DIFFERENTIABLE SHEAVES I: FRACTURED ∞-TOPOSES AND COMPACTNESS

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Abstract. In this note we endow the ∞-topos $\mathbf{Diff}^r$ of sheaves on the category of C^r-manifolds with the structure of a fractured ∞-topos and use this structure to give a simple proof that closed manifolds are categorically compact in $\mathbf{Diff}^r$. Moreover, surprisingly, we show that any manifold with non-empty corners is *not* categorically compact.

This is the first of three articles discussing $\mathbf{Diff}^r$ and its useful formal properties.

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In an ∞-topos of sheaves on a site of geometric objects such as

(1) manifolds (see [Car20, §6.1] and §2.1 of this article),
(2) schemes (ordinary, derived or spectral with either the Zariski or étale topology; see [DAG V, §4.2 & §4.3], [DAGVII], [Car20, §6.2], [Lur18, §2.6.4])
(3) derived complex analytic spaces (see [DAG IX, §11 & §12] & [Por19]), or
(4) derived manifolds (see [CS19], [Ste23]),

it is often possible to identify a suitable class of “étale morphisms”, giving rise to the attendant notion of Deligne-Mumford stack. In [Car20] Carchedi shows that, surprisingly, in many such cases the ∞-category of Deligne-Mumford stacks (with relaxed finiteness and separatedness conditions) and étale morphisms form an ∞-topos. The structure of an ∞-topos together with a class of Deligne-Mumford stacks and étale morphisms is axiomatised by *fractured ∞-toposes* in [Lur18, Def. 20.1.2.1]. Moreover, in the above example the slice ∞-topos of Deligne-Mumford stacks and étale morphisms over of a fixed Deligne-Mumford stack recovers its petit ∞-topos, so that fractured ∞-toposes also axiomatise the relationship between petit and gros ∞-toposes.

This notion of fractured ∞-topos has not received much attention and in this article we demonstrate a simple yet powerful application thereof to differentiable manifolds. Denote by $\mathbf{Diff}^r$ the ∞-topos of $\mathcal{S}$-valued sheaves on the category $\mathbf{Mfd}^r$ of C^r-manifolds. We equip $\mathbf{Diff}^r$ with the structure of a fractured ∞-topos. As the petit ∞-topos of any C^r-manifold X is equivalent to the ∞-topos of sheaves on its underlying topological space, we may express properties of X viewed as an object of $\mathbf{Diff}^r$ in terms of its underlying topological space. This allows for a simple proof of Theorem 2.15 that any closed manifold is categorically compact in $\mathbf{Diff}^r$. The surprising fact that any manifold with non-empty corners is *not* compact in $\mathbf{Diff}^r$ follows by contradiction from an explicit computation. Moreover, the classical results that $\mathbf{Diff}^r$ has enough points and is local is further clarified by considering the fractured ∞-topos structure.

This article has two sequels [Clo24a] and [Clo24b], in which, as in the present article, we refine some existing toposic or homotopical technology and apply it to $\mathbf{Diff}^r$. In [Clo24a] we use the fractured ∞-topos structure established here to give a new, conceptual, proof that $\mathbf{Diff}^r$ is locally contractible, and then develop a suitable notion of cofinal functors between ∞-toposes in order to show that various

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constructions which extract homotopy types from differentiable sheaves coincide with the shape functor. In [Clo24b] we develop homotopical calculi on locally contractible ∞-toposes and show that for a large class of differentiable sheaves the internal mapping sheaf functor is suitably compatible with the shape functor.

The present article is structured into two parts, §1 & §2, where the first part discusses the basics of fractured ∞-toposes, and the second applies this technology to differentiable manifolds.

Throughout the whole article, r denotes some element of $\mathbf{N} \cup \{\infty\}$.

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1. Fractured ∞-toposes

In §1.1 we provide a definition of fractured ∞-toposes equivalent to [Lur18, Def. 20.1.2.1] which highlights the salient properties necessary for us in the present article (as well as [Clo24a] & [Clo24b]). After this, we discuss some useful properties of fractured ∞-toposes. Then, in §1.2 we discuss the notion of *geometric sites* which allow us to construct fractured ∞-toposes (such as $\mathbf{Diff}^r$). Finally, in §1.3 we prove the equivalence between our definition of fractured ∞-topos and Lurie's.

1.1. Basic definitions and properties.

Definition 1.1. A ***fractured ∞-topos*** is an adjunction

$$j_! : \mathcal{E}^{\mathrm{corp}} \rightleftarrows \mathcal{E} : j^*$$

between ∞-toposes $\mathcal{E}^{\mathrm{corp}}$ and $\mathcal{E}$ satisfying properties (a) - (d) below:

(a) The topos $\mathcal{E}$ is generated under colimits by the objects in the image of $j_!$.

(b) For every object U in $\mathcal{E}^{\mathrm{corp}}$, the left adjoint in

$$(j_!)_{/U} : \mathcal{E}^{\mathrm{corp}}_{/U} \rightleftarrows \mathcal{E}_{/j_!U} : (j^*)_{/U}$$

is fully faithful.

(c) The functor $\mathcal{E}^{\mathrm{corp}} \leftarrow \mathcal{E} : j^*$ preserves colimits.

(d) For any pullback square

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V' & \longrightarrow & V \end{array}$$

in which $U \to V$ and V' are in the image of $j_!$, the map $U' \to V'$ is in the image of $j_!$.

An object in $\mathcal{E}$ is referred to as ***corporeal*** if it is in the image of $j_!$. ⌟

For the rest of this section $j_! : \mathcal{E}^{\mathrm{corp}} \to \mathcal{E}$ refers to a fixed fractured ∞-topos. Observe that $j_!$ is faithful (but never full, unless it is an equivalence). The ∞-topos $\mathcal{E}^{\mathrm{corp}}$ will then often be identified with its image under $j_!$.

A morphism $U \to X$ in a fractured ∞-topos $\mathcal{E}$ is called ***admissible*** if for every pullback diagram

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

in which X' is in $\mathcal{E}^{\mathrm{corp}}$, the morphism $U' \to X'$ is in $\mathcal{E}^{\mathrm{corp}}$. Thus, $\mathcal{E}^{\mathrm{corp}}$ may be identified with the ∞-category of corporeal objects in $\mathcal{E}$ together with the admissible morphisms. Under mild conditions the structure of a fractured ∞-topos may be recovered from its class of admissible morphisms (see [Lur18, Rmk. 20.3.4.6]). Axiom (d) in Definition 1.1 can then be thought of as consisting of two parts:

(1) Admissible morphisms are closed under pullbacks.

(2) If $U \to V$ is an admissible morphism, and V is a corporeal object, then U is a corporeal object.

For any corporeal object U the subcategory of $\mathcal{E}_{/U}$ spanned by the admissible morphisms is then equivalent to the ∞-topos $\mathcal{E}^{\mathrm{corp}}_{/U}$, so that we obtain *gros* and *petit* ∞-toposes $\mathcal{E}_{/U}$ and $\mathcal{E}^{\mathrm{corp}}_{/U}$ of U, respectively.

Remark 1.2. In [DAG V] Lurie introduces the notion of a *geometry* (see Remark 1.10), which is a site with extra structure. A different interpretation of the notion of fractured ∞-topos is given in [Lur18, §21] as a "coordinate free" version of geometries (in the sense that a site may be viewed as providing "coordinates" or generators and relations for an ∞-topos.). In the examples 2 - 4 listed in the beginning of this section the ∞-toposes are all classifying ∞-toposes for various flavours of locally ringed ∞-toposes, such as strictly Henselian or locally C^∞-ringed ∞-toposes. The structure of a fractured ∞-topos then makes it possible to define locally ringed morphisms for the various flavours of locally ringed ∞-toposes. From this perspective the ∞-topos $\mathbf{Diff}^r$ is unusual in that we don't know of any insightful way of thinking of it as a classifying ∞-topos. ⌟

We conclude this subsection with some basic properties of fractured ∞-toposes, which exhibit some ways in which $\mathcal{E}^{\mathrm{corp}}$ controls certain properties of $\mathcal{E}$.

Proposition 1.3. *The functor $\mathcal{E}^{\mathrm{corp}} \leftarrow \mathcal{E} : j^*$ is conservative.*

Proof. Let $X \to Y$ be a morphism in $\mathcal{E}$ such that $j^*X \to j^*Y$ is an isomorphism, then for every object U in $\mathcal{E}^{\mathrm{corp}}$ the map $\mathcal{E}^{\mathrm{corp}}(U, j^*X) \to \mathcal{E}^{\mathrm{corp}}(U, j^*Y)$ is an isomorphism, so that for every object U in $\mathcal{E}^{\mathrm{corp}}$ the map $\mathcal{E}(j_!U, X) \to \mathcal{E}(j_!U, Y)$ is an isomorphism. As any object Z can be written as a colimit of objects in the image of $j_!$ it follows that $\mathcal{E}(Z, X) \to \mathcal{E}(Z, Y)$ is an isomorphism, so that $X \to Y$ is an isomorphism by the Yoneda lemma. □

Corollary 1.4. *If $\mathcal{E}^{\mathrm{corp}}$ is hypercomplete, then so is $\mathcal{E}$.*

Proof. Let $f : X \to Y$ be an ∞-connected morphism in $\mathcal{E}$, then $j^*(f_{\leq n}) = (j^*f)_{\leq n}$ as j^* is both cocontinuous and preserves finite limits (see [Lur09b, Prop. 5.5.6.28]). Thus, $j^*X \to j^*Y$ is an isomorphism in $\mathcal{E}^{\mathrm{corp}}$, so that by Proposition 1.3 $X \to Y$ is an isomorphism. □

Corollary 1.5. *If $\mathcal{E}^{\mathrm{corp}}$ has enough points, then so does $\mathcal{E}$.*

Proof. Denote by j_* the right adjoint to j^* (which exists by the adjoint functor theorem), then any point $p_* : \mathcal{S} \to \mathcal{E}^{\mathrm{corp}}$ yields a point $j_*p_* : \mathcal{S} \to \mathcal{E}$. Let $X \to Y$ be a morphism in $\mathcal{E}$ such that $p^*j^*X \to p^*j^*Y$ is an isomorphism for every point p of $\mathcal{E}^{\mathrm{corp}}$, then $j^*X \to j^*Y$ is an isomorphism by assumption, and thus also $X \to Y$ by Proposition 1.3. □

Recall that an ∞-topos $\mathcal{F}$ is ***local*** if the global sections functor $\pi_* : \mathcal{F} \to \mathcal{S}$ admits a right adjoint, which we denote by $\pi^!$.

Proposition 1.6. *If there exists an object Φ in $\mathcal{E}^{\mathrm{corp}}$ such that $j_!\Phi = \mathbf{1}_{\mathcal{E}}$, and moreover $\mathcal{E}^{\mathrm{corp}}_{/\Phi} = \mathcal{S}$, then $\mathcal{E}$ is local.*

Proof. As $j^*_{/\Phi} = (\pi_{\mathcal{E}})_*$ commutes with colimits, we obtain a triple adjunction

$$\mathcal{S} = \mathcal{E}^{\mathrm{corp}}_{/\Phi} \quad \begin{matrix} \xrightarrow{\;(j_!)_{/\Phi}\;} \\ \xleftarrow{\;(j^*)_{/\Phi}\;} \\ \xrightarrow{\;(j_*)_{/\Phi}\;} \end{matrix} \quad \mathcal{E} = \mathcal{E}_{/j_!\Phi}$$

where $(j_!)_{/\Phi} \dashv (j^*)_{/\Phi}$ is the unique geometric morphism $\mathcal{E} \to \mathcal{S}$. □

1.2. **Geometric sites.** We now discuss the notion of *geometric site*, culminating in Theorem 1.12, which will allow us to exhibit $\mathbf{Diff}^r$ as a fractured ∞-topos.

Definition 1.7 ([Lur18, Def. 20.2.1.1])**.** An ***admissibility structure*** on an ∞-category G is a subcategory G^{ad} whose morphisms are referred to as *admissible morphisms*, such that:

(a) Every equivalence in G is an admissible morphism.
(b) For any admissible morphism $U \to X$ and any morphism $X' \to X$ there exists a pullback square

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X, \end{array}$$

in which $U' \to X'$ is admissible.

(c) For any commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

in which $g : Y \to Z$ is admissible, the morphism $f : X \to Y$ is admissible iff $h : X \to Z$ is.

(d) Admissible morphisms are closed under retracts.

⌟

Example 1.8. The admissible morphisms in a fractured ∞-topos form an admissibility structure. ⌟

Definition 1.9 ([Lur18, Def. 20.6.2.1])**.** A ***geometric site*** is a triple $(G, G^{\mathrm{ad}}, \tau)$ consisting of

(i) a small ∞-category G,

(ii) an admissibility structure G^{ad} on G, and

(iii) a Grothendieck topology τ on G,

such that every covering sieve in τ contains a covering consisting of admissible morphisms. ⌟

Remark 1.10. A geometric site $(G, G^{\mathrm{ad}}, \tau)$ for which G is finitely complete is called a *geometry* in [DAG V].

⌟

Lemma 1.11 ([Lur18, Props. 20.6.1.1 & 20.6.1.3])**.** *Let $(G, G^{\mathrm{ad}}, \tau)$ be a geometric site, then there exists a Grothendieck topology on G^{ad} in which a sieve R in G^{ad} is a covering sieve iff the sieve generated by R in G is a covering sieve. Any sheaf on G restricts to a sheaf on G^{ad}.* □

Theorem 1.12 ([Lur18, Th. 20.6.3.4])**.** *Let $(G, G^{\mathrm{ad}}, \tau)$ be a geometric site, and denote by $\mathcal{E}$ the ∞-topos of sheaves on G, and, by $\mathcal{E}^{\mathrm{corp}}$ the ∞-topos of sheaves on G^{ad}, then the restriction functor $\mathcal{E}^{\mathrm{corp}} \leftarrow \mathcal{E} : j^*$ admits a left adjoint, and the resulting adjunction is a fractured ∞-topos.* □

Remark 1.13. In the same way that not every ∞-topos is the category of sheaves on a site, not every fractured ∞-topos is given as in the preceding theorem. However, it *is* true that every fractured ∞-topos may be realised as the localisation of a fractured presheaf ∞-topos, and that this presheaf ∞-topos may be obtained as in the preceding theorem with $\tau = \varnothing$. See [Lur18, Th. 20.5.3.4]. ⌟

1.3. Equivalence with Lurie's definition of fractured ∞-toposes.

Proposition 1.14. *Definitions 1.1 and* [Lur18, Def. 20.1.2.1] *are equivalent.*

Proof. Lurie defines a fractured ∞-topos to be an ∞-topos $\mathcal{E}$ together with a subcategory $\mathcal{E}^{\mathrm{corp}}$ (which by [Lur18, Prop. 20.1.3.3] is an ∞-topos) satisfying conditions (0) - (3), which we now recall:

(0) If X is in $\mathcal{E}^{\mathrm{corp}}$ and Y is isomorphic to X, then Y is in $\mathcal{E}^{\mathrm{corp}}$.

(1) The inclusion functor $j_! : \mathcal{E}^{\mathrm{corp}} \to \mathcal{E}$ preserves fibre products.

(2) The inclusion functor $j_! : \mathcal{E}^{\mathrm{corp}} \to \mathcal{E}$ has a right adjoint, denoted by $\mathcal{E}^{\mathrm{corp}} \leftarrow \mathcal{E} : j^*$, which preserves colimits and is conservative.

(3) For every morphism $U \to V$ in $\mathcal{E}^{\mathrm{corp}}$ the square

$$\begin{tikzcd} j^*U \arrow[r] \arrow[d] & j^*V \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

is a pullback in $\mathcal{E}$.

First we prove (a) - (d) $\implies$ (1) - (2):

(d) $\implies$ (1): Let

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

be a pullback square in which $U \to V$ and $V' \to V$ are in $\mathcal{E}^{\mathrm{corp}}$, then (1) follows from applying (d) first to the above pullback square, and then to the pullback square obtained by switching $U \to V$ and $V' \to V$.
(a) & (c) $\implies$ (2): Follows from Proposition 1.3.

We now prove (1) - (3) $\implies$ (a) - (c): Axioms (a) and (b) follow from [Lur18, Cor. 20.1.3.4] and [Lur18, Prop. 20.1.3.1], respectively, and axiom (c) is contained in axiom (2).

We conclude the proof by showing that (d) $\iff$ (0) & (3) under the assumption of (a) - (c). Recall that we refer to the image under $j_!$ of any corporeal object U again by U.
(d) $\implies$ (3): Observe that by (b) & (d) the map $j^*U \to j^*V \times_V U$ is corporeal, so that for every corporeal object W we obtain a commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{corp}}(W, j^*U) \arrow[d] \arrow[drr] & & \\
\mathcal{E}^{\mathrm{corp}}(W, j^*V \times_V U) \arrow[r] \arrow[d] & \mathcal{E}(W, j^*V \times_V U) \arrow[r] \arrow[d] & \mathcal{E}(W, U) \arrow[d] \\
\mathcal{E}^{\mathrm{corp}}(W, j^*V) \arrow[r] & \mathcal{E}(W, j^*V) \arrow[r] & \mathcal{E}(W, V).
\end{tikzcd}$$

The rightmost square is a pullback by the definition of $j^*V \times_V U$, and the leftmost square is a pullback square by (b). But $\mathcal{E}^{\mathrm{corp}}(W, j^*V) \to \mathcal{E}(W, V)$ and $\mathcal{E}^{\mathrm{corp}}(W, j^*U) \to \mathcal{E}(W, U)$ are isomorphisms by the universal properties of j^*V and j^*U, and thus $\mathcal{E}^{\mathrm{corp}}(W, j^*V \times_V U) \to \mathcal{E}(W, U)$ is an isomorphism, so that $\mathcal{E}^{\mathrm{corp}}(W, j^*U) \to \mathcal{E}^{\mathrm{corp}}(W, j^*V \times_V U)$ is an isomorphism.
(0) & (3) $\implies$ (d): The pullback square in (d) factors as

$$\begin{tikzcd}
U' \arrow[r] \arrow[d] & j^*U \arrow[r] \arrow[d] & U \arrow[d] \\
V' \arrow[r] & j^*V \arrow[r] & V.
\end{tikzcd}$$

The rightmost square is a pullback by (3), and the outer square is a pullback by assumption, so that the leftmost square is also a pullback. The morphism $U' \to V'$ is then in the image of $j_!$ by (0) & (3). ☐

2. The fractured ∞-topos of differentiable sheaves

In this section we formally define the ∞-topos $\mathbf{Diff}^r$ of r-times differentiable sheaves, apply the machinery of §1 to exhibit $\mathbf{Diff}^r$ as a fractured ∞-topos, and derive some of its basic properties in §2.1. Finally, in §2.2 we give our first application of the fractured ∞-topos structure on $\mathbf{Diff}^r$ and show that closed manifolds are categorically compact in $\mathbf{Diff}^r$. Moreover, we show that manifolds with non-empty boundary or corners are *not* categorically compact.

2.1. Differentiable sheaves.

1. $\mathbf{Mfd}^r$ denotes the category of r-times differentiable (2^{nd}-countable, Hausdorff) manifolds and r-times differentiable maps, and
2. $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces $\mathbf{R}^n$ $(0 \leq n < \infty)$.

On each of these small categories we denote by τ the Grothendieck topology in which a sieve on a manifold is a covering sieve iff it contains a covering consisting of jointly surjective open embeddings.

1. $\mathbf{Mfd}^r_{\text{ét}}$ denotes the category of r-differentiable manifolds and r-differentiable open embeddings.
2. $\mathbf{Cart}^r_{\text{ét}}$ denotes the full subcategory of $\mathbf{Mfd}^r_{\text{ét}}$ spanned by the spaces for $\mathbf{R}^n$ $(0 \leq n < \infty)$.

On each of these (essentially) small subcategories we denote the restriction of τ by $\tau_{\text{ét}}$.

Definition 2.1. An $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r$ is an ***r-times differentiable sheaf***, and the ∞-topos thereof is denoted by $\mathbf{Diff}^r$. Similarly, an $\mathcal{S}$-valued sheaf on $\mathbf{Cart}^r_{\text{ét}}$ is an ***étale r-times differentiable stack***, and the ∞-topos thereof is denoted by $\mathbf{Diff}^r_{\text{ét}}$. ⌟

Combining [Aok23, Cor. A.7] with Corollary 2.14 below, we obtain the following result.

Proposition 2.2. *Denote by $u : \mathbf{Cart}^r \hookrightarrow \mathbf{Mfd}^r$ (resp. $u : \mathbf{Cart}^r_{\text{ét}} \hookrightarrow \mathbf{Mfd}^r_{\text{ét}}$) the canonical inclusion, then the restriction functor $[(\mathbf{Cart}^r)^{\mathrm{op}}, \mathcal{S}] \leftarrow [(\mathbf{Mfd}^r)^{\mathrm{op}}, \mathcal{S}] : u^*$ (resp. $[(\mathbf{Cart}^r_{\text{ét}})^{\mathrm{op}}, \mathcal{S}] \leftarrow [(\mathbf{Mfd}^r_{\text{ét}})^{\mathrm{op}}, \mathcal{S}] : u^*$) induces an equivalence between $\mathbf{Diff}^r$ (resp. $\mathbf{Diff}^r_{\text{ét}}$) and the ∞-topos of sheaves on $\mathbf{Mfd}^r$ (resp. $\mathbf{Mfd}^r_{\text{ét}}$).* □

Remark 2.3. Observe that our proof of Proposition 2.2 does not use good open covers. The question of their existence and whether they refine all open covers is subtle (as discussed in detail in [nLa23]). Moreover, arguments using good open covers may not carry over to other settings such as the real analytic or complex ones, which we hope to explore in the future. ⌟

Lemma 2.4. *The triple $(\mathbf{Mfd}^r, \mathbf{Mfd}^r_{\text{ét}}, \tau)$ is a geometric site.*

Proof. Axioms (a) - (c) are clear. To prove (d), consider a retract

$$\begin{tikzcd} V' \arrow[r, hook] & U' \arrow[r, two heads] & V' \\ V \arrow[u] \arrow[r, hook] & U \arrow[u, hook] \arrow[r, two heads] & V, \arrow[u] \end{tikzcd} \tag{1}$$

where $U \hookrightarrow U'$ is an open subset inclusion. Axiom (d) follows from (b) after proving the following claim:

Claim: The leftmost square in (1) is a pullback.

First, as monomorphisms have the left cancelling property, the map $V \to V'$ is a monomorphism. Let $y' \in V' \cap U$, then y' coincides with its image under $U \to V$, which shows that the leftmost square induces a pullback on underlying sets. Next, consider a commutative square

$$\begin{tikzcd} V' \arrow[r, hook] & U' \\ W \arrow[u] \arrow[r] & U, \arrow[u, hook] \end{tikzcd}$$

then the canonical map of sets $W \to V$ is smooth, as it may be written as the composition of $W \to U \to V$. □

Applying Theorem 1.12 we obtain the key result of this subsection:

Theorem 2.5. *The ∞-category $\mathbf{Diff}^r$ is a fractured ∞-topos whose ∞-topos of corporeal objects is given by $\mathbf{Diff}^r_{\text{ét}}$.* □

Remark 2.6. Observe that for any smooth manifold, the ∞-topos $(\mathbf{Diff}^r_{\text{ét}})_{/M}$ is equivalent to the ∞-topos of sheaves on underlying topological space of M (and is thus independent of r). ⌟

By [Car16, Th. C.3], $\mathbf{Diff}^\infty_{\text{ét}}$ is equivalent to the ∞-topos of sheaves on the category of smooth manifolds and local diffeomorphisms, so that the fractured ∞-topos $\mathbf{Diff}^\infty$ coincides with the one considered in [Car20, §6.1]. Carchedi moreover shows that $\mathbf{Diff}^\infty_{\text{ét}}$ coincides with

(1) the ∞-category of ∞-toposes locally ringed in $\mathbf{R}$-algebras, which can étale-locally be covered by manifolds.
(2) sheaves in $\mathbf{Diff}^r$ which may be presented by étale groupoids.

It is prima facie surprising that these two ∞-categories are ∞-toposes. These observations (in the generality of [Car20, §5]) were key to the development of fractured ∞-toposes (see [Lur18, Rmk. 20.0.0.2]).

Warning 2.7. The functor $j_! : \mathbf{Diff}^r_{\text{ét}} \to \mathbf{Diff}^r$ does not preserve 0-truncated objects. For example, $\mathbf{1}_{\mathbf{Diff}^r_{\text{ét}}}$ is mapped to the Haefliger stack (see [Clo24a, Th. 2.22]), which is 1-truncated but not 0-truncated. ⌟

We finish this subsection by proving some basic properties about the fractured ∞-topos $\mathbf{Diff}^r$.

Proposition 2.8. *The ∞-topos $\mathbf{Diff}^r$ is local.*

Proof. This follows immediately from Proposition 1.6. □

Lemma 2.9. *Let M be a connected, paracompact Hausdorff r-times differentiable manifold M, then the covering dimension of M is smaller than the dimension of M.*

Proof. By [APG90, §II.6.2] the covering dimension of M is equal to the inductive dimension, which is $\leq \dim M$. $\square$

Proposition 2.10. *For any manifold M the ∞-topos $(\mathbf{Diff}^r_{\text{ét}})_{/M}$ is hypercomplete.*

Proof. By Lemma 2.9, [Lur09b, Th. 7.2.3.6], and Remark 2.6 the ∞-topos $(\mathbf{Diff}^r_{\text{ét}})_{/M}$ has homotopy dimension $\leq \dim M$, and is thus hypercomplete by [Lur09b, Cor. 7.2.1.12]. $\square$

By [Lur17, Lm. A.3.9.] we obtain the following corollary:

Corollary 2.11. *For any r-times differentiable manifold M the ∞-topos $(\mathbf{Diff}^r_{\text{ét}})_{/M}$ has enough points.* $\square$

Proposition 2.12. *The ∞-topos $\mathbf{Diff}^r_{\text{ét}}$ has enough points.*

Proof. By Remark 2.6 the adjunction $x_* : \mathcal{S} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} (\mathbf{Diff}^r_{\text{ét}})_{/M} : x^*$ at any point $x \in \mathbf{R}^d$ provides a point of $(\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^d}$, and thus a point $\mathcal{S} \xrightarrow{x_*} (\mathbf{Diff}^r_{\text{ét}})_{/\mathbf{R}^d} \xrightarrow{(\mathbf{R}^d)_*} \mathbf{Diff}^r_{\text{ét}}$. Thus, $\mathbf{Diff}^r_{\text{ét}}$ has enough points, as it is generated by the spaces $\mathbf{R}^d$ under colimits. $\square$

By Corollary 1.5 we obtain the following result:

Corollary 2.13 ([Dug98, Ex. 4.1.2], [ADH21, Prop. A.5.3])**.** *The topos $\mathbf{Diff}^r$ has enough points.* $\square$

By [Lur09b, Rmk. 6.5.4.7] we obtain the following corollary:

Corollary 2.14. *The ∞-topos $\mathbf{Diff}^r$ is hypercomplete.* $\square$

2.2. **Compact manifolds are compact.** In this subsection we discuss the categorical compactness of manifolds in $\mathbf{Diff}^r$. We use the fractured ∞-topos structure to show that any closed manifold is categorically compact, by relating it to its underlying topological space, which is compact when viewed as an ∞-topos. Then we discuss the categorical compactness of other manifolds in §2.2.1.

Theorem 2.15. *Let M be a closed manifold, then $\mathbf{Diff}^r(M, \_)$ commutes with filtered colimits.*

Proof. Let A be a small filtered ∞-category, and let $X : A \to \mathbf{Diff}^r$ be a diagram, then

$$\begin{aligned}
\operatorname{colim}_\alpha \mathbf{Diff}^r(M, X_\alpha) &= \operatorname{colim}_\alpha \mathbf{Diff}^r(j_! M, X_\alpha) \\
&= \operatorname{colim}_\alpha \mathbf{Diff}^r_{\text{ét}}(M, j^* X_\alpha) \\
&= \operatorname{colim}_\alpha (\mathbf{Diff}_{\text{ét}})^r_{/M}(M, M \times j^* X_\alpha) \\
&\to (\mathbf{Diff}_{\text{ét}})^r_{/M}(M, \operatorname{colim}_\alpha M \times j^* X_\alpha) \\
&= (\mathbf{Diff}_{\text{ét}})^r_{/M}(M, M \times j^* \operatorname{colim}_\alpha X_\alpha) \\
&= \mathbf{Diff}^r_{\text{ét}}(M, j^* \operatorname{colim}_\alpha X_\alpha) \\
&= \mathbf{Diff}^r(j_! M, \operatorname{colim}_\alpha X_\alpha) \\
&= \mathbf{Diff}^r(M, \operatorname{colim}_\alpha X_\alpha)
\end{aligned}$$

where the map in the fourth line is an isomorphism by [Lur09b, Th. 7.3.1.16 & Rmk. 7.3.1.5]. $\square$

2.2.1. *On the compactness of non-closed manifolds.*

Proposition 2.16. *Any non-compact manifold is not categorically compact in $\mathbf{Diff}^r$.*

Proof. By assumption any such manifold M admits a sequence (x_i) such that $\{x_i\}$ is a closed subset of M, then the identity map $M \xrightarrow{\text{id}} M = \operatorname{colim}_{n \in \mathbf{N}} M \setminus \{x_i\}_{i \geq n}$, does not factor through any of the manifolds $M \setminus \{x_i\}_{i \geq n}$ for $n \in \mathbf{N}$. $\square$

Theorem 2.17. *Any connected manifold with non-empty corners is not categorically compact in $\mathbf{Diff}^r$ for $r \geq 2$.*

Proof. By Proposition 2.16 it is enough to prove the theorem for (topologically) compact manifolds. Moreover, as any finite coproduct of categorically compact manifolds is again compact, we may restrict to connected manifolds. We first prove the theorem for $[0,1]$, from which we then deduce the general case.

Denote by I the collection of all finite families of r-times differentiable maps of the form $\{\mathbf{R}^{d_i} \to [0,1]\}_{i=0}^k$, where each map $\mathbf{R}^{d_i} \to [0,1]$ factors through either $[0,1)$ or $(0,1]$, then I becomes a filtered poset under inclusion. For any member $C = \{\mathbf{R}^{d_i} \to [0,1]\}_{i=0}^k$ of I denote by $[0,1]_C$ the diffeological spaces consisting of the set $[0,1]$ together with the coarsest diffeology making all maps in C differentiable, then by Proposition [Clo24b, Prop. 1.2.9] $\operatorname{colim}_{C\in E}[0,1]_C$ is diffeomorphic to $[0,1]$. We will show that the identity map $[0,1] \to [0,1]$ does not factor through $[0,1]_C$ for any $C \in I$, thus showing that $[0,1]$ is not compact.

Let us fix $C \in I$ as well as $f \in C$, which we assume w.l.o.g. factors through $[0,1)$. After reparametrising, we may view f as a function $\mathbf{R}^d \to [0,\infty)$. We will show that for any $n > d$ the restriction of the smooth map $\sigma_n : \mathbf{R}^n \to [0,\infty)$, $(x_1,\dots,x_n) \mapsto x_1^2 + \cdots + x_n^2$ to any neighbourhood of $0 \in \mathbf{R}^n$ does not factor through f. Thus, for sufficiently large n, σ_n does not locally factor through any of the functions in C, so that $[0,1]_C$ has a strictly coarser diffeology than $[0,1]$.

So, suppose to the contrary that there exists some neighbourhood U of $0 \in \mathbf{R}^d$ such that $\sigma_n|_U$ factors through f via a map $g : U \to \mathbf{R}^d$, and assume w.l.o.g. that $g(0) = 0$, then as $n > d$ the kernel of $\mathrm{d}g|_0$ is non-trivial, and we may assume w.l.o.g. that it contains $(1,0,\dots,0)$. Choose $\varepsilon > 0$ such that $(-\varepsilon,\varepsilon) \times \{0\} \times \cdots \times \{0\} \subseteq U$, and write $h : (-\varepsilon,\varepsilon) \to \mathbf{R}^d$ for the map $x \mapsto g(x,0,\dots,0)$, then, by assumption $f \circ h$ is given by $x \mapsto x^2$, so that $(f\circ h)'' = 2$. On the other hand we have

$$
\begin{aligned}
(f\circ h)''(t) &= \left(\sum_{i=1}^n h_i'(t)\,\partial_i f\big(h_1(t),\dots,h_n(t)\big)\right)' \\
&= \sum_{i=1}^n \left(h_i''(t)\,\partial_i f\big(h_1(t),\dots,h_n(t)\big) + \sum_{j=1}^n h_i'(t)\,h_j'(t)\,\partial_i\partial_j f\big(h_1(t),\dots,h_n(t)\big)\right)
\end{aligned}
$$

which evaluates to 0 for $t = 0$ because for all $1 \le i \le n$ we have $\partial_i f\big(h_1(0),\dots,h_n(0)\big) = \partial_i f(0) = 0$ (as f has a local minimum at 0) and $(h_1'(0),\dots,h_n'(0)) = \mathrm{d}g|_0(1,0,\dots,0) = 0$ by assumption, yielding a contradiction.

Now, let M be a manifold of dimension > 1 with non-empty corners, then by assumption M admits at least one chart $\mathbf{R}^{\dim M-1} \times [0,\infty) \hookrightarrow M$. Consider the embedding

$$\iota : [0,1] \hookrightarrow \mathbf{R}^{\dim M-1} \times [0,\infty), \quad \theta \mapsto (\cos\pi\theta, 0,\dots,0,\sin\pi\theta).$$

Denote by L the image of the embedding $[0,1] \xrightarrow{\iota} \mathbf{R}^{\dim M-1} \hookrightarrow M$, and denote by $f : M \to [0,1]$ an extension the diffeomorphism $L \to [0,1]$. With notation as above, assume that f factors through $[0,1]_C \hookrightarrow [0,1]$ for some C in I, then this implies that the identity map $\mathrm{id}_{[0,1]} : [0,1] \to [0,1]$ factors through $[0,1]_C \hookrightarrow [0,1]$, as $\mathrm{id}_{[0,1]}$ is equal to the composition of $[0,1] \xrightarrow{\iota} \mathbf{R}^{\dim M-1} \times [0,\infty) \hookrightarrow M \xrightarrow{f} [0,1]$, yielding a contradiction. □

It is possible to show that the category of r-times differentiable manifolds with corners, when equipped with the standard Grothendieck topology and open immersions, forms a geometric site, yielding a fractured ∞-topos by Theorem 1.12, in which all topologically compact manifolds are categorically compact by the same argument as in Theorem 2.15. Thus, in a sense, topologically compact manifolds with corners become categorically compact when corners are encoded as *structure* rather than as a property.

For $r = 0$ this geometric site yields a fractured ∞-topos, which is equivalent to $\mathbf{Diff}^0$, so in this case topologically compact manifolds with corners are also compact in $\mathbf{Diff}^0$. We don't know whether topologically compact C^1-manifolds with corners are categorically compact in $\mathbf{Diff}^1$.

Conventions and notation

- Canonical isomorphisms are often denoted by equality signs. (An isomorphism is canonical if it originates from a universal property. More precisely, let $u : X \to C$ be a right fibration, and x, x'

two final objects in X, then for any morphism $x \to x'$ the morphism $ux \to ux'$ is a canonical isomorphism, and we may write $x = x'$.)

- $[\_,\_]$ denotes the internal hom in the ∞-category of ∞-categories.
- ∞-categories (including ordinary categories) are denoted by C, D, ...
- Let C be an ∞-category and let $x, y \in C$ be two objects, then the homotopy type of morphisms from x to y is denoted by $C(x, y)$.
- A final object in an ∞-category C is denoted by $\mathbf{1}_C$, or simply by $\mathbf{1}$, when C is clear from context.
- For any ∞-category C we denote its subcategory of n-truncated objects by $C_{\leq n}$.
- For any two categories C, D, an arrow $C \hookrightarrow D$ denotes a fully faithful functor.
- We use the following notation for various ∞-categories:
  - $\mathcal{S}$ denotes the ∞-categories of homotopy types.
  - **Cat** denotes the ∞-category of ∞-categories.
  - **Top** denotes the ∞-category of ∞-toposes.
  - $\mathbf{Mfd}^r$ denotes the category of r-times differentiable smooth manifolds and smooth maps.
  - $\mathbf{Cart}^r$ denotes the full subcategory of $\mathbf{Mfd}^r$ spanned by the spaces of $\mathbf{R}^n$ $(0 \leq n < \infty)$.
  - $\mathbf{Diff}^r$ denotes, equivalently, the ∞-category of sheaves on $\mathbf{Mfd}^r$ or $\mathbf{Cart}^r$.
- We denote ∞-toposes by $\mathcal{E}, \mathcal{F}, \ldots$, when they are thought of as ambient settings in which to do geometry, and by $\mathcal{X}, \mathcal{Y}, \ldots$, when they are thought of as geometric objects in their own right.

References

[ADH21] Araminta Amabel, Arun Debray, and Peter J. Haine. Differential cohomology: Categories, characteristic classes, and connections. Preprint available at https://arxiv.org/abs/2109.12250, 2021.

[Aok23] Ko Aoki. Tensor triangular geometry of filtered objects and sheaves. *Math. Z.*, 303(3):27, 2023. Id/No 62.

[APG90] A. V. Arkhangel'skij, L. S. Pontryagin, and R. V. Gamkrelidze, editors. *General topology. I. Basic concepts and constructions. Dimension theory. Transl. from the Russian by D. B. O'Shea*, volume 17 of *Encycl. Math. Sci.* Berlin etc.: Springer-Verlag, 1990.

[Car16] David Carchedi. On the homotopy type of higher orbifolds and Haefliger classifying spaces. *Adv. Math.*, 294:756–818, 2016.

[Car20] David Joseph Carchedi. Higher orbifolds and Deligne-Mumford stacks as structured infinity-topoi. *Mem. Amer. Math. Soc.*, 264(1282):v+120, 2020.

[Clo24a] Adrian Clough. Differentiable sheaves ii: Local contractibility and cofinality. Preprint available at http://adrianclough.github.io, 2024.

[Clo24b] Adrian Clough. Differentiable sheaves iii: Homotopical calculi and the smooth oka princple. Preprint available at http://adrianclough.github.io, 2024.

[CS19] David Carchedi and Pelle Steffens. On the universal property of derived manifolds. Preprint available at https://arxiv.org/abs/1905.06195v1, 2019.

[Dug98] Daniel Dugger. Sheaves and homotopy theory. Incomplete draft available at https://pages.uoregon.edu/ddugger/cech.html, 1998.

[DAG V] Jacob Lurie. Derived algebraic geometry v: Structured spaces. Preprint available at https://arxiv.org/abs/0905.0459, 2009.

[Lur09b] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[DAG IX] Jacob Lurie. Derived algebraic geometry ix: Closed immersions. Preprint available at https://www.math.ias.edu/~lurie/papers/DAG-IX.pdf, 2011.

[DAGVII] Jacob Lurie. Derived algebraic geometry VII: Spectral schemes. Preprint available at https://www.math.ias.edu/~lurie/papers/DAG-VII.pdf, 2011.

[Lur17] Jacob Lurie. Higher algebra. Available online at http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf, 2017.

[Lur18] Jacob Lurie. Spectral algebraic geometry. Preprint notes available at http://www.math.harvard.edu/~lurie/papers/SAG-rootfile.pdf, 2018.

[nLa23] nLab authors. good open cover. https://ncatlab.org/nlab/show/good+open+cover, August 2023. Revision 48.

[Por19] Mauro Porta. GAGA theorems in derived complex geometry. *J. Algebr. Geom.*, 28(3):519–565, 2019.

[Ste23] Pelle Steffens. Derived C^∞-geometry I: Foundations. Preprint available at https://arxiv.org/abs/2304.08671, 2023.