us to exhibit $\mathbf{Diff}^r$ as a fractured ∞-topos.

Definition 3.1.3 ([Lur18, Def. 20.2.1.1]). Let G be an ∞-category, then an ***admissibility structure*** on G is a subcategory G^{ad}, whose morphisms are referred to as *admissible morphisms*, such that:

(a) Every equivalence in G is an admissible morphism.

(b) For any admissible morphism $U \to X$, and any morphism $X' \to X$ there exists a pullback square

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ X' \arrow[r] & X, \end{tikzcd}$$

in which $U' \to X'$ is admissible.

(c) For any commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

in which $g : Y \to Z$ is admissible, the morphism $f : X \to Y$ is admissible iff $h : X \to Z$ is.

(d) Admissible morphisms are closed under retracts.

⌟

Example 3.1.4. The admissible morphisms in a fractured ∞-topos form an admissibility structure. ⌟

Definition 3.1.5 ([Lur18, Def. 20.6.2.1]). A ***geometric site*** is a triple $(G, G^{\mathrm{ad}}, \tau)$ consisting of

(i) a small ∞-category G,

(ii) an admissibility structure G^{ad} on G, and

(iii) a Grothendieck topology τ on G,

such that every covering sieve in τ contains a covering sieve generated by admissible morphisms.

⌟

Lemma 3.1.6 ([Lur18, Props. 20.6.1.1 & 20.6.1.3]). *Let $(G, G^{\mathrm{ad}}, \tau)$ be a geometric site, then there exists a Grothendieck topology on G^{ad} in which a sieve R in G^{ad} is a covering sieve iff the sieve generated by R in G is a covering sieve. Any sheaf on G restricts to a sheaf on G^{ad}.* □

Theorem 3.1.7 ([Lur18, Th. 20.6.3.4]). *Let $(G, G^{\mathrm{ad}}, \tau)$ be a geometric site, and denote by $\mathcal{E}$ the ∞-topos of sheaves on G, and, by $\mathcal{E}^{\mathrm{corp}}$ the ∞-topos of sheaves on G^{ad}, then the restriction functor $\mathcal{E}^{\mathrm{corp}} \leftarrow \mathcal{E} : j^*$ admits a left adjoint, and the resulting adjunction is a fractured ∞-topos.* □

Remark 3.1.8. In the same way that not every ∞-topos is the category of sheaves on a site, not every fractured ∞-topos is given as in the preceding theorem. However, it *is* true that every fractured ∞-topos may be realised as the localisation of a fractured presheaf ∞-topos, and that this presheaf ∞-topos may be obtained as in the preceding theorem with $\tau = \varnothing$. See [Lur18, Th. 20.5.3.4]. ⌟

Proposition 3.1.9. *Definitions 3.1.1 and [Lur18, Def. 20.1.2.1] are equivalent.*

Proof. We will sweep condition (0) of [Lur18, Def. 20.1.2.1] under the rug. By [Lur18, Prop. 20.1.3.3] $\mathcal{E}^{\mathrm{corp}}$ is a topos. The following implications are clear.
(a) - (d) $\implies$ (1) & (2)

- (d) $\implies$ (1)
- (a) & (c) $\implies$ (2)

(1) - (3) $\implies$ (a) - (c)

- [Lur18, Cor. 20.1.3.4] $\implies$ (a)
- [Lur18, Prop. 20.1.3.1] $\implies$ (b)
- (2) $\implies$ (c)

Claim: Under the assumptions of (a) - (c), properties (d) and (3) are equivalent.
(3) $\implies$ (d) The pullback square in question factors as

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & j^*U \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & j^*V \arrow[r] & V. \end{tikzcd}$$

The rightmost square is a pullback by (3), and the outer square is a pullback by assumption, so that the leftmost square is also a pullback. The morphism $U' \to V'$ is then in the image of $j_!$, because $j_!$ preserves pullbacks.
(d) $\implies$ (3) Observe that by (b) the map $j^*U \to j^*V \times_V U$ is corporeal, so that for every corporeal object object W we obtain a commutative diagram:

$$\begin{tikzcd} \mathcal{E}^{\mathrm{corp}}(W, j^*U) \arrow[d] \arrow[rrd] & & \\ \mathcal{E}^{\mathrm{corp}}(W, j^*V \times_V U) \arrow[r] \arrow[d] & \mathcal{E}(W, j^*V \times_V U) \arrow[r] \arrow[d] & \mathcal{E}(W, U) \arrow[d] \\ \mathcal{E}^{\mathrm{corp}}(W, j^*V) \arrow[r] & \mathcal{E}(W, j^*V) \arrow[r] & \mathcal{E}(W, V). \end{tikzcd}$$

The rightmost square is a pullback by the definition of $U \times_V j^*V$, and the leftmost square is a pullback square by (b). But $\mathcal{E}^{\mathrm{corp}}(W, j^*V) \to \mathcal{E}(W, V)$ is an isomorphism by the universal property of j^*V, and thus $\mathcal{E}^{\mathrm{corp}}(W, j^*V \times_V U) \to \mathcal{E}(W, U)$ is an isomorphism, so that $\mathcal{E}^{\mathrm{corp}}(W, j^*U) \to \mathcal{E}^{\mathrm{corp}}(W, j^*V \times_V U)$ is an isomorphism. □

3.2 Fractured ∞-toposes and shapes

The results here may be viewed as a vast generalisation of the techniques underlying [Cis03, Lm. 6.1.5].

Theorem 3.2.1. *Let* $j_! : \mathcal{E}^{\text{corp}} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{E} : j^*$ *be a fractured topos, then for any corporeal object* X *the toposes* $\mathcal{E}^{\text{corp}}_{/X}$ *and* $\mathcal{E}_{/X}$ *have the same shape.* □

Thus, the cohomology of a geometric object such as a scheme with coefficients in a locally constant sheaf is the same when computed in its big or small topos. For us, the technology provides a way of showing that a topos is locally contractible. We have following corollary (with notation as in Theorem 3.2.1):

Theorem 3.2.2. *Let* $C \subseteq \mathcal{E}^{\text{corp}}$ *be a small subcategory, spanned by contractible objects, and generating* $\mathcal{E}^{\text{corp}}$ *under colimits, then* $j_!C \subseteq \mathcal{E}$ *is a small subcategory, spanned by contractible objects, generating* $\mathcal{E}$ *under colimits.* □

In other words, if $\mathcal{E}^{\text{corp}}$ is locally contractible, then so is $\mathcal{E}$.

Theorem 3.2.1 itself is a corollary of either of the following to propostions, either of which is of independent interest.

Proposition 3.2.3. *Any geometric morphism* $f : \mathcal{E} \to \mathcal{F}$ *such that* f^* *is fully faithful is a local shape equivalence.*

Proof. For every Y in $\mathcal{F}$ we have $\mathcal{E}\left(f^*Y, \pi^*_{\mathcal{E}}(\_)\right) = \mathcal{E}\left(f^*Y, f^*\pi^*_{\mathcal{F}}(\_)\right) = \mathcal{F}\left(Y, \pi^*_{\mathcal{E}}(\_)\right)$. □

Proposition 3.2.4. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a geometric morphism such that* f^* *admits a left adjoint* $f_!$ *which preserves finite limits (so that* $g := (f_!, f^*)$ *is a geometric morphism), then* f *is a shape equivalence.*

Proof. The unit and counit of the adjunction between f_* and $g_* = f^*$ furnish equivalences

$$\Pi_\infty g \circ \Pi_\infty f \sim \mathrm{id}_{\Pi_\infty \mathcal{X}}$$

and

$$\Pi_\infty f \circ \Pi_\infty g \sim \mathrm{id}_{\Pi_\infty \mathcal{Y}}$$

by the following proposition. □

Proposition 3.2.5. *Consider two geometric morphisms* $f, g : \mathcal{X} \to \mathcal{Y}$*, then any natural transformation* $f^* \to g^*$ *furnishes an equivalence between* $\Pi_\infty f$ *and* $\Pi_\infty g$.

Proof. The proposition follows from the following claim:

Claim: The diagram

$$\begin{tikzcd} \mathrm{id}_{\mathcal{Y}} \arrow[r] \arrow[d] & g_*g^* \arrow[d] \\ f_*f^* \arrow[r] & f_*g^* \end{tikzcd} \tag{10}$$

commutes.

Pre- and postcomposing all the functors in the diagram with $(\pi_{\mathcal{Y}})^*$ and $(\pi_{\mathcal{Y}})_*$ respectively yields the commutative diagram

$$\begin{tikzcd}
\Pi_\infty \mathcal{Y} & \Pi_\infty \mathcal{X} \arrow[l, "\Pi_\infty g"'] \\
\Pi_\infty \mathcal{X} \arrow[u, "\Pi_\infty f"] & \Pi_\infty \mathcal{X}, \arrow[l, "\sim"'] \arrow[u, "\sim"']
\end{tikzcd}$$

which concludes the proof.

Proof of claim:

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{Y}} \arrow[r] \arrow[d] & g_* g^* \arrow[d] \\
f_* f^* \arrow[r] \arrow[d] & f_* f^* g_* g^* \arrow[d] \\
f_* g^* \arrow[r] \arrow[dr, "="'] & f_* g^* g_* g^* \arrow[d] \\
 & f_* g^*
\end{tikzcd}$$

The two squares commutes by whiskering. The triangle commutes by the triangle identity. The rightmost vertical maps compose to the map $g_* g^* \to f_* g^*$ in (10) via the mates correspondence (see [HHLN21]). $\square$

4 Homotopy theory in locally contractible (∞-)toposes

Fix a ***relative** ∞**-category*** (C, W), i.e. an ∞-category C together with a subcategory containing all equivalences. It is then natural to study the relationship between C and its localisation $W^{-1}C$; in particular, one may ask which limits in $W^{-1}C$ may be obtained via constructions in C.

Definition 4.0.1. Let K be a simplicial set, then a functor $p : K^{\triangleleft} \to C$ is called a ***homotopy limit*** of $p|_K : K \to C$ if the composition of $K^{\triangleleft} \to C \to W^{-1}C$ is a limit of the composition of $K \xrightarrow{p|_K} C \to W^{-1}C$. A functor $K^{\triangleright} \to C$ is a ***homotopy colimit*** if $(K^{\triangleright})^{\mathrm{op}} \to C^{\mathrm{op}}$ is a homotopy limit. ⌟

In particular, a (co)limit in C is a homotopy (co)limit iff it is carried to a (co)limit by $C \to W^{-1}C$. While at the level of generality of the above theorem the theories of homotopy limits and colimits are dual to each other, in this article homotopy limits and colimits have very different flavours. The localisation functors under consideration are all $C \to \mathcal{S}$, where C is some subcategory of a locally contractible ∞-topos $\mathcal{E}$, and localisation functor restriction of $\pi_! : \mathcal{E} \to \mathcal{S}$. If $C = \mathcal{E}$, then all colimits are homotopy colimits. For $C \subsetneq \mathcal{E}$, any colimit in C which is preserved by the inclusion $C \hookrightarrow \mathcal{E}$ is a homotopy colimit. This approach is explored in §4.1 & §**??**, and further refined in §7.1.3 & §7.1.4.

Commuting limits past $(\pi_!)|_C$ will require different techniques which are the subject of §4.2.

4.1 Colimits of n-truncated objects ∞-toposes

Let $\mathcal{E}$ be a fixed ∞-topos, and $n \geq -2$. In this subsection we will show that many colimits of n-truncated objects in $\mathcal{E}$ are again n-truncated.

Proposition 4.1.1. *Consider a pushout square in $\mathcal{E}$*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, hook] & Y' \end{tikzcd}$$

for which the objects X, X', Y are n-truncated, and in which the top horizontal map (and thus the bottom horizontal map) is a monomorphism, then Y' is n-truncated. □

Proposition 4.1.2. *The inclusion $\mathcal{E}_{\leq n} \hookrightarrow \mathcal{E}$ commutes with filtered colimits.* □

Proposition 4.1.3. *The inclusion $\mathcal{E}_{\leq n} \hookrightarrow \mathcal{E}$ commutes with coproducts.* □

Proposition 4.1.4. *The subcategory of n-truncated objects in any finitely complete ∞-category is closed under retracts.* □

4.1.0.1 Discussion of the proofs of Propositions 4.1.1 - 4.1.4 All four propositions may be proved by first checking the statement for simplicial sets equipped with the Kan-Quillen model structure, so that they are true in $\mathcal{S}$. In any presheaf ∞-topos the statements can be checked pointwise. The general statements then follow from the fact that left exact functors preserve monomorphisms and truncation.

We believe that it would be conceptually pleasing to have proofs of these statements which rely on descent (similar to e.g. [ABFJ20, Prop. 2.2.6]) rather than the fact that every ∞-topos is a left exact localisation of a presheaf ∞-category.

Proposition 4.1.5. *Let C be a finitely complete ∞-category, then n-truncated maps in C are closed under retracts.*

Proof. Let

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & x' \arrow[d] \\ y' \arrow[r] & y \arrow[r] & y' \end{tikzcd}$$

be a retract diagram in which $x \to y$ is n-truncated, then we wish to show that $x' \to y'$ is likewise n-truncated. For $n = -2$ the statement is clear, so assume that $n > -2$. Then we obtain a new retract diagram

$$\begin{tikzcd} x' \arrow[r] \arrow[d] & x \arrow[r] \arrow[d] & x' \arrow[d] \\ x' \times_{y'} x' \arrow[r] & x \times_y x \arrow[r] & x' \times_{y'} x' \end{tikzcd}$$

and the general statement follows by induction. □

The following theorem reduces Proposition 4.1.2 to [Lur09, Ex. 7.3.4.7], which however itself relies on the same technique as above.

Lemma 4.1.6. *The n-truncated morphisms are closed under filtered colimits in the arrow ∞-category $\mathcal{E}^{\Delta^1}$.*

Proof. The case $n = -2$ clear, and the general case follow via induction by

$$\operatorname*{colim}_\alpha X_\alpha \to \operatorname*{colim}_\alpha X_\alpha \times_{\operatorname{colim}_\alpha Y_\alpha} \operatorname*{colim}_\alpha X_\alpha = \operatorname*{colim}_\alpha (X_\alpha \times_{Y_\alpha} X_\alpha).$$

□

Proposition 4.1.2 now follows from preceding lemma with $Y_\alpha = 1$ and the observation that filtered colimits of constant diagrams yield constant object.

4.1.1 Colimits indexed by relatively flat functors

Although we will not use any of the results in this paragraph in the remainder of this article we have included them as they fit in with the general theme of constructing homotopy (co)limits in a toposic setting. The results in this paragraph are inspired by results of Cisinski around the notion of relative flatness discussed in [Cis03, §3.3]. Throughout this paragraph we fix $0 \leq n < \infty$ and an n-localic hypercomplete topos $\mathcal{E}$. The following result should be compared to [Cis03, Th. 3.3.9].

Proposition 4.1.7. *The inclusion $\mathcal{E}_{\leq n} \hookrightarrow \mathcal{E}$ commutes with colimits of relatively flat functors.*

Proof. Let K be a small ordinary category, and assume $p : K \to \mathcal{E}_{\leq n}$ preserves pullbacks, then by assumption we obtain a geometric morphism $\widehat{K} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} (\mathcal{E}_{\leq n})_{/\operatorname{colim} p}$. Observe that $(\mathcal{E}_{\leq n})_{/\operatorname{colim} p} = (\mathcal{E}_{/\operatorname{colim} p})_{\leq n}$ by [Lur09, Lm. 5.5.6.14], so that by [Ane21] the inclusion from ordinary toposes into hypercomplete n-localic toposes yields a geometric morphism $\underline{\operatorname{Hom}}(K^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{E}_{/\operatorname{colim} p}$. The left adjoint preserve the final object, but this is equivalent to saying that the colimit of $K \to \mathcal{E}_{/\operatorname{colim} p}$ is the final object, which is the identity morphism $\operatorname{colim} p \to \operatorname{colim} p$. □

Example 4.1.8. The functor $p : \Lambda^2_0 \to \mathcal{E}_{\leq n}$ is preserves pullbacks iff p carries both legs of Λ^2_0 to monomorphisms, so that we recover a special case of Lemma 4.1.1. ⌟

Example 4.1.9. The functor $p : \mathbf{N} \to \mathcal{E}_{\leq n}$ preserves pullbacks iff p carries all morphisms in $\mathbf{N}$ to monomorphisms, so that we recover a special case of Lemma 4.1.2. ⌟

4.2 Basic theory of homotopical calculi on locally contractible (∞-)toposes

Let us begin with the simplest case of a homotopy (co)limit. Any localisation functor $\gamma : C \to W^{-1}C$ is both initial and final (see [Cis19, Prop. 7.1.10]), so that if x_0 is an initial or final object of C, then $\gamma(x_0)$ is an initial or final object of $W^{-1}C$. Thus, if C has a final object, then $W^{-1}C$ admits all finite limits iff it admits all pullbacks, and moreover admits all limits if it furthermore admits all products. Thus we will focus on the construction of homotopy pullbacks. From now on we will assume that C admits all finite limits and colimits, although this condition can be relaxed significantly in most of the following results. This leads us to consider the following definition.

Definition 4.2.1. A morphism is ***sharp*** if pullbacks along it are homotopy pullbacks (see Remark 4.2.5).

We abstract the properties of right proper model categories to recognise sharp morphisms.

Definition 4.2.2. An object x in C is called ***right proper*** if the universal functor

$$W_{/x}^{-1} C_{/x} \to (W^{-1}C)_{/x}$$

is an equivalence. The relative category (C, W) is called ***right proper*** if all objects in C are right proper. ⌟

Notation 4.2.3. If an object x in C is right proper, then we will denote the ∞-category $(W^{-1}C)_{/x}$ by $W^{-1}C_{/x}$. ⌟

Remark 4.2.4. A model category is right proper in the usual sense iff its underlying relative category is right proper. This may be seen by combining [Rez98, Prop. 2.7] with [Cis19, Cor. 7.6.13][1]. ⌟

Remark 4.2.5. Let $f : x' \to x$ be a morphism in C, then recall that it is *sharp* in the sense of Rezk (see [Rez98, §2]), if for every morphism $b \to x$ and every weak equivalence $a \xrightarrow{\sim} b$ there exists a diagram

$$\begin{tikzcd} a' \arrow[r] \arrow[d] & b' \arrow[r] \arrow[d] & x' \arrow[d] \\ a \arrow[r] & b \arrow[r] & x \end{tikzcd} \tag{11}$$

in which all squares are pullbacks and such that $a' \to b'$ is a weak equivalence. If (C, W) is right proper, then a morphism in C is sharp in our sense iff it is sharp in the sense of Rezk.

To see this, first assume that $x' \to x$ is sharp in our sense, then it is sharp in the sense of Rezk, because for every diagram of the form (11) the rightmost and outer squares are homotopy pullbacks, so that the leftmost square is a homotopy pullback. If $a \to b$ is a weak equivalence, then $a' \to b'$ is a weak equivalence.

Conversely, if $x' \to x$ is sharp in our sense, the functor $C_{/x'} \leftarrow C_{/x} : f^*$ preserves weak equivalences, so that [Cis19, Prop. 7.1.14] yields, canonically a commutative diagram

$$\begin{tikzcd} C_{/x'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] \arrow[d] & C_{/x} \arrow[l, shift left, "f^*"] \arrow[d] \\ W^{-1}C_{/x'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & W^{-1}C_{/x} \arrow[l, shift left, "f^*"] \end{tikzcd}$$

The pullback of any morphism $y \to x$ along f in C thus yields the pullback of $y \to x$ along f in $W^{-1}C$. ⌟

Proposition 4.2.6. *Let $\mathcal{E}$ be a locally ∞-connected ∞-topos, then $\mathcal{E}$ together with its class W of shape equivalences is a right proper relative ∞-category.*

Proof. By Proposition 2.1.2 the comparison functor $W_{/E}^{-1}(\mathcal{E})_{/\pi_! E} \to W^{-1}(\mathcal{E}_{/E})$ is given by $\mathcal{S}_{/\pi_! E \to \pi_! 1} \to \mathcal{S}_{/\pi_! E}$, which is an equivalence by [Lur09, Prop. 4.1.1.8] and the fact that $\Delta^{\{0\}} \to \Delta^1$ is initial. □

[1] Rezk's proof of [Rez98, Prop. 2.7] can be interpreted verbatim in model ∞-categories, so that the remark is in fact true for model ∞-categories.

Definition 4.2.7. A ***fibration structure*** on (C, W) consists of a subcategory $\mathrm{Fib} \subseteq C$, such that W and Fib satisfy the following conditions:

(a) Fib contains all equivalences in C.

(b) W satisfies the 2-out-of-3 property.

The morphisms in W, Fib, and $\mathrm{Fib} \cap W$ are called *weak equivalences*, *fibrations*, and *trivial fibrations* respectively. An object x for which some (and therefore any) morphism to a final object of C is a fibration is called *fibrant*. Furthermore:

(c) In any diagram

$$\begin{tikzcd} & x' \arrow[d, "f"] \\ y \arrow[r] & x \end{tikzcd}$$

such that f is either a fibration, or trivial fibration, the pullback is again a fibration of trivial fibration, respectively.

(d) Any morphism $x \to y$ admits a factorisation $x \to x' \to y$ such that $x' \to x'$ is a weak equivalence, and $x' \to y$ is a fibration.

An ∞-category equipped with a fibration structure is called a ***fibration ∞-category***.

Dually, a subcategory $\mathrm{Cof} \subseteq C$ is a ***cofibration structure*** on C, if $\mathrm{Cof}^{\mathrm{op}}$ is a fibration structure on $(C^{\mathrm{op}}, W^{\mathrm{op}})$. An ∞-category equipped with a cofibration structure is called a ***cofibration ∞-category***.

⌟

Remark 4.2.8. Our notion of *fibration structure* is slightly stronger than the notion of *∞-category with weak equivalences and fibrations* considered in [Cis19, Def. 7.4.12]. ⌟

Example 4.2.9. The classes of weak equivalences and fibrations of any ∞-model category (see [MG14]) form a fibration structure, which moreover satisfies the condition of Proposition 4.2.11 if it admits all limits. ⌟

From now on we assume that (C, W) is equipped with a fibration structure Fib.

Proposition 4.2.10. *Let*

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd} \tag{12}$$

be pullback square in C, where y and x are proper, and $x' \to x$ is a fibration, then the square is a homotopy pullback.

The following proof is similar to the last part of Remark **??**.

Proof. Equip the relative ∞-category $(C_{/y}, W_{/y})$ with the cofibration structure in which all morphisms are cofibrations, and $(C_{/y}, W_{/y})$ with the fibration structured induced by Fib, then, by [Cis19, Th. 7.5.30] the derived functors of $f_!$ and f^* exist and are canonically adjoint to each other. As $y \to x$ is fibrant, $y' \to x'$ is isomorphic to the derived pullback of $y \to x$ in $W^{-1}C_{/x}$. To show that (12) is the derived pullback square, however, we also need to show that $y' \to x'$ is the counit of the derived adjunction (evaluated at x'). Denote by $\gamma : C_{/x} \to W^{-1}C_{/x}$ the localisation functor. By the proof [Cis19, Th. 7.5.30] the derived counit is given by composing

$$\mathbf{L}f_! \circ \mathbf{R}f^* \xrightarrow{\sim} \mathbf{R}(\mathbf{L}f_! \circ \gamma \circ f^*) \xrightarrow{\sim} \mathbf{R}(f_! \circ f^*) \to \mathrm{id}_{W^{-1}C_{/x'}}$$

where the first morphism is an isomorphism, again by the proof of [Cis19, Th. 7.5.30]), and the second morphism is also an isomorphism, because $f_!$ preserves weak equivalences between all objects. The claim then follows because $f_! \circ f^*$ preserves weak equivalences between fibrant objects, so that by the construction of derived functors (see [Cis19, 7.5.25]) upon plugging in $x' \to x$ the last morphism computes to

$$\begin{tikzcd} y' \arrow[rr] \arrow[dr] & & x' \arrow[dl] \\ & x. & \end{tikzcd}$$

□

We now move onto arbitrary limits.

Proposition 4.2.11 ([Cis19, Prop. 7.7.4])**.** *If an arbitrary product of fibrant objects in C is again fibrant, and an arbitrary product of trivial fibrations is again a trivial fibration, then arbitrary products of fibrant objects are homotopy products.* □

Thus, if (C, W, Fib) satisfies the condition of Proposition 4.2.11, then $W^{-1}C$ has arbitrary limits. This fact will not be that useful in this article, because $W^{-1}C \simeq \mathcal{S}$ in all cases considered here. In fact our main application of our technology in §7.2, we will only ever need to calculate pullbacks. However, any diagram in C is levelwise weakly equivalent to one, whose limit is a homotopy limit (see [Cis19,]).

Remark 4.2.12. Model categories and model ∞-categories are frequently viewed as providing competing foundations for homotopy theory (see [MO78400]). In reality, the axioms for model categories can be interpreted without difficulty in the setting of ∞-categories, not just ordinary categories, and model structures constitute tools for studying localisation. Any ∞-category may be obtained as the localisation of an ordinary relative category (see [Cis19, Prop. 7.3.15], [BK12]), and any presentable ∞-category may be obtained as the localisation of a combinatorial simplicial model category (see [Lur09, Prop. A.3.7.6] & [Lur17, Thm. 1.3.4.20] & [Cis19, Thm. 7.5.18]). Before the work of Joyal, Simpson, Toën, Rezk, Lurie and many others it was simply not practical to present a given ∞-category in any other way than as an ordinary relative category (or a simplicially enriched category). Thus, nowadays, one has a *choice* of whether one wishes to work in a given ∞-category C, or whether one wishes to view C as the localisation of some other (∞-)category D. The optimal choice of D does not necessarily have to be an ordinary category, as seen in Mazel-Gee's generalisation of the Goerss-Hopkins obstruction theorem (see [MG16]), and in our applications to differentiable stacks in this thesis. ⌟

4.3 Constructing homotopical calculi in locally contractible toposes

We have now seen the utility of fibration structures on ∞-toposes. This subsection concerns their construction. Our strategy is the following: Let be A a test category, then presheaves on A admit model structure. Consider a functor $A \to \mathcal{E}$. Transferring fibration structure only involves checkin factorisation. A slightly more complicated version of this idea pursued in Theorem 7.2.16. Due to the prominent role played by monomorphisms in tests categories, it will turn out that it generally quite easy to transfer the full model structure, which is what we do here. In §4.3.1 first discuss the basics of test categories; model structure on sheaves of homotopy types; model toposes. In §4.3.2 prove our transfer theorems. We consider some simple recognition tools in the end.

4.3.1 Test categories

The main ideas discussed in this subsection are essentially all due to Grothendieck, and were first outlined in [Gro83]. A systematic account of Grothendieck's theory is given by Maltsiniotis in [Mal05]. The theory of test categories, and in particular its model categorical aspects, are further developed in [Cis06].

We fix a small ordinary category A. In Example **??** we saw how $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ models the ∞-category $\mathcal{S}_{/A_{\simeq}}$ in the sense that taking colimits produces a localisation $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{/A_{\simeq}}$. In the special case $A = \Delta$ something rather remarkable happens. The restriction of the functor $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{/\Delta_{\simeq}} \xrightarrow{\sim} \mathcal{S}$ to $\widehat{\Delta} \to \mathcal{S}$ is still a localisation. As the construction of the model category of simplicial sets is quite involved, one might expect this phenomenon to be particular to Δ, but it turns out to be surprisingly common. The starting point for understanding the above phenomenon is the following fact: Recall that the classifying space of an ∞-category is nothing but the homotopy type obtained by inverting all its arrows, and furthermore, that the classifying space construction is left adjoint to the inclusion of $\mathcal{S}$ into $\mathcal{Q}$, the ∞-category of ∞-categories. Then, paralleling the situation for $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$, the restriction of the classifying space functor to the ∞-category $\mathcal{C}$ of ordinary categories exhibits $\mathcal{S}$ as a localisation of $\mathcal{C}$, and since $\mathcal{C}$ is a localisation of **Cat**, the ∞-category $\mathcal{S}$ is likewise a localisation of **Cat**.

$$\mathbf{Cat} \longrightarrow \mathcal{C} \longrightarrow \mathcal{Q} \underset{\longleftarrow}{\overset{(\_)_{\simeq}}{\underset{\perp}{\longrightarrow}}} \mathcal{S}$$

Remark 4.3.1. This fact has been known in essence since [Ill72, Cor. 3.3.1] (specifically, that the category of elements of a simplicial set encodes the same homotopy type as the simplicial set itself is shown in [Ill72, Th. 3.3.ii]. Illusie attributes the ideas presented in [Ill72, §3.3] to Quillen; see also [Qui73]). ⌟

The relative category **Cat** can be shown to be right proper (see Definition 4.2.2), by exhibiting a right proper model structure on **Cat** by right transferring the Kan-Quillen model structure (which is right proper) along the functor $\mathrm{Ex}^2 \circ N : \mathbf{Cat} \to \widehat{\Delta}$ (see [Tho80]). Thus, the category $\mathbf{Cat}_{/A}$ is a model for $\mathcal{S}_{/A_{\simeq}}$; a model which turns out to be particularly convenient for determining conditions on A such that $\mathrm{colim} : \widehat{A} \to \mathcal{S}_{/A_{\simeq}}$ is a localisation. In a first instance, we will focus on the special case when $A_{\simeq} = 1$. Then, $\mathrm{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ factors as $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \xrightarrow{A_{/\_}} \mathcal{Q} \xrightarrow{(\_)_{\simeq}} \mathcal{S}$, which restricts to $\widehat{A} \xrightarrow{A_{/\_}} \mathbf{Cat} \xrightarrow{(\_)_{\simeq}} \mathcal{S}$. Thus, $A_{/\_}$ models the left adjoint of the adjunction $\mathrm{colim} : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}$. The functor $A_{/\_}$ also admits a right adjoint given by $N_A : C \mapsto (a \mapsto \mathrm{Hom}(A_{/a}, C))$. The category A is a ***weak test category*** if N_A sends homotopical equivalences to shape equivalence, and if the resulting adjunction $L\widehat{A} \rightleftarrows \mathcal{S}$ is an adjoint equivalence. We can now state the main definition of this subsection:

Definition 4.3.2. The category A is a ***local test category*** if $A_{/a}$ is a weak test category for all $a \in A$. ⌟

Definition 4.3.3. A small ordinary category A is a ***test category*** if it is a local test category, and if moreover $A_{\simeq} = 1$. ⌟

Theorem 4.3.4 ([Cis06, Cor. 4.4.20]). *If A is a local test category, then the composition of the functors $A_{/\_} : \widehat{A} \to \mathbf{Cat}_{/A} \to \mathcal{S}_{A_{\simeq}}$ is a localisation of $\widehat{A}$ along the shape equivalences.* □

One of the key properties of weak test categories is that they are easy to recognise.

Definition 4.3.5. Let M be an ordinary category, admitting a final object 1, then an object I in M with two morphisms $1 \rightrightarrows I$ is called an ***interval*** in M. If M admits an initial object 0, and the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & I \end{tikzcd}$$

is a pullback, then I is ***separating interval***. ⌟

Example 4.3.6. Let $\mathcal{E}$ be an ordinary topos, then the subobject classifier Ω in $\mathcal{E}$ canonically has the structure of a separating interval. The first morphism $1 \to \Omega$ is given by the universal monomorphism, and the second morphism $1 \to \Omega$ classifies the subobject $0 \to 1$. ⌟

Definition 4.3.7. Let A be a small ordinary category, then a presheaf X on A is called ***locally aspherical*** if $(a \times X)_{\simeq} = 1$ for all $a \in A$. ⌟

Theorem 4.3.8 ([Mal05, Th. 1.5.6] & [Cis06, Thms. 1.4.3 & 4.1.19 & 4.2.15]). *The following are equivalent:*

(I) *A is a local test category.*

(II) *The subobject classifier of $\widehat{A}$ is locally aspherical.*

(III) *The category $\widehat{A}$ admits a locally aspherical separating interval.*

(IV) *Any trivial fibration in $\widehat{A}$ is a shape equivalence.*

(V) *The category $\widehat{A}$ admits a (cofibrantly generated) model structure in which the weak equivalences are the shape equivalences, and the cofibrations are the monomorphisms.*

□

Proposition 4.3.9. *For a local test category A, the following are equivalent:*

(I) *A is is sifted.*

(II) *$A_{/\simeq} = 1$ and $(A_{/a \times a'})_{\simeq} = 1$ for all $a, a' \in A$.*

(III) $A_{/\simeq} = 1$ *and model structure from Theorem 4.3.8 Cartesian closed.*

□

Proposition 4.3.10. *Let A be a small category admitting finite products and a representable separating interval on $\widehat{A}$, then A is a strict test category.* □

In [Mal05, §1.8] Cisinski and Maltsiniotis develop sophisticated tools for recognising strict test categories, and produces some surprising examples thereof such as the monoid of increasing functions $\mathbf{N} \to \mathbf{N}$ (see [Mal05, Ex. 1.8.15]).

We begin by extending the canonical model structure from $\widehat{A}$ to $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$. For this we need to understand how to construct cofibrantly generated model structure on presentable ∞-categories.

Proposition 4.3.11 ([MG14, Th. 3.11])**.** *Let M be a presentable ∞-category, let $W \subseteq M$ be a subcategory, which is closed under retracts, and satisfies the 2-out-of-3 property. Suppose that I and J are* sets *of homotopy classes of maps. such that*

(a) ${}^{\boxslash}(J^{\boxslash}) \subseteq {}^{\boxslash}(I^{\boxslash}) \cap W$

(b) $I^{\boxslash} \subseteq J^{\boxslash} \cap W$

(c) *and either*

  (c$_1$) ${}^{\boxslash}(J^{\boxslash}) = {}^{\boxslash}(I^{\boxslash}) \cap W$*, or*

  (c$_1$) $I^{\boxslash} = J^{\boxslash} \cap W$*,*

then the I and J define a cofibrantly generated model structure (see [MG14, Def. 3.8]*) on M whose weak equivalences are W.* □

We can now extend the canonical model structure. The following proposition generalises [MG14, Th. 4.4].

Proposition 4.3.12. *Let A be a local test category, then there exists a (necessarily unique) cofibrantly generated model structure on $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ whose weak equivalences are the shape equivalences, and whose trivial fibrations are characterised by having the right lifting property against monomorphisms in $\widehat{A}$.*

Furthermore, if I and J are generating cofibrations and trivial cofibrations, respectively, of the canonical model structure on $\widehat{A}$, then these generate the model structure on $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$.

Proof. Let I and J be generating cofibrations and trivial cofibrations, respectively, of the canonical model structure on $\widehat{A}$. By Lemmas 4.1.1 - 4.1.4, ${}^{\boxslash}(I^{\boxslash})$ (constructed in $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$) coincide with the monomorphisms, so that applying Proposition 4.3.11 to W together with I, J will prove both statements in the proposition. We will now verify (a), and (b), (c$_2$).

Proof of (a): By Lemmas 4.1.1 - 4.1.4 all colimits involved in constructing the morphisms in ${}^{\boxslash}(J^{\boxslash})$ are homotopy colimits. As all morphisms in J are weak equivalences, the morphisms in ${}^{\boxslash}(J^{\boxslash})$ must be weak equivalences.

Proof of (b): The inclusion $I^{\boxslash} \subseteq J^{\boxslash}$ is clear as $J \subseteq {}^{\boxslash}(I^{\boxslash})$, so we need to show $I^{\boxslash} \subseteq W$. So, let $X \to Y$ be a morphism in $I^{\boxslash}$.

First, we show that it is enough to prove the statement in the case when Y is representable. For all objects a in A, and all maps $a \to Y$ the morphisms $a \times_Y X \to X$ are in $I^{\boxslash}$. If these morphisms are in W, then $X \to Y$ is in W by faithful descent, as the morphism can be written as a colimit indexed by $A_{/Y} \to A$.

So, assume that Y is representable. As a morphism in $A_{/Y}$ is a monomorphism iff it is a monomorphism in A, we may furthermore assume that A has a final object, and that Y is such a final object.

As the shape of the presheaf represented by the final object in A is contractible, it is enough to show that the shape of X is contractible. Now, the shape of X is given by $(A_{/X})_{\simeq} \simeq \operatorname{Ex}^{\infty} A_{/X}$, so that any map $S^k \to \pi_! X$ $(k \geq 0)$ may be represented by a map $\operatorname{Sd}^n \partial\Delta^k \to A_{/X}$ for some $n \geq 0$. If $n \geq 1$, then Sd^n is a finite poset, and therefore a finite direct category. We will show that for any finite direct category I and any functor $I \to A_{/X}$ we obtain a factorisation

$$\begin{tikzcd} I_{\simeq} \arrow[r] \arrow[d] & (A_{/X})_{\simeq} \\ * \arrow[ur, dashed] & \end{tikzcd} \tag{13}$$

Consider the diagram $f : I \to A$, and take a Reedy cofibrant replacement $\tilde{f} \xrightarrow{\sim} f$ in $\widehat{A}$ (see [Cis19, Prop. 7.4.19]), then by an inductive application of [Cis19, Cor. 7.4.4] and Lemmas 4.1.1 & 4.1.2 we see that the colimit of $\tilde{f}$ is 0-truncated. The map $I_{\simeq} \to (A_{/X})_{\simeq}$ corresponds to the map $\pi_! \operatorname{colim} \tilde{f} \to \pi_! X$. Consider a factorisation $\operatorname{colim} \tilde{f} \to c \to 1$ in $\widehat{A}$, where $\operatorname{colim} \tilde{f} \to c$ is a monomorphism, and $c \to 1$ is a trivial fibration, and thus a weak equivalence. By our assumption on X, we obtain a lift

$$\begin{tikzcd} \operatorname{colim} \tilde{f} \arrow[r] \arrow[d, hook] & X \\ c \arrow[ur, dashed] & \end{tikzcd}$$

Taking the shape of this diagram yields the desired lift in (13).

Proof of (c_2): The proof of this fact for $A = \Delta$ is given in [MG14, Prop. 7.9], and may be interpreted verbatim in our setting. □

4.3.1.1 Test toposes The theory of test toposes is developed in [Cis03].

We are now able to state the following generalisation of Theorem 4.3.8.

Theorem 4.3.13 ([Cis03, Th. 4.2.8]). *Let $\mathcal{E}$ be a locally contractible ordinary topos, then the following are equivalent:*

(I) *For any object X in $\mathcal{E}$ the projection map $X \times \Omega_{\mathcal{E}} \to X$ is a shape equivalence.*

(II) *Any trivial fibration is a shape equivalence;*

(III) *There exists a local test category C and a geometric embedding $\mathcal{E} \hookrightarrow \widehat{C}$ which is also a shape equivalence.*

(IV) *There exists a (necessarily unique as well as cofibrantly generated) model structure on $\mathcal{E}$ in which the weak equivalences are the shape equivalences, and in which the cofibrations are the monomorphisms.*

□

Definition 4.3.14. An ordinary topos satisfying the equivalent conditions of Theorem 4.3.13 is called a ***local test topos***. A local test topos with trivial shape is a ***test topos***. A test topos, whose shape functor commutes with finite products, is a ***strict test topos***. On any topos, the model structure given by Theorem 4.3.13 is referred to as the ***canonical model structure***. ⌟

Proposition 4.3.15 ([Cis03, Cor. 5.3.20 & Cor. 4.2.12])**.** *Any test topos is proper.* □

4.3.2 Transferring model structures to locally contractible (∞-)toposes

We now finally come to the theorem which allows us to produce fibrations on toposes.

Proposition 4.3.16. *Let M be a cofibrantly generated model ∞-category with generating cofibrations I and generating trivial cofibrations J, let N be a presentable ∞-category, and consider an adjunction $f : M \underset{\perp}{\rightleftarrows} N : u$. If the functor u takes relative fJ-cell complexes to weak equivalences, then*

(1) *the ∞-category N admits a cofibrantly generated model structure whose weak equivalences are those morphisms carried to weak equivalences by u, and with generating cofibrations and trivial cofibrations given by fI and fJ respectively, and*

(2) *the adjunction $f : M \underset{\perp}{\rightleftarrows} N : u$ is a Quillen adjunction.*

Sketch of proof. By [DAGX, Prop. 1.4.7], any morphism in N factors into a relative fI-complex (fJ-complex) followed by a morphism with the right lifting property w.r.t. fI (fJ). The rest of the proof now proceeds as in the proof or [Hir03, Th. 11.3.2]. □

Both should be compared to Theorem 2.1.8.

Proposition 4.3.17. *Let*

(i) *$\mathcal{E}$ be an ∞-topos, generated under small colimits by a small subcategory C consisting of contractible objects,*

(ii) *A, a small ∞-category, and*

(iii) *$u : A \to C$, a functor.*

Assume that

(a) *$u : A \to C$ is initial, and that*

(b) *$\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ admits a cofibrantly genreated model structure in which the weak equivalences are the shape equivalences,*

then for any sets I and J of, respectively, generating cofibrations and generating trivial cofibrations, there exists a cofibrantly generated model structure on $\mathcal{E}$ such that

(1) *the weak equivalences are precisely the shape equivalences,*

(2) *the sets $u_!I$ and $u_!J$ are generating sets for the cofibrations and trivial cofibrations, respectively, and*

(3) *the adjunction* $u_! : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{E} : u^*$ *is a Quillen equivalence.*

If moreover

(c) *the inclusions* $u\ell \hookrightarrow ud$ *admit retracts for all morphisms* $\ell \hookrightarrow d$ *in* J,

then

(4) *all objects in the resulting model structures on* $\mathcal{E}_{\leq 0}$ *and* $\mathcal{E}$ *are fibrant.*

Proof. By Theorem 2.1.8 the weak equivalences in $\mathcal{E}$ created by u^* are precisely the shape equivalence. The conditions of Proposition 4.3.16 are then trivially satisfied, because $\pi_! : \mathcal{E} \to \mathcal{S}$ commutes with all colimits, so that we obtain a Quillen adjunction. Again, by Theorem **??** u^* descends to an equivalence $\mathcal{S}_{/BA} \xrightarrow{\sim} \mathcal{S}_{/\pi_! 1_{\mathcal{E}}}$. □

Theorem 4.3.18. *Let*

(i) $\mathcal{E}$ *be an* ∞*-topos, generated under small colimits by a small subcategory* C *of* $\mathcal{E}_{\leq 0}$ *consisting of contractible objects,*

(ii) A*, a local test category, and*

(iii) $u : A \to C$*, a functor.*

Assume that

(a) C *is a local test category,*

(b) $u : A \to C$ *is initial,*

(c) $u_! : \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \to \mathcal{E}$ *preserves* 0*-truncated objects, and*

(d) $u_! : \widehat{A} \to \widehat{C}$ *preserves monomorphisms,*

then for any sets I *and* J *of, respectively, generating cofibrations and generating trivial cofibrations for the canonical model structure on* $\widehat{A}$*, there exists a cofibrantly generated model structure on* $\mathcal{E}_{\leq 0}$ *such that*

(1) *the weak equivalences are precisely the shape equivalences,*

(2) *the sets* $u_! I$ *and* $u_! J$ *are generating sets for the cofibrations and trivial cofibrations, respectively, and*

(3) *the adjunction* $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{E}_{\leq 0} : u^*$ *is a Quillen equivalence.*

If moreover

(e) *the inclusions* $u\ell \hookrightarrow ud$ *admit retracts for all morphisms* $\ell \hookrightarrow d$ *in* J,

then

(4) *all objects in the resulting model structures on* $\mathcal{E}_{\leq 0}$ *and* $\mathcal{E}$ *are fibrant.*

Proof. In the adjunction $u_! : \widehat{A} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{E}_{\leq 0} : u^*$ are again created by u^* by Theorem 2.1.8. The conditions of Proposition 4.3.16 below are satisfied by assumption (d) and Propositions 4.1.1 - 4.1.3. To show that the Quillen equivalence is a Quillen adjunction, we will show that $\widehat{A} \leftarrow \mathcal{E}_{\leq 0} : u^*$ induces an equivalence on localisations. The functor $\widehat{A} \leftarrow \widehat{C} : u^*$ induces an equivalence of localisation by assumption (b) and Theorem 4.3.4. We are left with showing that a_* induces an equivalence on localisations. As both a_* and a^* preserve weak equivalence we obtain an induced adjunction on localisations (see [Cis19, 7.1.14]), so it is enough to show that the unit and counit of $a_* \vdash a^*$ are both weak equivalences. The counit is an isomorphism. The unit $X \to a_*a^*X$ is a weak equivalence if and only if $a_*X \to_* a_*a^*X$, but this map is likewise an isomorphism. $\square$

Proposition 4.3.19. *Let $\mathcal{E}$ be an ordinary topos, then a cocontinuous functor $\widehat{\Delta} \to \mathcal{E}$ preserves monomorphisms iff the inclusion $\Delta^{\{0\}} \sqcup \Delta^{\{1\}} \hookrightarrow \Delta^1$ is sent to a monomorphism.*

Sketch of proof. The proof of [Cis06, Lm. 2.1.10] remains true mutatis mutandis for toposes. $\square$

Proposition 4.3.20. *Let $\mathcal{E}$ be an ordinary topos, then a cocontinuous functor $\widehat{\square} \to \mathcal{E}$ preserves monomorphisms iff the inclusions $(\delta_i^0, \delta_i^1) : \square^{n-1} \sqcup \square^{n-1} \hookrightarrow \square^n$ are sent to monomorphisms for all $n \geq i \geq 1$.*

Sketch of proof. The proof of [Cis06, Lm. 8.4.21] remains true mutatis mutandis for toposes. $\square$

The asymmetry between Propositions 4.3.19 & 4.3.20 disappears in the following situation:

Corollary 4.3.21. *Let $\mathcal{E}$ be an ordinary topos, then a cocontinuous* monoidal *functor $\widehat{\square} \to \mathcal{E}$ preserves monomorphisms iff the inclusion $(\delta^0, \delta^1) : \square^0 \sqcup \square^0 \hookrightarrow \square^1$ is sent to a monomorphism.* $\square$

Part II

Differentiable sheaves

5 Basic definitions and properties of differentiable sheaves

5.1 Differentiable sheaves

Throughout this subsection we fix some $0 \leq r \leq \infty$.

Notation 5.1.1.

1. $\mathbf{Mfd}^r$ denotes the category of r-differentiable manifolds and r-differentiable maps.
2. $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.

On each of these small categories we denote by τ the Grothendieck topology in which a sieve on a space M is a covering sieve iff it contains a subset $\{U_\alpha \to M\}$ consisting of jointly surjective open embeddings.

1. $\mathbf{Mfd}^r_{\text{ét}}$ denotes the category of r-differentiable manifolds and r-differentiable open embeddings.
2. $\mathbf{Cart}^r_{\text{ét}}$ denotes the full subcategory of $\mathbf{Mfd}^r_{\text{ét}}$ spanned by the spaces for $\mathbf{R}^n$ $(0 \leq n < \infty)$.

On each of these small subcategories we denote restriction of τ by $\tau_{\text{ét}}$. $\lrcorner$

Definition 5.1.2. An $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r$ is an ***r-differentiable sheaf***, and the category thereof is denoted by $\mathbf{Diff}^r$; ∞- and 0-differentiable stacks are called ***differentiable*** and ***continuous sheaves***, respectively. An object of $\mathbf{Diff}^r_{\leq 0}$ is then an ***r-differentiable space***, and, again for $r = \infty$ and $r = 0$ respectively, a ***differentiable space*** and ***continuous space***.

Similarly, an $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r_{\text{ét}}$ is an ***étale r-differentiable stack***, and the category thereof is denoted by $\mathbf{Diff}^r_{\text{ét}}$; étale ∞- and 0-differentiable stacks are called ***étale differentiable*** and ***étale continuous stacks***, respectively. An object of $\mathbf{Diff}^r_{\text{ét},\leq 0} := (\mathbf{Diff}^r_{\text{ét}})_{\leq 0}$ is then an ***étale r-differentiable space***, and, again for $r = \infty$ and $r = 0$ respectively, a ***étale differentiable space*** and ***étale continuous space***. $\lrcorner$

Observe that the restricted Yoneda embedding exhibits $\mathbf{Mfd}^r$ as a full subcategory of $\mathbf{Diff}^r$, containing $\mathbf{Cart}^r$, and also $\mathbf{Mfd}^r_{\text{ét}}$, as full a subcategory of $\mathbf{Diff}^r_{\text{ét}}$ containing $\mathbf{Cart}^r_{\text{ét}}$. Thus, $\mathbf{Diff}^r$ may also be characterised as the ∞-category of sheaves on $\mathbf{Mfd}^r$, and $\mathbf{Diff}^r_{\text{ét}}$ may also be characterised as the ∞-category of sheaves on $\mathbf{Mfd}^r_{\text{ét}}$.

Lemma 5.1.3. *The triple* $(\mathbf{Mfd}^r, \mathbf{Mfd}^r_{\text{ét}}, \tau)$ *is a geometric site.*

Proof. Axioms (a) - (c) are clear. To prove (d), consider the diagram

$$\begin{tikzcd} V' \arrow[r, hook] & U' \arrow[r, two heads] & V' \\ V \arrow[u] \arrow[r, hook] & U \arrow[u, hook] \arrow[r, two heads] & V, \arrow[u] \end{tikzcd} \tag{14}$$

where $U \hookrightarrow U'$ is an open subset inclusion. Axiom (d) follows from (b), after proving the following claim:

Claim: The leftmost square in (14) is a pullback.

First, as monomorphisms have the left cancelling property, the map $V \to V'$ is a monomorphism. Let $y' \in V' \cap U$, then y' coincides with its image under $U \to V$, which shows that the leftmost square induces a pullback on underlying sets. Next, consider a commutative square

$$\begin{tikzcd} V' \arrow[r, hook] & U' \\ W \arrow[u] \arrow[r] & U, \arrow[u, hook] \end{tikzcd}$$

then the canonical map of sets $W \to V$ is smooth, as it may be written as the composition of $W \to U \to V$. $\square$

Theorem 5.1.4. *The ∞-category $\mathbf{Diff}^r$ is a fractured topos, whose ∞-topos of corporeal objects is given by $\mathbf{Diff}^r_{\text{ét}}$.* $\square$

The following result is first sketched in [Dug98, Ex. 4.1.2]; a complete proof may be found in [ADH21, Prop. A.5.3].

Proposition 5.1.5. *The ∞-topos $\mathbf{Diff}^r$ has enough points.* $\square$

Proof. First, observe that $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^d}$ $(d \in \mathbf{N})$ is equivalent to the category sheaves on the topological space $\mathbf{R}^d$, and thus has enough points because it has homotopy dimension $\leq d$.

Next, observe that $\square$

Corollary 5.1.6. *The ∞-topos* $\mathbf{Diff}^r$ *is hypercomplete.* □

5.2 Diffeological spaces

5.2.1 Concrete objects

Here we collect the necessary background on concrete objects to discuss diffeological spaces. Throughout this subsection $\mathcal{E}$ denotes an ordinary topos.

Definition 5.2.1. The topos $\mathcal{E}$ is ***local*** if the right adjoint component π_* of the unique geometric morphism $\pi : \mathcal{E} \to \mathbf{Set}$ admits a further right adjoint $\mathcal{E} \leftarrow \mathbf{Set} : \pi^!$, which is fully faithful[2]. ⌟

From now on we assume that $\mathcal{E}$ is local.

Definition 5.2.2. An object X in $\mathcal{E}$ is ***concrete*** if the canonical morphism $X \to \pi^!\pi_*X$ is a monomorphism. The subcategory of $\mathcal{E}$ spanned by concrete objects is denoted by $\mathcal{E}_{\mathrm{concr}}$. ⌟

Remark 5.2.3. More general definitions than the above are possible, e.g., for $\infty \geq m \geq n \geq 0$ one could assume that $\mathcal{E}$ is only m-truncated and $X \to \pi^!\pi_*X$, n-truncated, but we are not aware of any such examples. ⌟

Example 5.2.4. For any small category A which admits a final object, the topos $\widehat{A}$ is local. To see this, observe that $\pi_* : \widehat{A} \to \mathbf{Set}$ is simply given by evaluating at the final object, and thus commutes with colimits; therefore, it admits a right adjoint by the adjoint functor theorem, which is given by sending any set X to $a \mapsto \mathbf{Set}(A(1_A, a), X)$. Concrete objects in $\widehat{A}$ are then referred to as **concrete presheaves on** A. A concrete presheaf on A is given by a set X together with a subset of $\mathbf{Set}(A(1_A, a), X)$ for every object a in A; these subsets are then required to be closed under precomposing by morphisms in A.

This observation applies to the topos of simplicial sets $\widehat{\Delta}$, with the functor $\pi^!$ being exhibited by $\mathrm{cosk}_0 : \mathbf{Set} \hookrightarrow \widehat{\Delta}$. The concrete objects are then those simplicial sets X such that for any $(n+1)$-tuple $(x_0, \ldots, x_n) \in X_0^{(n+1)}$ there exists at most one n-simplex with precisely these vertices. ⌟

Proposition 5.2.5. *The inclusion* $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ *admits a left adjoint.*

Proof. Recall that in any topos the epimorphisms and the monomorphisms form an orthogonal factorisation system. Let X be an object in $\mathcal{E}$, then $X \to \pi^!\pi_*X$ may be factored uniquely as $X \twoheadrightarrow X' \hookrightarrow \pi^!\pi_*X$. Consider any map $X \to Y$, where Y is a diffeological space, then the lifting problem

$$\begin{tikzcd} X \arrow[rr] \arrow[d, two heads] & & Y \arrow[d, hook] \\ X' \arrow[r, hook] & \pi^!\pi_*X \arrow[r] & \pi^!\pi_*Y \end{tikzcd}$$

admits a unique solution, exhibiting the universality of $X \to X'$. □

Definition 5.2.6. The left adjoint of the inclusion $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ (which exists by the preceding proposition) is called the ***concretisation***, and is denoted by $\mathcal{E}_{\mathrm{concr}} \twoheadleftarrow \mathcal{E} : (\_)^{\dagger}$. ⌟

[2]The right adjoint $\pi^!$ is, in fact, automatically fully faithful, as can be seen from [Joh02, Th. 3.6.1] and the observation that any geometric morphism is indexed over **Set**.

Proposition 5.2.7. *The category $\mathcal{E}_{\mathrm{concr}}$ is presentable.*

Proof. The pair $(\pi^!, \pi_*)$ is a geometric embedding, so that **Set** is a κ-accessible subcategory of $\mathcal{E}$ for some regular cardinal κ, i.e. $\pi^! : \mathbf{Set} \hookrightarrow \mathcal{E}$ commutes with κ-filtered colimits. We claim that $\mathcal{E}_{\mathrm{concr}} \hookrightarrow \mathcal{E}$ likewise commutes with κ-filtered colimits. Let A be a κ-filtered category, and consider a functor $X : A \to \mathcal{E}_{\mathrm{concr}}$; as filtered colimits, and a fortiori κ-filtered colimits preserve monomorphisms, the canonical map $\operatorname{colim} X \to \operatorname{colim} \pi^!\pi_* X \xrightarrow{\cong} \pi^!\pi_* \operatorname{colim} X$ is a monomorphism, so that $\operatorname{colim} X$ is concrete. $\square$

5.2.2 Diffeological spaces

The category $\mathbf{Cart}^\infty$ has a final object, so that $\widehat{\mathbf{Cart}^\infty}$ is local by Example 5.2.4. It is moreover easily verified that $\widehat{\mathbf{Cart}^\infty} \leftarrow \mathbf{Set} : \pi^!$ factors through $\mathbf{Diff}^r_{\leq 0}$, so that $\mathbf{Diff}^r_{\leq 0}$ is likewise local.

Definition 5.2.8. A ***diffeological space*** X is a concrete object in $\mathbf{Diff}^\infty_{\leq 0}$. A ***plot*** of X is a map $\mathbf{R}^n \to X$. The collection of all plots of X is called the ***diffeology*** of X. ⌟

Convention 5.2.9. Let X be a diffeological space, then plots of X are usually identified with their images under π_*. ⌟

A diffeological space is thus a concrete presheaf on $\mathbf{Cart}^\infty$ with underlying set S such that a map $p : \pi_*\mathbf{R}^n \to S$ is a plot iff for every point $t \in \mathbf{R}^n$ there exists an open embedding $u : \mathbf{R}^n \to \mathbf{R}^n$ sending 0 to t such that $p \circ u$ is a plot.

Definition 5.2.10. Let X be a diffeological space, and $Y \subseteq X$ a subset, then the ***subspace diffeology*** on Y is the diffeology in which a map $\mathbf{R}^n \to Y$ is a plot iff it is a plot viewed as a map to X. ⌟

Example 5.2.11. The standard simplex Δ^n with the subspace diffeology inherited from $\mathbf{R}^{n+1}$ is denoted by Δ^n_{sub}. ⌟

Proposition 5.2.12 ([Wat12, Lm. 2.64])**.** *Write* $\mathbf{R}^n_+ := \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n \;\middle|\; x_1, \ldots, x_n \geq 0 \right\}$, *and endow this set with the subspace diffeology inherited from $\mathbf{R}^n$. A map $f : \mathbf{R}^n_+ \to \mathbf{R}$ is smooth iff it is the restriction of a smooth map $U \to \mathbf{R}$, where U is an open neighbourhood of $\mathbf{R}^n_+$ in $\mathbf{R}^n$.*

Proof. Write $s : \mathbf{R}^n \to \mathbf{R}^n$, $(x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^2)$, then by assumption $f \circ s : \mathbf{R}^n \to \mathbf{R}$ is smooth, and moveover invariant under the action $(\mathbf{Z}^*)^n \times \mathbf{R}^n \to \mathbf{R}^n$, $\big((\sigma_1, \ldots, \sigma_n), (x_1, \ldots, x_n)\big) \mapsto (\sigma_1 x_1, \ldots, \sigma_n x_n)$. By [Sch75] there exists a smooth map $\widetilde{f} : \mathbf{R}^n \to \mathbf{R}$ such that $\widetilde{f} \circ s = f \circ s$. As s restricts to a bijection on the underlying sets of $\mathbf{R}^n_+ \to \mathbf{R}^n_+$, the maps f and $\widetilde{f}$ agree on $\mathbf{R}^n_+$, so that f is a restriction of $\widetilde{f}$. $\square$

Corollary 5.2.13. *Let M be a smooth manifold with corners, and N a smooth manifold* without *corners, then a map $M \to N$ is smooth iff there exists a manifold $\widetilde{M}$ without corners, and an open embedding $M \subseteq \widetilde{M}$ and a smooth map $\widetilde{M} \to N$ which restricts to $M \to N$. In particular, a map $\Delta^n_{\mathrm{sub}} \to N$ is smooth iff there exists an open neighbourhood U of Δ^n_{sub} in $\mathbf{R}^{n+1}$ and a smooth map $U \to N$ which restricts to $\Delta^n_{\mathrm{sub}} \to N$.* $\square$

Example 5.2.14. Consider the unique cocontinuous functor $\widehat{\Delta} \to \mathbf{Diff}^\infty_{\leq 0}$ carrying Δ^n to Δ^n_{sub} from Example 5.2.11 then this functor carries the simplicial sets $\partial\Delta^n$ and Λ^n_k to diffeological spaces. These diffeological spaces *are not* equipped with the subspace diffeology of Δ^n_{sub}. Write $\Lambda^2_1 := u_!\Lambda^2_1$ and $\Lambda^2_{1,\mathrm{sub}}$ for the 1-horn of Δ^2 with the subdiffeology. Any path passing through the corner of Λ^2_1 must restrict to a constant functor which takes values in that corner for a positive amount of time. A retract of $\Lambda^2_1 \hookrightarrow \Delta^2$ could then be composed with an injective path passing through the corner of $\Lambda^2_{1,\mathrm{sub}}$, yielding a contradiction. ⌟

6 Shapes, cofinality and differentiable sheaves

In this section we prove that $\mathbf{Diff}^r$ is a locally contractible ∞-topos, and we apply the results of §2. First we prove in §6.1 that various ways of extracting homotopy types from manifolds are equivalent. Then in §6.2 we exhibit three application, where we give simple and canonical proofs of well-known and interesting facts.

Theorem 6.0.1. *The spaces $\mathbf{R}^d$ have contractible shape in $\mathbf{Diff}^r_{\text{ét}}$, and thus the latter is locally contractible.*

Proof. By Theorem 3.2.1 we must show that the topos $\left(\mathbf{Diff}^r_{\text{ét},\leq 0}\right)_{/\mathbf{R}^d}$ has contractible shape for all $0 \leq d < \infty$. As $\left(\mathbf{Diff}^r_{\text{ét},\leq 0}\right)_{/\mathbf{R}^d}$ is the same for all $0 \leq r \leq \infty$, it suffices to consider the case $r = 0$, and show that the underlying topological space of $\mathbf{R}^d$ has contractible shape. We will show that $\mathbf{R}$ using Proposition 2.0.2 has constant shape, then the general cases follows from [Lur17, Lm. A.2.9].

Let X be a set, then $H^0(\mathbf{R}, X) = H^0(1, X) = X$, as $\mathbf{R}^1$ is connected.

Let G be any group, then $H^1(\mathbf{R}, G)$ equivalent to the set isomorphisms principle G-bundles on 1 in $\widetilde{\mathbf{R}}$, but these are locally constant, and thus constant by [Sch14, Lm. 5.1.2].

Finally, the higher cohomology groups vanish because $H^i(\mathbf{R}; M) = 0$ for all $i > 0$ and all Abelian (not necessarily (locally) constant) sheaves M on $\mathbf{R}$ by [Sch14, Lm. 5.1.1] (i.e. $\mathbf{R}$ has cohomological dimension 1).

To get arbitrary $\mathbf{R}^d$ use that tensoring local geometric morphisms yields local geometric morphisms. □

Remark 6.0.2. Alternatively, Theorem 6.0.1 could be proved using that the shape of a CW complex is given by its singular homotopy type (see [Lur09, §7.1]). The fact that the shape of $\mathbf{R}$ also follows from [Lur17, Prop. A.2.1] (or [Lur17, Lm. A.2.2]). Both of these results rely on results in [Lur09, §7.1], which in turn rely a generalisation of the Quillen equivalence between topological spaces and simplicial sets. As we are providing a new proof of this equivalence in §7.1.3, we have refrained from using these results of Lurie in order to avoid circular reasoning.

It is possible to eschew fractured toposes altogether as is done in [Pav22] and [Sch13, §4.4]. We give a short argument which avoids using simplicial (pre)sheaves. First, use that $\mathbf{A}^\bullet : \Delta \to \mathbf{Cart}^r$ is initial (see Proposition **??**.) Then, assuming that $r = 0$, one may use the Nerve theorem (see [Seg68, §4]) to show that shape equivalences in $\underline{\mathrm{Hom}}((\mathbf{Cart}^0)^{\mathrm{op}}, \mathcal{S})$ are sent to isomorphisms by a^*. By [Cis19, Th. 7.7.9 & Prop. 7.11.8 & Prop. 7.1.17]) we obtain a commutative triangle

$$\begin{tikzcd}
\underline{\mathrm{Hom}}((\mathbf{Cart}^0)^{\mathrm{op}}, \mathcal{S}) \arrow[rr, shift left, "a^*"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\pi_!"] \arrow[dr, phantom, "\perp"] & & \mathbf{Diff}^0 \arrow[ll, shift left, "a_*"] \arrow[dl, shift right, "\Pi"'] \arrow[dl, phantom, "\perp"] \\
& \mathcal{S} \arrow[ul, shift left, "\pi^*"] \arrow[ur, shift right, "C"'] &
\end{tikzcd}$$

We thus need to identify C with $\mathbf{Diff}^0 \hookleftarrow \mathcal{S} : \pi^*_{\mathbf{Diff}^0}$, but we have

$$\begin{aligned} C &= a^* \circ a_* \circ C \\ &= a^* \circ \pi^* \\ &= \pi^*_{\mathbf{Diff}^0} \end{aligned}$$

Dugger and Carchedi give different formula for shape: $\mathbf{Mfd}^r \to \mathbf{TSpc}$ induces functor $\mathbf{Mfd}^r \to \mathcal{S}$. Gives Yoneda extension $\underline{\mathrm{Hom}}((\mathbf{Mfd}^r)^{\mathrm{op}}, \mathcal{S}) \rightleftarrows \mathcal{S}$, which descends to adjunction $\mathbf{Diff}^r \rightleftarrows \mathcal{S}$ via Dugger and Isaksen's hypercovering theorem (see §**??**). Again, from discussion in §6.1.2 it is clear that left adjoint is shape functor.

The reason that we take the route of fractured ∞-toposes, is because it provides tools for concrete calculations such as in §6.2.1 but also for aesthetic reasons: The fact that the homotopy theory of topological spaces plays well with descent, as exhibited in §**??** & §6.2.1.1 (and in a more subtle manner in §7.1.3 & §7.1.4) may be derived from the purely Galois theoretic argument (and the fundamental properties of intervals of being connected and having the least upper bound property) used in the proof of Theorem 6.0.1. Dugger and Carchedi also first obtain descent, and then construct their shape functor.

Corollary 6.0.3. *The shape functor* $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ *preserves finite products.*

Proof. By Proposition 2.1.3 the shape of any sheaf in $\mathbf{Diff}^r$ may be computed as the colimit of the corresponding presheaf on $\mathbf{Cart}^r$, but $\mathbf{Cart}^r$ has finite products, and is thus sifted. □

Corollary 6.0.4. *The shape of* $\mathbf{Diff}^r$ *is contractible.* □

Proposition 6.0.5. *The* ∞*-category* $\mathcal{S}$ *is the localisation of* $\mathbf{Diff}^\infty$ *by inverting* $\mathbf{R}^1 \times X \to \mathbf{R}^0 \times X$ *for all objects* X *in* $\mathbf{Diff}^r$.

Proof. It is clear that under the localisation in the statement of the theorem, all maps $\mathbf{R}^m \to \mathbf{R}^n$ are inverted. We already know that $L\,\mathbf{Diff}^r \to \mathcal{S}$ is essentially surjective. The statement then follows from contemplating the following diagram:

$$\begin{tikzcd} \underline{\mathrm{Hom}}((\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}) \arrow[r, shift left, two heads] \arrow[d] \arrow[r, phantom, "\perp"] & \mathbf{Diff}^r \arrow[l, shift left, hook'] \arrow[d] \\ L\underline{\mathrm{Hom}}((\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}) \arrow[r, shift left, two heads] \arrow[r, phantom, "\perp"] & L\,\mathbf{Diff}^r \arrow[l, shift left, hook'] \end{tikzcd}$$

□

To our knowledge, the above proposition is new as stated. It is well-known that the $\mathbf{R}$-invariant sheaves in $\mathbf{Diff}^r$ are equivalent to $\mathbf{S}$; or stated, differently, that the cocontinuous localisation w.r.t. the above maps is equivalent to $\mathcal{S}$ (see [Cis19]).

6.1 Comparing methods of calculating underlying homotopy types of differentiable sheaves

We first compare different nerves in §6.1.1, and then show §6.1.2 that the underlying topological space of a manifold has the correct homotopy type.

6.1.1 Nerves

We illustrate Theorem 4.3.18 in five cases. In

§6.1.1.1 we discuss the *extended simplices*, denoted by $\mathbf{A}^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$;

§6.1.1.2 we discuss the *closed simplices*, denoted by $\Delta^\bullet_{\mathrm{sub}} : \Delta \to \mathbf{Diff}^r_{\leq 0}$;

§6.1.1.3 we discuss *Kihara's simplices*, denoted by $\Delta^\bullet : \Delta \to \mathbf{Diff}^r_{\leq 0}$;

§6.1.1.4 we discuss the *extended cubes*, denoted by $\boldsymbol{\square}^\bullet : \square \to \mathbf{Diff}^r_{\leq 0}$;

§6.1.1.5 we discus the *closed cubes*, denoted by $\square^\bullet : \square \to \mathbf{Diff}^r_{\leq 0}$.

6.1.1.1 Extended simplices

Definition 6.1.1. Consider the cosimplicial object

$$\begin{array}{rccl} \mathbf{A}^\bullet : & \Delta & \to & \mathbf{Diff}^r \\ & \Delta^n & \mapsto & \mathbf{A}^n := \left\{ \ (x_0, \dots, x_n) \in \mathbf{R}^{n+1} \ \middle| \ x_0 + \cdots + x_n = 1 \ \right\}, \end{array}$$

then the spaces $\mathbf{A}^n$ for $n \geq 0$ are referred to as ***extended simplices***. Moreover, we write

$$\begin{array}{rcll} \partial \mathbf{A}^n & := & \mathbf{A}^\bullet_! \partial \Delta^n, & n \geq 0 \\ \mathbf{\Lambda}^n_k & := & \mathbf{A}^\bullet_! \Lambda^n_k, & n \geq 1, \ n \geq k \geq 0. \end{array}$$

⌟

6.1.1.2 Closed simplices

Definition 6.1.2. Consider the cosimplicial object

$$\begin{array}{rccl} \Delta^\bullet_{\mathrm{sub}} : & \Delta & \to & \mathbf{Diff}^r \\ & \Delta^n & \mapsto & \Delta^n_{\mathrm{sub}}, \end{array}$$

(see Example 5.2.11), then the spaces Δ^n_{sub} for $n \geq 0$ are referred to as ***closed simplices***. Moreover, we write

$$\begin{array}{rcll} \partial \Delta^n_{\mathrm{sub}} & := & (\Delta_{\mathrm{sub}})^\bullet_! \partial \Delta^n, & n \geq 0 \\ \Lambda^n_{k,\mathrm{sub}} & := & (\Delta_{\mathrm{sub}})^\bullet_! \Lambda^n_k, & n \geq 1, \ n \geq k \geq 0. \end{array}$$

6.1.1.3 Kihara's simplices It has been a longstanding goal to establish a model structure on diffeological spaces (see e.g. [CW14] and [HS18]). To this end Kihara endows the standard simplices with a new diffeology in [Kih19, § 1.2]. With this diffeology the horn inclusions admit deformation retracts (see Proposition 6.1.3), allowing Kihara to mimic the construction of the model structure on topological spaces in [Qui67, §II.3], and show that the resulting model category is Quillen equivalent to simplicial sets with the Kan-Quillen model structure. We need Kihara's simplices in order to construct formally cofibrant objects in §**??**.

For the convenience of the reader, we repeat the construction of Kihara's simplices: For each $n \geq 1$ and each $0 \leq k \leq n$ we define the set

$$A^n_k := \left\{ \ (x_0, \dots, x_n) \in \Delta^n \ \middle| \ x_k < 1 \ \right\}.$$

We now proceed inductively: On Δ^0 and Δ^1 the diffeology is the subspace diffeology coming from $\mathbf{R}^1$ and $\mathbf{R}^2$, respectively. Let $n > 1$, and assume that the diffeologies on the simplices Δ^m for $m < n$ have been defined, then we define a diffeology on A^n_k by exhibiting this set as the underlying set of the quotient

$$\begin{tikzcd} \Delta^{n-1} \times \{0\} \arrow[r, hook] \arrow[d, two heads] & \Delta^{n-1} \times [0,1) \arrow[d, two heads] \\ 1 \arrow[r, hook] & A^n_k, \end{tikzcd}$$

where $\Delta^{n-1} \times [0,1) \to A^n_n$ is given by $(x_0, \ldots, x_{n-1}; t) \mapsto ((1-t) \cdot x_0, \ldots, (1-t) \cdot x_n, t)$. Finally, the diffeology on Δ^n is determined by the map $\coprod_{k=0}^n A^n_k \to \Delta^n$.

Proposition 6.1.3 ([Kih19, § 8])**.** *The horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $n = 2$ and $n \geq k \geq 0$ admit a deformation retract.* $\square$

Definition 6.1.4. We write

$$\begin{array}{rccl} \Delta^\bullet : & \Delta & \to & \mathbf{Diff}^r \\ & \Delta^n & \mapsto & \Delta^n \end{array}$$

for the cosimplicial object sending each simplex Δ^n to the standard n-simplex endowed with the diffeology described above. Moreover we write

$$\begin{array}{rcll} \partial\Delta^n & := & \Delta^\bullet_! \partial\Delta^n, & n \geq 0 \\ \Lambda^n_k & := & \Delta^\bullet_! \Lambda^n_k, & n \geq 1,\ n \geq k \geq 0. \end{array}$$

⌟

6.1.1.4 Extended cubes

Definition 6.1.5. Consider the cocubical object

$$\begin{array}{rccl} \boldsymbol{\square}^\bullet : & \square & \to & \mathbf{Diff}^r \\ & \square^n & \mapsto & \boldsymbol{\square}^n := \mathbf{R}^n \end{array}$$

then the spaces $\boldsymbol{\square}^n$ for $n \geq 0$ are referred to as ***extended cubes***. Moreover, we write

$$\begin{array}{rcll} \partial\boldsymbol{\square}^n & := & \boldsymbol{\square}^\bullet_! \partial\square^n, & n \geq 0 \\ \boldsymbol{\sqcap}^n_{i,\varepsilon} & := & \boldsymbol{\square}^\bullet_! \sqcap^n_{i,\varepsilon}, & n \geq i \geq 1 \text{ and } \varepsilon = 0, 1. \end{array}$$

⌟

6.1.1.5 Closed cubes

Definition 6.1.6. Consider the cocubical object

$$\begin{array}{rccl} \boldsymbol{\square}^\bullet : & \square & \to & \mathbf{Diff}^r \\ & \square^n & \mapsto & \boldsymbol{\square}^n := [0,1]^n \end{array}$$

then the spaces $\square^n$ for $n \geq 0$ are referred to as ***closed cubes***. Moreover, we write

$$\begin{aligned} \partial\square^n &:= \square^{\bullet}_{!}\partial\square^n, \quad n \geq 0 \\ \sqcap i, \varepsilon^n &:= \square^{\bullet}_{!}\sqcap^n_{i,\varepsilon}, \quad n \geq i \geq 1 \text{ and } \varepsilon = 0, 1. \end{aligned}$$

$\lrcorner$

6.1.2 Change of regularity

Let $0 \leq s \leq r$, then the forgetful functor $u : \mathbf{Cart}^r \to \mathbf{Cart}^s$ both preserves and reflects covering families, so that we obtain an essential geometric morphism

$$\begin{tikzcd} \mathbf{Diff}^r \arrow[r, shift left=3, "u_!"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "u_*"'] & \mathbf{Diff}^s \arrow[l, "u^*"'] \end{tikzcd}.$$

By Proposition 2.0.7

$$\begin{tikzcd} \mathbf{Diff}^r \arrow[rr, "u_!"] \arrow[dr, "\pi_!"'] & & \mathbf{Diff}^s \arrow[dl, "\pi_!"] \\ & \mathcal{S} & \end{tikzcd}$$

commutes. Let M be an r-times differentiable manifold. Consider a hyper cover of M by real spaces, then, as $u_!$ preserves colimits we see that $u_!$ sends M to its underlying s-times differentiable manifold. Thus M, and its underlying s-times differentiable manifold have the same shape. The functor $u : \mathbf{Cart}^r \to \mathbf{Cart}^s$ is initial, as its composition with $\mathbf{A}^{\bullet} : \Delta \to \mathbf{Cart}^r$ is initial, and initial functors are right cancelative. Thus the unit map $M \to u_! u^* M$ is a weak equivalence, and applying $(\mathbf{A}^{\bullet})^*$ produces the map of simplicial sets in the statement of the theorem. Choosing $r = \infty$ and $s = 0$ we obtain a classical smoothing theorem:

Theorem 6.1.7. *The map of simplicial sets* $\mathbf{Mfd}_{\infty}(\Delta^{\bullet}, M) \to \mathbf{Mfd}_0(\Delta^{\bullet}, M)$ *is a weak homotopy equivalence.* $\square$

6.2 Applications

6.2.1 The shape of the Haeflilger stack

In [Hae58] Haefliger introduces for each $d \in \mathbf{N}$ a topological groupoid Γ^d (see Definition **??**) with a view towards applications to the study foliations (which come into fruition in [Hae71]). Its classifying space (in the sense of [Seg68]) is first determined in [Seg78, Prop. 1.3].

Definition 6.2.1. The ***d-th Haefliger groupoid***, denote by Γ^d ($d \in \mathbf{N}$), is the groupoid in $\mathrm{Emb}^r_{\text{ét},d}$ defined in [Car16, Ex. 2.6], and the ***d-th topological Haefliger groupoid*** is the underlying topological groupoid of Γ^d, likewise denote by Γ^d. The ***d-th Haefliger stack*** is the colimit of Γ^d in $\mathrm{Emb}^r_{\text{ét},d}$ and is denoted by H^d. We likewise refer to $j_! H^d$ in $\mathbf{Diff}^r$ as the *d-th Haefliger stack.* $\lrcorner$

Proposition 6.2.2 ([Car20, Th. 6.1.6])**.** *The d-th Haefliger Stack H^d is the final object in* $\mathrm{Emb}^r_{\text{ét},d}$. $\square$

Proposition 6.2.3. *For every $d \in \mathbf{N}$ endow* $\operatorname{Emb}(\mathbf{R}^n, \mathbf{R}^n)$ *with the discrete topology (so that we obtain a discrete monoid in* $\mathbf{TSpc}$*), then*

$$\pi_! H^d = B \operatorname{Emb}(\mathbf{R}^n, \mathbf{R}^n).$$

Proof. We have commutative triangle

$$\begin{tikzcd}
\mathbf{Diff}^r_{\text{ét},d} \arrow[rr, shift left=3, "(j_d)_!"] \arrow[rr, shift right=3, "(j_d)_*"'] \arrow[dr, "\pi_!"'] & & \mathbf{Diff}^r_{\text{ét}} \arrow[ll, "(j_d)^*" description] \arrow[dl, "\pi_!"] \\
& \mathcal{S} &
\end{tikzcd}$$

so that the shape H^d is the same as the shape of $(j_d)_! H^d$. But $\mathbf{Diff}^r_{\text{ét}}$ has the same shape as $\underline{\operatorname{Hom}}((\mathbf{Cart}^r_{\text{ét},d})^{\mathrm{op}}, \mathcal{S})$, but $B(\mathbf{Cart}^r_{\text{ét},d}) \simeq B \operatorname{Emb}(\mathbf{R}^n, \mathbf{R}^n)$. ☐

6.2.1.1 Principal bundles

It is often taken for granted that the base space of a principal bundle in $\mathbf{TSpc}$ is a homotopy quotient of the total space (in a sense which we make precise in course of the discussion below)[3]. While this statement is true, it is not straightforward to show using classical methods. For CGWH spaces a proof may be obtained by combining [May75, Thm. 7.6] and [Shu09, Lm. 12.4], where the first reference relies on technical pointset topological arguments. Alternatively, one can exhibit $\mathbf{TSpc}$ as a model topos (in the sense of [Rez10, §6]), and use a descent argument, which, when performed with rigour, is again technically demanding. Both of these approaches however do not adequately address the relationship between the geometry and homotopy theory of principal bundles.
We now exhibit how this statement admits a natural proof using the theory developed in this thesis.

Definition 6.2.4. Let C be an ∞-category with a final object, and let $G : \Delta^{\mathrm{op}} \to C$ be a group object, then a ***G-object*** in C is a Cartesian natural transformation $\Delta^1 \times \Delta^{\mathrm{op}} \to C$ with target G. ⌟

Convention 6.2.5. In $\mathcal{S}$, $\mathbf{Set}$, $\mathbf{Mfd}^\infty$ etc. G-objects will, respectively, be referred to as *G-homotopy types, G-sets, G-smooth manifolds* etc. ⌟

Example 6.2.6. Let G be an ordinary group, and let $P_\bullet$ be a G-set, then P_0 may be equipped with both a left G-action and a right G-action in a canonical way. To obtain the left G-action, observe that for any map $\alpha : \Delta^0 \to \Delta^n$ in Δ the square

$$\begin{tikzcd}
P_n \arrow[r] \arrow[d, "\alpha^*"'] \arrow[dr, phantom, "\lrcorner" very near start] & G \times \overset{n\times}{\cdots} \times G \arrow[d] \\
P_0 \arrow[r] & 1
\end{tikzcd}$$

---
[3] In fact, this is often claimed, incorrectly, to be true of the quotient of *any* free group action. To see that this is not the case, consider any non-trivial ordinary group G, and equip it with the discrete topology. Let G act on copy of itself equipped with the trivial topology, then the quotient is a point. If the quotient were a homotopy quotient, it would have to model the classifying space of G.

is again a pullback diagram. For each $n \geq 1$, choosing $\alpha : 0 \mapsto n$ one obtains a bijection between P_n and the *Cartesian* product $G \times \overset{n\times}{\cdots} \times G \times P_0$. By transferring the remaining face and degeneracy maps along these isomorphisms one obtains a simplicial object encoding a left G-action. The right G-action on P_0 is obtained similarly and, writing $(g, x) \mapsto g \cdot x$ for the induced left action, the right action is given by $(x, g) \mapsto g^{-1} \cdot x$, and the isomorphisms between the left G-set and right G-set over $G_\bullet$ is given by

$$\begin{array}{rcl} G \times \overset{n\times}{\cdots} \times G \times P_0 & \to & P_0 \times G \times \overset{n\times}{\cdots} \times G \\ (g_1, \ldots, g_n, x) & \mapsto & (g_1 \cdots g_n \cdot x, g_1, \ldots, g_n). \end{array}$$

$\lrcorner$

Definition 6.2.7. Let (C, W) be a relative ∞-category, then an augmented simplicial object $(\Delta^{\mathrm{op}})^{\triangleright} \to C$ is called a ***homotopy quotient*** if it is a homotopy colimit of its restriction $\Delta^{\mathrm{op}} \to C$ (in the sense of Definition 4.0.1). $\lrcorner$

Thus the theorem that we wish to prove is that for any topological group G and any topological principal G-bundle $P \to B$ the space B is a homotopy quotient of P, viewed as a G-space.

7 Homotopical calculi on differentiable sheaves

In this section we finally construct homotopical calculi. We begin with our main motivation for doing so. Denote by $L : \mathbf{TSpc} \to \mathcal{S}$ the localisation functor. Let A, X be topological spaces of which A is cofibrant (and X compactly / delta generated) in the Quillen-Serre model structure, then it is convenient and frequently used fact that the set $\mathbf{TSpc}(A, X)$ together with the compact open topology is a model for $\mathcal{S}(LA, LX)$. This follows from the fact that $\mathbf{TSpc}$ is Cartesian closed. As all objects are fibrant $A \times \_$ descends to the functor $LA \times \_$ by [Cis19, Prop. 7.1.14]. The fact that the Quillen-Serre model structure is Cartesian closed says precisely that $A \times \_ \dashv \underline{\mathbf{TSpc}}(A, \_)$ is a Quillen adjunction. Again, as all objects are fibrant $\underline{\mathbf{TSpc}}(A, \_)$ descends to a functor $\mathcal{S} \leftarrow \mathcal{S}$, which is a right adjoint to $LA \times \_$, and thus must be canonical equivalent to $\mathcal{S}(LA, \_)$.

$$\begin{tikzcd} \mathbf{TSpc} \arrow[r, shift left, "A\times\_"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathbf{TSpc} \arrow[l, shift left, "\underline{\mathbf{TSpc}}(A,\_)"] \arrow[d] \\ \mathcal{S} \arrow[r, shift left, "LA\times\_"] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, shift left, "\mathcal{S}(LA,\_)"] \end{tikzcd}$$

Now, moving on to the differentiable setting, let M be a closed smooth manifold, and N an arbitrary smooth manifold, then the set of smooth maps $\mathbf{Diff}^{\infty}(M, N)$ admits a canonical structure of a Fréchet manifold (see [GG73, Th. 1.11]). Via smoothing theory it is then possible to show that the homotopy type of this Fréchet manifold is equivalent to the homotopy type of $\underline{\mathbf{TSpc}}(M, N)$ (where M, N now denote the underlying topological spaces of the smooth manifolds M, N), which is equivalent to $\mathcal{S}(LM, LN)$, which is equivalent to $\mathcal{S}\big((\pi_{\mathbf{Diff}^\infty})_! M, (\pi_{\mathbf{Diff}^\infty})_! N\big)$ by the discussion in §6.1.2.

By [Wal12, Lm A.1.7] the Fréchet manifold of smooth maps from M to N is canonically equivalent to $\underline{\mathbf{Diff}}^{\infty}(M, N)$. By Corollary 6.0.3 the shape functor $\pi_! : \mathbf{Diff}^{\infty} \to \mathcal{S}$ commutes with finite products so that we obtain a canonical map $\pi_! \underline{\mathbf{Diff}}(A, X) \to \mathcal{S}(\pi_! A, \pi_! X)$. Explicitly:

$$
\begin{aligned}
&\underline{\mathbf{Diff}}^\infty(A,X)\times A \to X \\
\rightsquigarrow\ &\pi_!(\underline{\mathbf{Diff}}^\infty(A,X)\times A) = \pi_!\underline{\mathbf{Diff}}^\infty(A,X)\times \pi_! A \to \pi_! X \\
\rightsquigarrow\ &\pi_!\underline{\mathbf{Diff}}^\infty(A,X) \to \mathcal{S}(\pi_! A, \pi_! X).
\end{aligned}
$$

Thus, it makes sense to ask for which differentiable sheaves A, X the induced map is an equivalence. The following definition is due to Hisham Sati and Urs Schreiber (see [SS21]).

Definition 7.0.1. A differentiable stack A satisfies the ***smooth Oka principle*** if for every differentiable stack X the canonical map $\pi_!\underline{\mathbf{Diff}}^\infty(A,X) \to \mathcal{S}(\pi_! A, \pi_! X)$ is an equivalence. ⌟

We will see in §7.1 that all the nerves defined in §6.1.1 satisfy the conditions of Theorem 4.3.18, and thus induce model structures on $\mathbf{Diff}^\infty$ (and $\mathbf{Diff}^\infty_{\leq 0}$). Using this theorem, it would thus be natural to try to construct model structure such that:

1. The model structure is Cartesian closed.
2. All objects are fibrant.
3. All manifolds are cofibrant.

It turns out that satisfying the smooth Oka principal is $\mathbf{R}$-homotopy invariant, so that the third point can be relaxed to the condition that all manifolds be $\mathbf{R}$-homotopy equivalent to cofibrant objects. However, unless $r = 0$ we are not able to get 1. and 2. simultaneously.

We therefore bring the theory of §4.2 to bear on our problem. Let us assume that we have already shown that a given differentiable sheaf A satisfies the smooth Oka principle, and that $S \to D$ is a map between differentiable sheaves which likewise satisfy the smooth Oka principle, which we think of as constituting a "cell inclusion", then, if we attach a "cell" $S \hookrightarrow D$ along a map $f : S \to A$, a natural way of showing that $A \cup_f D$ is also cofibrant is to show that the pullback

$$
\begin{tikzcd}
\underline{\mathbf{Diff}}(A \cup_f D, X) \arrow[r] \arrow[d] & \underline{\mathbf{Diff}}(D, X) \arrow[d] \\
\underline{\mathbf{Diff}}(A, X) \arrow[r] & \underline{\mathbf{Diff}}(S, X)
\end{tikzcd}
$$

is a homotopy pullback. Thus we would like to find morphisms $S \to D$ between objects satisfying the smooth Oka principle such that the morphism $X^D \to X^S$ is sharp for every differentiable stack X.

Remark 7.0.2. Our sharp morphisms correspond to the *shape fibrations* morphisms of Myers. [Mye22, Th. 1.2]. ⌟

Definition 7.0.3. A morphism $S \to D$ in $\mathbf{Diff}^\infty$ is called a ***formal cofibration*** if $X^D \to X^S$ is sharp for every differentiable stack X. ⌟

Our main strategy is then to show hat, while the Kihara model structure is not Cartesian closed, its horn inclusions are formal cofibrations between sheaves satisfying the smooth Oka principle. To do this we introduce the *squishy Fibration structure* in §7.2.1. By a result of Kihara, all manifolds are $\mathbf{R}$-homotopy equivalent to simplicial complexes built using Kihara's simplices.

We now give a short overview of this section: In §7.1 we show that $\mathbf{Diff}^r_{\leq 0}$ satisfies the necessary conditions of Theorem 4.3.18, so that we may construct various model structures on $\mathbf{Diff}^r_{\leq 0}$ and $\mathbf{Diff}^r$. Moreover, we give a new simpler proof for the existence of Kihara's model structure, and the Quillen equivalence to simplicial sets. We further refine this technique to give a new and simple proof that $\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{TSpc}$ is a Quillen equivalence. Strøm colimits. In §7.2 we finally construct the squishy fibration structure and prove that all manifolds satisfy the smooth Oka principle.

7.1 Model structures on $\mathbf{Diff}^\infty$ and related ∞-categories

We are now ready to apply the theory of the preceding sections to establish one of the key theorems of this article.

Proposition 7.1.1. *The category* $\mathbf{Cart}^r$ *is a strict test category.*

Proof. By Corollary 4.3.10 it is enough to observe that $\mathbf{R}$ together with the inclusions of $\{0\}$ and $\{1\}$ is a separating interval. □

Theorem 7.1.2. *The topos* $\mathbf{Diff}^r_{\leq 0}$ *is a strict test topos.*

Proof. Combine the preceding proposition with Theorems 4.3.13 and 6.0.1. □

Theorem 7.1.3. *The functors introduced in §6.1.1 all satisfy the assumptions of Theorem 4.3.18.* □

Proposition 7.1.4. *The singular equivalences and the locally constant equivalences in* $\mathbf{Diff}^r$ *agree. There exist cofibrantly generated model structures on* $\mathbf{Diff}^r$ *and* $\mathbf{Diff}^r_{\leq 0}$ *with the aforementioned weak equivalences and with generating cofibrations and trivial cofibrations given, respectively, by* $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ *and* $\{\Lambda^n_k \hookrightarrow \Delta^n\}_{n\geq 1,\, n\geq k\geq 0}$*. Moreover, all objects in these model categories are fibrant.*

Proof. The proof is exactly the same as for Proposition **??**, except that it is not obvious that the simplices Δ^n are Δ^1-contractible, but this is shown in [Kih19, Rmk. 9.3]. □

The model structures on $\mathbf{Diff}^r_{\leq 0}$ and $\mathbf{Diff}^r$ are both referred to as the ***Kihara model structure***.

Corollary 7.1.5. *The shape functor* $\pi_! : \mathbf{Diff}^r \to \mathcal{S}$ *commutes with* all *products.* □

Proposition 7.1.6. *The Kihara model structures on* $\mathbf{Diff}^r_{\leq 0}$ *and* $\mathbf{Diff}^r$ *are* not *Cartesian.*

Proof. Taking the pushout-product of $\delta : \partial\Delta^1 \hookrightarrow \Delta^1$ with itself produces the inclusion $\delta\Box\delta := (\Delta^1 \times \partial\Delta^1) \sqcup_{\partial\Delta^1\times\partial\Delta^1} \partial\Delta^1 \times \Delta^1 \hookrightarrow \Delta^1 \times \Delta^1$. If this inclusion were a cofibration, then the square

$$\begin{tikzcd} \delta\Box\delta \arrow[r, hook] \arrow[d, hook] & u_!(\Delta^1 \times \Delta^1) \arrow[d] \\ \Delta^1 \times \Delta^1 \arrow[r] & 1 \end{tikzcd}$$

would admit a lift. □

Not only is the Kihara model structure not Cartesian, but fibrations are hard to detect, because it is difficult to write down maps $\Delta^n \to X$, when X is an arbitrary differentiable stack, essentially for the same reason that it is hard to write down maps $\Delta^n_{\text{sub}} \to X$, as explained in §5.2.2. This issue will be addressed in §**??**.

Remark 7.1.7. The model structures transferred via $\mathbf{A}^\bullet$, $\Delta^\bullet_{\text{sub}}$ etc. are Cartesian closed via [Pav22, §8]. ⌟

7.1.1 Colimits of concrete objects in a local topos

Fix a local topos $\mathcal{E}$ (see Definition 5.2.1).

Definition 7.1.8. A monomorphism $X \hookrightarrow Y$ in $\mathcal{E}_{\text{concr}}$ is called an ***embedding*** if

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\ \pi^!\pi_* X \arrow[r, hook] & \pi^!\pi_* Y \end{tikzcd}$$

is a pullback square. ⌟

Proposition 7.1.9. *Consider a span in* $\mathcal{E}_{\text{concr}}$

$$\begin{tikzcd} & Z \\ X \arrow[ur, hook] \arrow[dr, hook] & \\ & Y \end{tikzcd}$$

in which $X \hookrightarrow Y$ is monomorphism, and $X \hookrightarrow Z$ is an embedding, then the pushout of the above diagram in $\mathcal{E}$ is again an object of $\mathcal{E}_{\text{concr}}$.

Proof. Consider the following cube,

$$\begin{tikzcd} & Z \arrow[rr, hook] \arrow[dd, hook] & & Y \sqcup_X Z \arrow[dd] \\ X \arrow[ur, hook] \arrow[rr, hook] \arrow[dd, hook] & & Y \arrow[ur, hook] \arrow[dd, hook] & \\ & \pi^!\pi_* Z \arrow[rr, hook] & & \pi^!\pi_*(Y \sqcup_X Z) \\ \pi^!\pi_* X \arrow[ur, hook] \arrow[rr, hook] & & \pi^!\pi_* Y \arrow[ur, hook] & \end{tikzcd}$$

then we must show that the map $Y \sqcup_X Z \to \pi_! \pi_*(Y \sqcup_X Z)$ is a monomorphism. Consider a pair of maps $f, g : T \rightrightarrows Y \sqcup_X Z$, then we will show that if their compositions with $Y \sqcup_X Z \to \pi_! \pi_*(Y \sqcup_X Z)$ are equal, then $f = g$.

First, we consider the special case in which f and g each factor through either $Y \hookrightarrow Y \sqcup_X Z$ or $Z \hookrightarrow Y \sqcup_X Z$. If *both* maps factor through either Y or Z, then $f = g$ because $Y \hookrightarrow \pi_! \pi_*(Y \sqcup_X Z)$ and $Z \hookrightarrow \pi_! \pi_*(Y \sqcup_X Z)$ are monomorphisms. Thus, assume w.l.o.g. that f factors through Y and g factors through Z. We thus obtain a commutative diagram

$$\begin{tikzcd} & \pi^! \pi_* Z \arrow[dr, hook] & \\ T \arrow[ur, hook] \arrow[dr, hook] & & \pi^! \pi_*(Y \sqcup_X Z) \\ & \pi^! \pi_* Y \arrow[ur, hook] & \end{tikzcd}$$

By Proposition 4.1.1 the square

$$\begin{tikzcd} & Z \arrow[dr, hook] & \\ X \arrow[ur, hook] \arrow[dr, hook] & & Y \sqcup_X Z \\ & Y \arrow[ur, hook] & \end{tikzcd}$$

is pushout square in the ∞-topos of $\mathcal{S}$-valued sheaves on $\mathcal{E}$, and thus also a pullback by [ABFJ20, Prop. 2.2.6]. Thus

$$\begin{tikzcd} & \pi^! \pi_* Z \arrow[dr, hook] & \\ \pi^! \pi_* X \arrow[ur, hook] \arrow[dr, hook] & & \pi^! \pi_*(Y \sqcup_X Z) \\ & \pi^! \pi_* Y \arrow[ur, hook] & \end{tikzcd}$$

is also a pullback square, so that we obtain a canonical map $T \to \pi^! \pi_* X$, and thus a commutative square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Z \arrow[d, hook] \\ \pi^! \pi_* X \arrow[r, hook] & \pi^! \pi_* Z \end{tikzcd}$$

which yields a canonical morphism $T \to X$, because $X \hookrightarrow Z$ is an embedding. The composition of $T \to X \to Z$ yields g by construction. To see that the composition of $T \to X \to Y$ yields f we further compose with the monomorphism $Y \hookrightarrow \pi^! \pi_*(Y \sqcup_X Z)$ which is equal to f composed with the same monomorphism.

To obtain the general statement, observe that $Y \sqcup Z \to Y \sqcup_X Z$ is an effective epimorphism, and thus there exists an effective epimorphism $\bigcup_{i \in E} U_i \to T$ such that the composition of $U_i \to T \xrightarrow{f} Y \sqcup_X Z$ factors through $Y \sqcup Z$ for each $i \in I$. As coproducts are disjoint in toposes, we may further assume that each such map factors through either Y or Z. Similarly, for every $i \in I$ we may choose an effective epimorphism $\bigcup_{j \in F_i} V_{ij} \to T$ such that the composition of

$V_{ij} \to U_i \to T \xrightarrow{g} Y \sqcup_X Z$ factors through either Y or Z. Now, repeat the above argument with the maps $V_{ij} \to Y \sqcup_X Z$. As the map $\bigcup_{i\in I} \bigcup_{j\in F_i} V_{ij} \to T$ is an effective epimorphism, and its compositions with f and g are equal, f and g must be equal. □

Proposition 7.1.10. *Let I be a filtered category and $X : I \to \mathcal{E}_{\text{concr}}$ a diagram such that $X_i \hookrightarrow X_j$ is a monomorphism for all morphisms $i \to j$ in I, then the colimit of X in $\mathcal{E}$ is concrete.*

Proof. Denote by X the colimit of $X : I \to \mathcal{E}_{\text{concr}}$, then we will show that for any pair of morphisms $f, g : T \rightrightarrows X$ if their composition with $X \to \pi^!\pi_* X$ are equal, then $f = g$. By the same technique as in we may assume that f and g each factor through $X_i \to X$ and $X_j \to X$ respectively, and by the filteredness of I we may assume w.l.o.g. that $i = j$. Consider the square

$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[d, hook] & X \arrow[d] \\ \pi^!\pi_* X_i \arrow[r, hook] & \pi^!\pi_* X \end{tikzcd}$$

where $X_i \hookrightarrow X$ is a monomorphism, and therefore also $\pi^!\pi_* X_i \hookrightarrow \pi^!\pi_* X$, as $\pi^!\pi_*$ preserves limit. The compositions of the lifts of f and g to $T \to X_i$ with $X_i \hookrightarrow \pi^!\pi_* X$ are equal by assumption, and there fore the lifts are equal because $X_i \hookrightarrow \pi^!\pi_* X$ is a monomorphism, and therefore f and g are equal. □

7.1.2 The Kihara model structure on diffeological spaces

Here we provide a simple proof (using the theory developed in this thesis) of a model structure, originally due to Kihara, on $\mathbf{Diff}^r_{\text{concr}}$, which is Quillen equivalent to the category of simplicial sets together with the Kan-Quillen model structure.

Theorem 7.1.11 ([Kih19, Th. 1.3] [Kih17, Th. 1.1])**.** *There exists a cofibrantly generated model structure on $\mathbf{Diff}^r_{\text{concr}}$ in which the weak equivalences are equivalently the shape equivalences or the singular equivalences, and in which the generating cofibrations and trivial cofibrations are given, respectively, by $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ and $\{\Lambda^n_k \hookrightarrow \Delta^n\}_{n\geq 1,\, n\geq k\geq 0}$. Moreover, all objects in this model categories are fibrant. The adjunction $\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^r_{\text{concr}}$ is a Quillen equivalence.*

Proof. We shall transfer the model structure from $\widehat{\Delta}$ using Proposition 4.3.16. Transfinite composition of monomorphisms between diffeological spaces taken in $\mathbf{Diff}^r$ remains in $\mathbf{Diff}^r_{\text{concr}}$ by Proposition **??**, so that the transfinite composition of monomorphisms which are also shape equivalences is again a shape equivalence, as shape equivalences in $\mathbf{Diff}^\infty$ are closed under taking colimits. Thus, it is enough to show for any diffeological space X and any map $f : \Lambda^n_k \to X$ ($n \geq 1$, $n \geq k \geq 0$), that the map $X \to X \cup_f \Delta^n$ is a weak equivalence, which follows from the fact that $X \to X \cup_f \Delta^n$ is a Δ^1-deformation retract, since $\Lambda^n_k \hookrightarrow \Delta^n$ is one.
The fact that all objects are fibrant follows from the fact that all inclusions $\Lambda^n_k \to \Delta^n$ ($n \geq 1$, $n \geq k \geq 0$) are deformation retracts.

To show that $\widehat{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Diff}^r_{\text{concr}}$ is a Quillen equivalence, we observe that by Propositions **??** & **??** the Yoneda extension of $\Delta \to \mathbf{Diff}^r_{\leq 0}$, $\Delta^n \mapsto \Delta^n$ factors through the inclusion

$\mathbf{Diff}^r_{\mathrm{concr}} \hookrightarrow \mathbf{Diff}^r_{\leq 0}$, thus both the unit and counit of $\widehat{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Diff}^r_{\mathrm{concr}}$ are weak equivalences, because this is true for $\widehat{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Diff}^r_{\leq 0}$. □

7.1.3 The Quillen model structure on topological spaces

We begin with brief history of the problem. That Ho(**Kan**) and Ho(**CW**) are equivalent was first shown by Milnor in [Mil57] using a variant of the Hurewicz theorem. Gabriel and Zisman gave a new proof of this fact in [GZ67] using that the topological realisation functor preserves minimal fibrations. This allows for to compare long exact sequences in **Kan** and **CW** so that the theorem follows from Whitehead's theorem. Lurie defines weak equivalences of topological spaces via total singular complex, and thus only needs to prove that the unit is equivalence. His proof relies on the idea that the nerve functor takes pushouts along open inclusions to homotopy pushouts. We don't need to show that topological realisation preserves any homotopy limits, nor that the nerve functor preserves any homotopy pushouts. In fact, our proof is completely elementary.

The following is an easy consequence of [Hir03, Th. 11.3.2].

Theorem 7.1.12. *Transferred model structure to topological spaces exists.* □

That this model structure is the same as the usual one follows from [Qui67, Lms. 2.3.1 & 2.3.2].

Theorem 7.1.13. *The Quillen adjunction*

$$\Delta_! : \widehat{\Delta} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{TSpc} : \Delta^*$$

is a Quillen equivalence.

Proof. As weak equivalences created by nerve, it is enough to show that unit weak equivalence. This in turn follows from the fact that for any simplicial set X the morphism $u_!X \to v_!X$ is an $\mathbf{R}$-homotopy equivalence. First, observe that while the map of underlying sets of $u_!X \to v_!X$ is a bijection, it is not true that the map in the other direction is smooth. For every $n \geq 1$ choose a homotopy $H^n : [0,1] \times \Delta^n \to \Delta^n$ (in **TSpc**) which deforms a neighbourhood of the boundary down to the boundary in such that way that H_n restricted to any face yields H^{n-1}. For any simplicial set X these homotopies assemble homotopies $H^X : [0,1] \times u_!X \to u_!X$ and $H^X : [0,1] \times v_!X \to v_!X$. Then we obtain a unique factorisation

$$\begin{tikzcd} & u_!X \arrow[d] \\ v_!X \arrow[ur] \arrow[r, "H_1^X"'] & v_!X \end{tikzcd}$$

on under lying sets, where the vertical map is the canonical map. The diagonal and vertical maps are homotopy inverse to each other, as their compositions yield $H_1^X : u_!X \to u_!X$ and $H_1^X : v_!X \to v_!X$, which are homotopic to the identity by construction. □

7.1.4 Strøm homotopy colimits are Serre homotopy colimits

A similar argument used for our proof of Theorem 7.1.13 may also be used to show that homotopy colimits in **TSpc** constructed using the bar construction without taking objectwise cofibrant replacements in the Quillen model structure. To our knowledge this was first proved in [DI04, Th. A.7]. This was surely known for simple cases such as double mapping cylinders before this, but we were not able to find earlier references to this fact.

We first illustrate the special case of double mapping cylinders. The general case is then proved below. Consider a span

$$\begin{tikzcd} B & & C \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in $\mathbf{\Delta TSpc}$, then the double mapping cylinder is obtained as colimit of

$$\begin{tikzcd} B & & \Delta^1 \times A & & \Delta^1 \times A & & C \\ & A \arrow[ul] \arrow[ur] & & A \arrow[ul] \arrow[ur] & & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Denote by M the colimit of this diagram in $\mathbf{Diff}^0$, and by M' the colimit of this diagram in $\mathbf{\Delta TSpc}$. Applying H^1 we obtain map in other direction $M' \to M$. To check that this map is continuous, consider a continuous map $\mathbf{R}^d \to M'$. If $0 \in \mathbf{R}^d$ gets sent to the image of one of the two copies of $\mathring{\Delta}^1 \times A$, then some open neighbourhood U of $0 \in \mathbf{R}^d$ lifts to $\Delta^1 \times A$. Otherwise lifts through A, B, C. Thus M and M' $\mathbf{R}$-homotopy equivalent.

Theorem 7.1.14. *Let $X : D \to \mathbf{TSpc}$ be a diagram, then $B(X, D, *)$ computes the homotopy colimit of X.*

Proof. First assume that $X : D \to \mathbf{TSpc}$ factors through $\mathbf{\Delta TSpc}$. The fattened up diagram $\Delta_D \to \mathbf{\Delta TSpc}$ is equivalent to skinny one upon composition with $\pi_! : \mathbf{\Delta TSpc} \to \mathcal{S}$. The, the fattened up diagram is Reedy cofibrant w.r.t. embeddings of diffeological spaces. Thus, the colimit of the fattened up digram $\widetilde{B}(X, D, *)$ is again a diffeological space.

We are thus lead to show that $\widetilde{B}(X, D, *) \to B(X, D, *)$ is an $\mathbf{R}$-homotopy equivalence. Using H^X from proof of Theorem 7.1.13, we obtain map $B(X, D, *) \to \widetilde{B}(X, D, *)$, which we claim is continuous. We then obtain commutative diagram

$$\begin{tikzcd} \widetilde{B}(X, D, *) \arrow[r] \arrow[d] & B(X, D, *) \arrow[d] \\ \widetilde{B}(*, D, *) \arrow[r] & B(*, D, *) \end{tikzcd}$$

Consider the continuous map $\mathbf{R}^d \to B(X, D, *)$, then we want to show that its composition with $B(X, D, *) \to \widetilde{B}(X, D, *)$ is continuous. Next, consider the composition $\mathbf{R}^d \to B(X, D, *) \to B(*, D, *)$, then, by the local compactness of $\mathbf{R}^d$, its restriction to any bounded open subset $U \subseteq \mathbf{R}^d$ will factor through a finite subcomplex K of $B(*, D, *)$. Pulling back, we obtain a new diagram

$$\begin{tikzcd} \widetilde{L} \arrow[r] \arrow[d] & L \arrow[d] \\ \widetilde{K} \arrow[r] & K \end{tikzcd}$$

and the diagram

$$\begin{array}{ccc} \widetilde{B}(X,D,*) & \longleftarrow & B(X,D,*) \\ \downarrow & & \downarrow \\ \widetilde{B}(*,D,*) & \longleftarrow & B(*,D,*) \end{array}$$

restricts to a diagram

$$\begin{array}{ccc} \widetilde{L} & \longleftarrow & L \\ \downarrow & & \downarrow \\ \widetilde{K} & \longleftarrow & K \end{array}$$

so we must show that the composition $\widetilde{L} \leftarrow L \leftarrow U$ is continuous. Now, $x \in U$, then the composition of $K \leftarrow L \leftarrow U$ sends x to a point $p \in K$, which is located in a cell $C \subseteq K$. By construction H^X sends some neighbourhood V of x to the same cell C. Denote by B the preimage of C under $L \to K$, then H^X sends V to B, and we are done.

Finally we observe that when $X : D \to \mathbf{TSpc}$ does not factor through $\mathbf{\Delta TSpc}$, then H^X fixes this problem too. $\square$

7.2 The smooth Oka principle

We now implement the plan described in the introduction of this section.

7.2.1 The squishy fibration structure on $\mathbf{Diff}^\infty$

Before we can define the desired fibration structure we must define a precursor, the ε-squishy model structure.

Notation 7.2.1. Let $0 < \alpha < \beta < \frac{1}{2}$, then $\lambda_\alpha^\beta : [0,1] \to [0,1]$ denotes any map such that

(a) $\lambda_\alpha^\beta|_{[0,\alpha]} \equiv 0$, $\lambda_\alpha^\beta|_{[1-\alpha,1]} \equiv 1$,

(b) $\lambda_\alpha^\beta(t) = t$ for all $t \in \left[\frac{1}{2}(\beta+\alpha), 1 - \frac{1}{2}(\beta+\alpha)\right]$, and

(c) $\dot{\lambda}_\alpha^\beta(t) > 0$ for all $t \in (\alpha, 1-\alpha)$.

⌟

7.2.1.1 ε-squishy intervals and cubes Throughout this subsection we fix $0 < \varepsilon < \frac{1}{2}$.

Definition 7.2.2. The pushout of the span

$$\begin{array}{ccc} [0,\varepsilon] \cup [1-\varepsilon, 1] & \twoheadrightarrow & \{0\} \cup \{1\} \\ \big\downarrow & & \\ \square_\varepsilon^1 & & \end{array}$$

(in $\mathbf{Diff}^r$) is called the ε-***squishy interval*** and is denoted by $\square_\varepsilon^1$. For any $n \in \mathbf{N}$ the n-fold product of $\square_\varepsilon^1$ is called the ε-***squishy n-cube***, and is denoted by $\square_\varepsilon^n$. ⌟

Proposition 7.2.3. *The ε-squishy n-cube $\square_\varepsilon^n$ is 0-truncated for all $n \in \mathbf{N}$.*

Proof. This is an immediate consequence of Lemma 4.1.1. □

Proposition 7.2.4. *The ε-squishy cubes generate* $\mathbf{Diff}^{\infty}$. □

Lemma 7.2.5. *The differentiable sheaf* $\boxminus^1_\varepsilon$ *is* $\boxminus^1_\varepsilon$*-contractible.*

Proof. Set $\alpha = \varepsilon$, fix any $\alpha < \beta < \frac{1}{2}$, and write $\lambda := \lambda^\beta_\alpha$. Also, define

$$\mu : s \mapsto \left(\frac{1}{2} - \varepsilon\right) \cdot \lambda\left(\frac{1}{\frac{1}{2} - \varepsilon}(s - \varepsilon)\right) + \varepsilon,$$

and

$$\nu : s \mapsto \left(\frac{1}{2} - \varepsilon\right) \cdot \lambda\left(\frac{1}{\frac{1}{2} - \varepsilon}\left(s - \frac{1}{2}\right)\right) + \frac{1}{2}.$$

Consider the map

$$\begin{array}{rcl} H: \; [0,1] \times [0,1] & \to & [0,1] \\ (s,t) & \mapsto & \begin{cases} t & \text{if } \; 0 \le s \le \varepsilon \\ \frac{1}{\frac{1}{2}-\varepsilon}\big((\lambda(t) - t)\cdot \mu(s) + \frac{1}{2}t - \lambda(t)\cdot\varepsilon\big) & \text{if } \; \varepsilon \le s \le \frac{1}{2} \\ \frac{1-\varepsilon-\nu(s)}{\frac{1}{2}-\varepsilon}\cdot \lambda(t) & \text{if } \; \frac{1}{2} \le s \le 1-\varepsilon \\ 0 & \text{if } \; 1-\varepsilon \le s \le 1. \end{cases} \end{array}$$

This map clearly factor through $\boxminus^1_\varepsilon \times [0,1] \to [0,1]$. Also, observe that $H(s,t) = \lambda(t)$ for $t \in \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right)$. We will check seperately that $H|_{[0,\frac{1}{2}+\delta)\times[0,1]}$ and $H|_{(\frac{1}{2}-\delta,1]\times[0,1]}$ factor through $\left[0, \frac{1}{2} + \delta\right) \times \boxminus^1_\varepsilon \to \boxminus^1_\varepsilon$ and $\left(\frac{1}{2} - \delta, 1\right] \times \boxminus^1_\varepsilon \to [0,1]$, respectively.

In the first case, $H(s,t) \in [0,\varepsilon) \cup (1-\varepsilon, 1]$ for all values $t \in [0,\varepsilon) \cup (1-\varepsilon,1]$, so that $H|_{[0,\frac{1}{2}+\delta)\times[0,\varepsilon)}$ and $H|_{[0,\frac{1}{2}+\delta)\times(1-\varepsilon,1]}$ composed with $[0,1] \to \boxminus^1_\varepsilon$ are constant.

In the second case, as $\lambda|_{[0,\varepsilon)} \equiv 0$ and $\lambda|_{(1-\varepsilon,1]} \equiv 1$, the map $H|_{(\frac{1}{2}-\delta,1]\times[0,1]}$ is independent of t for all $t \in [0,\varepsilon) \cup (1-\varepsilon,1]$. □

Proposition 7.2.6. *The ε-squishy equivalences and the locally constant equivalences in* $\mathbf{Diff}^r$ *agree. There exist cofibrantly generated model structures on* $\mathbf{Diff}^r$ *and* $\mathbf{Diff}^r_{\le 0}$ *with the aforementioned weak equivalences and with generating cofibrations and trivial cofibrations given, respectively, by* $\{\partial\boxminus^n_\varepsilon \hookrightarrow \boxminus^n_\varepsilon\}_{n\ge 0}$ *and* $\{\sqcap^n_{k,\varepsilon} \hookrightarrow \boxminus^n_\varepsilon\}_{n\ge 1,\, n\ge k\ge 0}$.

The model structures on $\mathbf{Diff}^r$ and $\mathbf{Diff}^r_{\le 0}$ constructed in Proposition 7.2.6 are both referred to as the ***ε-squishy model structure***. Unfortunately, it does not seem to be the case that all objects are fibrant in the ε-squishy model structure. Moreover, the model structure depends on the parameter $0 < \varepsilon < \frac{1}{2}$. Removing this dependence *does* produce a fibration structure, in which all objects are fibrant.

7.2.1.2 Squishy intervals and cubes

Definition 7.2.7. The pro-differentiable stack

$$\boxminus^1 := \text{“}\lim_{\varepsilon>0}\text{”}\boxminus^1_\varepsilon$$

is called the ***squishy interval***. For any $n \in \mathbf{N}$ the n-fold product of $\boxbox^1$ is called the ***squishy n-cube***, and is denoted by $\boxbox^n$. The resulting cocubical pro-object is denoted as follows:

$$\begin{array}{rccl} \boxbox^\bullet : & \square & \to & \mathrm{Pro}(\mathbf{Diff}^r) \\ & \square^n & \mapsto & \boxbox^n \end{array}$$

⌟

By Proposition C.0.3 the functor $\boxbox^\bullet : \square \to \mathrm{Pro}(\mathbf{Diff}^r)$ may thus be extended to a colimit preserving functor $\boxbox^\bullet_! : \underline{\mathrm{Hom}}(\square^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Pro}(\mathbf{Diff}^r)$.

Notation 7.2.8. We write

$$\begin{array}{rcl} \partial\boxbox^n & := & \boxbox^\bullet_! \partial\square^n, \quad n \geq 0 \\ \sqcap\!\!\!\sqcap^n_{k,\xi} & := & \boxbox^\bullet_! \sqcap^n_{i,\xi}, \quad n \geq 1, \ n \geq k \geq 0, \xi = 0, 1. \end{array}$$

⌟

Proposition 7.2.9. *There is a canonical isomorphism*

$$\boxbox^n \simeq \text{“}\lim_{\varepsilon > 0}\text{”} \boxbox^n_\varepsilon \quad n \geq 0.$$

Proof. There is an isomorphism $\boxbox^n \simeq \text{“}\lim_{(\varepsilon_1>0)\times\cdots\times(\varepsilon_n>0)}\text{”} \boxbox^1_{\varepsilon_1} \times \cdots \times \boxbox^1_{\varepsilon_n}$ by the proof of Proposition C.0.1. As the ordered set $\left(0, \frac{1}{2}\right)$ admits products it is sifted, and the diagonal map $\left(0, \frac{1}{2}\right) \to \left(0, \frac{1}{2}\right) \times \cdots \times \left(0, \frac{1}{2}\right)$ is initial so that the induced map $\text{“}\lim_{\varepsilon>0}\text{”} \boxbox^1_\varepsilon \times \cdots \times \boxbox^1_\varepsilon \to \text{“}\lim_{(\varepsilon_1>0)\times\cdots\times(\varepsilon_n>0)}\text{”} \boxbox^1_{\varepsilon_1} \times \cdots \times \boxbox^1_{\varepsilon_n}$ is an isomorphism. □

Proposition 7.2.10. *The differentiable sheaves $\boxbox^n, \partial\boxbox^n, \sqcap\!\!\!\sqcap^n_{k,\xi}$ are compact.*

Proof. Follows from Appendix C and [Lur09, Cor. 5.3.4.15]. □

Definition 7.2.11. A map $X \to Y$ of differentiable stacks is a ***squishy fibration*** if it has the right lifting property w.r.t. all inclusions $\sqcap\!\!\!\sqcap^n_{k,\xi} \hookrightarrow \boxbox^n$ $(n \geq 1, \ n \geq k \geq 0, \xi = 0, 1)$, and a ***trivial squishy fibration*** if it has the right lifting property w.r.t. the inclusions $\partial\boxbox^n \hookrightarrow \boxbox^n$ for all $n \geq 0$. ⌟

Lemma 7.2.12. *The diagonal map $X \to \underline{\mathbf{Diff}}^r(\Delta^1, X)$ is shape equivalence.* □

Lemma 7.2.13. *Let $0 < \varepsilon' < \varepsilon < \frac{1}{2}$, then the triangle*

$$\begin{tikzcd} \square^1 \arrow[rr, "\lambda^\varepsilon_{\varepsilon'}"] \arrow[dr, two heads] & & \square^1 \arrow[dl, two heads] \\ & \boxbox^1_\varepsilon & \end{tikzcd}$$

commutes.

Proof. It is enough to show that composing $[\varepsilon', 1-\varepsilon'] \to \square^1 \to \boxbox^1_\varepsilon$ yields an epimorphism, then the statement follows from the observation that the triangle

$$\begin{tikzcd} & {[\varepsilon', 1-\varepsilon']} \arrow[dl] \arrow[dr] & \\ \mathbb{\square}^1_\varepsilon \arrow[rr, "{[\lambda^\varepsilon_{\varepsilon'}]}"] & & \mathbb{\square}^1_\varepsilon \end{tikzcd}$$

commutes. To see this, let X be any differentiable space, then any map $f : \square^1 \to X$, which descends to a map $\mathbb{\square}^1_\varepsilon \to X$, may be obtained by glueing $f|_{(\varepsilon', 1-\varepsilon')} : (\varepsilon', 1-\varepsilon') \to X$ with $\big[0, \frac{1}{2}(\varepsilon' + \varepsilon)\big) \to 1 \xrightarrow{f(\varepsilon')} X$ and $\big(1 - \frac{1}{2}(\varepsilon' + \varepsilon), 1\big] \to 1 \xrightarrow{f(1-\varepsilon')} X$ along their common intersection. $\square$

Theorem 7.2.14. *Let X be a differentiable stack, then*

$$X^{\Delta^n} \to X^{\partial \Delta^n}$$

is a squishy fibration for any $n \geq 0$.

Proof. In this proof we use the following notation:

$$\begin{aligned} \square^k \star_{i,\xi} \Delta^n &:= \Big(\sqcap^k_{i,\xi} \times \Delta^n\Big) \sqcup_{\square^k \times \Delta^n} \Big(\square^k \times \partial \Delta^n\Big)\,; \\ \mathbb{\square}^k \star_{i,\xi} \Delta^n &:= \Big(\mathbb{\sqcap}^k_{i,\xi} \times \Delta^n\Big) \sqcup_{\mathbb{\square}^k \times \Delta^n} \Big(\mathbb{\square}^k \times \partial \Delta^n\Big)\,; \\ \mathbb{\square}^k_\varepsilon \star_{i,\xi} \Delta^n &:= \Big(\mathbb{\sqcap}^k_{i,\xi,\varepsilon} \times \Delta^n\Big) \sqcup_{\mathbb{\square}^k_\varepsilon \times \Delta^n} \Big(\mathbb{\square}^k_\varepsilon \times \partial \Delta^n\Big) \qquad \big(0 < \varepsilon < \tfrac{1}{2}\big)\,. \end{aligned}$$

We must show that for every $n \geq 1$, $n \geq k \geq 0$ and $\xi = 0, 1$

$$\begin{tikzcd} \mathbb{\square}^k \star_{i,\xi} \Delta^n \arrow[r] \arrow[d, hook] & X \\ \mathbb{\square}^n \times \Delta^n & \end{tikzcd} \tag{15}$$

admits a lift. The horizontal map is represented by a map

$$\square^k \star_{i,\xi} \Delta^n \to X$$

which factors through $\mathbb{\square}^k_\varepsilon \star_{i,\xi} \Delta^n$ for some $0 < \varepsilon < \frac{1}{2}$. Fix $0 < \varepsilon' < \varepsilon$, and write $\lambda := \lambda_{\varepsilon'}$. To prove the statement we define maps $\mu, \nu : \square^k \times \Delta^n \to \square^k \times \Delta^n$ such that the digram

$$\begin{tikzcd} \square^k \star_{i,\xi} \Delta^n \arrow[rrrr, "{\left(\lambda^k \times \mathrm{id}_{\Delta^n}|_{\square^k \star_{i,\xi} \Delta^n}\right)^2}"] \arrow[d, hook] & & & & \square^k \star_{i,\xi} \Delta^n \arrow[d, hook] \\ \square^k \times \Delta^n \arrow[r, "\lambda^k \times \mathrm{id}_{\Delta^n}"] & \square^k \times \Delta^n \arrow[r, "\mu"] & \square^k \times \Delta^n \arrow[r, "\nu"] & \square^k \times \Delta^n \arrow[r, "\lambda^k \times \mathrm{id}_{\Delta^n}"] & \square^k \times \Delta^n \end{tikzcd}$$

commutes and admits a diagonal lift. (Qualitatively, the first instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ ensures that the resulting lift factors through $\mathbb{\square}^k_{\varepsilon'} \times \Delta^n$, μ is a first approximation to the desired retract, next ν completes the retraction in the "Δ^n-direction", and, finally, the second instance of $\lambda^k \times \mathrm{id}_{\Delta^n}$ completes the retract in the "$\square^i$-direction".) Recall, that by Lemma 7.2.13 the map

$\lambda^k \times \mathrm{id}_{\Delta^n} : \square^k \times \Delta^n \to \square^k \times \Delta^n$ descends to the identity map $\mathrm{id} : \square^k_\varepsilon \times \Delta^n \to \square^k_\varepsilon \times \Delta^n$, so that the triangle obtained by postcomposing with $\square^k \star_{i,\xi} \Delta^n \to X$ in

$$\begin{tikzcd} \square^k \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & \square^k \star_{i,\xi} \Delta^n \arrow[r] & X \\ \square^k \times \Delta^n \arrow[ur] & & \end{tikzcd}$$

commutes, yielding a commutative diagram

$$\begin{tikzcd} \square^k_{\varepsilon'} \star_{i,\xi} \Delta^n \arrow[r] \arrow[d] & X \\ \square^k_{\varepsilon'} \times \Delta^n, \arrow[ur] & \end{tikzcd}$$

which descends to a lift of (15).

Construction of μ and ν: In order to ease the notational burden we will only define μ and ν for $i = k$ and $\xi = 1$.

To define μ, I require an auxiliary smooth function $\rho : \square^{k-1} \times \Delta^n \to \square^1$, such that

(a) $\rho(t_1, \ldots, t_k, s_0, \ldots, s_n) = 1$ if $t_1, \ldots, t_k > \frac{2}{3} \cdot \varepsilon'$ or $s_0 + \cdots + s_n > \frac{2}{3}$;

(b) $\rho(t_1, \ldots, t_k, s_0, \ldots, s_n) = 0$ if $t_1, \ldots, t_k < \frac{1}{3} \cdot \varepsilon'$ and $s_0 + \cdots + s_n < \frac{1}{3}$.

Then, we define

$$\begin{array}{rrcl} \mu : & \square^k \times \Delta^n & \to & \square^k \times \Delta^n \\ & ((t_1, \ldots, t_k), s) & \mapsto & ((t_1, \ldots, t_{k-1}, \rho(t_1, \ldots, t_{k-1}, s) \cdot t_k), s). \end{array}$$

Using partition of unity one can patch together the retractions $\Delta^n \to \Lambda^n_{k_2}$, $1 \leq k_2 \leq n$ to obtain a retract $\sigma : \left\{ (s_0, \ldots, s_n) \in \Delta^n \;\middle|\; s_0 + \cdots + s_n > \frac{1}{3} \right\} \to \partial\Delta^n$. Now, let $\tau : \square^1 \to \square^1$ be a smooth map such that

(a) $\tau(t) = 1$ for $t > \frac{2}{3} \cdot \varepsilon'$, and

(b) $\tau(t) = 0$ for $t < \frac{1}{3} \cdot \varepsilon'$.

Then, we define

$$\begin{array}{rrcl} \nu : & \square^k \times \Delta^n & \to & \square^k \times \Delta^n \\ & ((t_1, \ldots, t_k), s) & \mapsto & (\,(t_1, \ldots, t_k), \mathrm{id}_{\Delta^n} + (\,\mathrm{id}_{\Delta^n} + \tau(t_k) \cdot (\sigma - \mathrm{id}_{\Delta^n})\,)(s)\,). \end{array}$$

Proof of smoothness of lift: By construction, it is clear that the lift is smooth at any point which gets mapped to $\square^k \times \Delta^n \setminus \left(\square^{k-1} \times \{0\}\right) \times \partial\Delta^n$. Points which get mapped to $\left(\square^{k-1} \times \{0\}\right) \times \partial\Delta^n$ admit a neighbourhood which gets mapped to $\left(\square^{k-1} \times \{0\}\right) \times \Delta^n$, which concludes the proof. $\square$

Proposition 7.2.15. *The squishy and shape equivalences agree.*

Unfortunately we don't know how to prove this in a conceptually enlightening way.

Proof. Any shape equivalence is a squishy equivalence: For any squishy equivalence $X \to Y$ in the arrow ∞-category of $\underline{\mathrm{Hom}}(\Delta^{\mathrm{op}}, \mathcal{S})$ we have:

$$\begin{aligned}
&\underline{\mathbf{Diff}}^r(\square^\bullet, X) \to \underline{\mathbf{Diff}}^r(\square^\bullet, Y) \\
&= \underline{\mathbf{Diff}}^r(\operatorname*{colim}_{\varepsilon>0} \square^\bullet_\varepsilon, X) \to \underline{\mathbf{Diff}}^r(\operatorname*{colim}_{\varepsilon>0} \square^\bullet_\varepsilon, Y) \\
&= \operatorname*{colim}_{\varepsilon>0} \underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, X) \to \operatorname*{colim}_{\varepsilon>0} \underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, Y) \\
&= \operatorname*{colim}_{\varepsilon>0} \left(\underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, X) \to \underline{\mathbf{Diff}}^r(\square^\bullet_\varepsilon, Y)\right)
\end{aligned}$$

Conversely, let $X \to Y$ be a squishy equivalence. We will reduce ourselves down to the situation in which $X \to Y$ is a squishy trivial fibration between diffeological spaces, and then show that we obtain a bijection on homotopy groups. First we obtain a square

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, two heads, "\sim"'] & Y' \arrow[d, two heads, "\sim"] \\
X \arrow[r] & Y
\end{tikzcd}$$

in which X', Y' are cofibrant, and thus 0-truncated. The map $X' \to Y'$ is obtained by lifting cofibrant X' against trivial fibration $Y' \to Y$. The consider the counit of the Kihara adjunction:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\sim"'] & B \arrow[d, "\sim"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

Then we factor $A \xrightarrow{\sim} C \twoheadrightarrow B$ in the Kihara model structure on diffeological space. Next we factor $C \to C \times_B B^{\Delta^1} \to B$, then $C \times_B B^{\Delta^1} \to B$ is a squishy fibration. By the 2-out-of-3 property $C \to B$ is a squishy equivalence. Because $B \to C$ is a Kihara fibration $C \to C \times_B B^{\Delta^1}$ is a shape equivalence. Thus $C \times_B B^{\Delta^1} \to B$ is also a squishy equivalence.

Thus, we may assume that $X \to Y$ is a squishy trivial fibration between diffeological spaces. It is a surjection, so it is enough that show that it induces bijections on homotopy groups. Choose $x \in X$, which is mapped to $y \in Y$. Via simple reparametrisations it is easily seen for any diffeological space Z, any point $z \in Z$, and any $n \geq 1$ that $\pi_n(Z, z)$ defined in any of the equivalent ways discussed in [CW14, Th. 3.2] is canonically bijective to both

- the Δ^1-hompotopy classes of maps $(\square^n, \partial\square^n) \to (Z, z)$, and
- and the Δ^1-hompotopy classes of maps $(\partial\square^{n+1}, \{(0, \dots, 0)\}) \to (Z, z)$.

<u>The map $\pi_n(X, x) \to \pi_n(Y, y)$ is surjective:</u> Consider a smooth map $(\square^n, \partial\square^n) \to (Y, y)$. Then, for some $\varepsilon > 0$ we obtain the lifting problem described by the rightmost square

$$\begin{tikzcd}
\partial\square^n_{\varepsilon'} \arrow[r] \arrow[d, hook] & \partial\square^n_\varepsilon \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\
\square^n_{\varepsilon'} \arrow[r] & \square^n_\varepsilon \arrow[r] & Y
\end{tikzcd}$$

and by squishiness we know that the outer square admits a lift for some $0 < \varepsilon' < \varepsilon$, which yields a lift of $(S^n, *) \to (Y, y)$.

The map $\pi_n(X, x) \to \pi_n(Y, y)$ is injective: For $n = 0$ consider a path two points in X, which get mapped to $y_0, y_1 \in Y$, as well as a path $\Delta^1 \to Y$ connecting y_0 and y_1. This path can be reparametrised, and we end up with the same situation as above. For $n > 0$ consider a smooth map $(\partial\square^{n+1}, \{(0, \dots, 0)\}) \to (X, x)$ which is sent to the identity class in $\pi_n(Y, y)$, then we obtain the rightmost square in

$$\begin{tikzcd}
\partial\square^{n+1}_{\varepsilon'} \arrow[r] \arrow[d] & \partial\square^{n+1}_{\varepsilon} \arrow[r] \arrow[d] & X \arrow[d, two heads] \\
\square^{n+1}_{\varepsilon'} \arrow[r] & \square^{n+1}_{\varepsilon} \arrow[r] & Y
\end{tikzcd}$$

By squishiness, there exists a $0 < \varepsilon' < \varepsilon$ such that the outer square admits a lift. □

Theorem 7.2.16. *The shape equivalences and the squishy fibrations determine a fibration structure on* $\mathbf{Diff}^r$*; moreover, all objects in this fibration structure are fibrant.*

Proof. We must verify that weak equivalences and fibrations described in the statement of the theory satisfy conditions (a) - (d) of Definition 4.2.7. Conditions (a) & (b) are clear.

Proof of (c): Let us call a map of differentiable stacks which has the right lifting property w.r.t. the inclusions $\partial\square^n \hookrightarrow \square^n$ for $n \geq 0$ a *squishy acyclic fibration.* As squishy fibrations and squishy acyclic fibrations are defined via a right lifting property they are both closed under pullback. It thus remains to show that the squishy acyclic fibrations are precisely the squishy fibrations which are also shape equivalences, but this is completely formal: Let $X \to Y$ be a map of differentiable stacks, then $X \to Y$ is an squishy acyclic fibration iff $(\square^\bullet)^* X \to (\square^\bullet)^* Y$ is a trivial fibration in $\underline{\mathrm{Hom}}(\square^{\mathrm{op}}, \mathcal{S})$ iff $(\square^\bullet)^* X \to (\square^\bullet)^* Y$ is both a fibration and a weak equivalence iff $X \to Y$ is both a squishy fibration and a shape equivalence.

Proof of (d): Let $X \to Y$ be a morphism of differentiable stacks. Denote by $X \to X_3 \to Y$ a factorisation of $X \to Y$ into a trivial cofibration followed by a fibration in the $\frac{1}{3}$-squishy model structure, then denote by $X_3 \to X_4 \to Y$ a factorisation of $X_3 \to Y$ into a fibration followed by a fibration in the $\frac{1}{4}$-squishy model structure and so on; finally, denote by $X \to X' \to Y$ the factorisation of $X \to Y$ obtained by taking the transfinite composition of the maps $X \to X_3 \to X_4 \to \cdots$. The map $X \to X'$ is a shape equivalence, as these are preserved by colimits. We claim that $X' \to X$ is a squishy fibration. To see this, consider a lifting problem

$$\begin{tikzcd}
\sqcap^n_{k,\xi} \arrow[r] \arrow[d] & X' \arrow[d] \\
\square^n \arrow[r] & Y
\end{tikzcd}$$

for some $n \geq 1,\ n \geq k \geq 0, \xi = 0, 1$, then by Proposition A.3.3 and the fact that compact objects are closed under finite colimits, we see that $\sqcap^n_{k,\xi} \to X'$ must factor through $X_n \to X'$ for some $n \geq 3$. We then obtain a new lifting problem

$$\begin{tikzcd}
\sqcap^n_{k,\xi,\varepsilon} \arrow[r] \arrow[d] & X_n \arrow[d] \\
\square^n_{\varepsilon} \arrow[r] & Y
\end{tikzcd}$$

for some $0 < \varepsilon < \frac{1}{n}$, so that we obtain a lift.
Finally, all differentiable stacks are seen to be fibrant by applying Theorem 7.2.14 for $n = 0$. $\square$

Definition 7.2.17. The fibration structures on $\mathbf{Diff}^\infty_{\leq 0}$ and $\mathbf{Diff}^\infty$ are both referred to as the ***squishy fibration structure***. $\lrcorner$

7.2.2 Formally cofibrant objects

Formally cofibrant objects are closed under various operations.

Proposition 7.2.18. *The subcategory $\mathbf{C} \subseteq \mathbf{Diff}^\infty$ of formally cofibrant objects is closed under arbitrary coproducts.*

Proof. All objects in $\mathbf{Diff}^\infty$ are fibrant in the squishy fibration structure. $\square$

Proposition 7.2.19. *Let $A : \mathbf{N} \to \mathbf{Diff}^\infty$ be a diagram such that each object A_i is formally cofibrant, and such that $A_i \to A_{i+1}$ is a cofibration in the Kihara model structure for all $i \in \mathbf{N}$, then* $\operatorname{colim} A$ *is formally cofibrant.* $\square$

Proposition 7.2.20. *The subcategory $\mathbf{C} \subseteq \mathbf{Diff}^\infty$ of formally cofibrant stacks is closed under finite products.*

Proof. Let A, B be formally cofibrant stacks, and let X be any stack, then one obtains the following series of canonical equivalences:

$$\begin{array}{rcl} \pi_! \underline{\mathbf{Diff}}^\infty(A \times B, X) & \simeq & \pi_! \underline{\mathbf{Diff}}^\infty(A, \underline{\mathbf{Diff}}^\infty(B, X)) \\ & \xrightarrow{\simeq} & \mathcal{S}(\pi_! A, \pi_! \underline{\mathbf{Diff}}^\infty(B, X)) \\ & \xrightarrow{\simeq} & \mathcal{S}(\pi_! A, \mathcal{S}(\pi_! B, \pi_! X)) \\ & \simeq & \mathcal{S}(\pi_! A \times \pi_! B, \pi_! X) \\ & \simeq & \mathcal{S}(\pi_!(A \times B), \pi_! X). \end{array}$$

$\square$

Proposition 7.2.21. *The ∞-category of formally cofibrant objects is closed under $\mathbf{R}$-, Δ^1-, and $\square^1$-homotopy equivalence.* $\square$

7.2.3 Proof of the smooth Oka principle

Let X be a diffeological space, and let $U = \{U_\alpha\}_{\alpha \in A}$ be a cover of X then there exists a $\mathbf{Diff}^\infty_{\mathrm{concr}}$-enriched category X_U with

$$\begin{array}{rcl} \operatorname{Obj} X_U & = & \coprod_\sigma U_\sigma \\ \operatorname{Mor} X_U & = & \coprod_{\sigma \supseteq \tau} U_\sigma \end{array}$$

where σ, τ denote non-empty finite subsets of A such that $U_\sigma := \bigcap_{\alpha \in \sigma} U_\alpha \neq \varnothing$. The geometric realisation of (the nerve of) X_U is denoted by BX_U. The space BX_U may be constructed in stages using the pushouts

$$\begin{tikzcd} \coprod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} U_{\sigma_n} \times \partial\Delta^n \arrow[r, hook] \arrow[d, hook] & BX_U^{(n-1)} \arrow[d, hook] \\ \coprod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} U_{\sigma_n} \times \Delta^n \arrow[r, hook] & BX_U^{(n)} \end{tikzcd} \tag{16}$$

At each stage one can construct inductively an obvious commutative square obtained by replacing $BX_U^{(n)}$ by X in (16), thus producing a canonical map $BX_U \to X$. As the pushouts at each step satisfy the conditions Proposition **??**, each stage $BX_U^{(n)}$ is a diffeological space; the object BX_U is then a diffeological space by Proposition **??**, as it is a filtered colimit of diffeological spaces along monomorphisms.

Definition 7.2.22. A covering on a diffeological space is called ***numerable*** if it admits a subordinate partition of unity. ⌟

The original formulation of the following lemma in the setting of topological spaces is due to Segal [Seg68, §4] and tom Dieck [tD71, Th. 4]. Translating these results into the smooth setting is very technical, and is carried out by Kihara in [Kih20, §9].

Lemma 7.2.23 ([Kih20, Prop. 9.5])**.** *Let X be a diffeological space, and let U be a numerable cover of X, then the canonical map $BX_U \to X$ is a Δ^1-homotopy equivalence.* □

Theorem 7.2.24. *Let X be a diffeological space, and let U be a numerable cover of X. If each member of U is formally cofibrant, then so is X.*

Proof. By Lemma 7.2.23 and Proposition 7.2.21 the space X is formally cofibrant iff BX_U is. We will show that each stage $BX_U^{(n)}$ is formally cofibrant, and then conclude that BX_U is formally cofibrant by Proposition 7.2.19. The diffeological space $BX_U^{(0)}$ is formally cofibrant by Proposition 7.2.18. Applying $\underline{\mathbf{Diff}}^\infty(\_, X)$ to the square (16) yields the pullback

$$\begin{tikzcd} \underline{\mathbf{Diff}}^\infty(BX_U^{(n)}, X) \arrow[r] \arrow[d] & \prod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} \underline{\mathbf{Diff}}^\infty(U_{\sigma_n}, X)^{\Delta^n} \arrow[d] \\ \underline{\mathbf{Diff}}^\infty(BX_U^{(n-1)}, X) \arrow[r] & \prod\limits_{\sigma_n \supsetneq \cdots \supsetneq \sigma_0} \underline{\mathbf{Diff}}^\infty(U_{\sigma_n}, X)^{\partial\Delta^n} \end{tikzcd}$$

in which the vertical morphism to the right is sharp as it is a squishy fibration by Theorem 7.2.14 and Proposition **??**. □

Corollary 7.2.25. *Any Hausdorff paracompact manifold is formally cofibrant.*

Proof. Such manifolds admit numerable covers in which all intersections are either empty or diffeomorphic to $\mathbf{R}^n$ for some $n \geq 0$. (2022-10-04: Intersection property only needed to show that manifolds are $\mathbf{R}$-*equivalent* to complexes). □

Remark 7.2.26. The above corollary may be extended to infinite dimensional manifolds, as studied in [KM97]; see [Kih20, Th. 11.1]. ⌟

7.2.4 Counterexamples

There are many directions in which it is not possible to extend Corollary 7.2.25.

Example 7.2.27. $B\mathbf{Z} = \pi_! \underline{\mathbf{Diff}}(1, S^1) = \pi_! \underline{\mathbf{Diff}}(\pi^* B\mathbf{Z}, S^1) \neq \mathcal{S}(B\mathbf{Z}, B\mathbf{Z}) = \mathbf{Z}$. ⌟

One must be careful when dropping the Hausdorfness requirement:

Example 7.2.28. Denote by $\mathbf{R}_{\bullet\bullet}$ the real line with two origins, then

$$\begin{aligned} B\mathbf{Z} &= \pi_! \underline{\mathbf{Diff}}(\mathbf{R}, S^1) \\ &= \pi_! \underline{\mathbf{Diff}}(\mathbf{R}_{\bullet\bullet}, S^1) \\ &\neq \mathcal{S}(\pi_! \mathbf{R}_{\bullet\bullet}, \pi_! S^1) \\ &= \mathcal{S}(B\mathbf{Z}, B\mathbf{Z}) \\ &= \mathbf{Z}. \end{aligned}$$

⌟

Example 7.2.29. Denote by $\mathbf{R}_{||}$ the space obtained by glueing two copies of $\mathbf{R}$ along the subspace $(-\infty, -1) \cup (1, \infty)$, then $\mathbf{R}_{||}$ is $\mathbf{A}^1$-homotopy equivalent to S^1, so that it is formally cofibrant. In particular,

$$\pi_! \underline{\mathbf{Diff}}(\mathbf{R}_{||}, S^1) = \pi_! \underline{\mathbf{Diff}}(S^1, S^1) = \mathcal{S}(\pi_! S^1, \pi_! S^1) = \mathcal{S}(\pi_! \mathbf{R}_{||}, \pi_! S^1).$$

⌟

Non-paracompact manifolds may not be formally cofibrant:

Example 7.2.30. Let $\mathbf{L}$ denote the long line. It has trivial shape but is not contractible. Thus $\mathcal{S}(\pi_! \mathbf{L}, \pi_! \mathbf{L}) = \mathcal{S}(1, 1) = 1$, while $\underline{\mathbf{Diff}}(\mathbf{L}, \mathbf{L})$ has at least two path components. ⌟

Appendix

A Compact manifolds are compact

Here we show that closed manifolds and modified notions of closed intervals are both compact in the categorical sense.

The only general paradigm that we are aware of for proving categorical compactness in (∞-)toposes involves n-coherent ∞-toposes (see [Lur18, §A.2.3.]). These are ∞-toposes satisfying strong boundedness conditions. Applying this theory to the case at hand would amount to showing that $\mathbf{Diff}^\infty$ is generated by sufficiently compact objects (in the topological sense), and that these are closed under pullback. This has no hope of being true; while, for instance, closed manifolds *do* indeed generate $\mathbf{Diff}^\infty$, it is possible to obtain the Cantor set (with the discrete differential structure) as a pullback of closed manifolds. Fortunately, the pullback stability of compact manifolds is close enough to being true by virtue of all manifolds being locally compact. In order to exploit this property we must acquire a good understanding of the sheafification procedure, which we do in the following subsection.

A.1 Sheafification in one step

Throughout this subsection C denotes a small (ordinary) site. Let $X : C^{\mathrm{op}} \to \mathcal{S}_{\leq n}$ be a presheaf, then for $n = 0$ the *plus-construction* (recalled below), first introduced in [SGA 4_{I}, §4.ii.3], is a functor $\widehat{C} \to \widehat{C}$ such that a double application thereof to X produces the sheaf universally associated to X. For arbitrary $n < \infty$ the plus-construction must be applied $n+1$ times in order to obtain the stack universally associated to X (see [Lur09, §6.5.3]). The plus-construction is built using Čech coverings; an analogous construction using hypercovers produces the universally associated stack in one step. Considering again the case $n = 0$, methods of constructing associated sheaves using a *variant* of hypercovers, has been part of the folklore for decades, but only appeared in writing in [Yuh07], following ideas of Dubuc; this is the construction that we will use in this section.

We will first recall the plus-construction so that we may better contrast it with the sheafification construction in one step. For more details, we refer the reader to [Yuh07, §3.2]. From now on the word *presheaf* refers to a presheaf valued in sets.

Let X be a presheaf on C, let c be an object of C, and let $U = \{u \to c\}$ be a covering of c. A ***matching family of*** X ***at*** U is a family of maps $\{u \to X\}$, satisfying the following property: For any pair of morphisms $a \to u$, $a \to u'$, if the square

$$\begin{tikzcd} & u \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & c \\ & u' \arrow[ur] & \end{tikzcd}$$

commutes, then so does

$$\begin{tikzcd} & u \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & X \\ & u' \arrow[ur] & \end{tikzcd}$$

Given another covering $V = \{v \to c\}$, then a matching family of X at V is a ***refinement*** of the matching family of X at U if for each span $c \leftarrow v \to X$ in the matching family at V there exists a span $c \leftarrow u \to X$ in the matching family at U together with a morphism $v \to u$ such that the resulting diagram

$$\begin{tikzcd} & v \arrow[dl] \arrow[dd] \arrow[dr] & \\ c & & X \\ & u \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. The ***plus-construction*** applied to X yields a presheaf X^+, which sends any object c in C to the set of matching families of X at coverings of c modulo the equivalence relation generated by refinement.

Dubuc and Yuhjtman's construction is obtained by modifying the notion of matching family. For a covering $U = \{u \to c\}$ a ***hyper-matching family of*** X ***at*** U is a family of maps $\{u \to X\}$ satisfying the following property: For any pair of morphisms $a \to u$, $a \to u'$, if the square

$$\begin{tikzcd} & u \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & c \\ & u' \arrow[ur] & \end{tikzcd}$$

commutes, there exists a covering $\{v \to a\}$ such that the squares

$$\begin{tikzcd} & u \arrow[dr] & \\ v \arrow[ur] \arrow[dr] & & X \\ & u' \arrow[ur] & \end{tikzcd}$$

obtained from composing the morphisms $v \to a$ in the covering with $a \to u$ and u' commute. Refinement of hyper-matching families is defined as for matching families. The assignment of equivalence classes of hyper-matching families to objects in C is functorial. We denote resulting presheaf on C by $X^\ddagger$.

Proposition A.1.1 ([Yuh07, §3.2])**.** *The presheaf $X^\ddagger$ is the associated sheaf of X.* $\square$

A.2 Closed manifolds are compact

Throughout this subsection we fix $0 \leq r \leq \infty$. We now apply Proposition A.1.1 to prove the following theorem.

Theorem A.2.1. *Let M be a closed manifold, then $\mathbf{Diff}^r_{\leq 0}(M, \_)$ commutes with filtered colimits.*

Proof. Let A be a filtered category, and consider a functor $X : A \to \mathbf{Diff}^\infty_{\leq 0}$, then we must show that the canonical morphism

$$\operatorname*{colim}_\alpha \mathbf{Diff}^\infty_{\leq 0}(M, X_\alpha) \to \mathbf{Diff}^\infty_{\leq 0}(M, \operatorname*{colim}_\alpha X_\alpha) \tag{17}$$

is invertible. Recall, that the category $\mathbf{Diff}^\infty_{\leq 0}$ is equivalently given by the category of sheaves on $\mathbf{Mfd}^\infty$. Denote by F the colimit of X in $\widehat{\mathbf{Mfd}}_\infty$, then $\mathbf{Diff}^\infty_{\leq 0}(M, \operatorname{colim}_\alpha X_\alpha) = F^\ddagger(M)$.
<u>The map (17) is injective:</u> Let $f : M \to X_\alpha$, $g : M \to X_{\alpha'}$ be two maps which get mapped to the same equivalence class by (17), then there exists a covering of M by open subsets $\{U_i\}_{i \in I}$, such that for each $i \in I$ there exists an object α_i and maps $\alpha \to \alpha_i$, $\alpha' \to \alpha_i$ such that $f|_{U_i} = g|_{U_i} : U_i \to X_{\alpha_i}$. Because M is compact, the covering $\{U_i\}_{i \in I}$ admits a finite subcovering (which is a refinement of $\{U_i\}_{i \in I}$); choose an object β in A with morphisms $\alpha_i \to \beta$ for all $i \in I'$, then f and g determine the same matching family of X_β at M, which must descend to a morphism $M \to X_\beta$, because X_β is a sheaf, and this morphism is in the same equivalence class as f and g, so that f and g are equivalent.
<u>The map (17) is surjective:</u> Let $\{U_i\}_{i \in I}$ be a covering of M, and let $\{f_i : U_i \to F\}_{i \in I}$ be a

hyper-matching family of F, then one may again restrict to a finite subcovering $\{U_i\}_{i\in I'}$. For each $i, j \in I'$ choose a covering $\{W_{ijk}\}_{k\in K_{ij}}$ of $U_i \cap U_j$. The covering $\{U_i\}_{i\in I'}$ can be further refined to a covering $\{V_i\}_{i\in I'}$ such that $\overline{V}_i \subseteq U_i$ for each $i \in I'$. Each intersection $\overline{V}_i \cap \overline{V}_j$ lies in $\bigcup_{k\in K'_{ij}} W_{ijk}$ for a finite subset $K'_{ij} \subseteq K_{ij}$. The intersection $V_i \cap V_j$ is then covered by $\{W_{ijk} \cap V_i \cap V_j\}_{k\in K'}$. We then obtain a hyper-matching family $\{f|_{V_i} : V_i \to F\}_{i\in I'}$; as I' is finite all maps in the hyper-matching family may be represented by maps $f|_{V_i} : V_i \to X_\alpha$ for some fixed α in A. For each pair i, j' in I we can then find an object α_{ij} in A and a map $\alpha \to \alpha_{ij}$ such that for each $k \in K'_{ij}$ the resulting square

$$\begin{tikzcd}
 & V_i \arrow[dr] & \\
W_{kij} \arrow[ur] \arrow[dr] & & X_{\alpha_{ij}} \\
 & V_j \arrow[ur] &
\end{tikzcd}$$

commutes. Chose an object β in A and a morphism $\alpha_{ij} \to \beta$ for each pair i, j in I', then the resulting compositions of $f|_{V_i} : V_i \to X_\alpha \to X_\beta$ for all $i \in I'$ form a hyper-matching family of M at X_β, and thus a map $M \to X_\beta$, because X_β is a sheaf. □

A.3 Special intervals are compact

Throughout this subsection we fix $0 \leq r \leq \infty$. As discussed in §5.2.2, ordinary intervals seem to be determined by infinitely many plots, and are thus unlikely to be categorically compact. However, this also means that for any differentiable stack X which is not a manifold without boundary it is very hard to construct maps $[0, 1] \to X$, even if one has a good grasp of the maps $\mathbf{R}^n \to X$. It thus stands to reason that the *extendable* and *squishy intervals* considered here are likely to be more useful in practice, and luckily these *are* categorically compact.
Let M be a smooth manifold (without boundary), then by Proposition 5.2.12, maps $[0, 1] \to M$ are precisely the maps which are the restriction of maps $(-\varepsilon, 1 + \varepsilon) \to M$. This motivates the following definition.

Definition A.3.1. The functor

$$\begin{array}{rcl}
\mathbf{Diff}^{\infty}_{\leq 0} & \to & \mathbf{Set} \\
X & \mapsto & \left\{ f : [0,1] \to X \;\middle|\; \exists\, \varepsilon > 0, \exists\, \tilde{f} : (-\varepsilon, 1+\varepsilon) \to X, \text{ s.t. } f = \tilde{f}|_{j[0,1]} \right\}
\end{array}$$

is called the ***extendable interval***, and is denoted by $\mathbf{Diff}^{\infty}(\llbracket 0, 1\rrbracket, \_)$. ⌟

Proposition A.3.2. *The extendable interval preserves filtered colimits.*

Proof. The comparison map

$$\operatorname*{colim}_{\alpha} \mathbf{Diff}^{\infty}_{\leq 0}(\llbracket 0, 1\rrbracket, X_\alpha) \to \mathbf{Diff}^{\infty}_{\leq 0}(\llbracket 0, 1\rrbracket, \operatorname*{colim}_{\alpha} X_\alpha) \tag{18}$$

is given by the restriction of

$$\operatorname*{colim}_{\alpha} \mathbf{Diff}^{\infty}_{\leq 0}([0, 1], X_\alpha) \to \mathbf{Diff}^{\infty}_{\leq 0}([0, 1], \operatorname*{colim}_{\alpha} X_\alpha).$$

The proof is very similar to the proof of Proposition A.2.1. Again, denote by F the colimit of X in $\widehat{\mathbf{Mfd}}_\infty$.

The map (18) is injective: Let $f : [\![0,1]\!] \to X_\alpha$, $g : [\![0,1]\!] \to X_{\alpha'}$ be two maps which get mapped to the same equivalence class by (18). W.l.o.g. we may assume that there exists a single value $\varepsilon > 0$, such that the maps f, g are represented by maps $\widetilde{f} : (-\varepsilon, 1+\varepsilon) \to X_\alpha$, $\widetilde{g} : (-\varepsilon, 1+\varepsilon) \to X_{\alpha'}$. There exists some $0 < \varepsilon' < \varepsilon$ and a covering of $(-\varepsilon', 1+\varepsilon')$ by open subsets $\{U_i\}_{i\in I}$, such that for each $i \in I$ there exists an object α_i and maps $\alpha \to \alpha_i$, $\alpha' \to \alpha_i$ such that $f|_{U_i} = g|_{U_i} : U_i \to X_{\alpha_i}$. For any $0 < \varepsilon'' < \varepsilon'$ there exists a finite subset $I' \subseteq I$ such that $(-\varepsilon'', 1+\varepsilon'') \subseteq \bigcup_{i\in I'} U_i$, and this part of the proof proceeds precisely as in the corresponding part of the proof Proposition A.2.1.

The map (18) is surjective: Consider $\varepsilon > 0$, and let $\{U_i\}_{i\in I}$ be a covering of $(-\varepsilon, 1+\varepsilon)$, and let $\{f_i : U_i \to F\}_{i\in I}$ be a hyper-matching family of F. By restricting to $(-\varepsilon', 1+\varepsilon')$ for some $0 < \varepsilon' < \varepsilon$ one may again restrict to a finite subcovering $\{U_i\}_{i\in I'}$ and proceed as in the second half of the proof of Proposition A.2.1. □

Proposition A.3.3. *The squishy interval $\boxdot^1$ is compact.*

Proof. Similar to the proof of Proposition A.3.2. □

Corollary A.3.4. *Squishy (trivial) fibrations are closed under filtered colimits.* □

Vista A.3.5. It is possible to define extendable and squishy variants of compact manifolds with boundary (and possibly even manifolds with corners), which are then also compact. ⌟

B The cube category

Here we discuss some background material on the cube category needed in §**??**.

Definition B.0.1. The ***cube category*** $\square$ is the subcategory of **Set** whose objects are given by $\{0,1\}^n$ $(n \geq 0)$, and whose morphisms are generated by the maps

$$\begin{array}{rrcl} \delta_i^\xi : & \square^{n-1} & \to & \square^n \\ & (x_1,\dots,x_{n-1}) & \mapsto & (x_1,\dots,x_{i-1},\xi,x_i,\dots,x_{n-1}) \end{array}$$

for $n \geq i \geq 1$ and $\varepsilon = 0, 1$, and

$$\begin{array}{rrcl} \sigma_i : & \square^{n+1} & \to & \square^n \\ & (x_1,\dots,x_{n+1}) & \mapsto & (x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1}) \end{array}$$

for $n \geq 0$ and $n \geq i \geq 1$. The category of ***cubical sets*** is the category $\widehat{\square}$ of presheaves on $\square$. ⌟

The cube category $\square$ admits a (strict) monoidal structure given by $(\square^m, \square^n) \mapsto \square^{m+n}$ which extends to cubical sets via Day convolution. This monoidal structure is denoted by $\otimes$.

We denote by $\square^{\leq 1}$ the full subcategory of $\square$ spanned by $\square^0, \square^1$.

Proposition B.0.2 ([Cis06, Prop. 8.4.6])**.** *Let M be a monoidal category, then the restriction functor*

$$\underline{\mathrm{Hom}}(\square, M) \to \underline{\mathrm{Hom}}(\square^{\leq 1}, M)$$

induces an equivalence of categories between the full subcategory of $\mathbf{Cat}(\square, M)$ *spanned by monoidal functors, and the full subcategory of* $\mathbf{Cat}(\square^{\leq 1}, M)$ *spanned by functors sending* $\square^0$ *to the monoidal unit of* M*.* $\square$

Definition B.0.3. For every $n \geq 0$ the ***boundary of*** $\square^n$ is the subobject $\partial\square^n := \cup_{(j,\zeta)}\mathrm{Im}_{\delta_j^\zeta} \subset \square^n$, and for every $n \geq i \geq 1$ and $\xi = 0, 1$ the (i,ξ)***-th horn of*** $\square^n$ is the subobject $\sqcap^n_{i,\xi} := \cup_{(j,\zeta)\neq(i,\xi)}\mathrm{Im}_{\delta_j^\zeta} \subset \square^n$. ⌟

Proposition B.0.4 ([Cis06, Lm. 8.4.36])**.** *For* $m \geq 1$*,* $n \geq k \geq 1$ *and* $\varepsilon = 0, 1$ *the universal morphisms determined by the pushouts of the spans contained in the commutative squares*

$$\begin{tikzcd}
\sqcap^n_{i,\varepsilon} \otimes \partial\square^m \arrow[r] \arrow[d] & \square^n \otimes \partial\square^m \arrow[d] \\
\sqcap^n_{i,\varepsilon} \otimes \square^m \arrow[r] & \square^n \otimes \square^m
\end{tikzcd}
\qquad
\begin{tikzcd}
\partial\square^m \otimes \sqcap^n_{i,\varepsilon} \arrow[r] \arrow[d] & \partial\square^n \otimes \square^m \arrow[d] \\
\square^m \otimes \sqcap^n_{i,\varepsilon} \arrow[r] & \square^n \otimes \square^m
\end{tikzcd}$$

recover the canonical inclusions $\sqcap^{n+m}_{i,\varepsilon} \hookrightarrow \square^{n+m}$ *and* $\sqcap^{n+m}_{i+m,\varepsilon} \hookrightarrow \square^{n+m}$ *and the universal morphism determined by the pushout of the span contained in the commutative square*

$$\begin{tikzcd}
\partial\square^m \otimes \partial\square^n \arrow[r] \arrow[d] & \partial\square^m \otimes \square^n \arrow[d] \\
\square^m \otimes \partial\square^n \arrow[r] & \square^m \otimes \square^n
\end{tikzcd}$$

recovers the inclusion $\partial\square^{m+n} \hookrightarrow \square^{m+n}$*.* $\square$

Theorem B.0.5 ([Cis06, Cor. 8.4.13 or Prop. 8.4.27])**.** *The cube category* $\square$ *is a test category.* $\square$

Theorem B.0.6 ([Cis06, Th. 8.4.38])**.** *The maps*

(i) $\partial\square^n \hookrightarrow \square^n$ *(*$n \geq 0$*), and*

(ii) $\sqcap^n_{i,\varepsilon} \hookrightarrow \square^n$ *(*$n \geq i \geq 1$*,* $\varepsilon = 0, 1$*)*

generate, respectively, the cofibrations and acyclic cofibrations of the test model structure on $\widehat{\square}$*.* $\square$

C Pro-objects in ∞-categories

Proposition C.0.1. *Let* C *be an* ∞*-category admitting finite products, then* $\mathrm{Pro}(C)$ *admits finite products.*

Proof. Let $x_0, \dots, x_n$ be objects in $\mathrm{Pro}(C)$ then for each $0 \leq i \leq n$ there exists a filtered small ordinary category A_i and a functor $x_{i\bullet} : A_i \to C$ such that $x_i \simeq$ “$\lim_{\alpha \in A_i}$”$x_{i\alpha}$ (see [Lur09, Prop. 5.3.1.16]). The category $A_0 \times \cdots \times A_n$ is filtered, and we claim that “$\lim_{\alpha \in A_0 \times \cdots \times A_n}$”$x_{0\bullet} \times \cdots \times x_{n\bullet}$ pro-represents the product of $x_0, \dots, x_n$. To see this, let y be any objects of C, then the isomorphisms

$$
\begin{aligned}
\mathrm{Pro}(C)(\text{“}\lim_{A_0\times\cdots\times A_n}\text{”}x_{0\bullet}\times\cdots\times x_{n\bullet},y) &\simeq \lim_{A_0\times\cdots\times A_n} C(x_{0\bullet}\times\cdots\times x_{n\bullet},y)\\
&\simeq \lim_{A_0\times\cdots\times A_n} C(x_{0\bullet},y)\times\cdots\times C(x_{n\bullet},y)\\
&\simeq \lim_{A_0}\cdots\lim_{A_n} C(x_{0\bullet},y)\times\cdots\times C(x_{n\bullet},y)\\
&\simeq \lim_{A_0}\cdots\lim_{A_{n-1}} C(x_{0\bullet},y)\times\cdots\times C(x_{n-1\bullet},y)\times C(x_n,y)\\
&\cdots\\
&\simeq \lim_{A_0} C(x_{0\bullet},y)\times C(x_1,y)\times\cdots\times C(x_n,y)\\
&\simeq C(x_0,y)\times\cdots\times C(x_n,y)
\end{aligned}
$$

are natural in y. □

Lemma C.0.2. *Let I be a set, and for each element $i\in I$ consider a small filtered category A_i and a functor $X_i : A_i \to \mathcal{S}$, then the canonical morphism*

$$\operatorname*{colim}_{(\alpha_i)\in\prod A_i}\prod_{i\in I} X_{i,\alpha_i}\to\prod_{i\in I}\operatorname*{colim}_{\alpha_i\in A_i} X_{i,\alpha_i} \tag{19}$$

is an equivalence.

Proof. By [KS06, Prop. 3.1.11.ii] the statement is true in **Set**. Then, by [Cis19, Cor. 7.9.9] we may lift the functors $X_i : A_i \to \mathcal{S}$ to functors $A_i \to \widehat{\Delta}$, which we may then compose with the Ex^∞ functor to obtain functors valued in Kan complexes. The morphism in $\widehat{\Delta}$ corresponding to (19) is then an isomorphism, and the statement follows from the fact that Kan complexes as well as weak equivalences are closed under filtered colimits (see [Cis19, Lm. 3.1.24 & Cor. 4.1.17]). □

Proposition C.0.3. *Let C be an accessible ∞-category admitting finite limits and coproducts, then the ∞-category $\mathrm{Pro}(C)$ is cocomplete.*

Proof. We show that $\mathrm{Pro}(C)$ admits pushouts and small coproducts.
<u>$\mathrm{Pro}(C)$ admits pushouts:</u> Recall that $\mathrm{Pro}(C)$ may be identified with the full subcategory of $\underline{\mathrm{Hom}}(C,\mathcal{S})^{\mathrm{op}}$ spanned by the left exact functors $f : C \to \mathcal{S}$ such that $C_{/f}$ is accessible ([DAGXIII, Prop. 3.1.6]). Consider a pullback square

$$
\begin{tikzcd}
p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & f \arrow[d]\\
g \arrow[r] & h
\end{tikzcd}
$$

of functors in $\underline{\mathrm{Hom}}(C,\mathcal{S})$ with f,g,h in $\mathrm{Pro}(C)$. As limits of functors are computed pointwise, $p : C \to \mathcal{S}$ commutes with finite limits. Moreover, the above diagram induces a homotopy pullback diagram

$$
\begin{tikzcd}
C^{\mathrm{op}}_{/p} \arrow[r] \arrow[d] & C^{\mathrm{op}}_{/f} \arrow[d]\\
C^{\mathrm{op}}_{/g} \arrow[r] & C^{\mathrm{op}}_{/h}
\end{tikzcd}
$$

in $\widehat{\Delta}$ w.r.t. the Joyal model structure. The morphisms $C^{\mathrm{op}}_{/f} \to C^{\mathrm{op}}_{/g}$ and $C^{\mathrm{op}}_{/h} \to C^{\mathrm{op}}_{/g}$ are colimits preserving, so that $C_{/p}$ is accessible by [Lur09, Prop. 5.4.6.6].

Pro(C) admits small coproducts: Let I be a small set, and consider a family of objects $x_\bullet : I \to \mathrm{Pro}(C)$, then for each i there exists a filtered small ordinary category A_i and a functor $x_{i\bullet} : A_i \to C$ such that $x_i \simeq$ " $\lim_{\alpha \in A_i}$ "$x_{i\alpha}$ (see [Lur09, Prop. 5.3.1.16]). We obtain canonical isomorphisms

$$\coprod_{i\in I} x_i \simeq \coprod_{i \in I} \text{“} \lim_{\alpha_i \in A_i} \text{”} x_{i\alpha_i} \simeq \text{“} \lim_{(\alpha_i)\in \prod A_i} \text{”} \coprod_{i\in I} x_{i\alpha_i},$$

in $\underline{\mathrm{Hom}}(C^{\mathrm{op}}, \mathcal{S})$, as limits and colimits in presheaf categories are computed pointwise. □

Proposition C.0.4. *Let x be an object of* Pro(C), *which may be realised as the cofiltered limit of compact objects in C, then* $\mathrm{Pro}(C)(x, \_) : C \to \mathcal{S}$ *commutes with filtered colimits.*

Proof. Let A be a small cofiltered category and $x_\bullet : A \to C$ a diagram, such that "$\lim_{\alpha\in A} x_\alpha \simeq x$, the for any small filtered category B and diagram $y_\bullet : B \to C$ we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Pro}(C)\big(x, \operatorname*{colim}_{\beta\in B} y_\beta\big) &\simeq \mathrm{Pro}(C)\big(\lim_{\alpha\in A} x_\alpha, \operatorname*{colim}_{\beta\in B} y_\beta\big)\\
&\simeq \operatorname*{colim}_{\alpha\in A} C\big(x_\alpha, \operatorname*{colim}_{\beta\in B} y_\beta\big)\\
&\simeq \operatorname*{colim}_{\alpha\in A} \operatorname*{colim}_{\beta\in B} C\big(x_\alpha, y_\beta\big)\\
&\simeq \operatorname*{colim}_{\beta\in B} \operatorname*{colim}_{\alpha\in A} C\big(x_\alpha, y_\beta\big)\\
&\simeq \operatorname*{colim}_{\beta\in B} \mathrm{Pro}(C)\big(x, y_\beta\big)
\end{aligned}$$

□

Conventions and notation

C.0.0.1 Linguistic conventions In order to facilitate readability we use the following contractions:

- We write "iff" instead of "if and only if".
- We write "w.l.o.g." instead of "without loss of generality".
- We write "w.r.t." instead of "with respect to".

C.0.0.2 Editorial conventions

- Propositions stated without proof are marked with the symbol "□".

C.0.0.3 Mathematical conventions

- We identify ordinary categories with their nerves, and consequently do not notationally distinguish between ordinary categories and their nerves.
- ∞-categories (including ordinary categories) are denoted by C, D, ...

- Let C be an ∞-category and let $x, y \in C$ be two objects, then the homotopy type of morphisms from x to y is denoted by $C(x, y)$.

- For any Cartesian closed ∞-category C and any two objects x, y in C the internal hom object in C is denoted by $\underline{C}(x, y)$ or sometimes y^x.

- For any ∞-category C we denote it subcategory of n-truncated objects by $C_{\leq n}$.

- For A any small ordinary category $\widehat{A}$ denotes the category of (set-valued) presheaves on A.

- For any two categories C, D, an arrow $C \hookrightarrow D$ denotes a fully faithful functor.

- We use the following notation for various ∞-categories: ★★★add all categories!
  - $\mathcal{S}$ denotes the ∞-categories of homotopy types.
  - $\mathcal{Q}$ denotes the ∞-category of ∞-categories.
  - $\mathcal{C}$ denotes the ∞-category of ordinary categories.
  - **Set** denotes the category of sets.
  - **TSpc** denotes the category of topological spaces.
  - Δ**TSpc** is the full subcategory of **TSpc** spanned by the Δ-generated topological spaces.
  - $\mathbf{Mfd}^r$ denotes the category of r-times differentiable smooth manifolds and smooth maps.
  - $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.
  - $\mathbf{Diff}^r$ denotes, equivalently, the ∞-category of sheaves on $\mathbf{Mfd}^r$ or $\mathbf{Cart}^r$.

- $\underline{\mathrm{Hom}}$ denotes the internal hom in $\widehat{\Delta}$, the category of simplicial sets.

- Let X be a simplicial set, then $X_{\simeq}$ denotes the classifying space of X, given e.g. by $\mathrm{Ex}^{\infty} A$.

- We denote ∞-toposes by $\mathcal{E}, \mathcal{F}, \ldots$, when they are thought of as ambient settings in which to do geometry, and by $\mathcal{X}, \mathcal{Y}, \ldots$, when they are thought of as geometric objects in their own right.

References

[ABFJ20] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. A generalized blakers-massey theorem. Preprint available at `https://arxiv.org/abs/1703.09050`, 2020.

[ADH21] Araminta Amabel, Arun Debray, and Peter J. Haine. Differential cohomology: Categories, characteristic classes, and connections. Preprint available at `https://arxiv.org/abs/2109.12250`, 2021.

[Ane21] Mathieu Anel. Enveloping ∞-topoi. In preparation, 2021.

[Aya09] David Ayala. *Geometric cobordism categories.* PhD thesis, Stanford University, 2009.

[BEBP19] Daniel Berwick-Evans, Pedro Boavida de Brito, and Dmitri Pavlov. Classifying spaces of infinity-sheaves. Preprint available at `https://arxiv.org/abs/1912.10544v2`, 2019.

[BK12] C. Barwick and D. M. Kan. Relative categories: another model for the homotopy theory of homotopy theories. *Indag. Math. (N.S.)*, 23(1-2):42–68, 2012.

[Car16] David Carchedi. On the homotopy type of higher orbifolds and Haefliger classifying spaces. *Adv. Math.*, 294:756–818, 2016.

[Car20] David Joseph Carchedi. Higher orbifolds and Deligne-Mumford stacks as structured infinity-topoi. *Mem. Amer. Math. Soc.*, 264(1282):v+120, 2020.

[Cis03] Denis-Charles Cisinski. Faisceaux localement asphériques. Definitive preliminary version available at `http://www.mathematik.uni-regensburg.de/cisinski/publikationen.html`, 2003.

[Cis06] Denis-Charles Cisinski. Les préfaisceaux comme modèles des types d'homotopie. *Astérisque*, (308):xxiv+390, 2006.

[Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2019.

[CW14] J. Daniel Christensen and Enxin Wu. The homotopy theory of diffeological spaces. *New York J. Math.*, 20:1269–1303, 2014.

[CW21] J. Daniel Christensen and Enxin Wu. Smooth classifying spaces. *Isr. J. Math.*, 241(2):911–954, 2021.

[DI04] Daniel Dugger and Daniel C. Isaksen. Topological hypercovers and $\mathbb{A}^1$-realizations. *Math. Z.*, 246(4):667–689, 2004.

[Dot14] Emanuele Dotto. A relative h-principle via cobordism-like categories. In *An alpine expedition through algebraic topology*, volume 617 of *Contemp. Math.*, pages 133–155. Amer. Math. Soc., Providence, RI, 2014.

[Dug98] Daniel Dugger. Sheaves and homotopy theory. Incomplete draft available at `https://pages.uoregon.edu/ddugger/cech.html`, 1998.

[Fra11] John Francis. The h-principle. Lecture notes of a course taught at Northwestern, 2011.

[GG73] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*. Springer-Verlag, New York-Heidelberg, 1973. Graduate Texts in Mathematics, Vol. 14.

[GL12] Richard Garner and Stephen Lack. Grothendieck quasitoposes. *J. Algebra*, 355:111–127, 2012.

[Gro83] Alexander Grothendieck. Pursuing stacks. Unpublished letter to D. Quillen. TEXed version available at `https://thescrivener.github.io/PursuingStacks/`, 1983.

[Gro86] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

[GTMW09] Søren Galatius, Ulrike Tillmann, Ib Madsen, and Michael Weiss. The homotopy type of the cobordism category. *Acta Math.*, 202(2):195–239, 2009.

[GZ67] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[Hae58] André Haefliger. Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes. *Comment. Math. Helv.*, 32:248–329, 1958.

[Hae71] A. Haefliger. Homotopy and integrability. Manifolds - Amsterdam 1970, Proc. NUFFIC Summer School Manifolds 1970, Lect. Notes Math. 197, 133-163 (1971)., 1971.

[HHLN21] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. Lax monoidal adjunctions, two-variable fibrations and the calculus of mates. Preprint available at `https://arxiv.org/abs/2011.08808`, 2021.

[Hir59] Morris W. Hirsch. Immersions of manifolds. *Trans. Amer. Math. Soc.*, 93:242–276, 1959.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[Hoy18] Marc Hoyois. Higher Galois theory. *J. Pure Appl. Algebra*, 222(7):1859–1877, 2018.

[HS18] Tadayuki Haraguchi and Kazuhisa Shimakawa. A model structure on the category of diffeological spaces. Preprint available at `https://arxiv.org/abs/1311.5668v7`, 2018.

[Ill72] Luc Illusie. *Complexe cotangent et déformations. II.* Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin-New York, 1972.

[Joh02] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*, volume 44 of *Oxford Logic Guides.* The Clarendon Press, Oxford University Press, Oxford, 2002.

[Kih17] Hiroshi Kihara. Quillen equivalences between the model categories of smooth spaces, simplicial sets, and arc-gengerated spaces. Preprint available at `https://arxiv.org/abs/1702.04070v1`, 2017.

[Kih19] Hiroshi Kihara. Model category of diffeological spaces. *J. Homotopy Relat. Struct.*, 14(1):51–90, 2019.

[Kih20] Hiroshi Kihara. Smooth homotopy of infinite-dimensional c^{∞}-manifolds. Preprint available at `https://arxiv.org/abs/2002.03618v1`, 2020.

[KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1997.

[KS06] Masaki Kashiwara and Pierre Schapira. *Categories and Sheaves*, volume 332 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag Berlin Heidelberg, 2006.

[Kup19] Alexander Kupers. Three applications of delooping to h-principles. *Geom. Dedicata*, 202:103–151, 2019.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[DAGX] Jacob Lurie. Derived algebraic geometry X: Formal moduli problems. Preprint available at `https://www.math.ias.edu/~lurie/papers/DAG-X.pdf`, 2011.

[DAGXIII] Jacob Lurie. Derived algebraic geometry XIII: Rational and p-adic homotopy theory. Preprint available at `http://www.math.harvard.edu/~lurie/papers/DAG-XIII.pdf`, 2011.

[Lur17] Jacob Lurie. Higher algebra. Available online at `http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`, 2017.

[Lur18] Jacob Lurie. Spectral algebraic geometry. Preprint notes available at `http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf`, 2018.

[Mal05] Georges Maltsiniotis. La théorie de l'homotopie de Grothendieck. *Astérisque*, (301):vi+140, 2005.

[MO78400] Akhil Mathew. Do we still need model categories? MathOverflow. URL:https://mathoverflow.net/q/78400 (version: 2011-10-17).

[May75] J. Peter May. Classifying spaces and fibrations. *Mem. Amer. Math. Soc.*, 1(1, 155):xiii+98, 1975.

[MG14] Aaron Mazel-Gee. Model ∞-categories I: some pleasant properties of the ∞-category of simplicial spaces. Preprint available at `http://arxiv.org/abs/1412.8411`, 2014.

[MG16] Aaron Mazel-Gee. *Goerss-Hopkins obstruction theory via model infinity-categories*. ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–University of California, Berkeley.

[Mil57] John W. Milnor. The geometric realization of a semi-simplicial complex. *Ann. Math. (2)*, 65:357–362, 1957.

[Mye22] David Jaz Myers. Good fibrations through the modal prism. *High. Struct.*, 6(1):212–255, 2022.

[Pav22] Dmitri Pavlov. Projective model structures on diffeological spaces and smooth sets and the smooth oka principle. Preprint available at `https://arxiv.org/abs/2210.12845`, 2022.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Rez98] Charles Rezk. Fibrations and homotopy colimits of simplicial sheaves. Preprint available at `http://www.math.uiuc.edu/~rezk/rezk-sharp-maps.pdf`, 1998.

[Rez10] Charles Rezk. Toposes and homotopy toposes. Preprint available at `http://www.math.uiuc.edu/~rezk/homotopy-topos-sketch.pdf`, 2010.

[RW11] Oscar Randal-Williams. Embedded cobordism categories and spaces of submanifolds. *Int. Math. Res. Not. IMRN*, (3):572–608, 2011.

[Sch75] Gerald W. Schwarz. Smooth functions invariant under the action of a compact Lie group. *Topology*, 14:63–68, 1975.

[Sch13] Urs Schreiber. Differential cohomology in a cohesive ∞-topos. Preprint available at `https://arxiv.org/abs/1310.7930v1`, 2013.

[Sch14] Pierre Schapira. Abelian sheaves. Lecture notes, available online at `http://www.math.jussieu.fr/~schapira/lectnotes`, 2014.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[Seg78] Graeme Segal. Classifying spaces related to foliations. *Topology*, 17:367–382, 1978.

[SGA 4_{I}] M. Artin, A. Grothendieck, and J.L. Verdier, editors. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos.* Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[Shu09] Michael Shulman. Homotopy limits and colimits and enriched homotopy theory. Preprint available at `http://arxiv.org/abs/math/0610194`, 2009.

[Sma59] Stephen Smale. The classification of immersions of spheres in Euclidean spaces. *Ann. of Math. (2)*, 69:327–344, 1959.

[Spa63] E. Spanier. Quasi-topologies. *Duke Math. J.*, 30:1–14, 1963.

[SS21] Hisham Sati and Urs Schreiber. Equivariant principal ∞-bundles. Preprint available at `https://arxiv.org/abs/2112.13654`, 2021.

[SW57] E. H. Spanier and J. H. C. Whitehead. The theory of carriers and S-theory. Algebraic geometry and topology. In *A symposium in honor of S. Lefschetz*, pages 330–360. Princeton University Press, Princeton, N.J., 1957.

[tD71] Tammo tom Dieck. Partitions of unity in homotopy theory. *Compositio Math.*, 23:159–167, 1971.

[Tho80] R. W. Thomason. Cat as a closed model category. *Cahiers Topologie Géom. Différentielle*, 21(3):305–324, 1980.

[Wal12] Konrad Waldorf. Transgression to loop spaces and its inverse, I: Diffeological bundles and fusion maps. *Cah. Topol. Géom. Différ. Catég.*, 53(3):162–210, 2012.

[Wat12] Jordan Watts. *Diffeologies, Differential Spaces, and Symplectic Geometry*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–University of Toronto (Canada).

[Yuh07] Sergio A. Yuhjtman. *Bilímites* 2*-cofiltrantes de topos*. PhD thesis, Universidad de Bueno Aires, December 2007. Available online at http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/yuhjtman.pdf.