Bordisms as a quasi-unital flagged ∞-category

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Abstract

We arrange the collection of all bordisms of a given dimension into a quasi-unital flagged ∞-category, a new ∞-categorical structure differing from that of ∞-categories in two ways: Identity morphisms are no longer given as structure, but rather as a property, and objects are equipped with an additional way in which they may be equivalent. These modifications reflect the facts that cylinders of "arbitrary length" may serve as units, and that two closed manifolds may naturally be viewed as being equivalent either by jointly forming the boundary of an invertible bordism, or by being diffeomorphic. A crucial advantage of quasi-unital flagged ∞-categories is that these may be modelled by semi-simplicial topological spaces; as we do not have to specify degeneracies, our model is particularly simple, and makes it straightforward to equip bordisms with extra structures more refined than tangential structures. We illustrate this latter point by constructing a quasi-unital flagged ∞-category of bordisms with a continuously varying Riemannian metric. These are equivalent to the quasi-unital flagged ∞-category or ordinary bordisms, allowing for a precise statement of the idea that a topological quantum field theory is the same thing as a Euclidean quantum field theory, invariant under perturbation of the metric.

We provide extensive differential topological and homotopical background, hopefully making this work accessible to the non-specialist.

Contents

1 Introduction

1.1 Overview

There are by now several models of the ∞-category of bordisms. This ∞-category has as objects closed manifolds of a fixed dimension, and for any two such manifolds M, N the mapping space from M to N is given by the classifying space of bordisms whose ingoing and outgoing boundary are M and N respectively. As discussed in §1.2 these models typically consist of simplicial spaces satisfying the Segal condition (see Definition 2.2.1).

Working with such simplicial models however leads to the following problem: As these have *strict* units we are forced to allow bordisms of "length 0", which leads to certain subtleties, as these bordisms are qualitatively different from bordisms of length > 0. These subtleties can then for example make it difficult to equip bordisms with extra structure, at least if these are not tangential, such as symplectic structures. A more philosophical issue is that one ought to be able to think of *any* cylinder as a (homotopical) unit. As we illustrate in §4.3.4, there are (putative) models of ∞-categories with *non-strict* units, which allow for models of the ∞-category of bordisms, where the units are indeed given by specified cylinders of length > 0; we pursue a course which allows for *all* cylinders to functions as units:

An ordinary category can equivalently be viewed as a semi-category with the *property* of admitting units for all objects. Similarly, Lurie claims in [Lur09b, §2.2] that any ∞-semi-category, modelled as a semi-simplicial space, admitting units up to homotopy, can be promoted to an ∞-category, i.e., is levelwise equivalent to a simplicial space. Seizing upon this observation Harpaz constructs a model structure on the category of marked semi-simplicial spaces such that a forgetful functor to the category of simplicial spaces together with the complete Segal space model structure (due to Rezk; see [Rez00]) induces a Quillen equivalence (see [Har15]).

The goal of this article is to construct a semi-simplicial topological space of bordisms in the spirit of the topological category first considered by Madsen and Tillmann in [MT01], and to exhibit this semi-simplicial space as an appropriate ∞-categorical structure admitting units up to homotopy; we provide a sketch of our construction in §1.2.5 and give a precise definition in §4.3.2. We can thus avoid the difficulties arising from a Segalic approach to units by simply ignoring them!

This leads to some simplifications, in particular, equipping bordisms with extra structures, as mentioned above, becomes significantly easier. Another simplification is the following: Providing a map between two simplicial spaces entails specifying a sequence of maps of spaces which are compatible with the face and degeneracy maps. In the quasi-categorical setting Tanaka observes that, given a sequence of maps constituting a putative map of simplicial sets, it is often easier to check that these are compatible with the face maps than with the degeneracy maps (see [Tan17]). In the semi-simplicial setting we are not required to check compatibility with the degeneracy maps at all; it suffices to check that homotopical units are sent to homotopical units. This can often be done by hand, or by applying criteria such as Proposition 2.3.10.

The ∞-categorical structure that we obtain in this manner is however not an ∞-category. This is because our semi-simplicial space does not satisfy the completeness condition (see §2.2.2), so we cannot directly apply Harpaz' theory. This reflects the fact that our semi-simplicial space encodes two ways in which any two closed manifolds can be equivalent:

- They can be diffeomorphic.
- They can be linked by an invertible bordism.

Every diffeomorphism f from a closed manifold M to itself gives rise to an invertible bordism by identifying M with either the ingoing or the outgoing boundary of $M \times [0,1]$ along f.
It is an observation of Ayala and Francis that the theory of simplicial spaces satisfying the Segal condition is equivalent to the theory of *flagged* ∞*-categories*; these are pairs consisting of an ∞-category C together with an ∞-groupoid X, and a functor $X \to C^{\simeq}$ inducing a surjection on connected components (see [AF18]). Bordisms of dimension ≥ 5 are invertible precisely if they are h-bordisms by [Sta65]; thus, the existence of non-trivial h-bordisms in certain dimensions tells us that the semi-simplicial space of bordisms is generally not complete. One of the key steps in this work is to modify Harpaz' theory to produce a theory of quasi-unital flagged ∞-categories in §2.3. We exhibit our semi-simplicial space of bordisms as a quasi-unital flagged ∞-category, and, as expected, the flagging precisely exhibits the map from diffeomorphisms to invertible bordisms described above; in §4.3.3 we exhibit the flagging in a different, more geometrically pleasing way.
Having figured out the structure into which we wish to arrange our bordisms, we must then be able to construct the constituent spaces of our semi-simplicial space. This requires us to understand spaces of smooth maps and transversality, to which we give a streamlined introduction in §3. Most of the theory in this section is well known; §3.3 contains results on smooth approximations which are part of the oral tradition, to which we produce detailed proofs. We hope that §3 may provide a nice introduction to mapping spaces, and may thus be of independent interest.
In §5 we conclude this article with a modest sample application: Using our model it becomes straightforward to construct a quasi-unital flagged ∞-category of bordisms with a continuously varying Riemannian metric. We show that the forgetful functor to the quasi-unital flagged ∞-category of ordinary bordisms is a levelwise equivalence. This makes it possible to give a precise meaning to the idea that functorial quantum field theories defined on Riemannian bordisms which are invariant under perturbations of the metric are the same as topological quantum field theories.

1.2 History and details

The history of bordism categories is tangled. We trace two strands of this history, and then describe precisely how our model relates to previous models.

Warning 1.2.1. When we wish to remain vague about precisely the sort of structure into which we are arranging our bordisms, we shall simply speak of *bordism categories*. These are typically *not* ordinary categories, and we will explicitly state when this is the case. ⌟

1.2.1 Beginnings

The late 80's witnessed the emergence of several notions of bordism categories, which were introduced in order to rigorously define functorial quantum field theories, i.e. symmetric monoidal functors from a bordism category to a category of vector spaces (or some suitable variant

thereof), where the symmetric monoidal structure on bordisms[1] is given by disjoint union. These field theories come in two flavours: conformal and topological. The former were introduced almost simultaneously in [Seg04], [Dij89], and [Vaf87] with a view towards applications to string theory, and have as domain the following topologically enriched semi-category: Its objects are given by non-negative integers. The mapping space from m to n is given by $\bigsqcup_{[F]} S(F)/\, \mathbf{Diff}_\partial F$, where the disjoint union ranges over the equivalence classes of all orientable surfaces F with $m+n$ boundary components, $S(F)$ denotes the space of conformal structures on F, and $\mathbf{Diff}_\partial F$ denotes the topological group of diffeomorphism preserving the boundary pointwise. Following [Seg04], we denote this topologically enriched semi-category by $\mathscr{C}$. For any $g, n \in \mathbf{N}$ we denote by $F_{g,n}$ the orientable genus g surface with n boundary components. Recall that $S(F_{g,n})/\, \mathbf{Diff}_\partial F_{g,n}$ is the *moduli space of Riemann surfaces*, and is denoted by $\mathcal{M}_{g,n}$. Thus $\mathscr{C}$ assembles all moduli spaces of Riemann surfaces into one structure.
Witten describes certain quantum field theories in [Wit88, §3] with a view towards applications to Donaldson invariants. He observes that the quantum field theories under investigation are often invariant under scaling the metric, and dubs quantum field theories with this property *topological quantum field theories*. Atiyah proposes an axiomatisation of topological quantum field theories in [Ati88] using an ordinary category of bordisms obtained by taking connected components of the mapping spaces of the ∞-category sketched in the open paragraph of §1.1. It is a crucial observation that in low dimensions topological quantum field theories parametrise algebraic structures and are determined by minimal data:

- 1-dimensional field theories are determined by where they map the point. The image of the point is dualisable, and conversely any dualisable object together with the data exhibiting its dualisability determines a 1-dimensional field theory.

- 2-dimensional field theories are determined by where they map the circle. The image of the circle is canonically endowed with the structure of a Frobenius algebra, and any such structure determines a 2-dimensional field theory.

It is not clear how to extend this pattern to higher dimensions using Atiyah's bordism categories. Building on ideas by Freed, Lawrence, Quinn and Walker, in [BD95] Baez and Dolan conjecture how *extended field theories* might lead to a general statement of this sort. The n-category of bordisms has compact 0-manifolds (i.e. finite sets) as objects, 1-bordisms as 1-morphisms, 2-bordisms between 1-bordism as 2-morphisms, etc. An extended field theory is then a symmetric monoidal n-functor from the n-category of bordisms to a suitable linear n-category, and is determined by where it sends a point. If we moreover equip the constituent manifolds of the n-category of bordisms with suitable framings, then the algebraic structure parametrised by such a field theory is a so-called fully dualisable object (again together with its dualisability structure).
For a much more comprehensive account of the history of (extended) topological quantum field theories than we give here, we direct the reader to the introduction of [SP14].

[1]We will not consider a symmetric monoidal structure on our quasi-unital flagged ∞-category of bordisms, but the symmetric monoidal structure introduced in [Ngu17] can easily be adapted to our model.

1.2.2 Mumford's conjecture

We now return to the conformal picture: In [Til97] the bordism category $\mathscr{C}$ finds an unexpected application outside of field theory, and bordism categories become an object of interest in their own right.
One can show that the spaces $S(F_{g,n})$ defined in §1.2.1 are contractible for $n > 0$, and if moreover $g > 1$, the action of $\mathbf{Diff}(F_{g,n})$ is principal, so that $\mathcal{M}_{g,n} = F_{g,n}/\mathbf{Diff}_\partial F_{g,n} = B\,\mathbf{Diff}_\partial F_{g,n}$ (see [Til13, §2.2]). Furthermore, $\mathbf{Diff}_\partial F_{g,n}$ is 1-truncated; writing $\Gamma_{g,n} = \pi_0\,\mathbf{Diff}_\partial F_{g,n}$, we thus have $\mathcal{M}_{g,n} = B\Gamma_{g,n}$. Given a surface $F_{g,n}$, attaching a pair of trousers along the waist or the legs yields the surfaces $F_{g,n+1}$ and $F_{g+1,n-1}$ respectively; these operations induce maps $H^i(\Gamma_{g,n+1},\mathbf{Q}) \to H^i(\Gamma_{g,n},\mathbf{Q})$ and $H^i(\Gamma_{g+1,n-1},\mathbf{Q}) \to H^i(\Gamma_{g,n},\mathbf{Q})$ which, by a celebrated result of Harer [Har85], are isomorphisms when $g \gg i$. Thus the cohomology groups $H^i(\Gamma_{g,1},\mathbf{Q})$ stabilise as $g \to \infty$, and Mumford famously conjectured that $H^*(\Gamma_{\infty,1},\mathbf{Q}) \cong \mathbf{Q}[\kappa_1,\kappa_2,\ldots]$, where $\deg\kappa_i = 2i$. For a riveting account of the history of Mumford's conjecture and its eventual proof we recommend [Til13]. We will only touch upon a few key events which are of particular importance to us.
Denote by $\mathscr{C}_b$ the subcategory of $\mathscr{C}$ consisting of those bordisms whose outgoing boundary component are non-empty. From the above discussion we know that $\mathscr{C}_b$ can equivalently be modelled as a category enriched in groupoids. Using such a model, Tillmann shows in [Til97] that $\mathbf{Z} \times B\Gamma^+_{\infty,1} \simeq \Omega B\mathscr{C}_b$. The space $B\Gamma^+_{\infty,1}$ has the same homology (and thus rational cohomology) groups as $B\Gamma_{\infty,1}$. In [MT01] Madsen and Tillmann construct a topological category $\mathcal{Y}$, whose mapping spaces are classifying spaces for oriented surfaces. In fact, $\mathcal{Y}$ can be obtained by endowing **Bord**2 (sketched in §1.2.5 and defined precisely in §4.3.2) with the tangential structure encoding orientations, and then freely adding units (see [ERW19, §3.2]). This bordism category is used to define a map $\alpha_\infty : \mathbf{Z} \times B\Gamma^+_\infty \to \Omega^\infty\mathbf{CP}^\infty_{-1}$, where the latter space is known to have the cohomology ring predicted by Mumford, and is conjectured to being an equivalence. This is shown to be the case in [MW07], thus proving Mumford's conjecture.
In [GTMW09] the classifying spaces of topological categories of bordisms $(\mathcal{C}_d,\tau_u)$ (which we discuss in the next section) of arbitrary dimension are shown to be equivalent to certain Thom spectra. In dimension 2 with the tangential structure encoding orientations, this Thom spectrum is again given by $\Omega^\infty\mathbf{CP}^\infty_{-1}$. Moreover, it is shown that for any one of the bordism categories under consideration the subcategory given by bordisms, whose outgoing boundary component is non-empty, has the same classifying space. Independent proofs of the main results in [GTMW09] are given in [GRW10].

1.2.3 A taxonomy of bordism categories

In this subsection we discuss eight bordism categories, and explain how these relate to the models in the works discussed in the previous section. To this end we fix $d \in \mathbf{N}$ for the remainder of this introduction, and define two ordinary categories, $\mathcal{C}_d$ and $\mathcal{D}_d$. We then equip both of these categories with two topologies and two diffeologies, which we denote respectively by τ, τ_u, and δ, δ_u. We recall some relevant facts about diffeologies below. The bordism categories considered in [GTMW09] are $(\mathcal{C}_d,\tau_u)$, $(\mathcal{C}_d,\delta_u)$, and $(\mathcal{D}_d,\delta_u)$ (or rather the sheaves of categories represented by the latter two).
The category $\mathcal{C}_d$ has objects consisting of pairs (M,t), where $t \in \mathbf{R}$ and $M \subseteq \mathbf{R}^\infty$ is a compact $(d-1)$-dimensional submanifold without boundary. The morphisms are given by

triples $(W; t_0, t_1)$, where $t_0 < t_1$, and $W \subseteq \mathbf{R}^\infty \times [t_0, t_1]$ is an embedded submanifold with boundary, such that $W \cap (\mathbf{R}^\infty \times \{t_0, t_1\}) = \partial W$, and such that $W \cap (\mathbf{R}^\infty \times [t_0, t_0 + \varepsilon)) = W \cap (\mathbf{R}^\infty \times \{t_0\}) \times [t_0, t_0 + \varepsilon)$ and $W \cap (\mathbf{R}^\infty \times (t_1 - \varepsilon, t_1]) = W \cap (\mathbf{R}^\infty \times \{t_1\}) \times (t_1 - \varepsilon, t_1]$ for some $0 < \varepsilon < t_1 - t_0$.

The category $\mathcal{D}_d$ is a poset with elements consisting of pairs (M, t), where $t \in \mathbf{R}$ and $M \subseteq \mathbf{R}^\infty \times \mathbf{R}$ is an embedded submanifold contained in $\mathbf{R}^n \times \mathbf{R}$ for some $n \in \mathbf{N}$, such that the projection $M \to \mathbf{R}$ is proper, and t is a regular value. The order relation $(M, t) \leq (M', t')$ is satisfied iff $M = M'$ and $t \leq t'$.

There is an inclusion

$$\mathcal{C}_d \hookrightarrow \mathcal{D}_d \tag{1}$$

obtained by adjoining infinite cylinders on both sides of any manifold or bordism.

The subsequent discussion requires us to define the auxiliary maps $a : (\mathcal{C}_d)_0 \to \mathbf{R}$, $a : (\mathcal{D}_d)_0 \to \mathbf{R}$ given by $(M, t) \mapsto t$, as well as maps $a_i : (\mathcal{C}_d)_1 \to \mathbf{R}$, $a_i : (\mathcal{D}_d)_1 \to \mathbf{R}$ given by $(W; t_0, t_1) \mapsto t_i$ for $i = 0, 1$. Similarly, we define maps $\pi : (\mathcal{C}_d)_0 \to \mathcal{P}(\mathbf{R}^n)$, $\pi : (\mathcal{D}_d)_0 \to \mathcal{P}(\mathbf{R}^n \times \mathbf{R})$ given by $(M, t) \mapsto M$, as well as maps $\pi : (\mathcal{C}_d)_1 \to \mathcal{P}(\mathbf{R}^n \times [t_0, t_1])$, $\pi : (\mathcal{D}_d)_1 \to \mathcal{P}(\mathbf{R}^n \times \mathbf{R})$ given by $(W; t_0, t_1) \mapsto W$.

We begin by describing the bordism category closest to the one considered in this article, and probably most familiar to the reader. We then discuss the other models in the most efficient order. For simplicity we do not consider tangential structures.

The topology τ_u on $\mathcal{C}_d$ The topological category $(\mathcal{C}_d, \tau_u)$ is first considered in [GTMW09, §2.1], and is the model used to state the main theorem of loc. cit. The set $(\mathcal{C}_d)_0$ is topologised via the canonical bijection to

$$\left(\coprod_{[M]} \mathrm{Emb}(M, \mathbf{R}^\infty) / \mathbf{Diff}(M) \right) \times \mathbf{R},$$

where the coproduct ranges over all diffeomorphism classes of closed $(d-1)$-dimensional manifolds, and the sets of embeddings and the diffeomorphism groups are topologied using the CO^∞-topology as explained in §4.1. The set $(\mathcal{C}_d)_1$ is topologised via the canonical bijection to

$$\left(\left(\coprod_{[M]} \mathrm{Emb}(M, \mathbf{R}^\infty) / \mathbf{Diff}(M) \right) \times \mathbf{R} \right) \amalg \left(\left(\coprod_{[W]} \mathrm{Emb}(W, \mathbf{R}^\infty \times [0,1]) / \mathbf{Diff}(W) \right) \times \left\{ \ (t_0, t_1) \in \mathbf{R}^2 \ \middle| \ t_0 < t_1 \ \right\} \right),$$

where the coproduct to the right ranges over all compact bordisms of dimension d, the embeddings and diffeomorphism respect a collaring on W as explained in §4.2, and these are again equipped with the CO^∞-topology.

Diffeologies Recall that a diffeological space is a set X together with a specified subset of $\mathbf{Set}(M, X)$, called *smooth maps*, for every manifold M, which are required to be closed under composition with smooth maps between manifolds, and the resulting presheaf on the site of smooth manifolds must furthermore satisfy descent. Diffeological spaces then form a full subcategory of all sheaves on the site of manifolds (see [BH11]). There exists a forgetful functor from diffeological spaces to topological spaces given by assigning to the underlying set of any

diffeological space the finest topology making all smooth maps into it continuous. Any smooth manifold has the canonical structure of a diffeological space, and the underlying topological space of this diffeological space is precisely the underlying topological space of the manifold.
Two ways of extracting a homotopy type from a diffeological space X are to either build a simplicial set from smooth simplices in X (whose geometric realisation is called *representing space* in [GTMW09]), or to consider its underlying topological space. If continuous maps into the underlying topological space of X admit appropriate "smooth approximations" (see §3.3), then these two homotopy types coincide; all diffeological spaces considered below have this property.
Similarly, given a category object in the category of diffeological spaces, or *diffeological category*, we can extract from it a homotopy type by either

- constructing the total singular complex of the spaces of objects and morphisms individually using smooth simplices, and then constructing the classifying space of the resulting simplicial category, or
- constructing the classifying space of the underlying topological category.

For the diffeological categories considered below these two procedures produce equivalent homotopy types.

The diffeology δ on $\mathcal{D}_d$ As $\mathcal{D}_d$ is a poset, it suffices to describe the diffeology on $(\mathcal{D}_d)_0$. For any manifold M (without boundary), a map $M \to (\mathcal{D}_d)_0$ is smooth iff

(a) $\Gamma_f := \bigcup_{x\in M}\{x\} \times \pi \circ f(x) \subseteq M \times \mathbf{R}^\infty \times \mathbf{R}$ is a smooth submanifold,

(b) the projection map $\Gamma_f \to M$ is a smooth submersion, and

(c) the map $a \circ f : M \to \mathbf{R}$ is smooth.

The diffeology δ on $\mathcal{C}_d$ The diffeology on δ on $\mathcal{C}_d$ is given by the restriction of the diffeology δ on $\mathcal{D}_d$. The diffeology on $(\mathcal{C}_d)_0$ admits a more direct description: For any manifold M (without boundary) a map $f : M \to (\mathcal{C}_d)_0$ is smooth iff

(a) $\Gamma_f := \bigcup_{x\in M}\{x\} \times \pi \circ f(x) \subseteq M \times \mathbf{R}^\infty$ is a smooth submanifold,

(b) the projection map $\Gamma_f \to M$ is a smooth submersion (and thus a fibre bundle by Ehresmann's theorem), and

(c) the map $a \circ f : M \to \mathbf{R}$ is smooth.

The inclusion (1) is a homotopy equivalence. We describe the idea on the level of objects; the idea for morphisms is similar. Any object $(M,t) \in (\mathcal{D}_d)_0$ is homotopic to an object (M',t) which is cylindrical over $(t-\varepsilon, t+\varepsilon)$ for some $\varepsilon > 0$. Let $h : [0,1] \times \mathbf{R} \to \mathbf{R}$ be an isotopy (which is constant near the endpoints) from the identity map on $\mathbf{R}$ to some embedding $\mathbf{R} \subseteq (t-\varepsilon, t+\varepsilon)$, such that $h_s(t) = t$ for all $t \in [0,1]$, then the map $\mathbf{R} \to (\mathcal{D}_d)_0$ given by $s \mapsto (h_s^{-1}(M'), t)$ is smooth, where h has been extended to $\mathbf{R}$ by requiring $h_s = h_0$ for $s \leq 0$, and $h_s = h_1$ for $s \geq 1$.

The diffeology δ_u on $\mathcal{D}_d$ and $\mathcal{C}_d$ One might hope that $(\mathcal{C}_d, \tau_u)$ is the underlying topological category of $(\mathcal{C}_d, \delta)$, but this is not the case, as for a given closed $(d-1)$-dimensional submanifold $M \subseteq \mathbf{R}^n$, any cylinder $(M \times [t_0, t_1]; t_0, t_1)$ with $t_0 < t_1$ is connected to $(M \times [t_0, t_0]; t_0, t_0)$ by a smooth homotopy in $(\mathcal{C}_d, \delta)_1$, yet these two morphisms lie in different connected components in $(\mathcal{C}_d, \tau_u)_1$.
This leads us to modify the diffeology δ on $\mathcal{D}_d$ to produce a new diffeology δ_u, which we again restrict to $\mathcal{C}_d$. The above statement then becomes true when we replace $(\mathcal{C}_d, \delta)$ with $(\mathcal{C}_d, \tau_u)$. Observe that part of the structure of a smooth map $M \to (\mathcal{D}_d, \delta)_1$ are two smooth functions $a_0, a_1 : M \to \mathbf{R}$ such that $a_0(x) \leq a_1(x)$ for all $x \in M$. We obtain the diffeology δ_u by imposing the additional requirement on δ that the preimage of $\{0\}$ under $a_1 - a_0$ be open, and thus a union of connected components. We note that δ and δ_u coincide on $(\mathcal{C}_d)_0$ and $(\mathcal{D}_d)_0$.
Finally, the inclusion (1) is again a homotopy equivalence w.r.t. δ_u.

The topology τ on $\mathcal{D}_d$ In [GRW10] Galatius and Randal-Williams extend the topology on the space of all compact d-dimensional submanifolds of $\mathbf{R}^n$ to the space of all closed d-dimensional submanifolds. This allows them to equip $\mathcal{D}_d$ with a topology which precisely coincides with the underlying topology of $(\mathcal{D}_d, \delta)$, as shown in [SP17, §A]. This topology thus has to capture the fact that the inclusion (1) is a homotopy equivalence, and is quite subtle. A very detailed introduction to the bordism category from [GRW10] is given in [Kra15].
Like the bordism categories considered in [GTMW09], $(\mathcal{D}_d, \tau)$ has a well-developed theory of tangential structures.

The topology τ on $\mathcal{C}_d$ This topology is equivalently given by the underlying topology of δ or as the restriction of the topology on $(\mathcal{D}_d, \tau)$.

The topology τ_u on $\mathcal{D}_d$ This topology is given as the underlying topology of δ_u or by modifying the topology τ in $\mathcal{D}_d$ in an obvious way.

Classifying spaces All models of the bordism category considered in this subsection have the same classifying space; this is clear a priori for the categories $(\mathcal{C}_d, \delta)$, $(\mathcal{C}_d, \tau)$, $(\mathcal{D}_d, \delta)$, $(\mathcal{D}_d, \tau)$ and for $(\mathcal{C}_d, \delta_u)$, $(\mathcal{C}_d, \tau_u)$, $(\mathcal{D}_d, \delta_u)$, $(\mathcal{D}_d, \tau_u)$.
We denote by $\mathring{\mathcal{C}}_d$ and $\mathring{\mathcal{D}}_d$ the semi-categories obtained by removing the identity morphisms form $\mathcal{C}_d$ and $\mathcal{D}_d$ respectively, then $(\mathcal{C}_d, \tau_u)$ and $(\mathcal{D}_d, \tau_u)$ are obtained from $(\mathring{\mathcal{C}}_d, \tau)$ and $(\mathring{\mathcal{D}}_d, \tau)$ by freely adjoining units, and [ERW19, Prop. 3.8] states that this procedure produces a semi-simplicial space with an equivalent classifying space. Finally, it is easily checked using [GTMW09, Crit. 2.5], that the inclusions $(\mathring{\mathcal{C}}_d, \delta) \hookrightarrow (\mathcal{C}_d, \delta)$ and $(\mathring{\mathcal{D}}_d, \delta) \hookrightarrow (\mathcal{D}_d, \delta)$ induce weak equivalences on the spaces of objects and morphisms.

1.2.4 On the classification of topological field theories

The two strands of our story come together in [Lur09b]. Lurie observes that some (1-categorical) topological quantum field theories are constructed using (co-)homology groups, and that it should be possible to refine these field theories to ones taking values in chain complexes, which arrange into an ∞-category; thus the indexing bordism category could reasonably be taken to be one of the models considered in the preceding subsection. Lurie first considers (a slight variant) of the topological semi-category $(\mathring{\mathcal{C}}_d, \tau)$, noting that while this semi-category does not possess units on the nose, it does possess units up to homotopy. Moreover, this guarantees that there exists a simplicial space satisfying the Segal condition, which is levelwise equivalent to $(\mathring{\mathcal{C}}_d, \tau)$. Such a simplicial space, namely $(\mathcal{D}_d, \tau)$, had just been proposed by Galatius in a (preliminary version) of [Gal11]. A more important upshot of working with a model of the sort described in the preceding subsection is that these can be modified to produce an (∞, n)-categorical version of the n-category envisaged by Baez and Dolan. Just as $(\mathring{\mathcal{C}}_d, \tau)$ contains manifolds with boundary, the corresponding (∞, n)-category would contain manifolds with corners; this problem is avoided by working with an (∞, n)-categorical version of $(\mathcal{D}_d, \tau)$, which provides a further reason to work with this model. Lurie then states and proves a more general (∞, n)-categorical version of Baez and Dolan's cobordism hypothesis.

1.2.5 The quasi-unital flagged ∞-category of bordisms

Looking back at §1.2.3 we see that the simplest topological category discussed therein is $(\mathcal{C}, \tau)$. This topological category does not quite model the bordism category that we are interested in, as in addition to the homotopical units which are already present, it contains strict units that have been adjoined formally. As touched upon in §1.1 Harpaz constructs a model category whose fibrant objects are precisely (Reedy fibrant) semi-simplicial spaces satisfying the Segal condition as well as a completeness condition (see 2.3.3); we generalise this theory in §2 allowing us to drop the completeness (and the Reedy fibrancy) condition. This permits us to consider $(\mathring{\mathcal{C}}_d, \tau)$ as a valid bordism category; however, we add a further simplification producing a homotopically better behaved model, $\mathbf{Bord}d, n$. The objects of $\mathbf{Bord}d, n$ are given by the space of compact $(d-1)$-dimensional submanifolds of $\mathbf{R}^n$ (without boundary), and the morphisms are given by the space of pairs (W, t) where $t > 0$, and $W \subseteq \mathbf{R}^\infty \times [0, t]$ is an embedded bordism respecting some collaring. These spaces are topologised in a way entirely analogous to $(\mathring{\mathcal{C}}_d, \tau)$, and there is a functor $(\mathring{\mathcal{C}}_d, \tau) \to \mathbf{Bord}d, \infty$ given by $(M, t) \mapsto M$ on objects, and by $(W; t_0, t_1) \mapsto (W, t_1 - t_0)$ on morphisms, which are both homotopy equivalences.

We prove that $\mathrm{Emb}(M, \mathbf{R}^n)$ and $\mathrm{Emb}(W, \mathbf{R}^n \times [0,1])$ are highly connected for $n \gg d$ in Theorems 4.1.9 and 4.2.30 respectively. The former result appears to be well known, but we were unable to find a reference in the literature. Both spaces are contractible for $n = \infty$, as can be seen by a simpler independent method (see [nLa19]). We hope that $\mathbf{Bord}d, n$ may serve as a useful simpler approximation to $\mathbf{Bord}d, \infty$.

In §1.1 we mention that it is particularly easy to equip $\mathbf{Bord}d, n$ with extra structure; this is because we can do so using a straightforward associated bundle construction. In particular there is no extra difficulty in equipping $\mathbf{Bord}d, n$ with extra structures satisfying some integrability condition, such as a symplectic structure. This would be difficult to accomplish using $(\mathcal{D}_d, \tau)$: Given any compact $(d-1)$-dimensional submanifold $M_0 \subseteq \mathbf{R}^\infty$, the subspace of $(\mathcal{D}_d)_0$ consisting of objects $(M, 0)$ such that $M|_0 = M_0$ is contractible. There exists a topology on the variant of $\mathcal{D}_d$, where the constituent manifolds are equipped with a tangential structure, which again has this property (see [GRW10]). We are however unaware how to specified such topologies for more general structures.

1.2.6 A remark on classifying spaces

We briefly return to the question of classifying spaces. The classifying space functor is invariant both under changing of flaggings, and, in the case of bordism categories, under forgetting integrality conditions. This can be seen as both a bug and a feature: On the one hand this means that we cannot produce new examples of infinite loop spaces by changing the flagging or adding integrability conditions to a bordism category with structure; on the other hand, given an infinite loop space that is known to be the classifying space of some structured bordism category, changing the flagging or structure might allow for new calculations.
Invariance under change of flaggings can be seen formally: Geometric realisation forms a homotopy left adjoint to the inclusion of the category of spaces into (semi-)simplicial spaces. Constant (semi-)simplicial spaces satisfy the Segal condition and are complete. Furthermore, the functor taking any (semi-)simplicial space satisfying the Segal condition to its completion is also a (homotopy) reflection. As adjoints to inclusions are localisations, the result follows. A direct proof for topological (semi-)categories satisfying a mild fibrancy condition (which is satisfied by $\mathbf{Bord}d, n)$ is given in [ERW19, Th. 5.2].
Let $\mathcal{F}$ be an equivariant sheaf (see [Aya09, Def. 2.14]), e.g. the sheaf associating to any d-manifold the space of symplectic structures, then Ayala defines a topological category $\mathsf{Cob}_d^{\mathcal{F}}$, based on $(\mathcal{C}_d, \tau_u)$, where all non-identity bordisms W are equipped with a section of $\mathcal{F}(W)$. Using McDuff's scanning construction (see [Mac77]) he then obtains an equivariant sheaf $\tau\mathcal{F}$ which is equivalent to a tangential structure, as well as a map $\mathcal{F} \to \tau\mathcal{F}$, which induces a functor $\mathsf{Cob}_d^{\mathcal{F}} \to \mathsf{Cob}_d^{\tau\mathcal{F}}$. Ayala's main theorem (stated in [Aya09, §3.6]) is then that this map induces an equivalence on classifying spaces. In our example where $\mathcal{F}$ is the sheaf of symplectic structures, $\tau\mathcal{F}$ is the sheaf of almost complex structures (see [Aya09, §9.1]).

1.3 Leitfaden and some remarks on the exposition

The overall structure of this article can be described as follows: Section 4 is the main part and contains the construction of $\mathbf{Bord}d, n$. Sections 2 and 3 contain background material and are independent of each other: Section 2 describes what sort of thing it is that we wish to construct, namely a quasi-unital flagged ∞-category, and Section 3, which focuses on the differential topology, provides the necessary theory to construct the constituent building blocks. Thus, in §4 we will constantly refer back to §3, and only in the end will we verify that our bordism category satisfies the properties outlined in §2. In §5 we present our modest sample application, where we equip $\mathbf{Bord}d, n$ with the structure of continuously varying Riemannian metrics.
We now describe §§2 & 3 in more depth. Both sections are detailed and relatively self-contained, hopefully providing useful introductions to their respective subjects in their own right. In §2.1 we set up the homotopical foundations used in the rest of this paper, which in turn are based on the theory of quasi-categories. We emphasise, however, that almost all statements as well as most arguments in §2 can be understood with a basic knowledge of model categories and a smattering of enriched category theory; the quasi-category theory only becomes visible upon peeking under the hood, so to speak. We then develop the theory of quasi-unital flagged ∞-categories in two steps:

1. In §2.2 we develop the theory of flagged ∞-categories.

2. In §2.3 we then move from the unital to the quasi-unital setting.

Warning 1.3.1. The terms *quasi-category* and ∞-*category* are often used interchangeably. We do not follow this convention, as a different model, which would typically be referred to as (not necessarily fibrant) complete Segal spaces, takes centre stage in the present work, and we call these ∞-*categories*. ⌟

In §2.2.1 we give an expository introduction to the idea that simplicial spaces satisfying the Segal condition model a homotopy theoretical generalisation of ordinary categories, and then rederive in the model categorical setting Ayala's and Francis' result that such structures model flagged ∞-categories in §2.2.2.
Section 2.3.2 is the main part of §2.3; here we discuss ∞-categorical quasi-unitality, and state our main theorem. We work in the setting of topological spaces rather than simplicial sets, allowing us to simplify certain parts of the discussion in [Har15], and hopefully make them more accessible. Sections 2.3.1 and 2.3.3 are expository. Finally, §2.3.4 contains the proof of the main theorem. A large part of the proof consists in transferring Harpaz' theory from the simplicial to the topological setting, (as well as from the Reedy fibrant to the objectwise fibrant setting), and is straightforward but tedious.
The main building blocks of our topological semi-category of bordisms are various spaces of smooth maps, and such spaces are the focus of §3.2. We develop the theory for manifolds with corners, (as these are required in the proof of Theorem 4.2.30), and are defined using jet spaces, which we discuss in §3.1. We examine the homotopy type of certain spaces of embeddings in §§4.1 & 4.2 using transversality arguments, which we discuss in §3.4. Moreover, throughout this article we often need to be able to perturb a given continuous map $L \to \mathbf{Man}(M, N)$ to a smooth one (see Definition 3.2.12); we prove a smooth approximation theorem in §3.3, which appears to be part of the oral tradition.
Section 4 on the construction of $\mathbf{Bord}d, n$ consists of three parts. In §4.1 we construct the constituent spaces of $(\mathbf{Bord}d, n)_0$, in §4.2 we construct the constituent spaces of $(\mathbf{Bord}d, n)_1$, and these are then assembled in §4.3. The spaces which we construct in §4.1 are the quotients of $\mathrm{Emb}(M, \mathbf{R}^n) \circlearrowleft \mathbf{Diff}(M)$, where M is a closed $(d-1)$-manifold; we show that these actions are principal, and that the spaces $\mathrm{Emb}(M, \mathbf{R}^n)$ are highly connected for $n \gg d$. These connectedness results are well known, but we were not able to find a proof in the literature. In §4.2 we establish analogous results for collared embeddings of bordisms in $\mathbf{R}^n \times [0, 1]$; the proofs generally become more tedious, as we must respect the bordism structure. Moreover, our connectedness result is proved for *non-collared* embeddings, and we then prove that the spaces of collared and non-collared embeddings are weakly equivalent. In §4.3 we carefully prove that the structure morphisms of $\mathbf{Bord}d, n$ are continuous, and that $\mathbf{Bord}d, n$ satisfies appropriate fibrancy properties. We conclude §4 with §4.3.4, where we sketch two alternative model for a flagged ∞-category of bordisms, where units are given by cylinders of "length 1".
In §5 we describe a sample application of our theory. We begin §5 by outlining quite generally how to associate extra structure to $\mathbf{Bord}d, n$. Moreover, we prove that the associated bundle construction preserves weak equivalences; a result that is widely used, but for which we were not able to find a proof in the literature. The remainder of the section is then devoted to making the aforementioned sketch precise for Riemannian bordisms, where the metric may vary continuously. In our setup this is now completely routine. The forgetful functor from Riemannian to ordinary bordisms is a levelwise weak equivalence, which, of course, follows from the contractibility of the

space of Riemannian metrics.

2 Homotopy theory

2.1 Homotopical foundations

In this section we briefly explain our homotopical setup. The theory of quasi-categories serves as our foundation for homotopy theory. Within the body of this article this will hardly be felt, however, as all essential homotopy theoretical discussions outside of a small number of proofs will take place in the context of model categories, or some variation thereof.

To the expert, this approach may, a priori, seem contradictory or at least unnecessarily convoluted, as the theories of quasi-categories and model categories are typically viewed as living in different worlds, requiring some (often technically demanding) process of translation to move from one to the other. In a series of papers starting with [MG14], Mazel-Gee observes that a model structure on a category is useful not only for understanding its localisation in the realm of categories, but (upon taking its nerve) also for understanding its *quasi-categorical* localisation, and that it in fact makes perfect sense to define model structures even on quasi-categories which are not necessarily nerves of categories. Cisinski then further extends and systematises the theory of localisations of quasi-categories in [Cis19, Ch. 7][2]. He focuses on one half of model structures, on *quasi-categories with weak equivalences and (co-)fibrations* (see [Cis19, Defs. 7.4.12 & 7.4.6]), carefully exhibiting which components of the structure allow us to do which constructions. Many difficulties which arise when trying to do homotopy theory using classical localisation stem from the paucity of (co-) limits in many important homotopy categories, most strikingly in the homotopy category of topological spaces, or equivalently, simplicial sets. Of course, the quasi-categorical localisations of the aforementioned categories produce the quasi-category of homotopy types, which is both complete and cocomplete. Cisinski then views Homotopical Algebra to a large extent as the study of the compatibility of localisations with (co-)limits.

We now turn to the definitions and theorems which we will need in the rest of this work.

Definition 2.1.1. A category (quasi-category) with weak equivalences is a category $\mathcal{C}$ (quasi-category C) together with a subcategory $\mathcal{W}$ (simplicial subset $W \subseteq C$). ⌟

Convention 2.1.2. From now on we shall no longer distinguish carefully between categories and their nerves. ⌟

Definition 2.1.3. Let (C, W) and (C', W') be quasi-categories with weak equivalences, then a functor $C \to C'$ *preserves weak equivalences* if it restricts to a map $W \to W'$. ⌟

Definition 2.1.4. Let (C, W) be a quasi-category with weak equivalences. A quasi-category $C \to W^{-1}C$ together with a functor $\gamma_C : C \to W^{-1}C$ sending edges in W to equivalences is called the *localisation of C by W*, if for every quasi-category D the induced map of quasi-categories $\gamma_C^* : \underline{\mathrm{Hom}}(W^{-1}C, D) \to \underline{\mathrm{Hom}}_W(C, D)$ is an equivalence, where $\underline{\mathrm{Hom}}_W(C, D)$ denotes the quasi-category spanned by functors sending edges in W to equivalences. ⌟

[2] The only earlier treatments of localisation that we are aware of are [Lur09a, §3.1], where Lurie develops a model structure of (a slight variant of) quasi-categories with weak equivalences, and various parts of [Lur09a, Ch. 5], where he discusses the special case of reflexive subcategories with a focus on presentable quasi-categories.

Notation 2.1.5. With notation as in the preceding definition, if W is clear from context, we shall often denote $W^{-1}C$ by LC. ⌟

Definition 2.1.6. Let C and C' be quasi-categories with weak equivalences, and let $f : C \to C'$ be a functor which preserves weak equivalences, then we say that f *induces an equivalence of homotopy theories* if it induces an equivalence $LC \xrightarrow{\sim} LC'$. ⌟

Definition 2.1.7. With notation as above, given a diagram $F : I \to C$, a (co-)cone on F is called a *homotopy (co-)limit* if its composition with the localisaiton functor $C \to L(C)$ is a (co-)limit. ⌟

Proposition 2.1.8. *Any functor between quasi-categories with weak equivalences, which preserves weak equivalences and induces an equivalence on homotopy theories, preserves and reflects homotopy (co-)limits.* □

Proposition 2.1.9. *Let C, C' be quasi-categories with weak equivalences, and let $f : C \to C'$, $f' : C' \to C$ be functors which preserve weak equivalences. If there exist natural transformations* $\mathrm{id} \to f'f$ *and* $ff' \to \mathrm{id}$ *which are levelwise weak equivalences, then f and f' are weak equivalences.* □

Sketch of proof. By [Cis19, Prop. 7.1.13] localisation is compatible with products of quasi-categories with weak equivalences. We apply this fact to $\Delta^1 \times C$ and $\Delta^1 \times C'$. □

The main advantage of our chosen approach is that the theory of derived functors becomes significantly cleaner. Moreover, these can be obtained using weaker structures than model structures. In [Mal07] Maltsiniotis generalises and clarifies Quillen's classical theorem that Quillen adjunctions induce adjunctions between homotopy categories (see [Qui67, Thm. I.4.3]). Cisinski then observes that Maltsiniotis' proof is robust enough to carry over essentially verbatim to the quasi-categorical setting.

For the rest of this subsection we fix quasi-categories with weak equivalences C and D as well as an adjunction $F : C \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} D : G$. A Kan extension is called *absolute* if its composition with any functor is again a Kan extension.

Theorem 2.1.10 ([Cis19, Th.. 7.5.30])**.** *If the derived functors of F and G exist*[3] *and are absolute Kan extensions, then the derived functors $\mathbf{L}F : LC \rightleftarrows LD : \mathbf{R}G$ admit the canonical structure of an adjunction.* □

We will be interested in two cases of derived functors, which are absolute Kan extensions:

(1) If F and G preserve weak equivalences, then the functors induced by the universal property of localisation are absolute derived functors.

(2) Assume C is equipped with weak equivalences and cofibrations, and D, with weak equivalences and fibrations. If F sends weak equivalences between cofibrant objects to weak equivalences, and G sends weak equivalences between fibrant objects to weak equivalences, then the derived functors exist and are absolute (see [Cis19, 7.5.25]).

Recall that an adjunction of quasi-categories is called a *(co-)reflection* if its (co-)unit is an equivalence.

[3] Through set theoretical trickery we can always assume that the localisations of quasi-categories exists, but this is not the case for Kan extensions, and thus derived functors.

Proposition 2.1.11. *With assumptions as in (2) above, if for any cofibrant object $c \in C$ there exists a fibrant replacement $Fc \xrightarrow{\sim} \overline{Fc}$ such that the composition of $c \to UFc \to U\overline{Fc}$ is a weak equivalence, then $\mathbf{L}F : LC \overset{\longrightarrow}{\underset{\perp}{\longleftarrow}} LD : \mathbf{R}G$ is a reflection.*
Dually, if for any fibrant object $d \in D$ there exists a cofibrant replacement $\underline{Ud} \xrightarrow{\sim} Ud$ such that the composition of $F\underline{Ud} \to FUd \to d$ is a weak equivalence, then $\mathbf{L}F : LC \overset{\longrightarrow}{\underset{\perp}{\longleftarrow}} LD : \mathbf{R}G$ is a coreflection.
If both conditions are satisfied, then $\mathbf{L}F : LC \overset{\longrightarrow}{\underset{\perp}{\longleftarrow}} LD : \mathbf{R}G$ is an adjoint equivalence.

Proof. We will only prove the first statement, as the second is dual, and the third follows immediately from the first two. We denote by $\iota : C_{\mathrm{cof}} \hookrightarrow C$ the subcategory of C spanned by cofibrant objects, by η, ε the unit and counit of the adjunction $F \dashv U$, and by $\overline{\eta}, \overline{\varepsilon}$ the unit and counit of the adjunction $\mathbf{L}F \dashv \mathbf{R}U$. We obtain a diagram

$$
\begin{tikzcd}
C_{\mathrm{cof}} \arrow[r, hook, "\iota"] \arrow[d] & C \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & D \arrow[r] \arrow[d] \arrow[u, phantom, "\Downarrow\eta"] & C \arrow[d] \arrow[dl, Rightarrow, "\psi"'] \\
LC_{\mathrm{cof}} \arrow[r, "\simeq", "\mathbf{L}\iota"'] \arrow[ur, Rightarrow, "\simeq"] & LC \arrow[r] \arrow[ur, Rightarrow, "\varphi"] \arrow[rr, bend right, "\mathrm{id}"'] & LD \arrow[r] \arrow[d, phantom, "\Uparrow\overline{\eta}"] & LC.
\end{tikzcd}
$$

Then

$$
\begin{array}{lll}
 & \overline{\eta} & \text{is an equivalence} \\
\Leftrightarrow & \overline{\eta}\,\mathbf{L}\iota & \text{is an equivalence (because } \mathbf{L}\iota \text{ is an equivalence)} \\
\Leftrightarrow & \overline{\eta}\,\mathbf{L}\iota\,\gamma_{\mathrm{cof}} & \text{is an equivalence (because } \gamma_{\mathrm{cof}} \text{ is essentially surjective)} \\
\Leftrightarrow & \overline{\eta}\,\gamma_C\,\iota & \text{is an equivalence (because } \mathbf{L}\iota\,\gamma_{\mathrm{cof}} \simeq \gamma_C\,\iota) \\
\Leftrightarrow & (\mathbf{R}U\,\varphi\,\iota)(\overline{\eta}\,\gamma_C\,\iota) & \text{is an equivalence (because } \mathbf{R}U\,\varphi\,\iota \text{ is an equivalence),}
\end{array}
$$

but the last natural transformation $\gamma_C\,\iota \Rightarrow \mathbf{R}U\,\gamma_D\,F\,\iota$ is equivalent to $(\psi\,F\,\iota)(\eta\,\iota)$ by the construction of the derived adjoint.
Now choose a cofibrant object $c \in C$ and a weak equivalence $Fc \xrightarrow{\sim} \overline{Fc}$ such that the map $c \to UFc \to U\overline{Fc}$ is a weak equivalence, then we obtain a commutative diagram

$$
\begin{tikzcd}
\gamma_C\,c \arrow[r] & \gamma_C\,U\,Fc \arrow[r] \arrow[d] & \gamma_C\,U\,\overline{Fc} \arrow[d, "\simeq"] \\
 & \mathbf{R}U\,\gamma_D\,Fc \arrow[r, "\simeq"] & \mathbf{R}U\,\gamma_D\,\overline{Fc},
\end{tikzcd}
$$

so that $\gamma_C\,c \to \gamma_C\,UFc \to \mathbf{R}U\,\gamma_D\,Fc$ is an equivalence by the 2-out-of-3 property. $\square$

We conclude this subsection by discussing a potential point of confusion. Working within our homotopical framework we develop a theory in §2 whose central objects are called ∞-categories as well as several variations thereof. Our ∞-categories are a slight (point-set theoretical) generalisation of what are typically called complete Segal spaces (see [Rez00]). This conflicts with the standard usage of the term ∞-categories to refer to quasi-categories. The theory of quasi-categories and complete Segal spaces are, however, equivalent by [JT07]. The fact that we develop one theory of ∞-categories within another, is entirely analogous to how one could develop category theory within category theory, e.g. as the theory of category objects in **Set**.

2.2 Flagged ∞-categories and flagged ∞-semi-categories

Complete Segal spaces offer one of many equivalent models of ∞-categories. In this section we will generalise complete Segal spaces in several directions to finally arrive at the notion of a quasi-unital flagged ∞-category, the structure into which we will arrange our bordisms.

2.2.1 Definitions and basic properties

A category can be viewed as a simplicial set X such that the squares in the diagram (2) below are pullbacks. As ∞-categories are the homotopical incarnation of ordinary categories, one might hope to obtain the correct notion of ∞-category by translating the above characterisation of ordinary categories into a homotopical setting.

Definition 2.2.1. A (semi-)simplicial topological space X satisfies the *Segal condition* if for every $m, n > 0$ the diagram

$$\begin{tikzcd} X_{m+n} \arrow[r, "d_0\circ\cdots\circ d_0"] \arrow[d, "d_{m+1}\circ\cdots\circ d_{m+n}"'] & X_n \arrow[d, "d_1\circ\cdots\circ d_n"] \\ X_m \arrow[r, "d_0\circ\cdots\circ d_0"] & X_0 \end{tikzcd} \tag{2}$$

is a homotopy pullback. ⌟

The following section concerns mainly semi-simplicial spaces, and, as the above definition illustrates, we often discuss simplicial and semi-simplicial spaces in tandem in this section.

Definition 2.2.2. A *flagged ∞-(semi-)category* is a (semi-)simplicial topological space satisfying the Segal condition. ⌟

The qualifier "flagged" indicates that we have not yet obtained quite the correct notion of ∞-category. This issue will be dealt with in the following subsection. This phenomenon is not present in the groupoidal analogue of flagged ∞-categories.

Definition 2.2.3. An *∞-(semi-)groupoid* is a *flagged ∞-(semi-)category*, such that all face maps are weak equivalences. ⌟

Remark 2.2.4. The characterisation of ∞-(semi-)groupoids in the previous definitions is heavily overdetermined. We obtain the same notion if, e.g.

- we drop the Segal condition;
- we keep the Segal condition but only require the face maps $X_1 \to X_0$ to be weak equivalences.

⌟

Definition 2.2.5. A *functor* of flagged ∞-(semi)-categories is a natural transformation in $\mathbf{Top}^{\Delta^{\mathrm{op}}}$. Such a natural transformation is called an *equivalence of flagged ∞-categories* if it is a levelwise weak equivalence. ⌟

Example 2.2.6.

- *The nerve of any topological (semi-)category $\mathfrak{C}$, whose morphisms $d_0, d_1 : \mathfrak{C}_1 \to \mathfrak{C}_0$ are fibrations, is a flagged ∞-(semi)-category. In particular the nerve of any ordinary (semi-) category is a flagged ∞-(semi)-category.*

- *Constant (semi-)simplicial topological spaces are examples of ∞-(semi-)groupoids, so we may view any topological space A as a ∞-(semi-)groupoid, which we again denote by A.*

⌟

Reedy fibrant bisimplicial sets satisfying the Segal condition are typically called *Segal spaces*, and were first systematically studied by Rezk in [Rez00]. They are the fibrant objects of a Bousfield localisation of the Reedy model structure on bisimplicial sets. We will also refer to Reedy fibrant flagged ∞-categories in the topological setting as Segal spaces. These are again the fibrant objects of a model structure, which by [Hir03, Prop. 15.4.1. & Thm. 3.3.20] is Quillen equivalent to the corresponding model structure on bisimplicial sets. The semi-simplicial topological space of bordisms which we construct is not Reedy fibrant. This model has interesting geometric properties, which are lost upon taking a Reedy fibrant replacement, which is why we consider this more general notion. Fortunately, this is not a problem, because the Segal condition is preserved by levelwise weak equivalences, so the inclusion of Segal spaces into ∞-categories and functorial Reedy fibrant replacement induce inverse equivalences of homotopy theories by Proposition 2.1.9.

Mapping spaces For a flagged ∞-(semi)-category X, elements of X_0 are called *objects*, and for any $(n+1)$-tuple of objects $(x_0, \dots, x_n)$ the *mapping space* of $(x_0, \dots, x_n)$, denoted by $\mathrm{Map}(x_0, \dots, x_n)$, is the homotopy pullback of

$$\begin{tikzcd} & X_n \arrow[d, "(a_0,\dots,a_n)"] \\ \{(x_0,\dots,x_n)\} \arrow[r, hook] & X_0 \times \cdots \times X_0, \end{tikzcd}$$

and the *hom space*, denoted by $\mathrm{Hom}(x_0, \dots, x_n)$, is the categorical pullback of the above diagram. Throughout this article we will use the classical construction (see e.g. in [Shu09, Defs. 6.6 & 8.2]) of the above homotopy pullback given by the limit of

$$\begin{tikzcd} & & X_n \arrow[d, "(a_0,\dots,a_n)"] \\ & X_0^{\Delta^1} \times \cdots \times X_0^{\Delta^1} \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_0"] & X_0 \times \cdots \times X_0 \\ \{(x_0,\dots,x_n)\} \arrow[r, hook] & X_0 \times \cdots \times X_0. & \end{tikzcd}$$

We then have a canonical inclusion $\mathrm{Hom}(x_0,\dots,x_n) \hookrightarrow \mathrm{Map}(x_0,\dots,x_n)$.
A *morphism* $f : x \to y$ in X is an element of $\mathrm{Map}(x,y)$; an *edge* $f : x \to y$ is an element of $\mathrm{Hom}(x,y)$.

Notation 2.2.7. By our above choice of model for the homotopy limit in the definition of mapping spaces, any morphism $f : x \to y$ consists of paths $\alpha : x \rightsquigarrow x'$, $\beta : y' \rightsquigarrow y$ together with an element $f' \in \mathrm{Hom}_X(x',y')$. We will often denote the morphism f by (α, f', β). ⌟

If the map $X_1 \xrightarrow{(d_1,d_0)} X_0 \times X_0$ is a fibration, then for any two objects $x, y \in X_0$ the inclusion $\mathrm{Hom}(x,y) \hookrightarrow \mathrm{Map}(x,y)$ is a weak homotopy equivalence. Note that in this case, the canonical maps $d_0, d_1 : X_1 \to X_0$ are fibrations; thus topological (semi-)categories $\mathfrak{C}$, with the property that $\mathfrak{C}_1 \xrightarrow{(d_1,d_0)} \mathfrak{C}_0 \times \mathfrak{C}_0$ is a fibration, are particularly nice examples of flagged ∞-(semi-)categories. Our main objet of interest, $\mathbf{Bord}d,n$, is such a semi-simplicial topological category. Moreover, for any two objects $M, N \in (\mathbf{Bord}d,n)_0$ we will even be able to construct an explicit homotopy inverse of the canonical map $\mathrm{Hom}(M,N) \hookrightarrow \mathrm{Map}(M,N)$ in §4.3.3.

Homotopy categories Given a flagged ∞-(semi)-category X we now show how to extract a (semi-)category enriched in $\mathrm{Ho}(\mathbf{Top})$, called its *homotopy category*, and denoted by $\underline{\mathrm{Ho}(X)}$. The underlying category $\mathrm{Ho}(X)$ (in the enriched sense) of $\underline{\mathrm{Ho}(X)}$ may be obtained by taking the connected components of the mapping homotopy types.
We set $\left(\underline{\mathrm{Ho}(X)}\right)_0$ to be the underlying set of X_0 and for any $x, y \in X_0$ we set $\underline{\mathrm{Ho}(X)}(x,y) = \mathrm{Map}(x,y)$. Note that because all topological spaces are fibrant, any product in $\mathbf{Top}$ descends to a product in $\mathrm{Ho}(\mathbf{Top})$. For any triple $(x_0,x_1,x_2) \in X_0 \times X_0 \times X_0$ composition is then given by the map in $\mathrm{Ho}(\mathbf{Top})$ presented by

$$\mathrm{Map}(x_0,x_1) \times \mathrm{Map}(x_1,x_2) \xleftarrow[\sim]{(d_2,d_0)} \mathrm{Map}(x_0,x_1,x_2) \xrightarrow{d_1} \mathrm{Map}(x_0,x_2).$$

For any 4-tuple $(x_0,\dots,x_3) \in X_0 \times X_0 \times X_0 \times X_0$ associativity is expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
 & & \mathrm{Map}(x_0,x_1)\times\mathrm{Map}(x_1,x_3) & & \\
 & \mathrm{Map}(x_0,x_1)\times\mathrm{Map}(x_1,x_2,x_3) \arrow[ur, "\mathrm{id}\times d_1"] \arrow[dl, "\mathrm{id}\times(d_2,d_0)"', "\sim"] & & \mathrm{Map}(x_0,x_1,x_3) \arrow[ul, "{(d_2,d_0,)}"', "\sim"] \arrow[dr, "d_1"] & \\
\mathrm{Map}(x_0,x_1)\times\mathrm{Map}(x_1,x_2)\times\mathrm{Map}(x_2,x_3) & & \mathrm{Map}(x_0,x_1,x_2,x_3) \arrow[ul, "{(a_0,d_0)}"', "\sim"] \arrow[ur, "d_2"] \arrow[dl, "{(d_3,a_2)}", "\sim"'] \arrow[dr, "d_1"] & & \mathrm{Map}(x_0,x_3) \\
 & \mathrm{Map}(x_0,x_1,x_2)\times\mathrm{Map}(x_2,x_3) \arrow[ul, "{(d_2,d_0)\times\mathrm{id}}", "\sim"'] \arrow[dr, "d_1\times\mathrm{id}"'] & & \mathrm{Map}(x_0,x_2,x_3) \arrow[ur, "d_1"'] \arrow[dl, "{(d_2,d_0)}", "\sim"'] & \\
 & & \mathrm{Map}(x_0,x_1)\times\mathrm{Map}(x_1,x_3) & &
\end{tikzcd} \tag{3}$$

It is easily checked that the above diagram restricts to a diagram of the same shape whose objects are (products of) hom spaces.
For any object $x \in X_0$, (the connected component corresponding to) $s_0 x$ is an identity morphism for x. To see this, it is enough to note that for any other object y of X the diagram

$$\begin{tikzcd} \mathrm{Map}(x,x,y) \arrow[r, "d_1"] \arrow[d, "(d_1,d_0)"', "\sim"] & \mathrm{Map}(x,y) \arrow[dl, "(\{s_0 x\},\mathrm{id})"] \\ \mathrm{Map}(x,x) \times \mathrm{Map}(x,y) & \end{tikzcd}$$

and its dual commute.
The construction of homotopy categories is functorial. A functor $X \to Y$ inducing an equivalence $\underline{\mathrm{Ho}(X)} \to \underline{\mathrm{Ho}(Y)}$ is called a *Dwyer-Kan equivalence.* Thus an equivalence of categories (in the classical sense) is precisely a functor inducing a Dwyer-Kan equivalence between their nerves.
If for X the squares (2) are also categorical pullbacks (i.e. X is a topological (semi-)category), then the backwards pointing maps in the restriction of (3) to (products of) hom spaces are isomorphisms. In this case, we obtain another $\mathrm{Ho}(\mathbf{Top})$-enriched category, constructed using the hom spaces rather than the mapping spaces, which we denote by $\mathfrak{Ho}(X)$, together with a canonical $\mathrm{Ho}(\mathbf{Top})$-functor $\mathfrak{Ho}(X) \to \underline{\mathrm{Ho}(X)}$. If $X_1 \xrightarrow{(d_1,d_0)} X_0 \times X_0$ is a fibration, then the $\mathfrak{Ho}(X) \to \underline{\mathrm{Ho}(X)}$ is an isomorphism of $\mathrm{Ho}(\mathbf{Top})$-enriched categories.

2.2.2 Flagged ∞-categories as ∞-categories with extra structure

We now describe the phenomenon giving rise to the qualifier "flagged", which is significantly more important in the homotopical setting than it is in the classical setting.
Let X be a flagged ∞-category. Given any two objects $x, y \in X_0$, we consider two notions of how they may be equivalent:

- x, y can be linked by a path in X_0.
- x, y can be linked by an invertible morphism in X_1,

where a morphism is invertible if it induces an invertible morphisms in $\underline{\mathrm{Ho}(X)}$. We then denote by $\mathrm{Map}_{\mathrm{inv}}(x, y)$ the subspace of $\mathrm{Map}(x, y)$ consisting of invertible morphisms. Recalling Notation 2.2.7 we obtain a map

$$\begin{array}{rcl} \mathrm{Path}(x,y) & \to & \mathrm{Map}(x,y) \\ \alpha & \mapsto & \left(\alpha|_{t\in[0,1/2]}(2\cdot t),\ \mathrm{id}_{\alpha(1/2)},\ \alpha|_{t\in[1/2,1]}(2\cdot t)\right). \end{array}$$

Definition 2.2.8. The flagged ∞-category X is an ∞-category if for all $x, y \in X_0$ the induced map $\mathrm{Path}(x, y) \to \mathrm{Map}_{\mathrm{inv}}(x, y)$ is a weak homotopy equivalence. ⌟

Reedy fibrant ∞-categories are typically called *complete Segal spaces* (introduced in [Rez00, §6]), or sometimes *Rezk spaces.* These are the fibrant objects of a model structure (see [Rez00, §7]), and the functor sending any simplicial space to its underlying simplicial set is the left adjoint of a Quillen equivalence between this model category and the model category of quasi-categories [JT07], the best studied model for ∞-categories[4]. The weak equivalences between

[4]As explained in §2.1, quasi-categories also provide the ambient framework in which we do homotopy theory in this article.

flagged ∞-categories in this model structure are precisely the Dwyer-Kan equivalences.
Taking the fibrant replacement of X in the complete Segal space model structure yields a Dwyer Kan equivalence $f : X \to X'$, where X' is a complete Segal space. We then obtain an induced map $X_0 \to X'$. The rest of this subsection is concerned with showing that in a precise sense $X_0 \to X'$ contains the same information as X. Thus we think of X as consisting of its "underlying" ∞-category X' together with the "flagging" $X_0 \to X'$. The flagging of $\mathbf{Bord}d$ has a geometric interpretation, which we study in §4.3.3. Taking the fibrant replacement of the nerve C of an ordinary category $\mathcal{C}$ yields a complete Segal space C' such that C'_0 is the classifying space of $\mathcal{C}^{\simeq}$. The flagging then amounts to a map $C_0 \to C'_0$. The classical notion of equivalences between categories offer a way of ignoring this flagging.

Notation 2.2.9. We denote by Δ_- the category given by the join $\Delta \star \{-1\}$. ⌟

Definition 2.2.10. A *coaugmented simplicial topological space* is a functor $\Delta_-^{\mathrm{op}} \to \mathbf{Top}$. ⌟

A coaugmented simplicial topological space $Y : \Delta_-^{\mathrm{op}} \to \mathbf{Top}$ contains the same information as the map of simplicial topological spaces $Y_{-1} \to Y|_{\Delta^{\mathrm{op}}}$.
We endow Δ_- with the unique Reedy structure which restricts to the standard Reedy structure on Δ, and whose degree function sends -1 to -1 and $(0 \to -1) \in \overleftarrow{\Delta}_-$. Denote by $\mathbf{STop}$ the category of simplicial topological spaces, and by $\mathbf{STop}_-$ the category of coaugmented simplicial topological spaces, both equipped with the respective Reedy model structures; by [Hir03, Thm. 15.3.4.3] these model structures are simplicial. The adjunction $i : \Delta \leftrightarrows \Delta_- : p$ induces a quadruple of adjoint functors:

$$\begin{tikzcd} \mathbf{STop} \arrow[r, hook, shift left=4.5, "i_!"] \arrow[r, leftarrow, shift left=1.5, "i^*=p_!"] \arrow[r, hook, shift right=1.5, "i_*=p^*"] \arrow[r, leftarrow, shift right=4.5, "p_*"] & \mathbf{STop}_- \end{tikzcd} .$$

The functor $i_!$ coaugments a simplicial topological space with the empty space. The functor p_* is given by taking the enriched nerve w.r.t. $\Delta \overset{F}{\hookrightarrow} \mathbf{STop} \xrightarrow{p^*} \mathbf{STop}_-$. This follows from the observation that the Yoneda adjunction (see [Cis19, Thm. 1.1.10]) can be extended to the enriched setting by combining the characterisation of enriched adjunctions given in [Shu09, Prop. 14.6] and the characterisation of the enriched Yoneda extension in [Hin16, §4]. We have in fact shown that $p^* \dashv p_*$ is a topologically enriched adjunction, and we see by a standard argument, that $p^* \dashv p_*$ induces a simplicial adjunction w.r.t. the simplicial enrichment already considered. With this description of p_* in hand, we see that for any coaugmented simplicial topological space X, the topological space $(p^*X)_n$ fits into the pullback diagram

$$\begin{tikzcd} (p^*X)_n \arrow[r] \arrow[d] & X_n \arrow[d, "(d_0,\dots,d_n)"] \\ X_{-1} \times \cdots \times X_{-1} \arrow[r] & X_0 \times \cdots \times X_0, \end{tikzcd}$$

for all $n \geq 0$.

Proposition 2.2.11. *The adjunction*

$$p^* : \mathbf{STop} \underset{\perp}{\rightleftarrows} \mathbf{STop}_- : p_*$$

is Quillen.

Proof. There exist general criteria for determining when a morphism of Reedy categories induces a Quillen adjunction (see [Bar07, §2]), but in our case this is easily established directly. The functor p^* clearly preserves weak equivalences. It also preserves cofibrations: The relative latching maps of a simplicial topological space X get sent the corresponding relative latching maps of p^*X in degree ≥ 1, the relative latching map in degree 0 gets sent to the relative latching in degree -1, and finally, the relative latching map in degree 0 of p^*X is the identity map of p^*X_0. $\square$

The adjunction $p^* \dashv p_*$ models the adjunction of quasi-categories

$$(8) : \mathsf{PShv}(\Theta_1) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \Gamma(\mathsf{PShv}(\Theta_\bullet)) : \mathsf{fN}$$

described in [AF18, §1.1].
Let $n \in \mathbf{N}$, then we denote by $F(n)$ the simplicial set Δ^n, viewed as a discrete simplicial topological space, and by $G(n) \subseteq F(n)$ the simplicial topological subspace spanned by its 0-simplices and those 1-simplices corresponding to a^i for $i = 0, \ldots, n-1$. We then denote by **SegSp** the category **STop** equipped with the Bousfield localisation of the Reedy model structure along the inclusions $\varphi_n : G(n) \hookrightarrow F(n)$ $(n \geq 0)$, and the category $\mathbf{STop}_{\_}$ equipped with the Bousfield localisation of the Reedy model structure along the maps $i_!\varphi_n : i_!G(n) \hookrightarrow i_!F(n)$ $(n \geq 0)$ by $\mathbf{SegSp}_{\_}$; these both exist and are simplicial by [Hir03, Thm. 4.1.1, Prop. 4.1.4, Thms. 4.3.8 & 15.7.6].

Proposition 2.2.12. *The adjunction*

$$p^* : \mathbf{SegSp} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{SegSp}_{\_} : p_*$$

is Quillen.

Proof. It is enough to check that $p^*\varphi_n$ is a weak equivalence in $\mathbf{SegSp}_{\_}$ for every $n \in \mathbf{N}$. As $p^* \dashv p_*$ is a simplicial adjunction, we can equivalently check that p_*X is local w.r.t. the maps $\varphi_n : G(n) \hookrightarrow F(n)$ for all $n \in \mathbf{N}$ and for all fibrant $X \in \mathbf{SegSp}_{\_}$, which is equivalent to showing that p_*X satisfies the Segal condition, which in turn is equivalent to showing that the canonical map $(p^*X)_m \times_{X'_0} (p^*X)_n \twoheadrightarrow (p^*X_{m+n})$ is a weak equivalence for all $m, n \in \mathbf{N}_{>0}$. As

$$\begin{tikzcd}
(p^*X)_m \times_{X'_0} (p^*X)_n \arrow[r] \arrow[d, two heads] & X_m \times_{X_0} X_n \arrow[d, two heads] \\
X_{-1} \underbrace{\times \cdots \times}_{n\times} X_{-1} \arrow[r] & X_0 \underbrace{\times \cdots \times}_{n\times} X_0
\end{tikzcd}$$

is a pullback, the diagram

$$\begin{tikzcd}
(p^*X)_m \times_{X_{-1}} \times (p^*X)_n \arrow[r] \arrow[d, two heads, "\sim"] & X_m \times_{X_0} X_n \arrow[d, two heads, "\sim"] \\
(p^*X)_{m+n} \arrow[r] & X_{m+n}
\end{tikzcd}$$

is a pullback by the pasting lemma, so the vertical map to the left is a trivial fibration, as these are preserved by pullbacks. $\square$

We denote by E the discrete nerve of the category $\bullet \rightleftarrows \bullet$, consisting of two inverse non-identity morphisms. By [Rez00, Prop. 6.4] a Segal space is complete iff it is local w.r.t. $F(0) \hookrightarrow E$. We then denote by $\mathbf{CSS}$ the localisation of $\mathbf{SegSp}$ by $F(0) \hookrightarrow E$, and, by $\mathbf{CSS}^{\text{⚐}}$ the localisation of $\mathbf{SegSp}_{-}$ along the morphism $i_!(F(0) \hookrightarrow E)$. As mentioned in the introduction of this subsection, the weak equivalences between Segal spaces in $\mathbf{CSS}$ are precisely the Dwyer-Kan equivalences (see [Rez00, Thm. 7.7]).

Theorem 2.2.13. *The derived unit of the Quillen adjunction*

$$p^* : \mathbf{SegSp} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{CSS}^{\text{⚐}} : p_*$$

is a weak equivalence. The image of the localisation[5] *of p^* is spanned by those fibrant coaugmented simplicial topological spaces Y such that $Y_{-1} \to Y_0$ induces a surjection on connected components.*

Proof. Let X be a Segal space, then we obtain a fibrant replacement of p^*X by taking a fibrant replacement of X' of X in $\mathbf{CSS}$ and coaugmenting X' with $X_0 \to X'_0$; we denote this fibrant replacement in $\mathbf{CSS}^{\text{⚐}}$ by Y. We obtain a commutative square

$$\begin{tikzcd}
\pi_0(\mathrm{Ho}(X_0)^{\simeq}) \arrow[r] \arrow[d, two heads] & \pi_0(\mathrm{Ho}(Y_0)^{\simeq}) \arrow[d, "\simeq"] \\
\pi_0(\mathrm{Ho}(X)^{\simeq}) \arrow[r, two heads] & \pi_0(\mathrm{Ho}(Y)^{\simeq}),
\end{tikzcd}$$

and see that $X_0 \to Y_0$ induces a surjection on connected components, which verifies one half of the second part of the theorem. We wish to show that

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & Y_n \arrow[d, "{(d_0,\dots,d_n)}"] \\
X_0 \times \cdots \times X_0 \arrow[r] & Y_0 \times \cdots \times Y_0,
\end{tikzcd}$$

is a homotopy pullback square for all $n \geq 0$. For $n = 2$ the fact that $X \to i^*Y$ is a Dwyer-Kan equivalence tells us precisely that X_2 is fiberwise equivalent to the homotopy pullback, and thus equivalent to the homotopy pullback. For $n > 2$ the fact that X_n is a homotopy pullback follows from the case $n = 2$ and the Segal condition.
To finish verifying the second part of the theorem, let Y be a fibrant object in $\mathbf{CSS}^{\text{⚐}}$ such that $Y_{-1} \to Y_0$ induces a surjection on connected components, then the surjectivity condition is precisely what is needed to guarantee that the induced map $p_*Y \to i^*Y$ is a Dwyer-Kan equivalence, thus $p^*p_*Y \to Y$ is a fibrant replacement in $\mathbf{CSS}^{\text{⚐}}$. □

By Proposition 2.1.11 and [RV19, Prop.13.4.5] we have recovered the following result:

Theorem 2.2.14 ([AF18, Thm. 0.26])**.** *The quasi-category of simplicial topological spaces satisfying the Segal condition is equivalent to the quasi-category of pairs consisting of an ∞-category X and an ∞-groupoid K together with an essentially surjective functor $K \to X$.* □

Remark 2.2.15. It would be interesting to know whether the model structure on $\mathbf{CSS}^{\text{⚐}}$ admits a right Bousfield localisation turning the adjunction $p^* \dashv p_*$ into a Quillen equivalence. ⌟

[5] Either classical or quasi-categorical.

2.3 Quasi-unital flagged ∞-categories

2.3.1 Quasi-unital categories

To acquaint ourselves with the main ideas we first discuss the situation for ordinary categories. From a set-theoretical perspective a category can be viewed as a semi-category satisfying the *property* of admitting identity morphisms, as these are uniquely determined. However, from a category theoretical perspective possessing identity morphisms is *not* a property[6], as these are not automatically preserved by functors of semi-categories, which means that categories do not form a full sub (2-)category of semi-categories.
Keeping with the convention in [Har15], [Har12] we shall refer to identity morphisms as *units*. A semi-category possessing units is then a *quasi-unital category* and a functor between quasi-unital categories is a *quasi-unital functor*. We also observe that the notion of natural transformation does not make any reference to units. This allows us to state the following obvious proposition:

Proposition 2.3.1. *The forgetful* 2*-functor from the* $(2,1)$*-category of categories and natural isomorphisms to the* $(2,1)$*-category of quasi-unital categories, quasi-unital functors, and natural isomorphisms is an equivalence*[7]. $\square$

One of the main goals of §2 is to obtain a flagged ∞-categorical version of the above proposition, Theorem 2.3.16. In the flagged ∞-categorical setting we will need an alternative characterisation of isomorphisms and units, which we now review in the setting of ordinary categories; the proofs are obvious. For any morphism $f : x \to y$ in any category we denote by f_* and f^* the natural transformations obtained by evaluating respectively the Yoneda and co-Yoneda embeddings at f.

Proposition 2.3.2. *Let* $\mathcal{C}$ *be a category, and let* x *be an object in* $\mathcal{C}$*. For a morphism* $q : x \to x$ *the following are equivalent:*

(I) q *is a unit.*

(II) *For every object* $z \in \mathcal{C}$ *the maps* $q_* : \mathcal{C}(z,x) \to \mathcal{C}(z,x)$ *and* $q^* : \mathcal{C}(x,z) \to \mathcal{C}(x,z)$ *are identities.*

$\square$

Proposition 2.3.3. *Let* $\mathcal{C}$ *be a category, and let* x, y *be an objects in* $\mathcal{C}$*. For a morphism* $f : x \to y$ *the following are equivalent:*

(I) f *is an isomorphism.*

(II) *For every object* $z \in \mathcal{C}$ *the maps* $f_* : \mathcal{C}(z,x) \to \mathcal{C}(z,x)$ *and* $f^* : \mathcal{C}(x,z) \to \mathcal{C}(x,z)$ *are bijections.*

$\square$

Proposition 2.3.4. *Any equivalence of semi-categories is unital.* $\square$

[6]In the sense of `stuff, structure, property`.

[7]We state the $(2,1)$-categorical versions rather than the 2-categorical version of the proposition in order to obtain a closer analogy with Theorem 2.3.16.

2.3.2 Quasi-unital flagged ∞-categories

Definition 2.3.5. Given a flagged ∞-category X and an object $x \in X_0$, the *unit* of x is simply the element $s_0 x$. It is viewed as an element of $\mathrm{Map}(x,x)$ via the inclusion $\mathrm{Hom}(x,x) \subseteq \mathrm{Map}(x,x)$. ⌟

Obtaining an analogues of Propositions 2.3.2 & 2.3.3 now amounts to transferring their statements to the $\mathrm{Ho}(\mathbf{Top})$-enriched setting.

Proposition 2.3.6. *For any endomorphism $q \in \mathrm{Map}(x,x)$ in a flagged ∞-category X the following are equivalent:*

(I) *The morphisms $q : x \to x$ is homotopic to the unit.*

(II) *The map $q_* : \mathrm{Map}(w,x) \to \mathrm{Map}(w,x)$ is equal to the identity map in $\mathrm{Ho}(\mathbf{Top})$ for each $w \in X$.*

(III) *The map $q^* : \mathrm{Map}(x,y) \to \mathrm{Map}(x,y)$ is equal to the identity map in $\mathrm{Ho}(\mathbf{Top})$ for each $y \in X$.*

(IV) *The map $q_* : \pi_0 \mathrm{Map}(w,x) \to \pi_0 \mathrm{Map}(w,x)$ is equal to the identity map for each $w \in X$.*

(V) *The map $q^* : \pi_0 \mathrm{Map}(x,y) \to \pi_0 \mathrm{Map}(x,y)$ is equal to the identity map for each $y \in X$.*

□

Proposition 2.3.7. *For any morphism $f \in \mathrm{Map}(x,y)$ in a flagged ∞-category X the following are equivalent:*

(I) *The morphisms $f : x \to y$ is an equivalence.*

(II) *The map $f_* : \mathrm{Map}(w,x) \to \mathrm{Map}(w,y)$ is an isomorphism in $\mathrm{Ho}(\mathbf{Top})$ for each $w \in X$.*

(III) *The map $f^* : \mathrm{Map}(y,z) \to \mathrm{Map}(x,z)$ is an isomorphism in $\mathrm{Ho}(\mathbf{Top})$ for each $z \in X$.*

(IV) *The map $f_* : \pi_0 \mathrm{Map}(w,x) \to \pi_0 \mathrm{Map}(w,y)$ is a bijection for each $w \in X$.*

(V) *The map $f^* : \pi_0 \mathrm{Map}(y,z) \to \pi_0 \mathrm{Map}(x,z)$ is a bijection for each $z \in X$.*

□

With this result in hand we proceed to define quasi-unital flagged categories.

Definition 2.3.8. An endomorphism $q : x \to x$ in a flagged ∞-semi-category X is called a *quasi-unit* if the map $q_* : \mathrm{Map}(w,x) \to \mathrm{Map}(w,x)$ is equal to the identity map in $\mathrm{Ho}(\mathbf{Top})$ for every object $w \in X$ and the map $q^* : \mathrm{Map}(x,y) \to \mathrm{Map}(x,y)$ is equal to the identity in $\mathrm{Ho}(\mathbf{Top})$ for every object $y \in X$. The flagged ∞-semi-category X is called *quasi-unital* if every object admits a quasi-unit, and X is then referred to as a *quasi-unital flagged ∞-category.* A map between flagged ∞-semi-categories is called *quasi-unital* if the induced map between homotopy categories carries quasi-units to quasi-units. ⌟

Remark 2.3.9. In spite of what conditions (IV) and (V) of Proposition 2.3.6 might seem to suggest, it is important that we work $\underline{\mathrm{Ho}(X)}$ rather than $\mathrm{Ho}(X)$. Consider the topological semi-category $\mathfrak{C}$ consisting of one object, $*$ with mapping space $\mathfrak{C}(*,*) = S^1$, and where composition given by a constant map, then this ∞-semi-category is certainly not quasi-unital, but the underlying category of its $\mathrm{Ho}(\mathbf{Top})$-enriched homotopy semi-category is precisely the final category. ⌟

The category of quasi-unital flagged ∞-categories is a non-full subcategory of $\mathbf{Top}^{\Delta^{\mathrm{op}}_{\mathrm{inj}}}$. We will spend the rest of this subsection understanding quasi-unital morphisms and then state the main theorem of §2.

Proposition 2.3.10. *For any map $F : X \to Y$ between quasi-unital flagged ∞-categories the following are equivalent:*

(I) *The map $F : X \to Y$ sends quasi-units to quasi-unitis, i.e. is quasi-unital.*

(II) *The map $F : X \to Y$ sends quasi-units to invertible maps.*

(III) *The map $F : X \to Y$ sends invertible maps to invertible maps.*

Sketch of proof. Both X and Y are quasi-unital, so their homotopy $\mathrm{Ho}(\mathbf{Top})$-semi-categories can unambiguously be viewed as $\mathrm{Ho}(\mathbf{Top})$-categories. As all objects of $\underline{\mathrm{Ho}(X)}$ and $\underline{\mathrm{Ho}(Y)}$ have units we needn't heed the warning in Remark 2.3.9, so we may deduce the equivalence of statements (I) - (III) from $\mathrm{Ho}(X) \to \mathrm{Ho}(Y)$, which is straightforward. □

In the next few propositions we show various useful properties of invertible morphisms.

Proposition 2.3.11. *Let X be an ∞-semi-category with*

(i) w, x, y, z *objects of* X*,*

(ii) $\gamma : w \rightsquigarrow x$ *and* $\delta : y \rightsquigarrow z$*, paths in* X*, and*

(iii) $(\alpha, f, \beta) \in \mathrm{Map}(x, y)$*, a morphism.*

Then

$$\begin{aligned} & (\alpha, f, \beta) \in \mathrm{Map}(x, y) && \textit{is invertible} \\ \iff\; & (\alpha\gamma, f, \beta) \in \mathrm{Map}(w, y) && \textit{is invertible} \\ \iff\; & (\alpha, f, \delta\beta) \in \mathrm{Map}(x, z) && \textit{is invertible.} \end{aligned}$$

Proof. For any $v \in X_0$ the action $\mathrm{Map}(v, w) \to \mathrm{Map}(v, y)$ of $(\alpha\gamma, f, \beta)$ is given by concatenating the actions $\mathrm{Map}(v, w) \to \mathrm{Map}(v, x) : (\zeta, g, \eta) \mapsto (\zeta, g, \alpha\eta)$ and $f_* : \mathrm{Map}(v, x) \to \mathrm{Map}(v, y)$. As the former action is invertible, the latter action is invertible iff the composed action is. The rest of the proof follows similarly. □

Corollary 2.3.12. *Let X be a ∞-semi-category. Consider a map $f := (\alpha, f', \beta) : x \to y$ in X, then f is invertible iff $(\mathrm{id}_{\alpha_1}, f', \mathrm{id}_{\beta_0})$ is.* □

We have shown that the following definition is meaningful.

Definition 2.3.13. In a flagged ∞-semi-category X, an edge $f \in X_1$ is *invertible* if it is invertible when viewed as a morphism under the inclusion $\mathrm{Hom}(d_1 f, d_0 f) \hookrightarrow \mathrm{Map}(d_1 f, d_0 f)$. ⌟

Corollary 2.3.14. *A map $X \to Y$ between quasi-unital flagged ∞-categories is quasi-unital iff carries invertible edges to invertible edges.* $\square$

Proposition 2.3.15. *Let X be a quasi-unital flagged ∞-category, then either all or non of the edges in any path-connected component of X_1 are invertible.*

Proof. Consider two edges $f, g \in X_1$ and a path $\alpha : f \rightsquigarrow g$. Assume f is invertible, then α induces a path in $\mathrm{Map}(d_1 f, d_0 f)$ from f to $(d_1\alpha, g, d_0\alpha)$. But by the preceding corollary $(d_1\alpha, g, d_0\alpha)$ is invertible iff g is. $\square$

We conclude this subsection with the statement of the main theorem of §2. We denote by $\mathbf{quCat}_\infty^{\text{⚐}}$ the category of quasi-unital flagged ∞-categories and quasi-unital maps, and, by $\mathbf{Cat}_\infty^{\text{⚐}}$ the category of flagged ∞-categories. Both categories are equipped with levelwise weak equivalences.

Theorem 2.3.16. *The forgetful functor $F : \mathbf{Cat}_\infty^{\text{⚐}} \to \mathbf{quCat}_\infty^{\text{⚐}}$ induces an equivalence of homotopy theories.*

2.3.3 Quasi-unital ∞-categories

Our work in §2 amounts to a slight extension of the work in [Har15] and [Har12]. Although not logically necessary for this article, in order to put our work into context we briefly discuss quasi-unital ∞-categories as well as Harpaz' main result in the aforementioned references. Harpaz works exclusively in the simplicial setting, so "space" means "simplicial set".

Notation 2.3.17. Let X be a semiSegal space, then the subspace of X_1 of invertible edges is denoted by X_{inv}. ⌟

Proposition 2.3.18. *Let X be a semiSegal space. The restriction of $d_0 : X_1 \to X_0$ to X_{inv} is an equivalence iff the restriction of d_1 is, in which case the restrictions of d_0 and d_1 are homotopic to each other. If these equivalent conditions are satisfied, then X is quasi-unital.* $\square$

Definition 2.3.19. A semiSegal space X, such that the restriction of the map d_0 (or equivalently d_1) to X_{inv} is an equivalence, is called a *complete semiSegal space.* ⌟

Proposition 2.3.20. *A map between complete semiSegal spaces is automatically unital.* $\square$

The category of complete semiSegal spaces can be identified with the full subcategory of fibrant objects of the category of marked semi-simplicial simplicial sets (discussed in §2.3.4) endowed with an appropriate model structure. We denote this model category by $\mathbf{CSS}_s$. There is an adjunction

$$RK^+ : \mathbf{CSS}_s \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{CSS} : F^+$$

(again discussed in §2.3.4) such that F^+ restricts to the forgetful functor on complete semiSegal spaces.

Theorem 2.3.21 ([Har15, Thm. 3.3.1])**.** *The adjunction*

$$RK^+ : \mathbf{CSS}_s \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{CSS} : F^+$$

is a Quillen equivalence.

Remark 2.3.22. A similar result to Theorem 2.3.21 exists for quasi-categories. By [Ste17] any semi-simplicial set satisfying the (semi-simplicial analogue) of the weak Kan condition and which admits an appropriate notion of quasi-units is the underlying semi-simplicial set of a quasi-category, and the main result [Tan17] is an analogue of Proposition 2.3.10.II. We are unaware of any results asserting the existence of model structure (or variation thereof) on semi-simplicial sets to describe the theory quasi-unital quasi-categories. ⌟

Unfortunately, according to [Har15, Rmk. 2.3.7] there does not exist a model structure on the category of marked semi-simplicial simplicial sets such that the quasi-unital semiSegal spaces are the fibrant objects[8], so Theorem 2.3.16 cannot be deduced from a Quillen equivalence.

2.3.4 Proof of Theorem 2.3.16

The proof proceeds by transferring the problem from the topological to the simplicial setting, where we can avail ourselves of Harpaz' theory.
We begin by establishing some notation. Let $\mathbf{Cat}_\infty^{⚐}(\mathbf{SSet})$ and $\mathbf{quCat}_\infty^{⚐}(\mathbf{SSet})$ denote the categories of flagged ∞-categories and quasi-unital flagged ∞-categories (with quasi-unital maps) in the simplicial setting, and let $\mathbf{Cat}_\infty^{⚐}(\mathbf{SSet})_{\text{fib}}$ and $\mathbf{quCat}_\infty^{⚐}(\mathbf{SSet})_{\text{fib}}$ denote the subcategories spanned by the Reedy fibrant objects. For clarity, we write $\mathbf{Cat}_\infty^{⚐}(\mathbf{Top})$ and $\mathbf{quCat}_\infty^{⚐}(\mathbf{Top})$ instead of $\mathbf{Cat}_\infty^{⚐}$ and $\mathbf{quCat}_\infty^{⚐}$ respectively.
We can then contemplate the solid arrows in the diagram

$$\begin{tikzcd}
\mathbf{Cat}_\infty^{⚐}(\mathbf{Top}) \arrow[r, shift left, dashed, leftarrow] \arrow[r, phantom, "\perp"] \arrow[r, shift right] \arrow[d] & \mathbf{Cat}_\infty^{⚐}(\mathbf{SSet}) \arrow[d] & \mathbf{Cat}_\infty^{⚐}(\mathbf{SSet})_{\text{fib}} \arrow[l, hook'] \arrow[d, shift left, "F"] \arrow[d, phantom, "\dashv"] \\
\mathbf{quCat}_\infty^{⚐}(\mathbf{Top}) \arrow[r, shift left, dashed, leftarrow] \arrow[r, phantom, "\perp"] \arrow[r, shift right] & \mathbf{quCat}_\infty^{⚐}(\mathbf{SSet}) & \mathbf{quCat}_\infty^{⚐}(\mathbf{SSet})_{\text{fib}}, \arrow[l, hook'] \arrow[u, shift left, dotted, "RK^+"]
\end{tikzcd} \tag{4}$$

which are all defined in the obvious way. We will show that all horizontal arrows as well as the rightmost vertical arrow induce equivalences of homotopy theories, establishing the statement of the theorem by the 2-out-of-3 property. This is achieved via the following claims:

Claim 1: The top and bottom horizontal arrows on the left side of (4) admit left adjoints (as indicated by the dashed arrows). Both adjoints respect weak equivalences and their respective units as well as their counits are weak equivalences.

Claim 2: Both $\mathbf{Cat}_\infty^{⚐}(\mathbf{SSet})$ and $\mathbf{quCat}_\infty^{⚐}(\mathbf{SSet})$ allow functorial fibrant replacement.

Claim 3: The rightmost vertical arrow in (4) admits a right adjoint (indicated by the dotted arrow). Both arrows in the adjunction respect weak equivalences and the counit is a weak equivalence.

The adjoint functors in Claim 1 induce equivalences of homotopy theories by Proposition 2.1.11. On both $\mathbf{Cat}_\infty^{⚐}(\mathbf{SSet})$ and $\mathbf{quCat}_\infty^{⚐}(\mathbf{SSet})$ we then denote the inclusion of fibrant objects by ι and the Reedy fibrant replacement functor from Claim 2 by Q. By design $\iota \circ Q$ and $Q \circ \iota$ respect weak equivalences and are weakly equivalent to the identity functor, so we may apply

[8] Although there does exist a variation thereof in the sense of [Cis19, Defs. 7.4.12 & 7.4.6].

Proposition 2.1.9. Finally, to see that Claim 3 implies that F induces an equivalence of homotopy theories it is enough to observe that F reflects weak equivalences: Given any object X in $\mathbf{quCat}_\infty^{\natural}(\mathbf{SSet})_{\mathrm{fib}}$, the maps $F(X) \to F \circ RK^+ \circ F(X) \to F(X)$ compose to the identity; the second arrow is a weak equivalence as these are preserved by F, so that the arrow on the left must be a weak equivalence by the 2-out-of-3 property, which shows that the unit is a natural weak equivalence, and we may again apply Proposition 2.1.11.
Both Claims 2 & 3 make sense in the topological setting, but there it is not clear how to obtain them. They rely on the existence of various model structures on marked semi-simplicial simplicial sets as well as the existence of a right adjoint to the functor F^+ (see Notation 2.3.30) from bisimplicial sets to marked semi-simplicial simplicial sets. The existence of all of these structures in turn relies crucially on the fact that the category marked semi-simplicial simplicial sets is presentable.

Marked semi-simplicial simplicial sets We now discuss marked semi-simplicial simplicial sets before moving on to the proofs of the claims.

Convention 2.3.23. As all statements take place in the simplicial setting we shall write *space* rather than *simplicial set*. ⌟

Definition 2.3.24. A *marked semi-simplicial space* is a pair (X, A) consisting of a semi-simplicial space X together with a subspace $A \subseteq X_1$. The category of marked semi-simplicial spaces is denoted by $\mathbf{SSet}_+^{\Delta_{\mathrm{inj}}^{\mathrm{op}}}$. ⌟

Definition 2.3.25. For any marked semi-simplicial space (X, A) we denote by $\overline{A}$ the image of the map $\pi_0(A) \to \pi_0(X_1)$, and call it the *set of marked connected components of* X_1. ⌟

Definition 2.3.26. A map $(X, A) \to (Y, B)$ of marked semi-simplicial spaces is a *marked equivalence* if the underlying map $X \to Y$ is a levelwise equivalence, and if the induced map $\overline{A} \to \overline{B}$ is a bijection. ⌟

Theorem 2.3.27 ([Har15, Thm. 1.6.7], [Lur09a, Prop. A.2.6.10])**.** *There exists a model structure on the category of marked semi-simplicial spaces where*

- *the weak equivalences are the marked equivalences,*
- *the cofibrations are the maps whose underlying maps semi-simplicial spaces are cofibrations, and*
- *the fibrations are the maps with the right lifting property w.r.t. the trivial fibrations.*

□

Definition 2.3.28. We call the model structure from the preceding theorem the *marked model structure*. We refer to fibrant objects in this model structure as being *marked-fibrant*. ⌟

Proposition 2.3.29 ([Har15, Lm. 1.6.11])**.** *A marked semi-simplicial space* (X, A) *is marked-fibrant iff*

- X *is Reedy fibrant;*
- A *is a union of connected components of* X_1.

$\square$

Notation 2.3.30. We denote by F^+ the functor $\mathbf{SSet}^{\Delta^{\mathrm{op}}} \to \mathbf{SSet}_+^{\Delta^{\mathrm{op}}_{\mathrm{inj}}}$ given by sending X to its underlying semi-simplicial space, which we again denote by X, together with the subspace $s_0X_0 \subseteq X_1$. ⌟

By the special adjoint functor theorem the functor F^+ admits a right adjoint which we denote by RK^+.
We now discuss marked semi-simplicial spaces satisfying the Segal condition. Harpaz only ever considers the Segal condition for marked-fibrant spaces. As such, he defines hom spaces, but not mapping spaces, and thus only discusses invertability for edges rather than for arbitrary maps. Our proof in the topological setting that a map of quasi-unital ∞-categories is quasi-unital iff it carries invertible edges to invertible edges (see Corollary 2.3.14) is robust enough to carry over to the setting of quasi-unital flagged ∞-categories which are objectwise fibrant. An alternative proof for quasi-unital semiSegal spaces (i.e. where we require Reedy fibrancy) is given in [Har15, Prop. 1.4.6] . We extend the notion of invertible edges to non-fibrant quasi-unital flagged ∞-categories in the simplicial setting as follows: Let X be a not necessarily fibrant quasi-unital flagged ∞-category, and consider some fibrant replacement $X \xrightarrow{\sim} X'$, then we obtain an induced bijection $\pi_0 X_1 \xrightarrow{\cong} \pi_0 X_1'$; we declare an edge of X to be invertible if it lies in a connected component which gets sent to a connected component of invertible edges in X'. It is easily checked that this notion of invertible edge is independent of the choice of the fibrant replacement. It is then straightforward to show that with this notion of invertible edge a map of quasi-unital flagged ∞-categories is quasi-unital iff it carries quasi-units to quasi-units. We can thus identify $\mathbf{Cat}_\infty^{\wp}(\mathbf{SSet})$ with the full subcategory of $\mathbf{SSet}_+^{\Delta^{\mathrm{op}}_{\mathrm{inj}}}$ spanned by the quasi-unital marked flagged ∞-categories whose invertible edges are all marked.

Proofs of Claims 1 - 3

<u>Proof of Claim 1:</u>

As the unit and counit of

$$| \ | : \mathbf{SSet} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Top} : S$$

are weak equivalences, and both constituent functors preserve weak equivalences, the same is true of

$$\mathbf{SSet}^{\Delta_{\mathrm{inj}}} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Top}^{\Delta_{\mathrm{inj}}}$$
$$\mathbf{SSet}^{\Delta} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Top}^{\Delta} ,$$

and these then restrict to adjunctions between flagged ∞-(semi)-categories in the simplicial and topological setting respectively by Proposition 2.1.8.
It thus remains to show that the adjoints between flagged ∞-semi-categories restrict to adjoints between quasi-unital flagged ∞-categories and quasi-unital functors. Given any quasi-unital semi-simplicial topological space satisfying the Segal condition X, denote by SX its image under $\mathbf{Top}^{\Delta^{\mathrm{op}}_{\mathrm{inj}}} \to \mathbf{SSet}^{\Delta^{\mathrm{op}}_{\mathrm{inj}}}$. We then obtain a Ho($\mathbf{SSet}$)-enriched category associated to SX.

For any x, x' in X_0 the diagram

$$\begin{tikzcd}
S\operatorname{Map}_X(x,x') \times S\operatorname{Map}_X(x',x') \arrow[r] \arrow[d, "\cong"] & S\operatorname{Map}_X(x,x') \arrow[d, "\cong"] \\
\operatorname{Map}_{SX}(x,x') \times \operatorname{Map}_{SX}(x',x') \arrow[r] & \operatorname{Map}_{SX}(x,x')
\end{tikzcd}$$

in $\operatorname{Ho}(\mathbf{SSet})$ commutes irrespectively of whether we view the horizontal maps as compositions or projections, as they are induced by d_1 and d_2 respectively. As the top two horizontal maps restrict to the same map on $S\operatorname{Map}_X(x,x') \times \{\mathrm{id}_X\}$, we see that id_X must be sent to an identity map in $\operatorname{Map}_{SX}(x',x')$. By a similar argument we see that quasi-unital maps are sent to quasi-unital maps.

<u>Proof of Claim 2:</u> Both the Reedy model structure on bisimplicial sets and the marked model structure are combinatorial and thus admit functorial fibrant replacement via the small object argument.

<u>Proof of Claim 3:</u> We continue viewing $\mathbf{quCat}_\infty^{\text{⚐}}(\mathbf{SSet})$ as a full subcategory of $\mathbf{SSet}_+^{\Delta_{\mathrm{inj}}^{\mathrm{op}}}$. The forgetful functor, F, is then given by first applying F^+, and then marking all the equivalences. It is clear that F preserves and reflects weak equivalences, and RK^+, being part of a Quillen adjunction, preserves weak equivalences between fibrant objects.
For the rest of the discussion we fix a Segal space X and a quasi-unital semiSegal space Y. By the argument used in the proof of [Har15, Prop. 1.4.6] a map between quasi-unital semiSegal spaces is quasi-unital iff it sends units to invertible edges (see also Proposition 2.3.10). Thus any map $F^+X \to Y$ factors uniquely through $FX \to Y$. The unit is then given as the composition of

$$X \to RK^+\,F^+\,X \to RK^+\,F\,X,$$

and the counit is the unique factorisation of $F^+\,RK^+\,Y \to Y$ through $F\,RK^+\,Y \to Y$. The triangle identities are established via the commutativity of the following two diagrams

$$\begin{tikzcd}
FX \arrow[rr] & & F\,RK^+\,F\,X \arrow[rr] & & FX \\
& & & F^+\,RK^+\,F\,X \arrow[ul] \arrow[ur] & \\
F^+X \arrow[uu] \arrow[rr] & & F^+\,RK^+\,F^+\,X \arrow[ur] \arrow[rr] & & F^+X \arrow[uu]
\end{tikzcd}$$

$$\begin{tikzcd}
RK^+Y \arrow[rr] \arrow[dr] & & RK^+\,F\,RK^+\,Y \arrow[rr] & & RK^+Y \\
& RK^+\,F^+\,RK^+\,Y \arrow[ur] \arrow[urrr] & & &
\end{tikzcd},$$

where for the top diagram we observe that we only need to compose to the identity on the underlying unmarked semi-simplicial simplicial sets to obtain an identity in the marked setting.

The fact that the counit is a weak equivalence is the content of [Har15, Prop. 6.2.10].

3 Differential topology

3.1 Manifolds with corners

In this subsection we introduce manifolds with corners and jet bundles. In the following three sections we then discuss topologies on sets of smooth maps between such manifolds, smooth approximations, and transversality. Our treatment is mostly expository and we closely follow [Mic80][9].

Definition 3.1.1. A *quadrant* Q is a subset of $\mathbf{R}^n$ ($n \in \mathbf{N}$) of the form

$$\left\{ \, x \in \mathbf{R}^n \;\middle|\; \ell_1(x) \geq 0, \ldots, \ell_k(x) \geq 0 \, \right\},$$

where $0 \leq k \leq n$, and $\ell_1, \ldots, \ell_k \in (\mathbf{R}^n)^*$ are linearly independent functionals. We call n the *dimension* of Q, and k the *index* of Q. ⌟

Definition 3.1.2. Let Q, Q' be quadrants of dimension $n, n' \in \mathbf{N}$ respectively, and let $U \subseteq Q$, $U' \subseteq Q'$ be open subsets, then a map $f : U \to Q'$ is *smooth* if equivalently (see [Tou72])

(I) f can be extended to a smooth map $V \to \mathbf{R}^{n'}$, where V is an open neighbourhood of Q;

(II) all partial derivatives of f exist and are continuous.

The map f is a *diffeomorphism* if it has a smooth inverse. ⌟

With notation as in the definition above, note that if f is a diffeomorphism, then U and U' have the same index.

Definition 3.1.3. Let M be a set, then a *chart* on M is a triple (U, u, Q) consisting of a subset $U \subseteq M$, a quadrant Q, and an injective map $u : U \hookrightarrow Q$, whose image is an open subset of Q. The chart is called *full*, if $u : U \to Q$ is a bijection. ⌟

Definition 3.1.4. Let M be a set, then an *atlas* on M is a family of charts $\mathcal{A}$ on M, such that $M = \bigcup_{(U,u,Q) \in \mathcal{A}} U$, and such that for any two charts $(U, u, Q), (U', u', Q) \in \mathcal{A}$ the map $u' \circ (u|_{u(U \cap U')})^{-1}$ is smooth. The atlas $\mathcal{A}$ is called *maximal* if for any given chart (V, v, P) on M, such that for all charts $(U, u, Q) \in \mathcal{A}$ the map $v \circ (u|_{u(U \cap V)})^{-1}$ is smooth, the chart (V, v, P) is already in $\mathcal{A}$. ⌟

Definition 3.1.5. A *manifold* is a pair $(M, \mathcal{A})$ consisting of a set M together with a maximal atlas. The manifold $(M, \mathcal{A})$ is called a *manifold with boundary*, if $\mathcal{A}$ contains an atlas consisting of charts (U, u, Q) such that Q has index ≤ 1. The manifold $(M, \mathcal{A})$ is called a *manifold without boundary*, if $\mathcal{A}$ contains an atlas consisting of charts (U, u, Q) such that Q has index $= 0$, and a compact manifold without boundary is called a *closed manifold*. ⌟

Convention 3.1.6. We will often refer to manifolds as *manifolds with corners*. Manifolds without corners will sometimes be referred to as *ordinary manifolds*. ⌟

Remark 3.1.7. Just as for ordinary manifolds, it is always possible to find an atlas for any manifold with corners consisting of full charts. ⌟

[9]Another important source which we sometimes cite is [MROD92]. This is the only other expository account of manifolds with corners that we are aware of.

3.1.1 Jet bundles over manifolds with corners

Jets on manifolds with corners are defined as for ordinary manifolds. Nevertheless, we collect the necessary definitions required for the rest of this article.
Throughout this subsection M, N denote manifolds, $\pi : E \to B$ denotes a surjective submersion, and $k \in \mathbf{N}$.

Definition 3.1.8. Consider a point $x \in M$, then two maps $f, g : M \to N$ have *contact of order k at x*, if $f(x) = g(x)$, and if for any pair of charts (U, u, Q) and (V, v, P) containing x and $f(x) = g(x)$ respectively, the k-th order Taylor polynomials of $v \circ f \circ u^{-1}$ and $v \circ g \circ u^{-1}$ agree. ⌟

The notion of contact of order k at x defines an equivalence relation on $\mathbf{Man}(M, N)$.

Definition 3.1.9. Consider a point $x \in M$, a neighbourhood U of x, and a map $f : U \to N$, then the *k-jet of f at x*, denoted by $j^k_x(f)$, is the equivalence class of pairs (g, V) consisting of a neighbourhood V of x together with a map $g : V \to N$, such that $f|_{U\cap V}$ and $g|_{U\cap V}$ have contact of order k at x . ⌟

Definition 3.1.10. A *k-jet from M to N* is a pair $(x, \mathfrak{f})$, where $x \in M$ is a point, and $\mathfrak{f}$ is the k-jet of some smooth map defined on an open neighbourhood of x with values in N. We call x the *source* of $(x, \mathfrak{f})$, and $f(x)$ its *target*, where $(f, U) \in \mathfrak{f}$. More generally, a *k-jet of $E \to B$* is a k-jet of a section of $E \to B$. ⌟

Definition 3.1.11. The set of k-jets of $E \to B$ is denoted by $J^k(E \to B)$. If $M \times N \to M$ is the projection map, then we write $J^k(M, N) := J^k(M \times N \to M)$. The map

$$\begin{array}{rcl} j^k : \Gamma(E \to B) & \to & \Gamma\left(J^k(E \to B) \to B\right) \\ s & \mapsto & \left(x \mapsto (x, j^k_x s)\right) \end{array}$$

is called the *k-th order jet prolongation map.* ⌟

Convention 3.1.12. The set $J^k(M, N)$ will be canonically identified with the set of all k-jets from M to N. ⌟

Observe that there exist canonical maps

$$J^k(E \to B) \to B \text{ and } J^k(E \to B) \to E$$

given by $(j^k_x s, b) \mapsto b$ and $(j^k_x s, b) \mapsto s(b)$ respectively.

Example 3.1.13. *We have canonical bijections*

$$J^0(M, N) \cong M \times N$$

and

$$J^1(M, N) \cong \coprod_{(x,y)\in M\times N} \mathbf{Vect}_{\mathbf{R}}(T_x M, T_y N).$$

⌟

Proposition 3.1.14. *Let Q, Q' be quadrants of dimension $n, n' \in \mathbf{N}$ respectively, and $U \subseteq Q$ and $U' \subseteq Q'$ open subsets, then the map taking any k-jet from U to U' to its k-th order Taylor polynomial defines a bijection*

$$J^k(U, U') \cong U \times U' \times P^k(n, n').$$

□

For any point $x \in E$ there exist full charts $(U, u, Q), (U', u', Q')$ of E and B respectively, such that $x \in U$ and $\pi(x) \in U'$, and such that π restricts to a map $U \to U'$. By the definition of smoothness we may find open neighbourhoods V and V' of Q and Q' in $\mathbf{R}^e$ and $\mathbf{R}^b$ respectively such that $u' \circ \pi \circ u^{-1}$ extends to a surjective submersion $\tilde{\pi} : V \to V'$. We may then restrict $\tilde{\pi}$ to neighbourhoods of $u(x)$ and $u'\pi(x)$, and find charts thereof, such that this restriction of $\tilde{\pi}$ is given by the projection $\mathbf{R}^b \times \mathbf{R}^{e-b} \to \mathbf{R}^b$. The diagram

$$\begin{tikzcd} E \arrow[d] & Q \cap \left(\mathbf{R}^b \times \mathbf{R}^{e-b}\right) \arrow[l, hook'] \arrow[d] \\ B & Q' \cap \mathbf{R}^b \arrow[l, hook'] \end{tikzcd}$$

then allows us to construct a chart

$$J^k(E \to B) \hookleftarrow Q \cap \left(\mathbf{R}^b \times \mathbf{R}^{e-b}\right) \times P^k(e, b).$$

Proposition 3.1.15. *The collection of charts of $J^k(E \to B)$ constructed in this manner form an atlas.* □

Corollary 3.1.16. *The k-th order jet prolongation map $j^k : \Gamma(E \to B) \to \Gamma\left(J^k(E \to B) \to B\right)$ is smooth.* □

Convention 3.1.17. We will always consider $J^k(E \to B)$ to be equipped with the manifold structure determined by the atlas from the preceding proposition. ⌟

3.2 Topologies on maps of manifolds

Given a compact manifold (without boundary) M we will want to topologise the set of diffeomorphisms $\mathbf{Diff}(M)$ in such a way that it presents the correct ∞-group of diffeomorphisms (see the introduction to §4 for details). A subsequent goal will be to topologise the set $\Psi^\infty(M)$ of submanifolds of $\mathbf{R}^\infty$ diffeomorphic to M in such a way that it models the classifying space of $\mathbf{Diff}(M)$. Moreover, we will want to make analogous constructions of classifying spaces of bordisms.

The tools used for exhibiting these desired properties rely heavily on the concepts of genericness and stability, which we now discuss. We then move on to discussing topologies on sets of continuous and smooth maps.

3.2.1 Genericness and stability

If X is some type of space on which it makes sense to speak of perturbations, then a property characterising a subspace $Y \subseteq X$ is *stable* if any point in Y remains in Y under sufficiently

small perturbations, and, *generic*, if any point in X can be perturbed into a point in Y by an arbitrarily small perturbation.
In our setting, X is simply a topological space, and a property is stable if the corresponding subset $Y \subseteq X$ is open. We are, however, interested in *two* incarnations of genericness:

- *Geometric genericness:* For any point $x \in X$ there exists a path $\alpha : [0,1] \to X$ starting at x such that $\alpha|_{(0,1]}$ factors through Y.
- *Point-set genericness:* The subset Y is dense in X.

The first notion of genericness clearly implies the second: Consider any open subset $U \subset X$ and a point $x \in U$, then we can choose a path α as above and $\alpha^{-1}(U) \cap (0,1]$ is non-empty; the desired path is then obtained by restricting α to a shorter interval starting at 0. If X is locally path-connected, then we obtain a weak converse: Assume Y is dense, then for any point $x \in X$ there exists a path starting at x and finishing in Y.
Given two manifolds M, N, we will introduce two topologies on $\mathbf{Man}(M,N)$ in §3.2.3: the CO$^\infty$- and the WO$^\infty$-topology. In general the WO$^\infty$-topology is finer than the CO$^\infty$-topology, but the two topologies coincide if M is compact. In Theorem 3.2.19 we will see that various subsets of $\mathbf{Man}(M,N)$ are open in WO$^\infty$-topology, such as embeddings. If L is another manifold, we will show in §3.3 that the set of smooth maps $L \to \mathbf{Man}(M,N)$ (see Definition 3.2.12) is geometrically dense w.r.t. the CO$^\infty$-topology. Thus if $L \to \mathbf{Man}(M,N)$ factors through any of the open subsets of Theorem 3.2.19, then we can perturb it to a smooth map without leaving the open subset. In §3.4 we will show that subspaces of $\mathbf{Man}(M,N)$ satisfying certain transversality conditions are residual. In particular, if M is compact, and the dimension of N is sufficiently much greater than that of M, then embeddings are determined by such transversality conditions.

3.2.2 Topologies on continuous maps

Throughout this subsection X, Y denote topological spaces.
A common desideratum of any topology on $\mathbf{Top}(X,Y)$ is for the evaluation map $\mathbf{Top}(X,Y) \times X \to Y$ to be continuous for sufficiently nice X, so that the collection of evaluation maps, as Y varies, form the constituent maps of the counit of an adjunction between $\_ \times X$ and $\mathbf{Top}(X, \_)$. Note that the topology on $\mathbf{Top}(X,Y)$ is uniquely determined by this requirement. For many topological spaces Y of interest, for example spaces which are locally compact, this is achieved by the compact open topology[10] (see Proposition 3.2.3), which we discuss below.
A first attempt at defining a topology on $\mathbf{Top}(X,Y)$ could be made by requiring that any open neighbourhood of a continuous map $X \to Y$ contain small perturbations of its graph:

Definition 3.2.1. The *graph or WO-topology on* $\mathbf{Top}(X,Y)$ is the topology generated by the basis:

$$\left\{ \left\{ f \in \mathbf{Top}(X,Y) \;\middle|\; \mathrm{graph}(f) \subseteq U \right\} \;\middle|\; U \subseteq X \times Y \text{ open} \right\}.$$

⌟

[10] A systematic and comprehensive discussion of when a topological space Y has continuous evaluation maps forming a counit can be found in [ELS04].

Consider now the example of the map $\varphi : \mathbf{R} \to \mathbf{Top}(\mathbf{R}, \mathbf{R}),\ t \mapsto (x \mapsto t)$. Defying expectations, this map is not continuous w.r.t. the WO-topology, as the basic open subset corresponding to the open subset $\left\{ (x, y) \in \mathbf{R} \times \mathbf{R} \;\middle|\; |y| < e^{-x^2} \right\}$ contains the graph of φ_0 but not of φ_t for any $t \neq 0$. We see, however, that if $K \subseteq \mathbf{R}$ is compact, then the composition of φ with the restriction map $\mathbf{Top}(\mathbf{R}, \mathbf{R}) \to \mathbf{Top}(K, \mathbf{R})$ *is* continuous. So we learn that open subsets of $\mathbf{Top}(X, Y)$ should only contain sufficiently small perturbations of graphs over a bounded subdomain. This motivates the following definition.

Definition 3.2.2. The *compact-open or CO-topology on* $\mathbf{Top}(X, Y)$ is the topology generated by the subbasis:

$$\left\{ \left\{ f \in \mathbf{Top}(X, Y) \;\middle|\; \operatorname{graph}(f|_K) \subseteq U \right\} \;\middle|\; K \subseteq X \text{ compact, } U \subseteq X \times Y \text{ open} \right\}.$$

⌟

It is easily checked that our definition of the compact-open topology coincides with the usual one (see e.g. [ELS04, p. 122]). We also note that if compact subsets of X are closed (e.g. if X is Hausdorff), then our subbasis for the compact-open topology is even a basis.
For a large class of spaces the CO-topology satisfies the desiderata discussed above:

Theorem 3.2.3 ([Bor94, Prop. 7.1.5])**.** *If X is locally compact, and $\mathbf{Top}(X, Y)$ carries the compact-open topology for all Y, then the evaluation maps $\mathbf{Top}(X, Y) \times X \to Y$, as Y varies, are continuous, and form the constituent maps of the counit of an adjunction between $\_ \times X$ and $\mathbf{Top}(X, \_)$.* □

Remark 3.2.4. In the situation of Theorem 3.2.3 one can show that the CO-topology is finest topology making all maps $Z \to \mathbf{Top}(X, Y)$ continuous, which have a continuous adjoint. See [ELS04, Prop. 5.1]. ⌟

Proposition 3.2.5. *If compact subsets are closed in X, then the WO-topology on $\mathbf{Top}(X, Y)$ is finer than the CO-topology, and the two topologies coincide if X is compact.* □

Although we presented the CO-topology as remedying certain defects of the WO-topology, we will use the smooth analogue of the WO-topology to prove the key results about transversality in §3.4, as we will need a sufficient supply of open subsets.

3.2.3 The CO$^\infty$- and WO$^\infty$-topologies

Throughout this subsection L, M, N denote manifolds, and $\pi : E \to B$ denotes a surjective submersion; moreover, we fix $k \in \mathbf{N}$. Our goal is to topologise the set of smooth sections of $E \to B$. In the preceding subsection we saw that it is useful to think of maps in terms of their graphs. This perspective is automatically enforced when considering sections: A section of the projection map $M \times N \to M$ is the same as a smooth map $M \to N$, and the image of this section is its graph. More generally, a section of $E \to B$ is the same as a submanifold S of E such that the restriction of $E \to B$ to S is a diffeomorphism. The k-th order jet prolongation map of a section $s : B \to E$ can be viewed as an enhancement of s; it is the section of the submersion $J^k(E \to B) \to B$ and its composition with the projection $J^k(E \to B) \to E$ recovers s. We thus obtain natural adaptions of the WO- and CO-topologies to the smooth case:

Definition 3.2.6. The WO^k-*topology* on $\Gamma(E \to B)$ is the topology generated by the basis

$$\Big\{ \Big\{ \ f \in \Gamma(E \to B) \ \Big|\ \mathrm{Im}_{j^k f} \subseteq U \ \Big\} \ \Big|\ U \subseteq J^k(E \to B) \ \Big\},$$

and the CO^k-*topology* on $\Gamma(E \to B)$ is the topology generated by the basis

$$\Big\{ \Big\{ \ f \in \Gamma(E \to B) \ \Big|\ \mathrm{Im}_{j^k f|_K} \subseteq U \ \Big\} \ \Big|\ U \subseteq J^k(E \to B),\ K \subseteq B \text{ compact} \ \Big\}.$$

The WO$^\infty$- and CO$^\infty$-topologies are respectively the union of the WO^k- and CO^k-topologies for $k \to \infty$. ⌟

Remark 3.2.7. The WO^k- and WO$^\infty$-topologies are often called the *strong topology*, the *strong Whitney topology*, or simply the *Whitney topology*. The CO^k- and CO$^\infty$-topologies are often called the *weak topology* or the *weak Whitney topology*. See e.g. [Hir94]. ⌟

Remark 3.2.8. The CO^0- and WO^0-topologies on $\mathbf{Man}(M, N)$ are respectively the restrictions to smooth maps of the CO- and WO-topologies from the last section. ⌟

We present an alternative description of the WO^k- and CO^k-topologies in Proposition 3.2.11, which is better suited to showing various results in this article. We require two lemmas:

Lemma 3.2.9. *Let $F \subseteq E$ be a submanifold such that $F \to B$ is a surjective submersion, then the induced map $J^k(F \to B) \to J^k(E \to B)$ is a submanifold such that $J^k(F \to B)$ is a surjective submersion.*

Proof. Consider a point $x \in F$, and choose charts (U, u, Q), (U', u', Q'), and (U'', u'', Q'') of F, E, and B respectively such that $x \in U \subseteq U'$, $\pi(x) \in U''$, and such that π restricts to $U' \to U''$. In the same way as in the construction of charts of jet bundles in the discussion preceding Proposition 3.1.15, we can extend the maps $(u')^{-1} \circ u : U \hookrightarrow U'$ and $(u'')^{-1} \circ \pi|_{U'} \circ u' : U' \to U''$ to maps between open subsets of Euclidean space, and again the diagram

$$\begin{tikzcd} F \arrow[rr, hook] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$

then reduces to

$$\begin{tikzcd} \big(\mathbf{R}^b \times \mathbf{R}^{f-b}\big) \cap Q \arrow[rr, hook] \arrow[dr] & & \big(\mathbf{R}^b \times \mathbf{R}^{f-b} \times \mathbf{R}^{e-f-b}\big) \cap Q' \arrow[dl] \\ & \mathbf{R}^b \cap Q'', & \end{tikzcd}$$

so that $J^k(F \to B) \to J^k(E \to B)$ restricts to a map of charts

$$\Big(\Big(\mathbf{R}^b \times \mathbf{R}^{f-b}\Big) \cap Q\Big) \times P^k(b, f-b) \hookrightarrow \Big(\Big(\mathbf{R}^b \times \mathbf{R}^{f-b} \times \mathbf{R}^{e-f-b}\Big) \cap Q'\Big) \times P^k(b, d-b).$$

□

Lemma 3.2.10. *Let $F \subseteq E$ be a submanifold such that $F \to B$ is a submersion, then the CO^0- and WO^0-topologies on $\Gamma(F \to B)$ are the restrictions of the CO^0- and WO^0-topologies respectively on $\Gamma(E \to B)$.*

Proof. This follows immediately from the fact that F inherits its topology from E. □

Proposition 3.2.11. *The WO^k- and CO^k-topologies on $\Gamma(E \to B)$ coincide, respectively, with the restriction of the WO^k- and CO^k-topologies on $\mathbf{Man}(B, E)$.*

Proof. Apply the preceding lemma to the submanifold $E \subseteq B \times E$. □

Definition 3.2.12. A map $L \to \Gamma(E \to B)$ is called *smooth* if its transpose $L \times B \to E$ is smooth. ⌟

Consider a jet $((w, x), \mathfrak{f}) \in J^k(L \times M, N)$ represented by a smooth map $f : L \times M \to N$, then we obtain a jet with source x in $J^k(M, N)$ represented by $f|_{\{w\}\times M}$. Working in local coordinates we see that this produces well-defined continuous map $J^k(L \times M, N) \to J^k(M, N)$.

Proposition 3.2.13. *Any smooth map $L \to \mathbf{Man}(M, N)$ is continuous w.r.t. the CO^∞-topology.*

Proof. Showing that $L \to \mathbf{Man}(M, N)$ is smooth amounts, by Theorem 3.2.3, to showing that the adjoint map $L \times M \to J^i(M, N)$ is continuous for all $i \in \mathbf{N}$. But this map is the composition of the continuous maps $L \times M \to J^i(L \times M, N) \to J^i(M, N)$. □

Remark 3.2.14. It can be shown that a smooth map $F : L \to \mathbf{Man}(M, N)$ is continuous w.r.t. WO^k-topology iff for every $z \in L$ there exists a neighbourhood $U \ni z$, a compact subset $A \subseteq M$, and a smooth map $f : M \to N$ such that $F_{z'}|_{M\setminus A} = f|_{M\setminus A}$ for all $z' \in U$. ⌟

We now state the smooth analogue of Proposition 3.2.5:

Proposition 3.2.15. *The WO^k-topology on $\Gamma(E \to B)$ is finer than the CO^k-topology.* □

Proposition 3.2.16. *Let $A \subseteq M$ be a compact subset, and let $g : M \to N$ be a fixed smooth map, then the restrictions of the CO^k- and WO^k-topologies to $\left\{ f \in \mathbf{Man}(M, N) \;\middle|\; f|_{M\setminus A} = g \right\}$ agree. In particular the two topologies coincide if M is compact.*

Proof. By the preceding proposition it is enough to show that the restriction of the CO^k-topology is finer than the restriction of the WO^k-topology. Let $U \subseteq J^k(M, N)$ be an open subset for some $k \in \mathbf{N}$, so that

$$\left\{ f \in \mathbf{Man}(X, Y) \;\middle|\; \mathrm{Im}_{j^k f} \subseteq U \right\}$$

is a basic open neighbourhood in the WO$^\infty$-topology, then

$$\left\{ f \in \mathbf{Man}(X, Y) \;\middle|\; j^k f(A) \subseteq U \right\}$$

is a basic open neighbourhood in the CO$^\infty$-topology, and their restrictions to

$$\left\{ f \in \mathbf{Man}(M, N) \;\middle|\; f|_{M\setminus A} = g \right\}$$

agree. □

Closing the circle of ideas revolving around Cartesian closedness from the beginning of Section 3.2.2, we remark that if B is compact, and both B and E have no boundary, then the set $\Gamma(E \to B)$ admits the structure of Fréchet manifold (described in [GG73, Prop. III.1.11]) whose underlying topology is the CO$^\infty$-topology, and that a map $L \to \Gamma(E \to B)$, is smooth in the sense of Definition 3.2.12 iff it is smooth as a map of Fréchet manifolds. The importance of smooth maps is further highlighted by noting that the CO$^\infty$-topology is the finest topology making all smooth maps $L \to \Gamma(E \to B)$ continuous, as L varies (see [CSW14]).

Some properties of the WO$^\infty$-topology

Theorem 3.2.17 ([Mic80, 4.4.5]). *The set* $\mathbf{Man}(M, N)$ *equipped with the WO$^\infty$-topology is a Baire space.* ☐

Theorem 3.2.18 ([Mic80, 4.4.5]). *Any* CO$^\infty$*-closed subset of* $\mathbf{Man}(M, N)$ *with the inherited* WO$^\infty$*-topology is a Baire space.* ☐

Theorem 3.2.19 ([Mic80, Ch. 5]). *The following subsets are open in* $\mathbf{Man}(M, N)$ *with respect to the WO$^\infty$-topology:*

(1) *The set* $\mathrm{Imm}(M, N)$ *of all embeddings.*

(2) *The set* $\mathrm{Emb}(M, N)$ *of all embeddings.*

(3) *The set* $\mathbf{Diff}(M, N)$ *of all diffeomorphisms restricting to a diffeomorphism* $\partial M \xrightarrow{\cong} \partial N$.

☐

Proposition 3.2.20. *Let* $U \subseteq N$ *be an open subset, then* $\mathbf{Man}(M, U)$ *is open in* $\mathbf{Man}(M, N)$ *in the WO$^\infty$-topology.* ☐

Proposition 3.2.21. *Let* M, N *be manifolds, and* $N' \subseteq N$ *a submanifold, then the map*

$$\mathbf{Man}(M, N') \hookrightarrow \mathbf{Man}(M, N)$$

is a homeomorphism onto its image w.r.t. both the CO$^\infty$- and the WO$^\infty$-topology. ☐

Proof. This follows immediately from Lemmas 3.2.9 & 3.2.10. ☐

3.2.4 Categorical properties of mapping topologies

Throughout this subsection L, M, N denote smooth manifolds.

Proposition 3.2.22 ([MROD92, Props. 9.6.6]). *Let* $g : M \to N$ *be a fixed smooth map, then the map*

$$\begin{array}{ccc} \mathbf{Man}(L, M) & \to & \mathbf{Man}(L, N) \\ f & \mapsto & g \circ f \end{array}$$

is continuous w.r.t. the CO$^\infty$- and WO$^\infty$-topologies. ☐

Proposition 3.2.23 ([MROD92, Props. 9.6.3]). *The composition map*

$$\mathbf{Man}(L, M) \times \mathbf{Man}(M, N) \to \mathbf{Man}(L, N)$$

is continuous w.r.t. the CO$^\infty$-topology. ☐

Proposition 3.2.24 ([MROD92, Props. 9.6.5]). *The composition map*

$$\mathrm{Prop}(L,M) \times \mathbf{Man}(M,N) \to \mathbf{Man}(L,N)$$

is continuous w.r.t. the WO$^\infty$-topology, where $\mathrm{Prop}(L,M) \subseteq \mathbf{Man}(L,M)$ *denotes the subset of proper maps.* ☐

Proposition 3.2.25 ([MROD92, Props. 9.6.9]). *Let N_1, N_2 be smooth manifolds, then the map*

$$\mathbf{Man}(M, N_1 \times N_2) \to \mathbf{Man}(M, N_1) \times \mathbf{Man}(M, N_2)$$

is a homeomorphism w.r.t. the CO$^\infty$- and WO$^\infty$-topologies. ☐

Proposition 3.2.26 ([MROD92, Props. 9.6.17]). *The map $\mathbf{Diff}(M) \to \mathbf{Diff}(M)$, $f \mapsto f^{-1}$ is continuous w.r.t. both the CO$^\infty$- and the WO$^\infty$-topology.* ☐

3.2.5 Topologising sections of vector bundles

Throughout this subsection $E \to M$ denotes a smooth vector bundle.
The canonical homeomorphism $\mathbf{Man}(M,E) \times \mathbf{Man}(M,E) \cong \mathbf{Man}(M, E \times E)$ restricts to a homeomorphism $\Gamma_E(M) \times \Gamma_E(M) \cong \Gamma_{E \times_M E}(M)$ in both the CO$^\infty$- and the WO$^\infty$-topology by Propositions 3.2.25 and 3.2.21. The second map in the composition $\Gamma_E(M) \times \Gamma_E(M) \cong \Gamma_{E\times_M E}(M) \hookrightarrow \mathbf{Man}(M, E \times_M E) \to \mathbf{Man}(M,E)$ factors through $\Gamma_M(E) \hookrightarrow \mathbf{Man}(M,E)$ so that addition is continuous in both topologies.
If M is non-compact, the map $\mathbf{R} \times \Gamma_E(M) \to \Gamma_E(M)$, induced by scaling $\mathbf{R} \times E \to E$, is not continuous in the WO$^\infty$-topology, because, by Remark 3.2.14, locally in $\mathbf{R}$ such a parametrised family of maps would have to be constant outside of a compact subset of M. Scaling *is* continuous w.r.t. the CO$^\infty$-topology, because $\mathbf{R} \times \Gamma_E(M) \to \Gamma_E(M)$ is adjoint to the map $\mathbf{R} \to \mathbf{Top}(\mathbf{Man}(M,E), \mathbf{Man}(M,E))$ obtained by composing the continuous maps $\mathbf{R} \to \mathbf{Man}(E,E) \to \mathbf{Top}(\mathbf{Man}(M,E), \mathbf{Man}(M,E))$, which are respectively the adjoints of the scaling map $\mathbf{R} \times E \to E$ and the composition map $\mathbf{Man}(M,E) \times \mathbf{Man}(E,E) \to \mathbf{Man}(M,E)$.

3.3 Smooth approximations

Here we witness genericness in the strong sense, as discussed in the introduction of §3.2.1.

Theorem 3.3.1. *Let L, M, N be smooth manifolds, where L, N are without corners, and equip $\mathbf{Man}(M,N)$ with the CO$^\infty$-topology. For any continuous map $f : L \to \mathbf{Man}(M,N)$ there exists a continuous homotopy $H : [0,1] \times L \to \mathbf{Man}(M,N)$ starting at f, such that its restriction to $(0,1] \times L \to \mathbf{Man}(M,N)$ is smooth.*
Moreover, the homotopy can be chosen to satisfy any of the following two properties:

(I) *For any N: If $A \subseteq L$ is a closed subset on which the adjoint of f is already smooth, then the adjoint of the homotopy H is constant on $[0,1] \times A \times M$.*

(II) *For affine N: For any collection of pairs (B, C), consisting of a subset $B \subseteq M$ and convex subset $C \subseteq N$ such that f_w maps B to C for all $w \in L$, the map $H_{(t,w)}$ likewise takes B to C for all $w \in L$ and $t \in [0,1]$.*

Proof. We begin by proving a special case, and then we progressively generalise until we have proved the main statement in the theorem without properties (I) and (II). Finally, we exhibit the additional properties (I) and (II).
Case: $L = \mathbf{R}^\ell, M = Q, N = \mathbf{R}^n$, where Q is a quadrant of dimension m.
Choose a bump function χ on $L = \mathbf{R}^\ell$ with support in the unit ball, so that we may define a smooth family of bump functions:

$$\begin{array}{rrcl} \chi_t: & (0,1] \times L & \to & \mathbf{R} \\ & (t,w) & \mapsto & \frac{1}{t^\ell}\chi(\frac{w}{t}). \end{array}$$

We claim that the following homotopy satisfies the properties described in the proposition:

$$\begin{array}{rcl} [0,1] \times L \times M & \to & N \\ (t,w,x) & \mapsto & \begin{cases} \big(\chi_t * f_1(w,x), \dots, \chi_t * f_n(w,x)\big) & \text{if } t > 0 \\ f(w,x) & \text{if } t = 0. \end{cases} \end{array}$$

By Theorem 3.2.3 the adjoint $[0,1] \times L \to \mathbf{Man}(M,N)$ of the above map is continuous iff

$$\begin{array}{rcl} [0,1] \times L \times M & \to & J^k(M,N) \\ (t,w,x) & \mapsto & \begin{cases} \big(j^k(\chi_t * f_1(w,x)), \dots, j^k(\chi_t * f_n(w,x))\big) & \text{if } t > 0 \\ j^k f(w,x) & \text{if } t = 0 \end{cases} \end{array}$$

is for all $k \geq 0$. This map on the other hand is continuous iff for all m-tuples of non-negative integers $(k_1, \dots, k_m)$ with $k_1 + \cdots + k_m \leq k$ and for all $1 \leq i \leq \ell$ the map

$$\begin{array}{rcl} [0,1] \times L \times M & \to & \mathbf{R} \\ (t,w,x) & \mapsto & \begin{cases} \partial_{k_1} \cdots \partial_{k_m} \chi_t * f_i(w,x) & \text{if } t > 0 \\ \partial_{k_1} \cdots \partial_{k_m} f_i(w,x) & \text{if } t = 0 \end{cases} \end{array}$$

is continuous. The map is clearly smooth for all points (t,w,x) with $t > 0$, so it remains to show continuity for $t = 0$. Choose a value (s_0, x_0), then for any triple (t, h, r) with $t > 0$ we have

$$
\begin{aligned}
&|\partial_{k_1} \cdots \partial_{k_m} \chi_t * f_i(w_0 + h, x_0 + r) - \partial_{k_1} \cdots \partial_{k_m} f_i(w_0, x_0)| \\
= &\left| \partial_{k_1} \cdots \partial_{k_m} \int_{|w| \leq t} f_i(w_0 + h - w, x_0 + r) \chi_t(w) \, dw - \partial_{k_1} \cdots \partial_{k_m} f_i(w_0, x_0) \right| \\
= &\left| \partial_{k_1} \cdots \partial_{k_m} \int_{|w| \leq t} f_i(w_0 + h - w, x_0 + r) \chi_t(w) \, dw - \int_{|w| \leq t} \partial_{k_1} \cdots \partial_{k_m} f_i(w_0, x_0) \chi_t(w) \, dw \right| \\
= &\left| \int_{|w| \leq t} \partial_{k_1} \cdots \partial_{k_m} f_i(w_0 + h - w, x_0 + r) \chi_t(w) \, dw - \int_{|w| \leq t} \partial_{k_1} \cdots \partial_{k_m} f_i(w_0, x_0) \chi_t(w) \, dw \right| \\
= &\left| \int_{|w| \leq t} \partial_{k_1} \cdots \partial_{k_m} (f_i(w_0 + h - w, x_0 + r) - f_i(w_0, x_0)) \chi_t(w) \, dw \right| \\
\leq &\sup_{w \in B_t(0)} |\partial_{k_1} \cdots \partial_{k_m} f_i(w_0 + h - w, x_0 + r) - \partial_{k_1} \cdots \partial_{k_m} f_i(w_0, x_0)| \, ,
\end{aligned}
$$

where the last term can be made arbitrarily small by choosing suitably small t, h, r.
Case: $N = \mathbf{R}^n$. Let $\{U_i, u_i : U_i \xrightarrow{\cong} \mathbf{R}^\ell\}_{i \in I}$ be an atlas of L, $\{V_j, v_j : V_j \xrightarrow{\cong} Q\}_{j \in I}$, an atlas of M, and let $\{\rho_{ij}\}_{i,j \in I}$ be a subordinate partition of unity of the resulting covering of $L \times M$. For every $i \in I$, $j \in J$ denote by $\widetilde{H}_{ij}$ the map

$$
\begin{array}{rcl}
[0,1] \times \mathbf{R}^\ell \times Q & \to & N \\
(t, w, x) & \mapsto & \big((\chi_t * f_1(u_i^{-1}, v_j^{-1}))(w, x), \, \ldots \, , (\chi_t * f_n(u_i^{-1}, v_j^{-1}))(w, x) \big),
\end{array}
$$

and by H_{ij} the continuous map obtained by glueing together

$$
\begin{array}{rcl}
[0,1] \times U_i \times V_j & \to & \mathbf{R}^n \\
(t, w, x) & \mapsto & \begin{cases} \rho_{ij}(w, x) \cdot \widetilde{H}_{ij}(t, u_i(w), v_j(x)) & t > 0 \\ f(w, x) & t = 0 \end{cases}
\end{array}
$$

and

$$
\begin{array}{rcl}
[0,1] \times (L \times M \setminus U_i \times V_j) & \to & \mathbf{R}^n \\
(t, w, x) & \mapsto & f(w, x).
\end{array}
$$

The map $\sum_{i \in I, j \in J} H_{ij}$ yields the desired homotopy.
General case: Assume N is embedded in $\mathbf{R}^p$ for some $p \in \mathbf{N}$, and let V be a tubular neighbourhood of N. Let $\{H_{ij}\}_{i \in I, j \in J}$ be a family of homotopies constructed as in the previous case, then these are locally compactly supported (viewed as maps $[0,1] \times L \to \mathbf{Man}(M, \mathbf{R}^p)$), so that they are continuous w.r.t. the WO$^\infty$-topology by Lemma 3.2.16. Now, by Proposition 3.2.20, for each $i \in I, j \in J$ there exists $t_0 > 0$ such that restriction of H_{ij} to $[0, t_0] \times L$ factors through $\mathbf{Man}(M, V) \subseteq \mathbf{Man}(M, \mathbf{R}^p)$, and thus we may assume w.l.o.g. that the homotopy $\sum_{i \in I, j \in J} H_{ij}$ factors through $\mathbf{Man}(M, V) \subseteq \mathbf{Man}(M, \mathbf{R}^p)$. Composing $\sum_{i \in I, j \in J} H_{ij}$ with the map $\mathbf{Man}(M, V) \to \mathbf{Man}(M, N)$ (given by postcomposing with the projection $V \to N$) yields a continuous map by Proposition 3.2.22, and we obtain the new desired homotopy.

Property (I): Let $\gamma : L \to [0,1]$ be a smooth function such that $A = \gamma^{-1}(\{0\})$. If H is the homotopy from the previous case, then

$$\begin{array}{rcl} [0,1] \times L \times M & \to & \mathbf{R} \\ (t, w, x) & \mapsto & H(\gamma(w) \cdot t, w, x) \end{array}$$

yields the desired homotopy.
Property (II): The homotopy produced in the case "$N = \mathbf{R}^n$" already satisfies this property: Let (B, C) be a pair as in the statement of (II), and fix a point $(w, x) \in L \times B$, then there are finitely many elements $(i_1, j_1), \ldots, (i_k, j_k)$ in $I \times J$, so that $\rho_{ij}(w, x) > 0$, and $H(t, w, x)$ is equal to the convex combination

$$\rho_{i_1 j_1}(w, x) \widetilde{H}_{i_1 j_1}(t, u_{i_1}(w), v_{j_1}(x)) + \cdots + \rho_{i_k j_k}(w, x) \widetilde{H}_{i_k j_k}(t, u_{i_k}(w), v_{j_k}(x)),$$

for any $t \in [0,1]$. The values $\widetilde{H}_{i_1 j_1}(t, u_{i_1}(w), v_{j_1}(x)), \ldots, \widetilde{H}_{i_k j_k}(t, u_{i_k}(w), v_{j_k}(x))$ lie in C, as they are obtained by integrating functions taking values in the convex subset C. $\square$

Remark 3.3.2. For $M = *$ Theorem 3.3.1 (without points (I) and (II)) appears to be a classical result, but we were not able to find a reference. The analogous result showing point-set genericness (i.e. density) is well documented; see e.g. [Hir94, Thm. 2.4]. ⌟

Remark 3.3.3. Property (II) of the theorem will be important when smoothing parametrised maps of bordisms. ⌟

Remark 3.3.4. One can use Theorem 3.3.1 to show that the homotopy type of $\mathbf{Man}(M, N)$ is determined by smooth maps into $\mathbf{Man}(M, N)$. More precisely: If we consider the simplicial set, which sends $[n]$ to the set of smooth maps $\Delta^n \to \mathbf{Man}(M, N)$, then the inclusion of this simplicial set into the ordinary total singular complex of $\mathbf{Man}(M, N)$ is an equivalence. ⌟

3.4 Transversality

By expressing embeddings as maps satisfying certain transversality conditions we shall show that they form a residual subset of all smooth maps from one manifold to another. This will be used in §§4.1.2 & 4.2.5 to show that embeddings of closed manifolds into Euclidean space of high dimension are highly connective.

3.4.1 Transversality and embeddings

Throughout this subsection L, M, N, Z denote smooth manifolds. Moreover, let $k \in \mathbf{N}$ and $s \in \mathbf{N}_{\geq 1}$.
Recall that two smooth maps $f : M \to L,\ g : N \to L$ are called *transverse* if $\mathrm{Im}_{\mathrm{d}f|_x} + \mathrm{Im}_{\mathrm{d}g|_y} = T_w L$ for any triple $(x, y, w) \in M \times N \times L$ such that $f(x) = g(y) = w$, in which case we write $f \pitchfork g$. Given subsets $A \subseteq M,\ B \subseteq N$ we say that f *and* g *are transverse over* $A \times B$ and write $f \pitchfork_{A \times B} g$ if $\mathrm{Im}_{\mathrm{d}f|_x} + \mathrm{Im}_{\mathrm{d}g|_y} = T_z L$ for any triple $(x, y, w) \in A \times B \times L$ such that $f(x) = g(y) = w$.
In this subsection we state Thom's transversality theorem as well as Mather's generalisation thereof, the Multi-Jet Transversality Theorem, and then a further generalisation which accounts for certain restrictions arising in §4. We give careful proofs of the Multi-Jet Transversality Theorem as well as its generalisation in the succeeding subsection, §3.4.2.

Theorem 3.4.1 (Thom's transversality theorem. [Mic80, Lm. 6.5. & Thm. 6.8])**.** *For any smooth map $f : Z \to J^k(M,N)$ the set of smooth maps $g : M \to N$ satisfying $j^k g \pitchfork f$ is residual in the WO$^\infty$-topology. This set is moreover open, if $f : Z \to J^k(M,N)$ is proper.* □

Corollary 3.4.2 (Elementary transversality theorem)**.** *For any smooth map $f : Z \to N$ the set of smooth maps $g : M \to N$ satisfying $g \pitchfork f$ is residual in the WO$^\infty$-topology. This set is moreover open, if $f : Z \to N$ is proper.* □

The following theorem, itself of individual interest, constitutes an important step in proving Thom's transversality theorem.

Theorem 3.4.3 (Parametrised transversality theorem. [Mic80, Lm. 6.4])**.** *Let $\Phi : L \times M \to N$ and $f : Z \to N$ be smooth maps. If $\Phi \pitchfork f$, then the set of elements $w \in L$ such that $\Phi(w, \_) \not\pitchfork f$ has Lebesgue measure zero, and therefore its complement is dense.* □

Lemma 3.4.4. *If $\dim N \geq 2\dim M$ there exists a finite set of submanifolds of $J^1(M,N)$ such that $f \in \mathbf{Man}(M,N)$ is an immersion iff $j^1 f$ intersects transversally with these submanifolds.*

Sketch of proof. See [Mic80, Cor. 6.13] for a complete proof. The key observations are that any jet in $J^1(M,N)$ has a well defined rank (see Example 3.1.13), and that the subsets consisting of all jets of a given rank form a submanifold. One can calculate the dimensions of these submanifolds and show that if $\dim N \geq 2\dim M$, then a map $M \to N$ is an immersion iff $j^1 f$ does not meet any of these submanifolds iff $j^1 f$ intersects these submanifolds transversely. □

Applying Thom's transversality theorem we then obtain the following theorem:

Theorem 3.4.5. *The set of immersions of M in N is residual in the WO$^\infty$-topology if $\dim N \geq 2\dim M$.*

In fact, by Theorem 3.2.19.1. the set of immersions is even open and dense.
To express embeddings in terms of transversality we require multi-jet transversality.

Definition 3.4.6. Write $M^{(s)} := M^s \setminus \Delta$. The *$s$-fold k-jet bundle* $J^k_s(M,N) \to M^{(s)} \times N^s$ is the pullback of the bundle $J^k(M,N)^s \to M^s \times N^s$ along $M^{(s)} \times N^s \hookrightarrow M^s \times N^s$. For any map $f : M \to N$ we denote by $j^k_s f$ the restriction of the map $(j^k f)^s : M^s \to J^k(M,N)^s$ to $M^{(s)}$. ⌟

Theorem 3.4.7 (Multi-jet transversality theorem. [Mic80, Thm. 6.12])**.** *Let $f : Z \to J^k_s(M,N)$ be a smooth map, then the subset of maps $g \in \mathbf{Man}(M,N)$ satisfying $j^k_s g \pitchfork f$ is residual in the* WO$^\infty$*-topology.*

We give a careful discussion of the proof of this theorem in the following subsection.

Lemma 3.4.8. *If $\dim N \geq 2\dim M + 1$, then a map $f : M \to N$ is injective iff the map $j^0_2 f : M^{(2)} \to N \times N$ intersects the diagonal transversely.*

Proof. The map $j^0_2 f : M^{(2)} \to N \times N$ intersects the diagonal transversely iff it does not meet it. □

Theorem 3.4.9. *The set of injective maps from M to N is residual in the WO$^\infty$-topology if $\dim N \geq 2\dim M + 1$.*

Proof. Combine the preceding lemma with Theorem 3.4.7 ☐

Corollary 3.4.10. *Assume M is compact, then the set of embeddings is open and dense if* $\dim N \geq 2\dim M + 1$. ☐

Proof. Combine the preceding theorem with Theorems 3.2.19.2 & 3.4.5. ☐

We now turn to the generalisation of the Multi-Jet Transversality Theorem alluded in the introduction of this subsection. Suppose $f : Z \to J^k(M, N)$ is a smooth map, L, a closed submanifold of M of the same dimension, and $K : L \to N$, a smooth map such that the induced map $L \to J^k(M, N)$ intersects f transversally, then the subset $\mathbf{Man}(M, N)_K \subseteq \mathbf{Man}(M, N)$ consisting of maps g such that $g \circ \alpha = K$ is closed in the CO$^\infty$-topology by Theorem 3.2.23, and the restriction of the WO$^\infty$-topology is Baire by Theorem 3.2.17.

Theorem 3.4.11. *The subset of maps $g \in \mathbf{Man}(M, N)_K$ satisfying $g \pitchfork f$ is residual.* ☐

Theorem 3.4.11 will be crucial for proving Theorem 4.1.9, the key theorem in §4.1.2. Unfortunately, Theorem 3.4.11 can't be applied directly to Theorem 4.2.30, the counterpart of Theorem 4.1.9 in §4.2.5; instead we apply a local version, Theorem 3.4.15, which is a key lemma in proving the Theorem 3.4.11.

3.4.2 Proof of the Multi-Jet Transversality Theorem

We keep the conventions from last subsection, and now, additionally, fix a smooth map $f : Z \to J^k(M, N)$.

The proof proceeds along the following local-to-global argument: We first obtain a local version of the result, namely that for any suitable pair consisting of a compact subset $A \subseteq M^{(s)}$ and a closed subset $B \subseteq N^s$, the subset of maps $g \in \mathbf{Man}(M, N)$ satisfying $j^k_s g \pitchfork_{A \cap g^{-1}(B) \times Z} f$ is residual. We then show that we can cover M by a countable family $\mathcal{A}$ of compact subsets and N by a countable family $\mathcal{B}$ closed subsets, so that the local statement holds for all pairs $(A, B) \in \mathcal{A} \times \mathcal{B}$, and then the set

$$\left\{ g \in \mathbf{Man}(M, N) \;\middle|\; j^k_s g \pitchfork f \right\} = \bigcap_{(A,B) \in \mathcal{A} \times \mathcal{B}} \left\{ g \in \mathbf{Man}(M, N) \;\middle|\; j^k_s g \pitchfork_{A \cap g^{-1}(B) \times Z} f \right\}$$

is again residual, since $\mathbf{Man}(M, N)$ is Baire.

The property of being a residual subset is itself the combination of two properties:

(1) Being dense.

(2) Being the intersection of countably many open subsets.

The following lemma is the principal reason we work with the WO$^\infty$-topology rather than the CO$^\infty$-topology.

Lemma 3.4.12 ([Mic80, Lm. 6.11]). *Let $C \subseteq Z$ be a compact subset, then the subset of $\mathbf{Man}(M, N)$ consisting of maps g satisfying $j^k_s g \pitchfork_{A \times C} f$ is open in the WO$^\infty$-topology.* ☐

Lemma 3.4.13. *Let $A \subseteq M^{(s)}$ be a subset, and $B \subseteq N^s$, a closed subset, then the subset of maps $g \in \mathbf{Man}(M, N)$ satisfying $j^k_s g \pitchfork_{A \cap g^{-1}(B) \times Z} f$ is the intersection of countably many open subsets in the WO$^\infty$-topology.*

Proof. We denote by ω the canonical map $J^k_s(M,N) \to N$. Let $C \subseteq Z$ be a compact subset. For any $g : M \to N$ smooth map and any pair $(x,z) \in A \times C$ such that $j^k_s g(x) = f(z)$ we have

$$\begin{aligned}
(x,z) \in g^{-1}(B) \cap A \times C \quad &\iff \quad (x,z) \in A \times C \ \wedge \ g^s(x) \in B \\
&\iff \quad (x,z) \in A \times C \ \wedge \ \omega \circ j^k_s g(x) \in B \\
&\iff \quad (x,z) \in A \times C \ \wedge \ \omega \circ f(z) \in B \\
&\iff \quad (x,z) \in A \times C \cap (\omega \circ f)^{-1}(B).
\end{aligned}$$

Thus the subset of smooth maps $g : M \to N$ satisfying $g \pitchfork_{g^{-1}(B) \cap A \times C} f$ is equal to the subset of smooth maps satisfying $g \pitchfork_{A \times C \cap (\omega \circ f)^{-1}(B)} f$, which is open by the previous lemma. We can cover Z by a countable family $\mathcal{C}$ of compact subsets, and then the subset of maps $g \in \mathbf{Man}(M,N)$ satisfying $j^k_s g \pitchfork_{A \cap g^{-1}(B) \times Z} f$ is equal to

$$\bigcap_{C \in \mathcal{C}} \left\{ \ g \in \mathbf{Man}(M,N)_G \ \middle| \ j^k_s g \pitchfork_{A \cap g^{-1}(B) \times C} f \ \right\}.$$

$\square$

To prove property (1) we need to choose our families $\mathcal{A}$ and $\mathcal{B}$ compatibly with an atlas of M and N respectively:

Theorem 3.4.14 (Local transversality theorem)**.** *Let*

$$\big((Q_1, u_1, U_1, A_1), \ \dots \ , (Q_s, u_s, U_s, A_s)\big)$$

be an s-tuple such that for each $i \in \{1, \dots, s\}$

- *Q_i is a quadrant,*
- *$A_i \subseteq U_i$,*
- *$M \supseteq U_i \xrightarrow{u_i} Q_i$ is a full chart,*
- *$u_i(A_i) \subseteq [0,\infty)^m \subseteq Q_i$ is compact,*

and such that the sets $U_1, \dots, U_s$ are pairwise disjoint. Let

$$\big((Q'_1, v_1, V_1, B_1), \ \dots \ , (Q'_s, v_s, V_s, B_s)\big)$$

be an s-tuple such that for each $i \in \{1, \dots, s\}$

- *Q'_i is a quadrant,*
- *$B_i \subseteq V_i$,*
- *$N \supseteq V_i \xrightarrow{v_i} Q'_i$ is a full chart,*
- *B_i is closed in N.*

Finally write $A := A_1 \times \cdots \times A_s$ and $B := B_1 \times \cdots \times B_s$. Then the subset of maps $g \in \mathbf{Man}(M,N)$ satisfying $j^k_s g \pitchfork_{A \cap g^{-1}(B) \times Z} f$ is dense in the WO^∞-topology.

Proof. Fix $g \in \mathbf{Man}(M,N)$. We shall produce a sequence of maps $(g_n)_{n\in\mathbf{N}}$ in $\mathbf{Man}(M,N)$ satisfying $j^k_s g_n \pitchfork_{g_n^{-1}(B)\cap A\times Z} f$, which converges to g.
We must introduce some further auxiliary subsets and functions: For each $i \in \{1,\dots,s\}$ define

- $U'_i := g^{-1}(V_i) \cap U_i$,
- $D_i := g^{-1}(B_i) \cap A_i$,

and choose

- an open subset $U''_i \subseteq M$ satisfying $D_i \subseteq U''_i \subseteq \overline{U''_i} \subseteq U'_i$ and $\overline{U''_i}$ compact;
- a smooth function $\rho_i : U'_i \to [0,\infty)$ satisfying $\rho_i|_{U''_i} \equiv 1$ and $\operatorname{supp}\rho_i \subseteq U'_i$.

Denote by R the set of degree k polynomial maps $\mathbf{R}^m \to \mathbf{R}^n$ with non-negative coefficients; this is a quadrant, and $\dim R \times M = \dim J^k(M,N)$.[11] Moreover polynomial maps in R restrict to maps $[0,\infty)^m \to [0,\infty)^n$.
Now denote by Φ the map obtained by glueing

$$\begin{array}{rcl} R^s \times (U_1 \cup \dots \cup U_s) & \to & N \\ ((\sigma_1,\dots,\sigma_s),x) & \mapsto & \{v_i^{-1}\big(v_i \circ g(x) + \rho_i(x)\cdot \sigma_i \circ u_i(x)\big) \text{ for } x \in U_i \end{array} \tag{5}$$

and

$$\begin{array}{rcl} R^s \times M \setminus (\operatorname{supp}\rho_1 \cup \dots \cup \operatorname{supp}\rho_s) & \to & N \\ ((\sigma_1,\dots,\sigma_s),x) & \mapsto & g(x). \end{array}$$

For any $\sigma \in R^s$ write Φ_σ for the map $x \mapsto \Phi(\sigma,x)$, then the map

$$\begin{array}{rcl} M^{(s)} \times R^n & \to & J^k_s(M,N) \\ (x,\sigma) & \mapsto & j^k_s\Phi_\sigma(x) \end{array}$$

is smooth, and restricts to a local embedding on $(U''_1 \times \dots \times U''_s)$, as can easily be checked. Thus, by Theorem 3.4.3 the set of s-tuples of polynomials $\sigma \in R^s$ satisfying $j^k_s\Phi_\sigma \pitchfork_{(U''_1\cup\dots\cup U''_s)\times Z} f$ is dense in R^s, and there exists a sequence (σ_n) in R^s converging to 0. The sequence $g_n := \Phi_{\sigma_n}$ and all its derivatives converge uniformly on $\operatorname{supp}\rho_1 \cup \dots \cup \operatorname{supp}\rho_s$, and is constant on its complement, and thus converges in the WO^∞-topology.
To finish the proof we will show the existence of a non-negative integer n_i for each $i \in \{1,\dots,s\}$ such that $j^k_s g_n|_{U'_i} \pitchfork_{g_n^{-1}(B)\cap A\times Z} f$ for $n \geq n_i$. Setting $\overline{n} := \max(n_1,\dots,n_s)$ we then obtain the desired sequence $(g_n)_{n\geq\overline{n}}$. So fix $i \in \{1,\dots,s\}$. For any smooth map $h : M \to N$ it is easily checked that $h^{-1}(B_i) \cap A_i \subseteq U''_i$ is equivalent to $h(A_i \setminus U''_i) \subseteq N \setminus B_i$. These equivalent statements are true for g. The sequence (g_n) converges in the WO^∞, so a fortiori converges in the CO-topology. Therefore, there exists $n_i \in \mathbf{N}$ such that $n \geq n_i$ implies that $g_n(A_i \setminus U''_i) \subseteq N \setminus B_i$, or equivalently $g_n^{-1}(B_i) \cap A_i \subseteq U''_i$. □

[11] In [Mic80, Thm. 6.12] a different quadrant of polynomial maps is considered: Given two quadrants Q, Q', denote by $E(Q,Q')$ the set of degree k polynomial maps carrying Q to Q'. This again naturally has the structure of a quadrant, but we may no longer have $\dim Q \times E(Q,Q') = \dim J^k(Q,Q')$. Consider for example the case $k=1$, $Q = \mathbf{R}$, $Q' = [0,\infty)$, then $E(Q,Q')$ consists of constant functions $x \mapsto c$ with $c \geq 0$, and $\dim \mathbf{R} \times E(\mathbf{R},[0,\infty)) = 2$, whereas $\dim J^1(\mathbf{R},[0,\infty)) = 3$.

Proof of Theorem 3.4.7. We use the same notation as in Theorem 3.4.14. It is then easily seen that we can find countable families of s-tuples

$$\big((Q_1, u_1, U_1, A_1), \ldots, (Q_s, u_s, U_s, A_s)\big)$$

and

$$\big((Q'_1, v_1, V_1, B_1), \ldots, (Q'_s, v_s, V_s, B_s)\big)$$

such that the sets $A_1 \times \cdots \times A_s$ and $B_1 \times \cdots \times B_s$ cover $M^{(s)}$ and N^s respectively, and we then apply the proof strategy discussed in the beginning of the subsection. $\square$

We now sketch how to modify the proof of Theorem 3.4.14 to obtain the local version of Theorem 3.4.11.

Theorem 3.4.15. *With notation as in Theorem 3.4.14 the subset of maps $g \in \mathbf{Man}(M, N)_K$ satisfying $j^k_s g \pitchfork_{A \cap g^{-1}(B) \times Z} f$ is residual in the WO^∞-topology.*

Sketch of proof. In order to fix the restriction of any map $g \in \mathbf{Man}(M, N)_K$ to K we choose a map $\gamma : M \to [0, \infty)$ with $\gamma|_L \equiv 0$ and $\gamma|_{L^c} > 0$, and then use it to modify (as indicated by the box) the map Φ from (5):

$$\begin{array}{rcl} R^s \times (U_1 \cup \cdots \cup U_s) & \to & N \\ ((\sigma_1, \ldots, \sigma_s), x) & \mapsto & \Big\{ v_i^{-1}\big(v_i \circ g(x) + \boxed{\gamma(x)} \cdot \rho_i(x) \cdot \sigma_i \circ u_i(x)\big) \;\text{ for } x \in U_i \end{array}$$

and

$$\begin{array}{rcl} R^s \times M \setminus (\operatorname{supp} \rho_1 \cup \cdots \cup \operatorname{supp} \rho_s) & \to & N \\ ((\sigma_1, \ldots, \sigma_s), x) & \mapsto & g(x). \end{array}$$

For any $\sigma \in R^s$ the map

$$\begin{array}{rcl} R^n \times M^{(s)} & \to & J^k_s(M, N) \\ (x, \sigma) & \mapsto & j^k_s \Phi_\sigma(x) \end{array}$$

is smooth, and restricts to a local embedding[12] on $(U''_1 \setminus L \times \cdots \times U''_s \setminus L)$. Thus, by Theorem 3.4.3 the set of s-tuples of polynomials $\sigma \in R^s$ satisfying $j^k_s \Phi_\sigma \pitchfork_{(U''_1 \cup \cdots \cup U''_s) \setminus L \times Z} f$ is again dense in R^s, and we can again construct a sequence (σ_ν) in R^s converging to 0. $\square$

4 The quasi-unital flagged ∞-category of bordisms

Throughout this section d denotes a positive integer.
We construct the quasi-unital flagged ∞-category as the nerve of a topological semi-category, whose space of objets consists of closed $(d-1)$-submanifolds of $\mathbf{R}^\infty$, and whose space of morphisms consists of suitably embedded compact d-bordisms in $[0, 1] \times \mathbf{R}^\infty$. The building blocks of the first space are constructed in §4.1, and those of the second space in §4.2. These are then assembled into a topological semi-category in §4.3.

[12] Note that $\gamma(x) = 0$ implies $\mathrm{d}\gamma|_x = 0$ for all $x \in M$.

4.1 Spaces of manifolds

Given a closed manifold M, we construct a classical model for $B\,\mathbf{Diff}(M)$, denoted by $\Psi^\infty(M)$, whose underlying set consists of all submanifolds of $\mathbf{R}^\infty$ diffeomorphic to M. Let us try to understand the homotopy type of $\mathbf{Diff}(M)$ a bit better (see Remark 3.3.4):

- A point $\Delta^0 \to \mathbf{Diff}(M)$ is a diffeomorphism.
- A path in $\Delta^1 \to \mathbf{Diff}(M)$ is an isotopy between the diffeomorphisms corresponding to its endpoints.
- Maps $\Delta^n \to \mathbf{Diff}(M)$ for $n > 1$ are higher isotopies and compositions thereof.

Correspondingly, for $\Psi^\infty(M)$:

- A point $\Delta^0 \to \Psi^\infty(M)$ is then a copy of M.
- A path in $\Delta^1 \to \Psi^\infty(M)$ determines a diffeomorphism up to isotopy.
- A map $\Delta^2 \to \Psi^\infty(M)$ determines the composition of two diffeomorphism together with an isotopy; all of this up to a higher isotopy.
- Etc.

4.1.1 The principal bundle of $(d-1)$-dimensional manifolds

Throughout this subsection M, L denote two manifolds without boundary, with M closed, of dimension $d-1$.
The space $\mathrm{Emb}(M, L)$ admits a continuous principal right action of $\mathbf{Diff}(M)$ by precomposition; we shall explicitly describe the topology on the quotient.

Notation 4.1.1. Let $\iota : M \hookrightarrow L$ be an embedding, and $NM \overset{\hat\iota}{\hookrightarrow} L$, a tubular neighbourhood of ι. We denote by:

$\Psi(M, L)$	The set of submanifolds of L diffeomorphic to M.
$\Psi(M, L)\vert_{\hat\iota}$	The subset of $\Psi(M, L)$ consisting of those submanifolds which are the image of the composition of some section of NM with $\hat\iota$.
$\mathrm{Emb}(M, L)\vert_{\hat\iota}$	The subspace of embeddings in $\mathrm{Emb}(M, L)$ whose image lie in $\Psi(M, L)\vert_{\hat\iota}$.
$\varphi_{\hat\iota}$	The bijective map $\Gamma NM \to \Psi(M, L)\vert_{\hat\iota}$ given by $s \mapsto \mathrm{Im}_{\hat\iota \circ s}$.

⌟

Lemma 4.1.2. *Let $\zeta : E \to M$ be a vector bundle, then a submanifold M' of E is the image of a section of $E \to M$ iff the map $\zeta|_{M'}$ is a diffeomorphism.* □

Lemma 4.1.3. *Let $\iota : M \hookrightarrow L$ be an embedding, and $\hat\iota : NM \hookrightarrow L$, a tubular neighbourhood, then* $\mathrm{Emb}(M, L)|_{\hat\iota}$ *is open in* $\mathrm{Emb}(M, L)$.

Proof. By Lemma 4.1.2 the set $\mathrm{Emb}(M,L)|_{\hat\iota}$ is given as the preimage of $\mathbf{Diff}(M)$ under the map $\mathrm{Emb}(M,\hat\iota(NM)) \to \mathbf{Man}(M,M)$, $f \mapsto \pi\circ(\hat\iota^{-1})\circ f$ (where π is the projection $NM \to M$). As $\mathbf{Diff}(M)$ is open in $\mathbf{Man}(M,M)$ by Theorem 3.2.19.3, and $\mathrm{Emb}(M,\hat\iota(NM))$ is open in $\mathrm{Emb}(M,L)$ by Proposition 3.2.20, the lemma follows. $\square$

Proposition 4.1.4. *For any two embeddings $\iota_1,\iota_2 : M \hookrightarrow L$ and tubular neighbourhoods $\hat\iota_i : N_iM \hookrightarrow L$ ($i=1,2$) the sets $\varphi_{\hat\iota_1}^{-1}(\Psi(M,L)|_{\hat\iota_2})$, $\varphi_{\hat\iota_2}^{-1}(\Psi(M,L)|_{\hat\iota_1})$ are open in ΓN_1M and ΓN_2M respectively. Furthermore, the induced bijection $\varphi_{\hat\iota_1}^{-1}(\Psi(M,L)|_{\hat\iota_2}) \to \varphi_{\hat\iota_2}^{-1}(\Psi(M,L)|_{\hat\iota_1})$ is a homeomorphism.*

Proof. For the remainder of this proof we shall assume that $\Psi(M,L)|_{\hat\iota_1}\cap\Psi(M,L)|_{\hat\iota_2}\neq\varnothing$; otherwise the proof is trivial. Let $i,j\in\{1,2\}$ with $i\neq j$. The image V of $(\hat{\iota_i})^{-1}\circ\hat{\iota_j}|_{(\hat{\iota_j})^{-1}(\mathrm{Im}_{\hat{\iota_i}})}$ is open in N_iM. By Lemma 4.1.2 the set $\varphi_{\hat\iota_i}^{-1}(\Psi(M,L)|_{\hat\iota_j})$ then consists of those sections s with image in V such that $(\iota_i)^{-1}\circ\iota_j\circ\pi_j\circ(\hat{\iota_j})^{-1}\circ\hat{\iota_i}|_{(\hat\iota_i)^{-1}(\mathrm{Im}_{\hat\iota_j})}\circ s$ is a diffeomorphism, and thus forms an open subset of ΓN_iM by Theorem 3.2.19.3 and Proposition 3.2.20 as in Lemma 4.1.3.
The map $\varphi_{\hat\iota_1}^{-1}(\Psi(M,L)|_{\hat\iota_2}) \to \varphi_{\hat\iota_2}^{-1}(\Psi(M,L)|_{\hat\iota_1})$ is given by $s\mapsto s\circ\iota_1^{-1}\circ\iota_2$, and its inverse, by $s\mapsto s\circ\iota_2^{-1}\circ\iota_1$. Both maps are continuous by Theorem 3.2.23. $\square$

We can thus topologise $\Psi(M,L)$ in the same way as one topologises manifolds using charts (see also Remark 4.1.8).

Convention 4.1.5. From now on the set $\Psi(M,L)$ will always be understood to be equipped with the topology described above. ⌟

Lemma 4.1.6. *The map* $\mathrm{Emb}(M,L)\to\Psi(M,L)$ *is continuous.*

Proof. By Lemma 4.1.3 it is enough to show that $\mathrm{Emb}(M,L)|_{\hat\iota}\to\Psi(M,L)|_{\hat\iota}$ is continuous for any embedding $\iota : M\hookrightarrow L$ and tubular neighbourhood. Identifying M with ιM we obtain a map $\alpha : \mathrm{Emb}(M,L)|_{\hat\iota}\to\mathbf{Diff}(M)$ given by composing any embedding $M\hookrightarrow NM$ with the projection $NM\to M$. Then $\mathrm{Emb}(M,L)|_{\hat\iota}\to\Psi(M,L)|_{\hat\iota}$ is the composition of

$$\mathrm{Emb}(M,L)|_{\hat\iota}\xrightarrow{(\mathrm{id},\alpha)}\mathrm{Emb}(M,L)|_{\hat\iota}\times\mathbf{Diff}(M)\xrightarrow{\mathrm{id}\times(\_)^{-1}}\mathrm{Emb}(M,L)|_{\hat\iota}\times\mathbf{Diff}(M)\xrightarrow{\circ}\Psi(M,L)|_{\hat\iota},$$

which is continuous by Propositions 3.2.23 & 3.2.26. $\square$

Note that for any embedding $\iota : M\hookrightarrow L$ and tubular neighbourhood $\hat\iota : NM\hookrightarrow L$ we obtain a continuous section $\Psi(M,L)|_{\hat\iota}\to\mathrm{Emb}(M,L)$, $M'\mapsto\hat\iota\circ(\varphi_{\hat\iota}^{-1}(M'))$.

Theorem 4.1.7. *The bundle* $\mathrm{Emb}(M,L)\to\Psi(M,L)$ *is a principal* $\mathbf{Diff}(M)$*-bundle which trivialises over $\Psi(M,L)|_{\hat\iota}$ for any embedding $\iota : M\hookrightarrow L$ and tubular neighbourhood $\hat\iota : NM\hookrightarrow L$.*

Proof. From the section $\Psi(M,L)|_{\hat\iota}\to\mathrm{Emb}(M,L)$ we obtain the trivialisation

$$\begin{tikzcd}
\Psi(M,L)|_{\hat\iota}\times\mathbf{Diff}\,M \arrow[r, "\cong"] \arrow[d, two heads] & \mathrm{Emb}(M,L)|_{\hat\iota} \arrow[d, two heads] \\
\Psi(M,L)|_{\hat\iota} \arrow[r, "="] \arrow[ur, hook] & \Psi(M,L)|_{\hat\iota}
\end{tikzcd}$$

of maps of sets. As $\Psi(M,L)|_{\hat\iota}\times\mathbf{Diff}\,M\to\mathrm{Emb}(M,L)|_{\hat\iota}$ is given by $(M',f)\mapsto\hat\iota\circ(\varphi_{\hat\iota}^{-1}(M'))\circ f$, it is continuous by Theorem 3.2.23. The map $\mathrm{Emb}(M,L)|_{\hat\iota}\to\Psi(M,L)|_{\hat\iota}\times\mathbf{Diff}\,M$ is the product of the projection $\mathrm{Emb}(M, L)|_{\hat{\imath}} \to \Psi(M, L)|_{\hat{\imath}}$, which is continuous by Lemma 4.1.6, and the map $f \mapsto \pi \circ \hat{\imath}^{-1} \circ f$, where π is the projection $NM \to M$, which is again continuous by Theorem 3.2.23. $\square$

Remark 4.1.8. One can show that the maps $\varphi_{\hat{\imath}}$ indexed by the tubular neighbourhoods $\hat{\imath}$ form an atlas of a Fréchet manifold structure on $\Psi(M, L)$. A map from a manifold into $\Psi(M, L)$ is then smooth iff (locally on the target) it is smooth in the sense of Definition 3.2.12. If we endow $\mathrm{Emb}(M, L)$ and $\mathbf{Diff}(M)$ with their natural Fréchet manifold structures we obtain a smooth principal $\mathbf{Diff}(M)$-bundle. For details see [BF81]. ⌟

4.1.2 Connectivity of the principal bundle of closed $(d-1)$-manifolds

The following statement appears to be well known (e.g. [Mad12]), but we are not aware of any proof in the literature.

Theorem 4.1.9. *The space* $\mathrm{Emb}(M, \mathbf{R}^{n+d})$ *is* $(n-d-2)$*-connected for* $n \in \mathbf{N}$.

Proof. We will show that the inclusion $\mathrm{Emb}(M, \mathbf{R}^{n+d}) \subseteq \mathbf{Man}(M, \mathbf{R}^{n+d})$ is $(n-d)$-connected, and the theorem then follows from the fact that $\mathbf{Man}(M, \mathbf{R}^{n+d})$ is contractible.
So let $0 \leq k \leq n-d$, and choose a continuous map $f : S^{k-1} \to \mathrm{Emb}(M, \mathbf{R}^{n+d})$. By Proposition 3.3.1, may assume w.l.o.g. that f is smooth.
As $\mathbf{Man}(M, \mathbf{R}^{n+d})$ is contractible, we may extend the map $f : S^{k-1} \to \mathrm{Emb}(M, \mathbf{R}^{n+d})$ to a map $F : D^k \to \mathbf{Man}(M, \mathbf{R}^{n+d})$, and moreover we may assume that $F(w) = F\left(\frac{w}{\|w\|}\right)$ for all $w \in L := D^k \setminus \frac{1}{2}\mathring{D}^k$. As for f, we may assume w.l.o.g. that F is smooth.
Denote by $\hat{F} : D^k \times M \to D^k \times \mathbf{R}^{n+d}$ the map $(r, x) \mapsto (r, F(r, x))$. The set of submersions $D^k \times M \to D^k$ is open, and thus by Proposition 3.2.23 its preimage V under the continuous map $\mathbf{Man}(D^k \times M, D^k \times \mathbf{R}^{n+d}) \to \mathbf{Man}(D^k \times M, D^k)$ is also open. We have just shown that the space $\mathbf{Man}(D^k \times M, D^k \times \mathbf{R}^{n+d})_{\hat{F}|_{L \times M}}$ is non-empty, and as the subspace of embeddings is dense by combining Lemmas 3.4.4 & 3.4.8 with Theorem 3.4.11, we obtain a non-trivial intersection with V.
We now choose a function G in this intersection. By Ehresmann's theorem $p_1 \circ G : D^k \times M \to D^k$ is a fibre bundle, and since D^k is contractible, it is in fact trivial. This endows $D^k \times M$ with a new product structure, which we denote by $D^k \times' M$. The projections $D^k \times' M \to D^k$ and $D^k \times M \to D^k$ agree, and thus have the same fibres, so that we obtain a fibrewise embedding of M into $\mathbf{R}^{n+d}$ over D^k, and the composition of $D^k \times' M \to D^k \times \mathbf{R}^{n+d} \to \mathbf{R}^{n+d}$ corresponds to a smooth map $D^k \to \mathrm{Emb}(M, \mathbf{R}^{n+d})$ extending f. $\square$

Convention 4.1.10. Recall that $\mathbf{R}^\infty := \varinjlim_{p \in \mathbf{N}} \mathbf{R}^p$. We can then endow $\mathrm{Emb}(M, \mathbf{R}^\infty) \cong \varinjlim_{p \in \mathbf{N}} \mathrm{Emb}(M, \mathbf{R}^p)$ with the colimit topology. ⌟

Corollary 4.1.11. *The space* $\mathrm{Emb}(M, \mathbf{R}^\infty)$ *is weakly contractible.*

Proof. By [DI04, Lm. A.3] $\pi_i \mathrm{Emb}(M, \mathbf{R}^\infty) \cong \varinjlim_{p \in \mathbf{N}} \pi_i \mathrm{Emb}(M, \mathbf{R}^p)$ for all $i \in \mathbf{N}$. $\square$

Notation 4.1.12. We write $\Psi^\infty(M) := \varinjlim_{p \in \mathbf{N}} \Psi^p(M)$. ⌟

Remark 4.1.13. The space $\Psi^\infty(M)$ is the homotopy colimit of $\Psi^{2d+1}(M) \hookrightarrow \Psi^{2d+2}(M) \hookrightarrow \cdots$. ⌟

Proposition 4.1.14. *The* $\mathbf{Diff}(M)$*-space* $\mathrm{Emb}(M, \mathbf{R}^\infty) \circlearrowleft \mathbf{Diff}(M)$ *is a principal* $\mathbf{Diff}(M)$*-bundle with base* $\Psi^\infty(M)$*.*

Proof. First we note that $\mathbf{Diff}(M)$ acts continuously on $\mathrm{Emb}(M, \mathbf{R}^\infty)$; the inclusions $\mathrm{Emb}(M, \mathbf{R}^{2d+1}) \hookrightarrow \mathrm{Emb}(M, \mathbf{R}^{2d+2}) \hookrightarrow \cdots$ are $\mathbf{Diff}(M)$-invariant, and since the forgetful functor from right $\mathbf{Diff}(M)$-spaces to topological spaces admits a right adjoint, we see that the colimit of the above diagram as $\mathbf{Diff}(M)$-spaces coincides with the colimit of the diagram of the underlying topological spaces together with the induced $\mathbf{Diff}(M)$-action.
Again, by virtue of colimits commuting with left adjoints, in this case the quotient functor, we see that $\Psi^\infty(M)$ is the quotient of $\mathrm{Emb}(M, \mathbf{R}^\infty) \circlearrowleft \mathbf{Diff}(M)$.
To conclude the proof that $\mathrm{Emb}(M, \mathbf{R}^\infty) \circlearrowleft \mathbf{Diff}(M)$ is principal we describe the local trivialisations: Let $M' \in \Psi^\infty(M)$; by assumption we have a chain of embeddings: $M' \overset{\iota}{\hookrightarrow} \mathbf{R}^p \hookrightarrow \mathbf{R}^\infty$ for some $p \in \mathbf{N}$. Let $\hat{\iota} : NM' \hookrightarrow \mathbf{R}^p$ be a tubular neighbourhood. Consider a continuous section of the bundle $NM' \times \mathbf{R}^\infty \to M'$; this is the same as a pair consisting of a continuous section of $NM' \to M'$ together with a continuous map $M' \to \mathbf{R}^\infty$, where the latter factors through $\mathbf{R}^{p'} \hookrightarrow \mathbf{R}^\infty$ by the compactness of M' for some $p' \in \mathbf{N}$. We denote by $\Gamma(NM' \times \mathbf{R}^\infty)$ those sections of $NM' \times \mathbf{R}^\infty \to M'$ such that the resulting sections of $NM' \to M'$ and $M' \times \mathbf{R}^{p'} \to M'$, as described above, are both smooth; we endow this set with the colimit topology of $\Gamma NM' \hookrightarrow \Gamma(NM' \times \mathbf{R}) \hookrightarrow \cdots$. We again denote the resulting map $NM' \times \mathbf{R}^\infty \hookrightarrow \mathbf{R}^\infty$ by $\hat{\iota}$; moreover, as in the finite dimensional case, the corresponding submanifolds in $\Psi^\infty(M)$ are again denoted by $\Psi^\infty(M)|_{\hat{\iota}}$, and the set of embeddings whose image lie in $\Psi^\infty(M)|_{\hat{\iota}}$ is similarly denoted by $\mathrm{Emb}(M', \mathbf{R}^\infty)|_{\hat{\iota}}$. It is easily checked that $\Gamma(NM' \times \mathbf{R}^\infty) \hookrightarrow \Psi^\infty(M)$ is an open embedding. As in the finite dimensional case we obtain a diagram of maps of sets

$$\begin{tikzcd}
\Psi^\infty(M)|_{\hat{\iota}} \times \mathbf{Diff}\, M \arrow[r, "\cong"] \arrow[d, two heads] & \mathrm{Emb}(M, \mathbf{R}^\infty)|_{\hat{\iota}} \arrow[d, two heads] \\
\Psi^\infty(M)|_{\hat{\iota}} \arrow[r, "="] \arrow[ur, hook] & \Psi^\infty(M)|_{\hat{\iota}};
\end{tikzcd}$$

checking that the maps are continuous, and verifying that the top horizontal map is a homeomorphism is simply a matter of writing out all spaces as the appropriate colimits. □

Corollary 4.1.15. *The space* $\Psi^\infty(M)$ *with a choice of base point is a model for* $B\,\mathbf{Diff}(M)$ *for any compact* d*-dimensional manifold* M*.* □

Remark 4.1.16. For any compact d-dimensional manifold M the space $\Psi^\infty(M)$ is paracompact by [Tri17, Prop. 4.2], and thus $\mathrm{Emb}(M, \mathbf{R}^\infty) \circlearrowleft \mathbf{Diff}(M)$ is a universal $\mathbf{Diff}(M)$-bundle in the classical sense (as described e.g. in [Hus94]). ⌟

4.2 Spaces of bordisms

4.2.1 Bordisms

Definition 4.2.1. A *bordism* is a triple $(W; W_0, W_1)$ consisting of a manifold with boundary W together with submanifolds W_0, W_1 such that $W_0 \cup W_1 = \partial W$ and $W_0 \cap W_1 = \varnothing$. The integer $\dim W$ is the *dimension of* $(W; W_0, W_1)$. ⌟

Synecdoche 4.2.2. We will often specify a given bordism only by its constituent manifold with boundary W, leaving it understood that the boundary of W has been partitioned into two components, which will consistently be denoted by W_0 and W_1. ⌟

Example 4.2.3. *For an $n \in \mathbf{N}$ the manifold $\mathbf{R}^n \times [0,1]$ is equipped with the canonical structure of a bordism:*

$$(\mathbf{R}^n \times [0,1]; \mathbf{R}^n \times \{0\}, \mathbf{R}^n \times \{1\}) .$$

⌟

Definition 4.2.4. Let

$$(W; W_0, W_1), \ (W'; W_0', W_1')$$

be two bordisms, then a *map of bordisms*

$$(W; W_0, W_1) \to (W'; W_0', W_1')$$

is a smooth map $W \to W'$ which restricts to a smooth map $W_i \to W_i'$ for $i = 0, 1$ and a smooth map $\mathring{W} \to \mathring{W}'$, and such that $W \to W'$ is transverse to W_i' for $i = 0, 1$.
We denote by

$$\mathbf{bMan}(W, W')$$

the topological space consisting of maps of bordisms $W \to W'$ equipped with the WO^∞-topology.
We denote by

$$\mathbf{Diff}(W)$$

the subspace of $\mathbf{bMan}(W, W)$ consisting of diffeomorphisms. For any $n \in \mathbf{N}$ we write

$$\mathbf{bMan}^n(W) := \mathbf{bMan}\left(W, \mathbf{R}^n \times [0,1]\right) .$$

⌟

4.2.2 Collared bordisms

Definition 4.2.5. Let $(W; W_0, W_1)$ be a bordism, then a *collaring of W* consists of

(i) a real numbers $\varepsilon > 0$ and

(ii) embeddings

$$\begin{array}{rrcl} c_0 : & W_0 \times [0, \varepsilon) & \hookrightarrow & W, \\ c_1 : & W_1 \times (-\varepsilon, 0] & \hookrightarrow & W, \end{array}$$

such that

(a) the map c_i restricts to the projection map $c_i|_{W_i \times \{0\}} : W_i \times \{0\} \xrightarrow{\cong} W_i$ for $i = 0, 1$,

(b) the map c_0 restricts to an embedding $W_0 \times [0, \varepsilon) \hookrightarrow W$,

(c) the map c_1 restricts to an embedding $W_1 \times (-\varepsilon, 0] \hookrightarrow W$,

(d) the images of the maps c_0 and c_1 are disjoint.

A bordism together with collaring is called *a collared bordism.* ⌟

Convention 4.2.6. Given a collared bordism, the data of the embeddings in (ii) in the preceding definition will almost always be suppressed from the notation. Moreover the domains and images of the embeddings will often be identified. $\lrcorner$

Example 4.2.7. *For any $n \in \mathbf{N}$ the bordism $\mathbf{R}^n \times [0,1]$ is equipped with an obvious collaring.* $\lrcorner$

Definition 4.2.8. Let W, W' be collared bordisms, then a map of bordisms $f : W \to W'$ is called a *map of collared bordisms*, if there exists $0 < \delta < \min(\varepsilon, \varepsilon')$ such that f restricts to a map of the form

$$\begin{array}{rcl} W_0 \times [0,\delta) & \to & W_0' \times [0,\delta) \\ (x,t) & \mapsto & (f(x),t), \end{array}$$

and

$$\begin{array}{rcl} W_1 \times (-\delta,0] & \to & W_1' \times (-\delta,0] \\ (x,t) & \mapsto & (f(x),t). \end{array}$$

We denote by

$$\mathbf{b\underset{\bowtie}{M}an}(W,W')$$

the subspace of $\mathbf{bMan}(W,W')$ consisting of collared bordisms.
We denote by

$$\mathbf{D\underset{\bowtie}{i}ff}(W)$$

the subspace of $\mathbf{b\underset{\bowtie}{M}an}(W,W')$ consisting of diffeomorphisms. For any $n \in \mathbf{N}$ we write

$$\mathbf{\underset{\bowtie}{M}an}^n(W) := \mathbf{b\underset{\bowtie}{M}an}\left(W, \mathbf{R}^n \times [0,1]\right).$$

$\lrcorner$

4.2.3 Embeddings of bordisms

For the rest of this section $(W; W_0, W_1)$ denotes some collared bordism, and $n \in \mathbf{N}$.

Definition 4.2.9. Let W' be a bordism, then a *bordism embedding of W in W'* is a map of bordisms which is an embedding. The subspace of $\mathbf{bMan}(W,W')$ consisting of embeddings is denoted by

$$\mathrm{bEmb}(W,W').$$

If W and W' are collared, then a *collared embedding of W in W'* is a map of collared bordisms which is an embedding. The subspace of $\mathbf{b\underset{\bowtie}{M}an}(W,W')$ consisting of collared embeddings is denoted by

$$\mathrm{b\underset{\bowtie}{E}mb}(W,W').$$

$\lrcorner$

Notation 4.2.10. We write

$$\mathrm{Emb}^n(W) := \mathrm{bEmb}\left(W, \mathbf{R}^n \times [0,1]\right)$$

and

$$\mathrm{\underset{\bowtie}{E}mb}^n(W) := \mathrm{b\underset{\bowtie}{E}mb}\left(W, \mathbf{R}^n \times [0,1]\right).$$

$\lrcorner$

Proposition 4.2.11. *Let W' be a bordism. If W is compact, then the inclusions*

$$\mathbf{b}\underset{\bowtie}{\mathbf{M}}\mathbf{an}(W, W') \hookrightarrow \mathbf{bMan}(W, W')$$

and

$$\mathrm{E}\underset{\bowtie}{\mathrm{m}}\mathrm{b}(W, W') \hookrightarrow \mathrm{Emb}(W, W')$$

are weak homotopy equivalences.

Proof. Let $k \in \mathbf{N}$, and consider a commutative diagram

$$\begin{tikzcd}
\mathbf{b}\underset{\bowtie}{\mathbf{M}}\mathbf{an}(W, W') \arrow[r, hook] & \mathbf{bMan}(W, W') \\
S^{k-1} \arrow[u] \arrow[r, hook] & D^k. \arrow[u, "f"']
\end{tikzcd}$$

We shall construct a smooth homotopy starting at the map $f : D^k \to \mathbf{bMan}(W, W')$, which is constant when restricted to S^{k-1}, and which terminates at the map which factors through the inclusion $\mathbf{b}\underset{\bowtie}{\mathbf{M}}\mathbf{an}(W, W') \hookrightarrow \mathbf{bMan}(W, W')$. We will see that if the vertical maps in the above diagram factor through

$$\begin{tikzcd}
\mathbf{b}\underset{\bowtie}{\mathbf{M}}\mathbf{an}(W, W') \arrow[r, hook] & \mathbf{bMan}(W, W') \\
\mathrm{E}\underset{\bowtie}{\mathrm{m}}\mathrm{b}(W, W') \arrow[u, hook] \arrow[r, hook] & \mathrm{Emb}(W, W'), \arrow[u, hook]
\end{tikzcd}$$

then the homotopy we construct likewise factors through $\mathrm{Emb}(W, W') \hookrightarrow \mathbf{bMan}(W, W')$. By the claim at the end of the proof, we may assume that f is smooth.
We shall only discuss the homotopy around W_0; the argument around W_1 is the same. Write $W_0 \times [0, \varepsilon) \subseteq W$ and $W'_0 \times [0, \varepsilon') \subseteq W'$ for the collarings around W_0 and W'_0 respectively. By compactness we may assume w.l.o.g. that $\bar{f}(D^k \times W_0 \times \varepsilon) \subseteq W'_0 \times \varepsilon'$.
Since D^k and W_0 are compact there exists $0 < \delta < \varepsilon$ such that $\partial_3 \bar{f}_2 : D^{n+1} \times W_0 \times [0, \delta) \to \mathbf{R}$ is positive. Again, by compactness there exists some $0 < \gamma < \delta$ such that $W'_0 \times [0, \gamma) \subseteq \bar{f}\left(D^{k+1} \times W_0 \times [0, \delta)\right)$. We now supply a collection of functions which we will assemble into the desired homotopy:

(i) Denote by $\varphi : D^k \times W_0 \times [0, \gamma) \to [0, \delta)$ the function which is uniquely determined by $\bar{f}_2(r, x_0, \varphi_r(x_0, s)) = s$ for all $(r, x_0, s) \in D^{k+1} \times W_0 \times [0, \gamma)$. This function is smooth by the implicit function theorem.

(ii) Denote by $\psi : D^k \times W_0 \times [0, \gamma) \to W'_0$ the unique map such that $\bar{f}_1(r, \psi_r(x_0, s), s) = \bar{f}_1(r, x_0, 0)$ for all $(r, x_0, s) \in D^k \times W_0 \times [0, \gamma)$, which is again smooth by the implicit function theorem.

(iii) Let

$$\begin{array}{rccc}
\lambda : & [0, 1] & \to & [0, 1], \\
\mu : & [0, \gamma] & \to & [0, 1], \\
\nu : & [0, \gamma] & \to & [0, 1]
\end{array}$$

be smooth functions such that

$$\begin{aligned} \lambda|_{[0,\frac{1}{3}]} &\equiv 0 \quad \text{and} \quad \lambda|_{[\frac{2}{3},1]} \equiv 1 \\ \mu|_{[0,\frac{1}{3}\gamma]} &\equiv 1 \quad \text{and} \quad \mu|_{[\frac{2}{3}\gamma,\gamma]} \equiv 0 \\ \nu|_{[0,\frac{1}{3}\gamma]} &\equiv 0 \quad \text{and} \quad \nu|_{[\frac{2}{3}\gamma,\gamma]} \equiv 1. \end{aligned}$$

The desired homotopy $[0,1] \times D^{k+1} \times W \to W'$ is then given by first concatenating

$$\begin{aligned} [0,1] \times D^k \times W_0 \times [0,\gamma) &\to W' \\ (t,r,x_0,s) &\mapsto \bar{f}(r, x_0, s + \lambda(t)\mu(s)(\alpha_r(x_0,s) - s)) \end{aligned}$$

and

$$\begin{aligned} [0,1] \times D^k \times W_0 \times [0,\gamma) &\to W' \\ (t,r,x_0,s) &\mapsto \bar{f}(r, \psi_r(x_0, \nu(t)s), s), \end{aligned}$$

and then glueing together the resulting map $[0,1] \times D^k \times W_0 \times [0,\gamma) \to W'$ with

$$\begin{aligned} [0,1] \times D^k \times \left(W \setminus \left(W_0 \times [0, \tfrac{2}{3}\gamma]\right)\right) &\to W' \\ (t,r,x) &\mapsto \bar{f}(r,x). \end{aligned}$$

<u>Claim:</u> Let $U \subseteq \mathbf{bMan}(W,W')$ be an open subset, and consider a diagram of continuous maps

$$\begin{tikzcd} \mathbf{b}\underset{\bowtie}{\mathbf{Man}}(W,W') \cap U \arrow[r, hook] & U \\ S^{k-1} \arrow[u] \arrow[r, hook] & D^k, \arrow[u, "f"'] \end{tikzcd}$$

then there exists a homotopy $H : [0,1] \times D^k \to U$ such that

(a) its restriction $\{1\} \times D^k \to U$ is smooth, and

(b) its restriction to $[0,1] \times S^{k-1} \to U$ factors through $\mathbf{b}\underset{\bowtie}{\mathbf{Man}}(W,W') \cap U$.

<u>Proof of claim:</u> We shall write our homotopy as the composition of two homotopies: The first homotopy describes how we change f near the boundary S^{k-1}, and the second homotopy describes how we change f in the interior of D^k. All homotopies considered below are such that they automatically factor through U or can be restricted to a shorter homotopy which factors through U, as they are obtained by applying Theorem 3.3.1.

<u>First homotopy:</u> As S^{k-1} is compact, we may assume that for all $r \in S^{k-1}$ we can use the same $\delta > 0$ for $f(r,\_)$ in the sense of the Definition 4.2.8. Moreover, by precomposing f with a suitable homotopy $[0,1] \times D^k \to D^k$, we can moreover assume that $f(r,\_) = f\left(\frac{r}{\|r\|},\_\right)$ for all $r \in D^k \setminus \frac{1}{4}\mathring{D}^k$; we can then extend f to a function $\tilde{f}$ on $\frac{3}{2}\mathring{D}^k$ in the obvious way.
We can find a homotopy $[0,1] \times \left(\frac{3}{2}\mathring{D}^k \setminus \frac{1}{4}D^k\right) \times \left(W_0 \cup W_1\right) \to W_0' \cup W_1'$ whose restriction to $(0,1] \times \left(\frac{3}{2}\mathring{D}^k \setminus \frac{1}{4}D^k\right) \times \left(W_0 \cup W_1\right)$ is smooth. Let $\lambda : [0,\delta) \to [0,1]$ be a smooth function such that $\lambda|_{[0,\frac{1}{2}\delta]} \equiv 1$ and $\lambda|_{[\frac{3}{4}\delta,\delta)} \equiv 0$. We obtain a homotopy H_1 defined on $\left(\frac{3}{2}\mathring{D}^k \setminus \frac{1}{4}D^k\right) \times W$ by

glueing

$$\begin{aligned}[0,1] \times \left(\tfrac{3}{2}\mathring{D}^k \setminus \tfrac{1}{4}D^k\right) \times \left(W_0 \times [0,\delta) \cup W_1 \times (-\delta,0]\right) &\to W' \\ (t,r,x_0,s) &\mapsto (H_0(\lambda(s)t,r,x_0),s)\end{aligned}$$

and

$$\begin{aligned}[0,1] \times \left(\tfrac{3}{2}\mathring{D}^k \setminus \tfrac{1}{4}D^k\right) \times \left(W \setminus \left(W_0 \times \left[0,\tfrac{3}{4}\delta\right] \cup W_1 \times \left[-\tfrac{3}{4}\delta,0\right]\right)\right) &\to W' \\ (t,r,x) &\mapsto f(r,x).\end{aligned}$$

Let $H_2 : [0,1] \times \left(\frac{3}{2}\mathring{D}^k \setminus \frac{1}{4}D^k\right) \times \mathring{W} \to \mathring{W}'$ be a homotopy, starting at $H_1|_{\{1\}\times\left(\frac{3}{2}\mathring{D}^k\setminus\frac{1}{4}D^k\right)\times\mathring{W}}$, and which is smooth when restricted to $(0,1] \times \left(\frac{3}{2}\mathring{D}^k \setminus \frac{1}{4}D^k\right) \times \mathring{W}$. Let $\mu : [0,\delta) \to [0,1]$ be a smooth function such that $\mu|_{[0,\frac{1}{4}\delta]} \equiv 0$ and $\mu|_{[\frac{1}{2}\delta,\delta)} \equiv 1$. We define H_3 by glueing

$$\begin{aligned}[0,1] \times \left(\tfrac{3}{2}\mathring{D}^k \setminus \tfrac{1}{4}D^k\right) \times \left(W \setminus \left(W_0 \times \left[0,\tfrac{1}{2}\delta\right] \cup W_1 \times \left[-\tfrac{1}{2}\delta,0\right]\right)\right) &\to W' \\ (t,r,x) &\mapsto H_2(t,r,x),\end{aligned}$$

$$\begin{aligned}[0,1] \times \left(\tfrac{3}{2}\mathring{D}^k \setminus \tfrac{1}{4}D^k\right) \times \left(W_0 \times (0,\delta) \cup W_1 \times (\delta,0)\right) &\to W' \\ (t,r,x_0,s) &\mapsto H_2(\mu(s)t,r,x_0,s)\end{aligned}$$

and

$$\begin{aligned}[0,1] \times \left(\tfrac{3}{2}\mathring{D}^k \setminus \tfrac{1}{4}D^k\right) \times \left(W_0 \times \left[0,\tfrac{1}{2}\delta\right) \cup W_1 \times \left(-\tfrac{1}{2}\delta,0\right]\right) &\to W' \\ (t,r,x) &\mapsto H_1(1,r,x).\end{aligned}$$

We denote the homotopy obtained by composing the homotopies H_1 and H_3, and then restricting to $[0,1] \times D^k \setminus \frac{1}{4}D^k \times W$ by H_4.
Let $\nu : \left[\frac{1}{4},1\right] \to [0,1]$ be a smooth function such that $\nu|_{\left[\frac{1}{4},\frac{1}{2}\right]} \equiv 0$ and $\nu|_{\left[\frac{3}{4},1\right]} \equiv 1$. We define the homotopy H by glueing

$$\begin{aligned}[0,1] \times \left(D^k \setminus \tfrac{1}{4}D^k\right) \times W &\to W' \\ (t,r,x) &\mapsto H_5(t,\nu(\|r\|),x),\end{aligned}$$

and

$$\begin{aligned}[0,1] \times \tfrac{1}{2}\mathring{D}^k \times W &\to W' \\ (t,r,x) &\mapsto f(r,x).\end{aligned}$$

Second homotopy: To ease the notational burden we again write $f = H(1,\_,\_)$. By compactness we can find finitely many triples

$$\left((Q',u',U',A'),\ (Q,u,U,A),\ (R,v,V)\right),$$

where

(i) (Q',u',U') is a full chart on $\frac{15}{16}D^k$,

(ii) (Q,u,U) is a full chart on W,

(iii) (R,v,V) is a full chart on W', and

(iv) $A' \subseteq U'$ and $A \subseteq U$ are compact subsets, such that

(a) $f(U' \times U) \subseteq V$, and

(b) the subsets $A' \times A$ cover $\frac{7}{8}D^k \times W$.

Enumerating the above triples we can construct homotopies $H_1, H_2, \ldots : [0,1] \times D^k \times W \to W'$ such that $H_{i+1}(0,\_,\_) = H_i(1,\_,\_)$, and the restriction of H_i to $(0,1] \times (A'_1 \times A_1 \cup \cdots \cup A'_i \times A_i)$ is smooth for each i. The homotopy for this second part is then given by composing these homotopies. To obtain H_i we first construct a homotopy H'_i which is only assumed to be smooth when restricted to to $(0,1] \times A'_i \times A_i$; we then choose a smooth map $\gamma : D^k \times W \to [0,1]$ such that $A'_1 \times A_1 \cup \cdots \cup A'_{i-1} \times A_{i-1} = \gamma^{-1}\{0\}$ and set H_i to $(t,r,x) \mapsto H'_i(\gamma(r,x)t, r, x)$.
We have thus reduced the problem to a homotopy as described above for a single triple: We first assume that R has index 0, in which case Q also has index 0. Then we simply choose a homotopy $\breve{H} : [0,1] \times \mathring{U}' \times U \to V$ which is smooth on $(0,1] \times \mathring{U}' \times U \to V$ together with a map $\rho : \mathring{U}' \times U \to [0,1]$ with compact support and such that $\rho|_{A' \times A} \equiv 1$. Then H is given by glueing

$$\begin{array}{rcl} [0,1] \times \mathring{U}' \times U & \to & W' \\ (t,r,x) & \mapsto & \breve{H}(\rho(r,x)t, r, x) \end{array}$$

and

$$\begin{array}{rcl} [0,1] \times (D^k \setminus A') \times (W \setminus A) & \to & W' \\ (t,r,x) & \mapsto & f(r,x). \end{array}$$

If R has index 1, then we may assume w.l.o.g. that Q also has index 1 (Otherwise just replace R with $\mathring{R}$, which reduces it to the case of R having index 0.). Under the identifications $\mathring{U}' = \mathbf{R}^k, U = Q, V = R$, we can again apply Theorem 3.3.1 to produce a homotopy $\breve{H}$, this time assuming Property (II) from the theorem with pairs (B, C) given by $(\partial Q, \partial R)$ and $(\mathring{Q}, \mathring{R})$, so that we obtain H as in the case where the index of R was 0. □

4.2.4 Principal bundles of bordisms

Throughout this subsection W denotes a compact bordism, and $n \in \mathbf{N}$.
The right actions

$$\begin{array}{c} \mathrm{bEmb}^n(W) \circlearrowleft \mathbf{Diff}(W) \\ \underset{\bowtie}{\mathrm{bEmb}}^n(W) \circlearrowleft \underset{\bowtie}{\mathbf{Diff}}(W) \end{array}$$

are again continuous by Theorem 3.2.23. We shall show in a short discussion, which parallels the one in §4.1.1, that both actions are principal. The only new difficulty which arrises is obtaining tubular neighbourhoods which are suitably compatible with the (collared) bordism structure. The proofs are then exactly the same as in §4.1.1 (apart from restricting to subspace topologies), so we omit them.

Definition 4.2.12. Let $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ be a bordism embedding, and $NW \overset{\hat{\iota}}{\hookrightarrow} \mathbf{R}^n \times [0,1]$, a tubular neighbourhood of the underlying manifold W. The space NW inherits the structure of a bordism from W, and if W is equipped with a collaring, then this determines a collaring on NW.
We denote the space of sections of $NW \to W$ which are also maps of bordisms by ΓNW. Likewise, if W is collared, then we denote the space of sections of $NW \to W$ which are also

maps of collared bordisms by ΓNW.
The tubular neighbourhood $NW \overset{\hat{\iota}}{\hookrightarrow} \mathbf{R}^n \times [0,1]$ is called a *bordism tubular neighbourhood of* ι if it is a map of bordisms, and, when W is collared, it is called a *collared tubular neighbourhood of* ι if it is a map of collared bordisms. ⌟

Lemma 4.2.13. *Any bordism embedding $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$, admits a bordism tubular neighbourhood. If W is collared, and ι is a collared embedding, then it admits a collared tubular neighbourhood.*

Proof. If W is collared, and ι is a collared embedding, then we can just use the normal bundle and the tubular neighbourhood induced form the Riemannian metric on $\mathbf{R}^n \times [0,1]$. If ι is a bordism embedding, we will construct a normal bundle, which looks as if it was obtained from a collared embedding, and then again use the Riemannian metric on $\mathbf{R}^n \times [0,1]$ to construct our tubular neighbourhood. We do so by constructing normal bundles on $\mathring{W}$ and on an open neighbourhood of ∂W, and then splicing them together.
As W is compact, we may assume w.l.o.g. that the value $\varepsilon > 0$ in the collaring of W is such that $U := \big(W_0 \times [0,\varepsilon) \cup W_1 \times (-\varepsilon, 1]\big)$ are regular points of the projection map $W \to [0,1]$. Let $S \subseteq T(\mathbf{R}^n \times [0,1])$ be the subbundle consisting of vectors lying in $\mathbf{R}^n \times [0,1] \times \mathbf{R}^n \times \{0\}$ under the identification $T(\mathbf{R}^n \times [0,1]) = \mathbf{R}^n \times [0,1] \times \mathbf{R}^n \times \mathbf{R}$. Denote by $\pi' : \iota^* T(\mathbf{R}^n \times [0,1])|_U \to \iota^* S|_U$ and $\pi : \iota^* T(\mathbf{R}^n \times [0,1])|_{\mathring{W}} \to T\mathring{W}$ the orthogonal projections. Denote by $\tau : TW \to [0,1]$ the composition of the maps $TW \to W$ and $W \to [0,1]$, and let $\lambda : [-\varepsilon, \varepsilon] \to [0,1]$ be a smooth function such that $\lambda|_{[-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon]} \equiv 0$ and $\lambda|_{[-\varepsilon, -\frac{2}{3}\varepsilon] \cup [\frac{2}{3}\varepsilon, \varepsilon]} \equiv 1$, then we obtain the desired normal bundle as the kernel of the map constructed by glueing $\pi'|_U + (\lambda \circ \tau) \cdot (\pi - \pi')|_U$ and $\pi|_{W \setminus (W_0 \times [0, \frac{1}{3}\varepsilon) \cup W_1 \times (-\frac{1}{3}\varepsilon, 1])}$. It is easily checked that fibres of this map all have dimension $n + 1 - d$, so that they form a vector bundle by [Lee09, Prop. 6.28]. □

Notation 4.2.14. Let $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ be a bordism embedding, and

$$NW \overset{\hat{\iota}}{\hookrightarrow} \mathbf{R}^n \times [0,1],$$

a bordism tubular neighbourhood of ι. We denote by:

$\Psi^n(W)$	The set consisting of submanifolds of $\mathbf{R}^n \times [0,1]$ which are the image of some bordism embedding of W.
$\Psi^n(W)\|_{\hat{\iota}}$	The subset of $\Psi^n(W)$ consisting of those submanifolds which are the image of the composition of any section of NW with $\hat{\iota}$.
$\mathrm{bEmb}^n(W)\|_{\hat{\iota}}$	The subspace of $\mathrm{bEmb}^n(W)$ whose image lie in $\Psi^n(W)\|_{\hat{\iota}}$.
$\varphi_{\hat{\iota}}$	The bijective map $\Gamma NW \to \Psi^n(W)\|_{\hat{\iota}}$ given by $s \mapsto \mathrm{Im}_{\hat{\iota} \circ s}$.

⌟

Notation 4.2.15. If W is collared, let $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ be a collared bordism embedding, and

$$NW \overset{\hat{\iota}}{\hookrightarrow} \mathbf{R}^n \times [0,1],$$

a collared tubular neighbourhood of ι. We denote by:

$\underset{\bowtie}{\Psi}^n(W)$	The subset of $\Psi^n(W)$ consisting of submanifolds which are the image of some bordism embedding of W into $\mathbf{R}^n \times [0,1]$.
$\underset{\bowtie}{\Psi}^n(W)\|_{\hat{\iota}}$	The subset of $\underset{\bowtie}{\Psi}^n(W)$ consisting of those submanifolds which are the image of the composition of any section of NW with $\hat{\iota}$.
$\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W)\|_{\hat{\iota}}$	The subspace of $\mathrm{bEmb}^n(W)\|_{\hat{\iota}}$ whose image lie in $\underset{\bowtie}{\Psi}^n(W)\|_{\hat{\iota}}$.
$\varphi_{\hat{\iota}}$	The bijective map $\Gamma NW \to \Psi^n(W)\|_{\hat{\iota}}$ given by $s \mapsto \mathrm{Im}_{\hat{\iota} \circ s}$.

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Lemma 4.2.16. *Let $\zeta : E \to W$ be a vector bundle, then a submanifold W' of E is the image of a section of $E \to M$ iff the map $\zeta|_{W'}$ is a diffeomorphism.* □

Lemma 4.2.17. *Let $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ be a bordism embedding and $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$, a tubular neighbourhood, then $\mathrm{bEmb}^n(W)|_{\hat{\iota}}$ is open in $\mathrm{bEmb}^n(W)$.* □

Lemma 4.2.18. *Assume that W is collared, and let $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ be a collared embedding and $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$, a collared tubular neighbourhood, then $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W)|_{\hat{\iota}}$ is open in $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W)$.* □

Proposition 4.2.19. *For any two bordism embeddings $\iota_1, \iota_2 : W \hookrightarrow \mathbf{R}^n \times [0,1]$ and bordism tubular neighbourhoods $\hat{\iota}_1, \hat{\iota}_2 : N_iW \hookrightarrow \mathbf{R}^n \times [0,1]$ ($i = 1,2$) the sets $\varphi_{\hat{\iota}_1}^{-1}(\Psi^n(W)|_{\hat{\iota}_2})$ and $\varphi_{\hat{\iota}_2}^{-1}(\Psi^n(W)|_{\hat{\iota}_1})$ are open in ΓN_1M and ΓN_2M respectively. Furthermore, the induced bijection $\varphi_{\hat{\iota}_1}^{-1}(\Psi^n(W)|_{\hat{\iota}_2}) \to \varphi_{\hat{\iota}_2}^{-1}(\Psi^n(W)|_{\hat{\iota}_1})$ is a homeomorphism.* □

Proposition 4.2.20. *Assume that W is collared, then for any two collared bordism embeddings $\iota_1, \iota_2 : W \hookrightarrow \mathbf{R}^n \times [0,1]$ and collared tubular neighbourhoods $\hat{\iota}_1, \hat{\iota}_2 : N_iW \hookrightarrow \mathbf{R}^n \times [0,1]$ ($i = 1,2$) the sets $\varphi_{\hat{\iota}_1}^{-1}\left(\underset{\bowtie}{\Psi}^n(W)\right|_{\hat{\iota}_2})$ and $\varphi_{\hat{\iota}_2}^{-1}\left(\underset{\bowtie}{\Psi}^n(W)|_{\hat{\iota}_1}\right)$ are open in ΓN_1M and ΓN_2M respectively. Furthermore, the induced bijection $\varphi_{\hat{\iota}_1}^{-1}\left(\underset{\bowtie}{\Psi}^n(W)\right|_{\hat{\iota}_2}) \to \varphi_{\hat{\iota}_2}^{-1}\left(\underset{\bowtie}{\Psi}^n(W)|_{\hat{\iota}_1}\right)$ is a homeomorphism.* □

We again topologise $\Psi^n(W)$ and $\underset{\bowtie}{\Psi}^n(W)$ (if W is collared) by the charts described in the previous two propositions. We note that $\underset{\bowtie}{\Psi}^n(W)$ is endowed with the subspace topology of $\Psi^n(W)$.

Proposition 4.2.21. *The map $\mathrm{bEmb}^n(W) \to \Psi^n(W)$ is continuous.* □

Proposition 4.2.22. *Assume that W is collared, then the map $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W) \to \underset{\bowtie}{\Psi}^n(W)$ is continuous.* □

Note that for any bordism embedding $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ and bordism tubular neighbourhood $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$ we obtain a continuous section $\Psi^n(W)|_{\hat{\iota}} \to \mathrm{bEmb}^n(W)$, $W' \mapsto \hat{\iota} \circ (\varphi_{\hat{\iota}}^{-1}(W'))$.

Likewise, if W is collared then for any collared embedding $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ and collared tubular neighbourhood $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$ we obtain a continuous section $\underset{\bowtie}{\Psi}^n(W)|_{\hat{\iota}} \to \mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W)$, $W' \mapsto \hat{\iota} \circ (\varphi_{\hat{\iota}}^{-1}(W'))$.

Theorem 4.2.23. *The bundle* $\mathrm{bEmb}^n(W) \to \Psi^n(W)$ *is a principal* $\mathbf{Diff}(W)$*-bundle which trivialises over* $\Psi^n(W)|_{\hat{\iota}}$ *for any bordism embedding* $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ *and bordism tubular neighbourhood* $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$.

Theorem 4.2.24. *If* W *is collared, then the bundle* $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W) \to \underset{\bowtie}{\Psi}^n(W)$ *is a principal* $\underset{\bowtie}{\mathbf{Diff}}(W)$*-bundle which trivialises over* $\underset{\bowtie}{\Psi}^n(W)|_{\hat{\iota}}$ *for any collared embedding* $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ *and collared tubular neighbourhood* $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$.

We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W) \arrow[r, hook] \arrow[d, two heads] & \mathrm{bEmb}^n(W) \arrow[d, two heads] \\
\underset{\bowtie}{\Psi}^n(W) \arrow[r, hook] & \Psi^n(W),
\end{tikzcd}$$

which is compatible with the actions of $\underset{\bowtie}{\mathbf{Diff}}(W)$ and $\mathbf{Diff}(W)$ on $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W)$ and $\mathrm{bEmb}^n(W)$ respectively as well as the inclusion $\underset{\bowtie}{\mathbf{Diff}}(W) \hookrightarrow \mathbf{Diff}(W)$. We have already established that the top horizontal map is a weak homotopy equivalence in Proposition 4.2.11.

Remark 4.2.25. We note that $\underset{\bowtie}{\Psi}^n(W)$ is endowed with the subspace topology of $\Psi^n(W)$. ⌟

Proposition 4.2.26. *The map* $\underset{\bowtie}{\mathbf{Diff}}(W) \hookrightarrow \mathbf{Diff}(W)$ *is a weak homotopy equivalence.*

Proof. Any proper embedding is a diffeomorphism, so the proposition follows from Proposition 4.2.11. □

Corollary 4.2.27. *The map* $\underset{\bowtie}{\Psi}^n(W) \hookrightarrow \Psi^n(W)$ *is a weak homotopy equivalence.* □

Remark 4.2.28. One can show that the spaces $\underset{\bowtie}{\Psi}^n(W)$ and $\Psi^n(W)$ are ANRs, so that the inclusion in the preceding is even a homotopy equivalence. An explicit homotopy inverse is described in [GRW10, Lm. 3.4]. ⌟

4.2.5 Connectivity of principal bundles of bordisms

Throughout this subsection W denotes a compact d-dimensional bordism, and $n \in \mathbf{N}$.

Lemma 4.2.29. *The space* $\mathbf{Man}^{n+d}(W)$ *is contractible.*

Proof. Choosing a smooth retraction of $\mathbf{R}^{n+d}$ down to the origin, we obtain a retraction of $\mathbf{Man}^{n+d}(W)$ down to the subspace $\mathbf{Man}^0(W)$, which itself is seen to be contractible by choosing a fixed real valued function $\ell \in \mathbf{Man}^0(W)$, and considering the homotopy

$$\begin{array}{rcl}
\mathbf{Man}^0(W) \times [0,1] & \to & \mathbf{Man}^0(W) \\
(s,t) & \mapsto & (x \mapsto s(x) + t \cdot (\ell(x) - s(x))).
\end{array}$$

□

Theorem 4.2.30. *The space* $\mathrm{bEmb}^{n+d}(W)$ *is* $(n-d)$*-connected.*

Proof. The proof of the theorem follows the same steps as those of Theorem 4.1.9, except that the arguments take place in spaces of bordism maps rather than arbitrary smooth maps.
So we fix some $0 \leq k \leq n-d$, and choose a continuous map $f : S^{k-1} \to \mathrm{bEmb}^{n+d}(W)$. We can assume w.l.o.g. that f is smooth by Theorem 3.3.1, with the smoothing homotopy satisfying property (II) of the theorem[13], with pairs (B,C) consisting of $\big(W_0, \mathbf{R}^{n+d} \times \{0\}\big)$, $\big(\mathring{W}, \mathbf{R}^{n+d} \times (0,1)\big)$, $\big(W_1, \mathbf{R}^{n+d} \times \{1\}\big)$.
By Lemma 4.2.29 we may extend the map $f : S^{k-1} \to \mathrm{bEmb}^{n+d}(W)$ to $F : D^k \to \mathbf{Man}^{n+d}(W)$, and we may furthermore assume that we have $F(w) = F\left(\frac{w}{\|w\|}\right)$ for all $w \in L := D^k \setminus \frac{1}{2}\mathring{D}^k$, and again that F is smooth.
Denote by $\hat{F} : D^k \times W \to D^k \times \mathbf{R}^{n+d} \times [0,1]$ the map $(r,x) \mapsto (r, F(r,x))$. The subspaces of $\mathbf{Man}(D^k \times W, \mathbf{R}^{n+d+k} \times [0,1])$ and $\mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])$ consisting of maps which restrict, respectively, to $\hat{F}|_{L\times W}$ and $\hat{F}|_{\mathring{L}\times W}$ are homeomorphic, and we denote the restriction of the latter subspace to maps of bordisms by $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$, where we equip $\mathring{D}^k \times W$ with the obvious bordism structure.
Any map in $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ is injective (resp. an immersion) iff it is injective (resp. an immersion) on the subspaces $\mathring{D}^k \times W_0$, $\mathring{D}^k \times W_1$, $\mathring{D}^k \times \mathring{W}$ separately[14]. By the following three claims the subspace consisting of $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ embeddings is residual.

<u>Claim 1:</u> The space $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ has the Baire property.

<u>Claim 2:</u> The subspace of $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ consisting of maps which restrict to injective immersions on $\mathring{D}^k \times \mathring{W}$ is residual.

<u>Claim 3:</u> The subspace of $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ consisting of maps which restrict to injective immersions on $\mathring{D}^k \times W_i$ is open and dense for $i = 0,1$.

The set of submersions $D^k \times W \to D^k$ is open, and so is its preimage U under the map $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}} \to \mathbf{Man}(D^k \times W, D^k)$ by Proposition 3.2.23.
We now choose a map G in the intersection of U and the subspace of $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ consisting of embeddings. By Ehresmann's theorem the map $p_1 \circ G : D^k \times W \to D^k$ is a fibre bundle, and because D^k is contractible, it is in fact trivial. This endows $D^k \times W$ with a new product structure, which we denote by $D^k \times' W$. The projection $D^k \times' W \to D^k$ agrees with $D^k \times W \to D^k$, so that the fibres of $D^k \times' W \to D^k$ are again bordisms. Moreover we can easily verify that $D^k \times' \mathring{W} = D^k \times \mathring{W}$ and $D^k \times' W_i = D^k \times W_i$ for $i = 0,1$. These last two facts imply that we obtain a fibrewise bordism embedding of W into $\mathbf{R}^{n+d} \times [0,1]$ over D^k, and that the composition of the map $D^k \times' W \to D^k \times \mathbf{R}^{n+d} \times [0,1] \to \mathbf{R}^{n+d} \times [0,1]$ corresponds to a smooth map $D^k \to \mathrm{bEmb}^{n+d}(W)$ extending f.

<u>Proof of Claim 1:</u> We will exhibit $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$ as a locally closed subset of the space $\mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])$ (w.r.t. the CO^∞-topology), so that it inherits the Baire

[13]To apply the theorem we view $\mathbf{R}^{n+d} \times [0,1]$ as sitting inside $\mathbf{R}^{n+d} \times \mathbf{R}$.
[14]Here the transversality condition in Definition 4.2.4 is crucial.

property by Theorem 3.2.18.
The subset P of $\mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])$ consisting of smooth maps which restrict to maps $\mathring{D}^k \times W_i \to \mathbf{R}^{n+d+k} \times \{i\}$ for all $i = 0, 1$ is closed by Proposition 3.2.21.
The subset T of $\mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])$ consisting of maps which are transverse to $\mathbf{R}^{n+d+k} \times \{i\}$ for $i = 0, 1$ is open by Corollary 3.4.2.
The subset $\mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])|_{\hat{F}|_{L\times W}}$ of $\mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])$ is closed by Propposition 3.2.24.
We claim that the subset of $P \cap T \cap \mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])|_{\hat{F}|_{L\times W}}$ consisting of maps which send $\mathring{D}^k \times \mathring{W}$ to $\mathbf{R}^{n+d+k} \times (0,1)$ is (relatively) open. Fix such a map $g : \mathring{D}^k \times W \to \mathbf{R}^{n+d+k} \times [0,1]$, and equip $\mathring{D}^k \times W$ with a collaring; by the compactness of D^k we may assume w.l.o.g. that $\varepsilon > 0$ are such that $\mathring{D}^k \times W_0 \times [0,\varepsilon) \cup \mathring{D}^k \times W_1 \times (-\varepsilon, 0]$ are regular points for the map $\pi \circ g : \mathring{D}^k \times W \to \mathbf{R}$. This implies that for fixed $(r,x) \in \mathring{D}^k \times W_0$ the map $\pi \circ g|_{\{(r,x)\}\times[0,\varepsilon)}$ has positive derivative, so that $g(\mathring{D}^k \times W \times (0,\varepsilon)) \subseteq \mathbf{R}^{n+d+k} \times (0,1)$; similarly $g(\mathring{D}^k \times W \times (-\varepsilon, 0)) \subseteq \mathbf{R}^{n+d+k} \times (0,1)$. Again by (shrinking $\varepsilon > 0$ a little and) compactness we can find an open neighbourhood U of g such that $\mathring{D}^k \times W \times [0,\varepsilon) \cup \mathring{D}^k \times W_1 \times (-\varepsilon, 0]$ are regular points of $\pi \circ h$ for all $h \in U$. By compactness there exists a neighbourhood U' of g consisting of maps carrying $\left(\mathring{D}^k \times W\right) \setminus \left(\mathring{D}^k \times W_0 \times \left[0, \frac{1}{2}\varepsilon\right) \cup \mathring{D}^k \times W_1 \times \left(-\frac{1}{2}\varepsilon, 0\right]\right)$ to $\mathbf{R}^{n+d+k} \times (0,1)$. The locally open subset $U \cap U' \cap P \cap T \cap \mathbf{Man}(\mathring{D}^k \times W, \mathbf{R}^{n+d+k} \times [0,1])|_{\hat{F}|_{L\times W}}$ contains g, and is contained in $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$.

<u>Proof of Claim 2:</u> For $s = 1, 2$ we cover $(\mathring{D}^k \times \mathring{W})^{(s)}$ and $(\mathbf{R}^{n+d+k} \times (0,1))^s$ using the s-tuples described in Theorem 3.4.14, and the proof then follows from combining the preceding lemma and Theorem 3.4.15 with Lemma 3.4.4 for $s = 1$ and, with Lemma 3.4.8 for $s = 2$.

<u>Proof of Claim 3:</u> By Theorem 3.2.19.2 and Proposition 3.2.23 the restriction map

$$\xi_i : \mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}} \to \mathbf{Man}(\mathring{D}^k \times W_i, \mathbf{R}^{n+d+k} \times \{i\})_{\hat{F}|_{L\times W_i}}$$

is continuous for $i = 0, 1$. We will show that the map is also open, so that the claim then follows from observing that embeddings are open and dense in $\mathbf{Man}(\mathring{D}^k \times W_i, \mathbf{R}^{n+d+k} \times \{i\})_{\hat{F}|_{L\times W_i}}$ (see Theorem 3.2.19.2), and by combining Lemmas 3.4.4 & 3.4.8 with Theorem 3.4.11.
We begin by fixing a map $g \in \mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$, a full chart $W \supseteq U \xrightarrow{u} Q$, a compact subset $A \subseteq \mathring{D}^k \times Q$, a number $\ell \in \mathbf{N}$, and a value $\varepsilon > 0$, then

$$V := \left\{ \ h \in \mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}} \ \middle| \ \|j^\ell h(r, u^{-1}(x)) - j^\ell g(r, u^{-1}(x))\| < \varepsilon, (r,x) \in A \ \right\},$$

is an open subset of $\mathbf{Man}^{n+d+k}(\mathring{D}^k \times W)_{\hat{F}|_{L\times W}}$, and in fact such open subsets form a subbasis[15].
We want to show that $\xi_i V$ is open.
If $A \cap W_i = \varnothing$, then $\xi_i V = \mathbf{Man}(\mathring{D}^k \times W_i, \mathbf{R}^{n+d+k} \times \{a_0\})_{\hat{F}|_{L\times W_i}}$, so there is nothing to prove.
If $A \cap W_i \neq \varnothing$, then we claim that

$$\xi_i V = \left\{ \ h \in \mathbf{Man}(\mathring{D}^k \times W_i, \mathbf{R}^{n+d+k} \times \{i\})_{\hat{F}|_{L\times W_i}} \ \middle| \ \left\| j^\ell h(r, u^{-1}(x)) - j^\ell g|_{\mathring{D}^k \times W_i}(r, u^{-1}(x)) \right\| < \varepsilon, (r,x) \in A \ \right\}.$$

[15] Here we endow $\mathring{D}^k \times Q \times \mathbf{R}^{n+d+k} \times [0,1] \times J^\ell(d+k, n+d+k+1)$ with the ℓ^1-norm from the ambient vector space.

Indeed, given any h in the set on the right hand side of the above equation, we can extend h to a map $\tilde{h}$ on all of $\mathring{D}^k \times W_i$. First, we define $\overline{h}$ on $\mathring{D}^k \times Q$ by setting $\overline{h}(x_1, \dots, x_{d-1+k}) = h(x_1, \dots, x_{d-1})$. Let ρ be a bump function on $\mathring{D}^k \times W$ with support in a compact subset $B \subseteq \mathring{D}^k \times U$ and constantly equal to 1 on A, then we define $\tilde{h}$ by glueing $g|_{(\mathring{D}^k \times W)\setminus B}$ and

$$\begin{array}{rcl} \mathring{D}^k \times U & \to & \mathbf{R}^{n+d} \times [0,1] \\ (r,x) & \mapsto & g(r,x) + (\rho(r,x) \cdot \overline{h}(r, u(x)), p(x)). \end{array}$$

$\square$

By Proposition 4.2.11 we then obtain the following corollary:

Corollary 4.2.31. *If W is collared, the space $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^{n+d}(W)$ is $(n-d)$-connected.*

The proof of the following two corollaries is then the same as for Corollary 4.1.11.

Corollary 4.2.32. *The space $\mathrm{bEmb}^{\infty}(W)$ is weakly contractible.*

Corollary 4.2.33. *If W is collared, the space $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^{\infty}(W)$ is weakly contractible.* $\square$

Notation 4.2.34. We write

$$\Psi^{\infty}(W) := \varinjlim_{p\to\infty} \Psi^p(W),$$
$$\underset{\bowtie}{\Psi}^{\infty}(W) := \varinjlim_{p\to\infty} \underset{\bowtie}{\Psi}^p(W).$$

⌟

The proofs for the following two propositions are entirely analogous to the proof of Theorem 4.1.14.

Proposition 4.2.35. *The $\mathbf{Diff}(W)$-space $\mathrm{bEmb}^{\infty}(W) \circlearrowleft \mathbf{Diff}(W)$ is a principal $\mathbf{Diff}(W)$-bundle, and its base is $\Psi^{\infty}(W)$.*

Proposition 4.2.36. *The $\underset{\bowtie}{\mathbf{Diff}}(W)$-space $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^{\infty}(W) \circlearrowleft \underset{\bowtie}{\mathbf{Diff}}(W)$ is a principal $\underset{\bowtie}{\mathbf{Diff}}(W)$-bundle, and its base is $\underset{\bowtie}{\Psi}^{\infty}(W)$.*

4.3 The quasi-unital flagged ∞-category of bordisms

4.3.1 Continuity of some key maps between spaces of bordisms and spaces of closed manifolds

Throughout this subsection we fix $d \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{\infty\}$.

Lemma 4.3.1. *Let $(W; W_0, W_1)$ be a compact collared bordism, then the maps*

$$\begin{array}{rrcl} \partial_i : & \underset{\bowtie}{\Psi}^n(W) & \to & \Psi^n(W_i) \\ & W' & \mapsto & W' \cap (\mathbf{R}^n \times \{i\}) \end{array} \tag{6}$$

for $i = 0, 1$ are continuous.

Proof. For $i = 0, 1$ the diagram

$$\begin{tikzcd}
\underset{\bowtie}{\mathrm{Emb}}^n(W) \arrow[r] \arrow[d, two heads] & \mathrm{Emb}(W_i, \mathbf{R}^n) \arrow[d, two heads] \\
\underset{\bowtie}{\Psi}^n(W) \arrow[r] & \Psi^n(W_i).
\end{tikzcd}$$

clearly commutes, and the top horizontal arrow is continuous by Propositions 3.2.23 and 3.2.21 for $n < \infty$, and by considering the induced map between colimits for $n = \infty$, so that the bottom map is continuous by the universal property of quotient topologies. $\square$

Lemma 4.3.2. *Let W, W' be a compact collared bordisms such that $W_1 = W_0'(=: M)$ and denote by $W \cup W'$ the collared bordism obtained by glueing[16] W and W' along M. Then the map*

$$\begin{array}{rrcl}
\mu: & \underset{\bowtie}{\Psi}^n(W) \times_{\Psi^n(M)} \underset{\bowtie}{\Psi}^n(W') \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} & \to & \underset{\bowtie}{\Psi}^n(W \cup W') \\
 & (S, S', t, t') & \mapsto & m_{t/(t+t')}S \;\cup \left(\; m_{t'/(t+t')}S' + \dfrac{t}{t+t'}\, e_{n+1} \right)
\end{array}$$

is continuous.
Here m_s denotes the linear operator on $\mathbf{R}^{n+1}$ scaling the $(n+1)$-th coordinate by t for any $t > 0$, and e_{n+1}, the $(n+1)$-th basis vector of $\mathbf{R}^{n+1}$, where we view $\mathbf{R}^n \times [0,1]$ as sitting inside $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$.

Proof. We first assume $n < \infty$. For this case the proof naturally splits up into two parts:
In the first part we fix $t = t' = 1$, so the map μ reduces to

$$(S, S') \mapsto m_{1/2}S \;\cup\; \left(\; m_{1/2}S' + \frac{1}{2}\, e_{n+1} \right).$$

Fix (S, S') and consider its image $m_{1/2}S \;\cup\; \left(m_{1/2}S' + \frac{1}{2}\, e_{n+1}\right)$. Consider collared embeddings $\iota : W \hookrightarrow \mathbf{R}^n \times [0,1]$ and $\iota' : W' \hookrightarrow \mathbf{R}^n \times [0,1]$ with respective images S and S' together with collared tubular neighbourhoods $\hat{\iota} : NW \hookrightarrow \mathbf{R}^n \times [0,1]$ and $\hat{\iota}' : NW' \hookrightarrow \mathbf{R}^n \times [0,1]$. We then obtain an embedding

$$\begin{array}{rrcl}
\iota \cup \iota': & W \cup W' & \hookrightarrow & \mathbf{R}^n \times [0,1] \\
 & w & \mapsto & \begin{cases} m_{1/2} \circ \iota(x) & \text{if } \; x \in W \\ m_{1/2} \circ \iota(x) + \frac{1}{2}\, e_{n+1} & \text{if } \; x \in W' \end{cases}
\end{array}$$

and a tubular neighbourhood

$$\begin{array}{rrcl}
\hat{\iota} \cup \hat{\iota}': & N(W \cup W') & \hookrightarrow & \mathbf{R}^n \times [0,1] \\
 & v & \mapsto & \begin{cases} m_{1/2} \circ \hat{\iota}(v) & \text{if } \; v \in NW \\ m_{1/2} \circ \hat{\iota}(v) + \frac{1}{2}\, e_{n+1} & \text{if } \; v \in NW', \end{cases}
\end{array}$$

and observe that any basic neighbourhood in $\underset{\bowtie}{\Psi}^n(W \cup W')|_{\hat{\iota}\cup\hat{\iota}'}$ then pulls back to an open set in $\underset{\bowtie}{\Psi}^n(W) \times_{\Psi^n(M)} \underset{\bowtie}{\Psi}^n(W')$.

[16]Here we use the collaring to obtain a canonical smooth structure.

We now come to the second part. For any continuous map $\alpha : [0,1] \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} \to [0,1]$ such that $\alpha_{(t,t')}$ is an endpoint preserving diffeomorphism for each $(t,t') \in \mathbf{R}_{>0} \times \mathbf{R}_{>0}$ we obtain a commutative square (on underlying sets)

$$\begin{tikzcd} \underset{\bowtie}{\mathrm{Emb}}^n(W \cup W') \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} \arrow[r, "(\mathrm{id}_{\mathbf{R}^n} \times \alpha)^*"] \arrow[d, two heads] & \underset{\bowtie}{\mathrm{Emb}}^n(W \cup W') \arrow[d, two heads] \\ \underset{\bowtie}{\Psi}^n(W \cup W') \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} \arrow[r, "R_\alpha"] & \underset{\bowtie}{\Psi}^n(W \cup W'), \end{tikzcd} \tag{7}$$

where α in the top horizontal map is parametrised by the factor $\mathbf{R}_{>0} \times \mathbf{R}_{>0}$. If α is smooth, then $(\mathrm{id}_{\mathbf{R}^n} \times \alpha)^*$ is continuous by Propositions 3.2.13 & 3.2.23, and so is R_α, because the vertical maps in the diagram are quotient maps. If the map

$$\begin{array}{rrcl} \gamma : & [0,1] \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} & \to & [0,1] \\ & (r,t,t') & \mapsto & \begin{cases} 2\frac{t}{t+t'}r & r \in \left[0, \frac{1}{2}\right] \\ 2\frac{t'}{t+t'}r + \frac{1}{2} & r \in \left[\frac{1}{2}, 1\right], \end{cases} \end{array} \tag{8}$$

were smooth, we could simply compose R_γ with $(\mu_{(1,1)}, \mathrm{id}_{\mathbf{R}_{>0} \times \mathbf{R}_{>0}})$, and this would conclude the proof for $n < \infty$. Instead, we fix $(S,S') \in \underset{\bowtie}{\Psi}^n(W \cup W')|_{\iota \cup \iota'}$ and consider two smooth maps $\alpha, \beta : [0,1] \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} \to [0,1]$ with the properties described in the discussion of (7), and such that α and γ agree on their restriction to $\left(\left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2} + \varepsilon, 1\right]\right) \times \mathbf{R}_{>0} \times \mathbf{R}_{>0}$, and β and γ agree on their restriction to $\left(\left[0, \frac{1}{2} - \varepsilon\right] \cup \left[\frac{1}{2}, 1\right]\right) \times \mathbf{R}_{>0} \times \mathbf{R}_{>0}$, where $\varepsilon > 0$ is such that $\mu_{(1,1)}(S,S') \cap \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right] \times \mathbf{R}^n = \left(\mu_{(1,1)}(S,S') \cap \left\{\frac{1}{2}\right\} \times \mathbf{R}^n\right) \times \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$. Then $R_\alpha(\mu_{(1,1)}(S,S')) = R_\beta(\mu_{(1,1)}(S,S')) = R_\gamma(\mu_{(1,1)}(S,S'))$. For any neighbourhood V of $\mu_{(1,1)}(S,S')$ we can then find neighbourhoods U_α of (S,S') and U_β of (S,S') such that $R_\alpha(U_\alpha) \subseteq V$ and $R_\beta(U_\beta) \subseteq V$, and therefore $R_\gamma(U_\alpha \cap U_\beta) \subseteq V$.
The case $n = \infty$ follows from the observation that the map

$$\varinjlim_m \left(\underset{\bowtie}{\Psi}^m(W) \times_{\Psi^m(M)} \underset{\bowtie}{\Psi}^m(W')\right) \to \underset{\bowtie}{\Psi}^\infty(W) \times_{\Psi^\infty(M)} \underset{\bowtie}{\Psi}^\infty(W')$$

is a homeomorphism: The map clearly induces a bijection on underlying sets. For any two embeddings $\iota : W \hookrightarrow \mathbf{R}^m \times [0,1]$ and $\iota' : W' \hookrightarrow \mathbf{R}^m \times [0,1]$ we can consider the subspace of $\Gamma(NW \times \mathbf{R}^\infty \to W) \times \Gamma(NW' \times \mathbf{R}^\infty \to W')$ consisting of pairs of sections which agree on M is an open subspace of both $\varinjlim_m \left(\underset{\bowtie}{\Psi}^m(W) \times_{\Psi^m(M)} \underset{\bowtie}{\Psi}^m(W')\right)$ and $\underset{\bowtie}{\Psi}^\infty(W) \times_{\Psi^\infty(M)} \underset{\bowtie}{\Psi}^\infty(W')$; as both spaces can be covered by such subspaces, we are done. $\square$

Lemma 4.3.3. *Let $(W; W_0, W_1)$ be a compact collared bordism, then the map*

$$\underset{\bowtie}{\Psi}^n(W) \xrightarrow{(\partial_0, \partial_1)} \Psi^n(W_0) \times \Psi^n(W_1)$$

is a fibration.

Proof. We first assume $n < \infty$. Consider the subdiagram of

$$\begin{tikzcd}
{[0,1]^k} \arrow[r, dashed] \arrow[d, hook] \arrow[rr, bend left] & \underset{\bowtie}{\mathrm{Emb}}^n(W) \arrow[r, two heads] \arrow[d] & \underset{\bowtie}{\Psi}^n(W) \arrow[d, "{(\partial_0,\partial_1)}"] \\
{[0,1]^{k+1}} \arrow[r, dashed] \arrow[ur, dotted] \arrow[rr, bend right] & \mathrm{Emb}(W_0, \mathbf{R}^n) \times \mathrm{Emb}(W_1, \mathbf{R}^n) \arrow[r, two heads] & \Psi^n(W_0) \times \Psi^n(W_1)
\end{tikzcd}$$

given by the solid arrows. As the top right horizontal arrow is a surjective fibration we can lift the top curved arrow to the top left, dashed arrow. We obtain the bottom left dashed arrow as the solution to the lifting problem against the bottom right horizontal arrow. It is thus enough to show that the central vertical arrow is a fibration, in order to obtain the dotted arrow.
Using the isotopy extension theorem (in the sense of [Pal60]) we can in fact show that the map admits a local trivialisation. Fix two embeddings $\iota_i : W_k \hookrightarrow \mathbf{R}^n$ $(i = 0, 1)$. By [Pal60] there exist open neighbourhoods $\iota_i \in U_i \subseteq \mathrm{Emb}(W_i, \mathbf{R}^n)$ and smooth maps

$$\chi_i : U_i \to \mathbf{Diff}_c^\circ(\mathbf{R}^n)$$

such that $\kappa = \chi_\kappa \circ \iota_i$ for all $\kappa \in U_i$ (here $\mathbf{Diff}_c^\circ(\mathbf{R}^n)$ denotes the space of diffeomorphisms of $\mathbf{R}^n$ with compact support which are homotopic to the identity). As $\mathbf{Diff}_c^\circ(\mathbf{R}^n)$ is locally contractible, the neighbourhoods U_i and maps χ_i can be chosen such that these factor through contractible neighbourhoods V_i of the identity in $\mathbf{Diff}_c^\circ(\mathbf{R}^n)$. Choose homotopies

$$\eta_i : [0, 1] \times V_i \to V_i$$

between the identity map on V_i and the map, which maps constantly to the identity in $\mathbf{Diff}_c^\circ(\mathbf{R}^n)$. Finally, choose a map

$$\rho : [0, 1] \to [0, 1]$$

which is constantly equal to zero in some neighbourhoods of 0 and 1 and constantly equal to 1 in some neighbourhood of $\left[\frac{1}{4}, \frac{3}{4}\right]$. The continuous map

$$\begin{array}{ccccl}
U_0 \times U_1 & \to & \mathbf{Diff}(\mathbf{R}^n \times [0,1]) & & \\
(\kappa_0, \kappa_1) & \mapsto & (r, s) & \mapsto & \begin{cases} ((\eta_{\rho(s)} \circ \chi_0(\kappa_0))(r), s) & s \leq \frac{1}{4} \\ (r, s) & \frac{1}{4} \leq s \leq \frac{3}{4} \\ ((\eta_{\rho(s)} \circ \chi_1(\kappa_1))(r), s) & \frac{3}{4} \leq s \end{cases}
\end{array}$$

(viewing $\mathbf{R}^n \times [0, 1]$ as a collared bordism) induces the desired local trivialisation.

The case $n = \infty$ follows from noting that any square

$$\begin{tikzcd}
{[0,1]^k} \arrow[r] \arrow[d, hook] & \underset{\bowtie}{\Psi}^\infty(W) \arrow[d, "{(\partial_0,\partial_1)}"] \\
{[0,1]^{k+1}} \arrow[r] & \Psi^\infty(W_0) \times \Psi^\infty(W_1)
\end{tikzcd}$$

factors through

$$\begin{tikzcd}
{[0,1]^k} \arrow[r] \arrow[d, hook] & \underset{\bowtie}{\Psi}^m(W) \arrow[d, "{(\partial_0,\partial_1)}"] \\
{[0,1]^{k+1}} \arrow[r] & \Psi^m(W_0) \times \Psi^m(W_1)
\end{tikzcd}$$

for some $m \in \mathbf{N}$ by [DI04, Lm. A.3]. $\square$

4.3.2 The topological semi-category of ordinary bordisms

We are now able to define the topological semi-category of bordisms. Throughout this subsection $d \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{\infty\}$.

Definition 4.3.4. The *topological semi-category of ordinary d-dimensional bordisms embedded in $\mathbf{R}^n$* is the topological semi-category $\mathbf{Bord}d, n$, with objects

$$(\mathbf{Bord}d, n)_0 = \left\{ \; M \subseteq \mathbf{R}^n \;\middle|\; \text{M is a closed } (d-1)\text{-dimensional submanifold} \; \right\} \cong \coprod_{[M]} \Psi^n(M),$$

and morphisms

$$\begin{aligned}
(\mathbf{Bord}d, n)_1 &= \left\{ \; (W, t) \;\middle|\; W \subseteq \mathbf{R}^n \times [0, t] \text{ is a compact collar embedded } d\text{-dimensional bordism, } t > 0 \; \right\} \\
&\cong \coprod_{[W]} \underset{\bowtie}{\Psi}^n(W) \times \mathbf{R}_{>0},
\end{aligned}$$

both topologised via the indicated canonical bijections, together with source and target maps

$$\begin{array}{rrcl}
d_i : & (\mathbf{Bord}d, n)_1 & \to & (\mathbf{Bord}d, n)_0 \quad (i = 0, 1) \\
& (W, t) & \mapsto & W_{i+1 \mod 2}
\end{array}$$

and multiplication map

$$\begin{array}{rrcl}
\mu : & (\mathbf{Bord}d, n)_1 \times_{(\mathbf{Bord}d,n)_0} (\mathbf{Bord}d, n)_1 & \to & (\mathbf{Bord}d, n)_1 \\
& \big((W, t), (W', t')\big) & \mapsto & \Big(m_{t/(t+t')}W \;\cup\; \Big(m_{t'/(t+t')}W' + \frac{t}{t+t'}\, e_{n+1}\Big),\; t + t'\Big),
\end{array}$$

where m_s $(s > 0)$ and e_{n+1} are defined as in Lemma 4.3.2. ⌟

By Lemma 4.3.1 the target and source maps in Definition 4.3.4 are continuous, and by Lemma 4.3.2 the composition map is continuous. By Lemma 4.3.3 and Example 2.2.6 the nerve of $\mathbf{Bord}d, n$ is a flagged ∞-semi-category.

Remark 4.3.5. As discussed in the introduction, our model is very similar to the topological semi-category obtained by "removing the units" in the topological category sketched in [GTMW09, §2.1]. Such a topological semi-category would have a space of objects consisting of pairs $(M;t)$, where $M \subseteq \mathbf{R}^n$ is an embedded closed $(d-1)$-manifold and $t \in \mathbf{R}$, and a space of morphisms consisting of triples $(W;t_0,t_1)$, where $W \subseteq \mathbf{R}^n \times [0,1]$ is a collared embedded bordism and $t_0 < t_1$. The functor from this topological semi-category to ours, given by $(M,t) \mapsto M$ and $(W;t_0,t_1) \mapsto W, t_1 - t_0)$ is a Dwyer-Kan equivalence. Our topological semi-category has the advantage that $(\mathbf{Bord}d,n)_1 \to (\mathbf{Bord}d,n)_0 \times (\mathbf{Bord}d,n)_0$ is a fibration, which is not case for the topological semi-category from [GTMW09, §2.1]; this can be seen by noting that the space of maps from (M,t) to $(M,'t)$ is empty if $t > t'$. ⌟

Definition 4.3.6. The *flagged ∞-semi-category of ordinary d-dimensional bordisms embedded in* $\mathbf{R}^n$ is the semi-simplicial space given by the nerve of $\mathbf{Bord}d,n$, which we again denote by $\mathbf{Bord}d,n$. ⌟

Notation 4.3.7. Let $d \in \mathbf{N}$. When $n = \infty$ we write $\mathbf{Bord}d$ rather than $\mathbf{Bord}d,\infty$. ⌟

Proposition 4.3.8. *The flagged ∞-semi-category* $\mathbf{Bord}d,n$ *is quasi-unital.*

Proof. We claim that for any $M \in (\mathbf{Bord}d,n)_0$ the bordism $(M \times [0,1], 1)$ is a quasi-unit. In the discussion at the end of §2.2.1 we saw that ordinary composition for nice topological semi-categories (of which $\mathbf{Bord}d,n$ is an example) induces composition on their homotopy categories. We will now show that the map $(M \times [0,1],1)_* : \mathbf{Bord}d,n(M,N) \to \mathbf{Bord}d,n(M,N)$ is homotopic to the identity for any $N \in (\mathbf{Bord}d,n)_0$.

For any bordism W with $W_0 \cong M$ and $W_1 \cong N$ we denote by ${}_M\underset{\bowtie}{\Psi}^n(W)_N$ the subset of $\underset{\bowtie}{\Psi}^n(W)$ consisting of bordisms W' such that $W'_0 = M$ and $W'_1 = N$. The restriction of the desired homotopy to the clopen subset ${}_M\underset{\bowtie}{\Psi}^n(W)_N \times \mathbf{R}_{>0} \subseteq \mathbf{Bord}d,n(M,N)$ is given by

$$\begin{array}{rcl} [0,1] \times {}_M\underset{\bowtie}{\Psi}^n(W)_N \times \mathbf{R}_{>0} & \to & {}_M\underset{\bowtie}{\Psi}^n(W)_N \times \mathbf{R}_{>0} \\ (s,W',t) & \mapsto & \Big(m_{s/(s+t)}\Big(M \times [0,1]\Big) \ \cup \ \Big(m_{t/(s+t)}W' + \tfrac{s}{s+t}\, e_{n+1}\Big),\ s+t \Big). \end{array}$$

We first assume that $n < \infty$. Denote by $(M \times [-1,0]) \cup W$ the bordisms obtained by glueing $(M \times [-1,0])$ and W along $M \times \{0\}$ and $M = W_0$. Furthermore, denote by $\underset{\bowtie}{\Upsilon}^n((M \times [-1,0]) \cup W)$ the set consisting of submanifolds of $\mathbf{R}^n \times [-1,1]$ which are the image of some collared embedding of $(M \times [-1,0]) \cup W$, topologised as the quotient of $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}((M \times [-1,0]) \cup W,\ \mathbf{R}^n \times [-1,1])$ by the group action of $\underset{\bowtie}{\mathbf{Diff}}((M \times [-1,0]) \cup W)$ given by precomposition. By (an obvious modification of) Lemma 4.3.2 the map

$$\begin{array}{rcl} [0,1] \times \underset{\bowtie}{\Psi}^n(M \times [0,1]) \times_M \underset{\bowtie}{\Psi}^n(W) \times \mathbf{R}_{>0} & \to & \underset{\bowtie}{\Upsilon}^n(M \times [-1,0] \cup W) \\ (s,S,S',t) & \mapsto & \Big(m_{1+s/(s+t)}S - e_{n+1}\Big) \ \cup \ \Big(m_{t/(s+t)}S' + \tfrac{s}{s+t}\, e_{n+1}\Big) \end{array} \tag{9}$$

is continuous. The subspace

$$[0,1] \times \{M \times [0,1]\} \times_M \Big({}_M\underset{\bowtie}{\Psi}^n(W)_N \Big) \times \mathbf{R}_{>0} \subseteq [0,1] \times \underset{\bowtie}{\Psi}^n(M \times [0,1]) \times_M \underset{\bowtie}{\Psi}^n(W) \times \mathbf{R}_{>0}$$

is canonically homeomorphic to $[0,1] \times {}_M\underset{\bowtie}{\Psi}^n(W)_N \times \mathbf{R}_{>0}$. The restriction of (9) to this space factors through the subspace of $\underset{\bowtie}{\Upsilon}^n(M \times [-1,0] \cup W) \times \mathbf{R}_{>0}$ consisting of pairs (S,t) such that $S \cap (\mathbf{R}^n \times [-1,0]) = M \times [-1,0]$; this latter space is homeomorphic to ${}_M\underset{\bowtie}{\Psi}^n(W)_N$, so we are done with the case $n < \infty$; the case $n = \infty$ is obtained by considering the continuous map induced from colimits. $\square$

4.3.3 The flagged structure on bordisms.

Throughout this subsection $d \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{\infty\}$. We concluded the last subsection by showing that the map $s_0 : (\mathbf{Bord}d, n)_0 \to (\mathbf{Bord}d, n)_1, M \mapsto (M \times [0,1], 1)$ produces a quasi-unit for every $M \in (\mathbf{Bord}d, n)_0$. In the beginning of §2.2.2, for any flagged ∞-category X and any $x, y \in X_0$ we constructed a map $\mathrm{Path}_X(x,y) \to \mathrm{Map}_X(x,y)$, exhibiting the flagging of X. Even though $\mathbf{Bord}d, n$ is not a simplicial topological space, the map s_0 described above is good enough to allow us to exhibit the correct flagging in this sense.
We now construct a more geometrically meaningful variant of the map from path spaces to mapping spaces, which also makes apparent how one may use the h-cobordism theorem (and its variants) to conclude that $\mathbf{Bord}d$ is not complete for many d. The main ingredient is the *graph construction*: Let $\alpha : [0,1] \to (\mathbf{Bord}d, n)_0$ be a smooth map (see Remark 4.1.8), then the graph of α, denoted Γ_α, is the subset of $\mathbf{R}^n \times [0,1]$ given by $\left\{ (t,x) \in \mathbf{R}^n \times [0,1] \;\middle|\; x \in \alpha(t) \right\}$.

Proposition 4.3.9. *The subset $\Gamma_\alpha \subseteq \mathbf{R}^n \times [0,1]$ is a smooth submanifold.* $\square$

Let $X, Y \in (\mathbf{Bord}d, n)_0$, and denote by $\mathrm{Path}_\infty(X,Y)$ the subspace of $\mathrm{Path}(X,Y)$ consisting of smooth paths which are constant near the beginning and the end.

Proposition 4.3.10. *The inclusion $\mathrm{Path}_\infty(X,Y) \hookrightarrow \mathrm{Path}(X,Y)$ is a weak homotopy equivalence.*

Sketch of proof. Follows from the analogue of Theorem 3.3.1 given in [GRW10, Lm. 2.17]. $\square$

Our new map from paths to invertible bordisms is given by $\mathrm{Path}_\infty(X,Y) \to \mathbf{Bord}d, n(X,Y)$, $\alpha \mapsto (\Gamma_\alpha, 1)$. The graph construction can also be used to obtain a homotopy inverse to the map $\mathrm{Hom}(X,Y) \hookrightarrow \mathrm{Map}(X,Y)$, given by $(\alpha, W, \beta) \mapsto \Gamma_\beta \circ (W,t) \circ \Gamma_\alpha$, and we may conclude that the two maps from paths to maps are equivalent.

4.3.4 A remark on units

Throughout this subsection $d \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{\infty\}$. A central theme of this article has been that we cannot equip the semi-simplicial topological space $\mathbf{Bord}d, n$ with degeneracy maps making it into a simplicial topological space. In this subsection we hope to convince the reader that the problem lies with simplicial spaces rather than with bordisms. The issue is that our notion of flagged ∞-categories is not completely homotopical: Units are still required to be strict, which, incidentally, is true for essentially all models for (flagged) ∞-categories currently under active consideration (see [Ber18]). We proceed to sketch two variant models for $\mathbf{Bord}d, n$, where units are given explicitly.
Our first sketch is a very natural adaption of the definition we have already given: We denote by $\mathfrak{N}$ the homotopy coherent nerve functor from simplicially enriched categories to simplicial sets, and by $\mathfrak{C}$ its left adjoint. Writing $\mathfrak{C}\Delta := \mathfrak{C}\mathfrak{N}\Delta = \mathfrak{C}N\Delta$ it should be possible to model $\mathbf{Bord}d, n$ as a functor $\mathfrak{C}\Delta^{\mathrm{op}} \to \mathbf{Top}$, i.e. a homotopy coherent simplicial diagram in $\mathbf{Top}$, or equivalently

a diagram of quasi-categories $\Delta^{\mathrm{op}} \to \mathfrak{N}\mathbf{Top}$ (see [Rie14, §16.3]). The objects of $\mathfrak{C}\Delta$ are the same as Δ. The constituent objects of $\mathbf{Bord}d, n^{\mathfrak{C}}$ would then be given by

$$
\begin{aligned}
(\mathbf{Bord}d, n^{\mathfrak{C}})_0 &:= \left\{ M \subseteq \mathbf{R}^n \;\middle|\; \text{M is a closed } (d-1)\text{-dimensional submanifold} \right\} = (\mathbf{Bord}d, n)_0, \\
(\mathbf{Bord}d, n^{\mathfrak{C}})_1 &:= \left\{ W \subseteq \mathbf{R}^n \times [0,1] \;\middle|\; \text{W is a compact collar embedded } d\text{-dimensional bordism} \right\} = \coprod_{[W]} \Psi^n_{\bowtie}(W) \\
(\mathbf{Bord}d, n^{\mathfrak{C}})_{k_1+k_2} &:= (\mathbf{Bord}d, n^{\mathfrak{C}})_{k_1} \times_{(\mathbf{Bord}d,n)_0} (\mathbf{Bord}d, n^{\mathfrak{C}})_{k_2}.
\end{aligned}
$$

The face maps are then as for $\mathbf{Bord}d, n$, except that we no longer keep track of the "length" of the bordisms. The degeneracy maps are given by inserting cylinders in the appropriate way. This defines a functor $\mathrm{Ho}(\mathfrak{C}\Delta)^{\mathrm{op}} = \Delta^{\mathrm{op}} \to \mathrm{Ho}(\mathbf{Top})$, which can be shown to lift to a homotopy coherent diagram in $\mathbf{Top}$.

In our second sketch we combine ideas of Bénabou, May, and Horel. Given a category with finite limits $\mathcal{T}$, and an object C_0 in $\mathcal{T}$, there exists a monoidal structure on the category of graphs over C_0, $\mathcal{T}_{/C_0\times C_0}$, given by $(C_1 \rightrightarrows C_0, C_1' \rightrightarrows C_0) \mapsto C_1\, {}_t\!\times_{s'} C_1'$. It is an observation of Bénabou (for $\mathcal{T} = \mathbf{Set}$; see [Bén67]) that a category object in $\mathcal{T}$ with underlying graph $C_1 \rightrightarrows C_0$ is then a monoid in $\mathcal{T}_{/C_0\times C_0}$. We see that if $\mathcal{T}$ is moreover Cartesian closed, then $\mathcal{T}_{/C_0\times C_0}$ is enriched and tensored over $\mathcal{T}$.

For $\mathcal{T} = \mathbf{Top}$ we can adapt May's construction of operadic categories in [May01] to our setting, and define $\mathbf{Bord}d, n^{E_1}$ as an algebra over the E_1-operad. We then set $(\mathbf{Bord}d, n^{E_1})_i = (\mathbf{Bord}d, n^{\mathfrak{C}})_i$ $(i = 0, 1)$; the E_1-action is given by obvious scaling and composition operations. One can construct a relative category of E_1-categories, and by adapting ideas in [Hor15] it should be possible to show that the inclusion of the relative category of topological categories induces an equivalence of homotopy theories.

We conclude with the following observation, which may help clarify the relationship of $\mathbf{Bord}d, n^{\mathfrak{C}}$ and $\mathbf{Bord}d, n^{E_1}$ to $\mathbf{Bord}d, n$: Let X be a pointed topological space, then we can model the homotopy coherent group structure on ΩX as a functor $\mathfrak{C}\Delta^{\mathrm{op}} \to \mathbf{Top}$ such that $[0] \mapsto *$, $[1] \mapsto \Omega X$, $[2] \mapsto \Omega X \times \Omega X$, $\ldots$, or as an algebra over the E_1-operad (introduced precisely for this purpose in [BV68]). A different approach is to form the space of Moore loops given by $\Omega X \times [0, \infty)$, with composition given by sending $((\gamma_1, t_1), (\gamma_2, t_2))$ to $(\gamma_2 * \gamma_1, t_1 + t_2)$, where $\gamma_2 * \gamma_1$ has been parametrised as in the composition in $\mathbf{Bord}d, n$. We see that $\mathbf{Bord}d, n$ is to $\mathbf{Bord}d, n^{\mathfrak{C}}$ and $\mathbf{Bord}d, n^{E_1}$, as Moore loops are to the two homotopy coherent multiplications on ΩX described above.

5 Application: Riemannian bordisms

As discussed in the introduction, an advantage of our model over Galatius' model from [Gal11, §6] is that we can consider bordisms equipped with extra structure which are not tangential structures. This means for instance that we can distinguish Riemannian bordisms from ordinary bordisms, and also that we can consider structures satisfying integrality conditions (see below). The key observation is that $(\mathbf{Bord}d, n)_0$ and $(\mathbf{Bord}d, n)_1$ $(d \in \mathbf{N}_{\geq 1},\, n \in \mathbf{N} \cup \{\infty\})$ are disjoint unions of base spaces of principal bundles, so that the extra structure can be added using the associated bundle construction.

We sketch this idea on the level of $(\mathbf{Bord}d, n)_0$: Let M be a closed $(d-1)$-manifold, and let $(-\varepsilon, \varepsilon)$ represent some notion of infinitesimal interval or germ of an interval. Denote by

Struct : $(\mathbf{Man}^{\cong})^{\mathrm{op}} \to \mathbf{Top}$ a continuous functor, for example the functor associating to any manifold its space of

- Riemannian (the metric can vary continuously),
- Euclidean (the metric is fixed)[17],
- conformal,
- symplectic,
- complex,
- ...

structures. For any closed $(d-1)$-manifold M, the connected component of $(\mathbf{Bord}d, n^{\mathrm{Struct}})_0$ corresponding to M can then be written as $\mathrm{Emb}(M, \mathbf{R}^n) \times_{\mathbf{Diff}\, M} \mathrm{Struct}(M \times (-\varepsilon, \varepsilon))$.
The following lemma is essential for comparing different structures on bordisms, as we will see in §5.3.1.

Lemma 5.0.1. *Let $P \times G \rightrightarrows P \to B$ be a principal G-bundle of weakly Hausdorff k-spaces, and $X \xrightarrow{\sim} Y$ a weak equivalence of weakly Hausdorff k-spaces with a left G-action, then the map $P \times_G X \to P \times_G Y$ is a weak equivalence.*

Proof. We denote by G' the space obtained by whiskering G. The diagram $P \times G' \rightrightarrows P \to B$ is again a quotient, and we obtain a commutative diagram

$$\begin{tikzcd} P \times_{G'} X \arrow[r] \arrow[d, "\cong"] & P \times_{G'} Y \arrow[d, "\cong"] \\ P \times_G X \arrow[r] & P \times_G Y. \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} P \arrow[d] & B(P, G', G') \arrow[l] \arrow[d] \\ B & B(P, G', *). \arrow[l] \end{tikzcd}$$

The top horizontal arrow is a weak equivalence by [May75, Prop. 7.5]. The right vertical arrow is a quasi-fibration by [May75, Thm. 7.6], and moreover a quotient map by the monoid action of G' by [Shu09, Lm. 12.4]. Comparing long exact sequences of homotopy groups we see that the bottom horizontal arrow induces an equivalence on all homotopy groups of degree ≥ 1 and a surjection on connected components. To see that we obtain a bijection on connected components, we note that π_0 sends grouplike h-spaces to groups, and quotients by grouplike h-space actions to quotients by group actions.
For any left G'-space Z we have a canonical homeomorphism $B(P, G', G') \times_{G'} Z \cong B(P, G', Z)$ by [Shu09, Lm. 12.4], so that $B(P, G', G') \times_{G'} Z \to B(P, G', *)$ is a quasi-fibration, and we see

[17]Note that in this case we only obtain a flagged ∞-semi-category.

that the top horizontal arrow in

$$\begin{tikzcd} P \times_{G'} Z \arrow[d] & B(P,G',G') \times_{G'} Z \arrow[l] \arrow[d] \\ B & B(P,G',*). \arrow[l] \end{tikzcd}$$

is a weak equivalence by comparing long exact sequences of homotopy groups. Moreover, $B_*(P, G', Z)$ is proper, and thus good, so that we obtain a commuting square

$$\begin{tikzcd} P \times_{G'} X \arrow[d] & B(P,G',X) \arrow[l, "\sim"'] \arrow[d] \\ P \times_{G'} Y & B(P,G',Y), \arrow[l, "\sim"'] \end{tikzcd}$$

where the right vertical arrow is a weak equivalence by [Seg74, Prop. A.1], and thus also the left vertical arrow by the 2-out-of-3 property. ☐

Remark 5.0.2. Using the philosophy outlined in §2.1 regarding the interplay between model categories and their quasi-categorical localisations it is possible to show that the above proposition is true without the condition that all spaces be weakly Hausdorff k-spaces, by exploiting that $L\mathbf{Top}$ is an ∞-topos. ⌟

5.1 Germs of Riemannian metrics

Throughout this subsection we fix $d \in \mathbf{N}$, $n \in \mathbf{N} \cup \{\infty\}$, and a closed $(d-1)$-manifold M. We denote by Riem : $(\mathbf{Man}^{\cong})^{\mathrm{op}} \to \mathbf{Top}$ the functor which associates to any smooth manifold X its space of Riemannian metrics, topologised as a subspace of $\Gamma(T^*X \otimes T^*X \to X)$ equipped with the CO^∞-topology. We then define $\underset{\bowtie}{\mathrm{Riem}}(M) := \varinjlim_{\varepsilon \to 0} \mathrm{Riem}(M \times (-\varepsilon, \varepsilon))$, and denote by

$$\Psi^n_{\mathrm{Riem}}(M) := \left\{ \ (M', g) \ \middle| \ M' \in \Psi^n(M), \ g \in \underset{\bowtie}{\mathrm{Riem}}(M') \ \right\}$$

the space topologised via the natural bijection

$$\Psi^n_{\mathrm{Riem}}(M) \cong \mathrm{Emb}(M, \mathbf{R}^n) \times_{\mathrm{Aut}(M)} \underset{\bowtie}{\mathrm{Riem}}(M). \tag{10}$$

Any embedding $\iota : M \hookrightarrow \mathbf{R}^n$ together with a tubular neighbourhood $NM \hookrightarrow \mathbf{R}^n$ induces a local trivialisation of $\Psi^n_{\mathrm{Riem}}(M) \to \Psi^n(M)$, so that we obtain a basis for $\Psi^n_{\mathrm{Riem}}(M)$ given by subspaces of the form $\Psi^n(M)|_{\hat{\iota}} \times \underset{\bowtie}{\mathrm{Riem}}(M)$. To build some intuition for the topology on $\Psi^n_{\mathrm{Riem}}(M)$, we observe that a continuous path in $\Psi^n_{\mathrm{Riem}}(M)$ is locally just a continuous map $[a, b] \to \Psi^n(M)|_{\hat{\iota}} \times \underset{\bowtie}{\mathrm{Riem}}(M)$; so locally we are continuously moving around M in $\mathbf{R}^n$ while independently varying its metric.

Proposition 5.1.1. *The space* $\underset{\bowtie}{\mathrm{Riem}}(M)$ *is convex, and thus contractible.*

Proof. If we pick two points in $\underset{\bowtie}{\mathrm{Riem}}(M)$, then these may be represented by two metrics g_1 and g_2 on $M \times (-\varepsilon_1, \varepsilon_1)$ and $M \times (-\varepsilon_2, \varepsilon_2)$ respectively for some $\varepsilon_1, \varepsilon_2 > 0$. We may assume w.l.o.g. that $\varepsilon_1 \leq \varepsilon_2$, so that $[g_2] = [g_2|_{M \times (-\varepsilon_1, \varepsilon_1)}]$, and we may then further assume that $\varepsilon := \varepsilon_1 = \varepsilon_2$.

The path $[0,1] \to \underset{\bowtie}{\mathrm{Riem}}(M),\ t \mapsto [g_1 + t \cdot (g_2 - g_1)]$ exists because $\mathrm{Riem}(M \times (-\varepsilon, \varepsilon))$ is convex, and is equal to $t \mapsto [g_1] + t \cdot ([g_2] - [g_1])$, which concludes the first part of the proposition. Pick any point $x_0 \in \underset{\bowtie}{\mathrm{Riem}}(M)$, then $[0,1] \times \underset{\bowtie}{\mathrm{Riem}}(M) \to \underset{\bowtie}{\mathrm{Riem}}(M)$, $(t, x) \mapsto x + t \cdot (x_0 - x)$ contracts $\underset{\bowtie}{\mathrm{Riem}}(M)$ down to $\{x_0\}$, which exhibits the contractibility of $\underset{\bowtie}{\mathrm{Riem}}(M)$. □

Theorem 5.1.2. *The map* $\mathrm{Emb}(M, \mathbf{R}^n) \times_{\mathbf{Diff}(M)} \underset{\bowtie}{\mathrm{Riem}}(M) \to \Psi^n(M)$ *is a weak equivalence for all* $n \in \mathbf{N} \cup \{\infty\}$.

Proof. The theorem follows immediately from combining Lemma 5.0.1 with Theorem 4.1.7 for $n \in \mathbf{N}$, and with Theorem 4.1.14 for $n = \infty$. □

We now fix a compact d-bordism $(W; W_0, W_1)$. Denote the inclusions $[0, \varepsilon_0) \hookrightarrow (-\varepsilon_0, \varepsilon_0)$ and $(-\varepsilon_1, 0] \hookrightarrow (-\varepsilon_1, \varepsilon_1)$ by ι_0 and ι_1 respectively, and choose some $0 < \delta < \min(\varepsilon_0, \varepsilon_1)$. We write $W_\delta := (-\delta, \delta)\, {}_{\iota_0}{\times}_{c_0} W\, {}_{c_1}{\times}_{\iota_1} (-\delta, \delta)$. Similarly as for M we define $\underset{\bowtie}{\mathrm{Riem}}(W) := \varinjlim_{\delta \to 0} \mathrm{Riem}(W_\delta)$, and denote by

$$\Psi^n_{\mathrm{Riem}}(W) := \left\{ \ (W', g) \ \middle| \ W' \in \underset{\bowtie}{\Psi}^n(W) \ \ g \in \underset{\bowtie}{\mathrm{Riem}}(W') \ \right\}$$

the space topologised under the canonical bijection

$$\Psi^n_{\mathrm{Riem}}(W) \cong \mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W) \times_{\underset{\bowtie}{\mathbf{Diff}}(W)} \underset{\bowtie}{\mathrm{Riem}}(W). \tag{11}$$

The proofs of the following two results are the same as for the analogous statements for M.

Proposition 5.1.3. *The space* $\underset{\bowtie}{\mathrm{Riem}}(W)$ *is convex, and thus contractible.* □

Theorem 5.1.4. *The map* $\mathrm{b}\underset{\bowtie}{\mathrm{Emb}}^n(W) \times_{\underset{\bowtie}{\mathbf{Diff}}(W)} \underset{\bowtie}{\mathrm{Riem}}(W) \to \underset{\bowtie}{\Psi}^n(W)$ *is a weak equivalence for all* $n \in \mathbf{N} \cup \{\infty\}$. □

5.2 Continuous maps

In this subsection we consider analogous of the lemmas in §4.3.1 in the Riemannian setting.

Lemma 5.2.1. *Let* $(W; W_0, W_1)$ *be a collared bordism, then the maps*

$$\begin{array}{rccc} \partial_i: & \Psi^n_{\mathrm{Riem}}(W) & \to & \Psi^n_{\mathrm{Riem}}(W_i) \\ & W' & \mapsto & W' \cap (\mathbf{R}^n \times \{i\}) \end{array} \tag{12}$$

for $i = 0, 1$ *are continuous.*

Proof. For $i = 0, 1$ the diagram

$$\begin{tikzcd} \underset{\bowtie}{\mathrm{Emb}}^n(W) \times \underset{\bowtie}{\mathrm{Riem}}(W) \arrow[r] \arrow[d, two heads] & \mathrm{Emb}(W_i, \mathbf{R}^n) \times \underset{\bowtie}{\mathrm{Riem}}(W_i) \arrow[d, two heads] \\ \Psi^n_{\mathrm{Riem}}(W) \arrow[r] & \Psi^n_{\mathrm{Riem}}(W_i). \end{tikzcd}$$

clearly commutes. The top horizontal arrow is continuous by Propositions 3.2.23 & 3.2.21 and the observation that the map $\underset{\bowtie}{\mathrm{Riem}}(W) \to \underset{\bowtie}{\mathrm{Riem}}(W_i)$ is continuous. The bottom horizontal map is continuous by the universal property of quotient topologies. □

Lemma 5.2.2. *Let W, W' be collared bordisms such that $W_1 = W'_0 =: M$ and denote by $W \cup W'$ the collared bordism obtained by glueing W and W' along M. Then the map*

$$\begin{array}{rrcl} \mu: & \Psi^n_{\mathrm{Riem}}(W) \times_{\Psi^n_{\mathrm{Riem}}(M)} \Psi^n_{\mathrm{Riem}}(W') \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} & \to & \Psi^n_{\mathrm{Riem}}(W \cup W') \\ & \big((S,g),(S',g'),t,t'\big) & \mapsto & \Big(m_{t/(t+t')}S \ \cup \Big(\ m_{t'/(t+t')}S' + \tfrac{t}{t+t'}\, e_{n+1}\Big),\ g \cup g'\Big) \end{array}$$

is continuous.
Here, $m_{(\_)}$ and e_{n+1} are defined as in Lemma 4.3.2.

Sketch of proof. We indicate how to modify the two parts of the proof of Lemma 4.3.2. For the first part we again fix $t = t' = 1$, and $\big((S,g),(S',g')\big) \in \Psi^n_{\mathrm{Riem}}(W) \times_{\Psi^n_{\mathrm{Riem}}(M)} \Psi^n_{\mathrm{Riem}}(W')$. We again obtain a map

$$\Big(\underset{\bowtie}{\Psi}^n(W) \times \underset{\bowtie}{\mathrm{Riem}}(W)\Big) \times_{\Psi^n(M) \times \underset{\bowtie}{\mathrm{Riem}}(M)} \Big(\underset{\bowtie}{\Psi}^n(W') \times \underset{\bowtie}{\mathrm{Riem}}(W')\Big) \to \underset{\bowtie}{\Psi}^n(W \cup W') \times \underset{\bowtie}{\mathrm{Riem}}(W \cup W'),$$

as well as embeddings $\iota, \iota', \iota \cup \iota'$ and collared tubular neighbourhoods $\hat{\iota}, \hat{\iota}', \hat{\iota} \cup \hat{\iota}'$; these serve to trivialise various associated bundle structures, which allows to check that the above map is continuous.
For the second part we observe that for any continuous map $\alpha : [0,1] \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} \to [0,1]$ such that $\alpha_{(t,t')}$ is an endpoint preserving diffeomorphism for each $(t,t') \in \mathbf{R}_{>0} \times \mathbf{R}_{>0}$ we obtain a commutative square (on underlying sets)

$$\begin{array}{ccc} \underset{\bowtie}{\mathrm{Emb}}^n(W \cup W') \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} \times \underset{\bowtie}{\mathrm{Riem}}(W \cup W') & \xrightarrow{\alpha_* \times \mathrm{id}} & \underset{\bowtie}{\mathrm{Emb}}^n(W \cup W') \times \underset{\bowtie}{\mathrm{Riem}}(W \cup W') \\ \downarrow & & \downarrow \\ \Psi^n_{\mathrm{Riem}}(W \cup W') \times \mathbf{R}_{>0} \times \mathbf{R}_{>0} & \xrightarrow{\quad R_\alpha \quad} & \Psi^n_{\mathrm{Riem}}(W \cup W'), \end{array}$$

describing the action on $\Psi^n_{\mathrm{Riem}}(W \cup W')$, so we see that again, essentially the same argument goes through as for Lemma 4.3.2. □

5.3 The quasi-unital flagged ∞-category of Riemannian bordisms

Definition 5.3.1. Let $d \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{\infty\}$. The *topological semi-category of Riemannian d-dimensional bordisms embedded in $\mathbf{R}^n$* is the topological semi-category $\mathbf{Bord}d, n$, with objects

$$(\mathbf{Bord}d, n^{\mathrm{Riem}})_0 = \Big\{ \ (M,g) \ \Big| \ \text{M is a closed } (d-1)\text{-dimensional submanifold of } \mathbf{R}^n,\ g \in \underset{\bowtie}{\mathrm{Riem}}(M) \ \Big\} \cong \coprod_{[M]} \Psi^n_{\mathrm{Riem}}(M),$$

and morphisms

$$\begin{aligned} (\mathbf{Bord}d, n^{\mathrm{Riem}})_1 &= \Big\{ \ (W,g,t) \ \Big| \ W \subseteq \mathbf{R}^n \times [0,t] \text{ is a compact collar embedded } d\text{-dimensional bordism},\ g \in \underset{\bowtie}{\mathrm{Riem}}(W),\ t > 0 \\ &\cong \coprod_{[W]} \Psi^n_{\mathrm{Riem}}(W) \times \mathbf{R}_{>0}, \end{aligned}$$

both topologised via the indicated canonical bijections, together with source and target maps

$$\begin{array}{rrcl} d_i: & (\mathbf{Bord}d,n^{\mathrm{Riem}})_1 & \to & (\mathbf{Bord}d,n^{\mathrm{Riem}})_0 \quad (i=0,1) \\ & (W,g,t) & \mapsto & (W_{i+1}, g|_{W_{i+1}}) \end{array}$$

and multiplication map

$$\begin{array}{rrcl} \mu: & (\mathbf{Bord}d,n^{\mathrm{Riem}})_1 \times_{(\mathbf{Bord}d,n^{\mathrm{Riem}})_0} (\mathbf{Bord}d,n^{\mathrm{Riem}})_1 & \to & (\mathbf{Bord}d,n^{\mathrm{Riem}})_1 \\ & ((W,g,t),(W',g',t')) & \mapsto & (W + (W' + t \cdot e_{n+1}), g \cup g', t+t'), \end{array}$$

where e_{n+1} is the $(n+1)$-st canonical standard basis vector of $\mathbf{R}^{n+1}$, and $g \cup g' \in \underset{\bowtie}{\mathrm{Riem}}(W + (W' + t \cdot e_{n+1}))$ is obtained by glueing g and g' in the obvious way. $\lrcorner$

By Lemma 5.2.1 the target and source maps in Definition 5.3.1 are continuous, and by Lemma 5.2.2 the composition map is continuous.

Definition 5.3.2. Let $d \in \mathbf{N}$ and $n \in \mathbf{N} \cup \{\infty\}$. The *flagged ∞-semi-category of Riemannian d-dimensional bordisms embedded in* $\mathbf{R}^n$ is the semi-simplicial space given by the nerve of $\mathbf{Bord}d,n^{\mathrm{Riem}}$, which we again denote by $\mathbf{Bord}d,n^{\mathrm{Riem}}$. $\lrcorner$

Notation 5.3.3. Let $d \in \mathbf{N}$. When $n = \infty$ we write $\mathbf{Bord}d^{\mathrm{Riem}}$ rather than $\mathbf{Bord}d,\infty^{\mathrm{Riem}}$. $\lrcorner$

5.3.1 The quasi-unital flagged ∞-categories of ordinary and Riemannian bordisms are equivalent

Throughout this subsection n denotes a non-negative integer or infinity.

Theorem 5.3.4. *The map* $\mathbf{Bord}d,n^{\mathrm{Riem}} \to \mathbf{Bord}d,n$ *which forgets the Riemannian structure is a level-wise weak equivalence.*

Proof. This follows immediately from Theorems 5.1.2 & 5.1.4. $\square$

Corollary 5.3.5. *The semi-simplicial space* $\mathbf{Bord}d,n^{\mathrm{Riem}}$ *is a quasi-unital ∞-category.*

Proof. The properties of satisfying the Segal condition and being quasi-unital are manifestly invariant under levelwise weak equivalence. $\square$

Conventions and notation

Linguistic conventions In order to facilitate readability we use the following contractions:

- We write "iff" instead of "if and only if".
- We write "w.l.o.g." instead of "without loss of generality".
- We write "w.r.t." instead of "with respect to".

Editorial conventions

- Propositions stated without proof are marked with the symbol "$\square$".

Category theory

- Categories are denoted by $\mathcal{C}$, $\mathcal{D}$, ...
- Quasi-categories are denoted by C, D, ...
- Topological (semi-)categories are denoted by $\mathfrak{C}$, $\mathfrak{D}$, ...
- Let $\mathcal{C}$ be a category and let $X, Y \in \mathcal{C}$ be two objects, then the set of morphisms from X to Y is denoted by $\mathcal{C}(X, Y)$.
- For any two categories $\mathcal{C}, \mathcal{D}$, a hooked arrow $\mathcal{C} \hookrightarrow \mathcal{D}$ denotes a full subcategory.
- For any category $\mathcal{C}$, we denote by $\mathcal{C}^{\simeq}$ the core of $\mathcal{C}$, i.e. the wide subcategory consisting of all isomorphisms.
- We use the following notation for various categories:
  - **Set** denotes the category of sets.
  - **Top** denotes the category of topological spaces.
  - **SSet** denotes the category of simplicial sets.
  - $\mathbf{Vect}_{\mathbf{R}}$ denotes the category of $\mathbf{R}$-vector spaces.
  - **Man** denotes the category of smooth manifolds and smooth maps.

Algebra

- Let n, n' be non-negative integers, and j a positive integer, then $L^j_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^{n'})$ denotes the vector space of j-multilinear symmetric mappings $\mathbf{R}^n \to \mathbf{R}^{n'}$, and we write $P^k(n, n') := \prod_{j=1}^k L^j_{\mathrm{sym}}(\mathbf{R}^n, \mathbf{R}^{n'})$ for any $k \geq 1$. This later space should be thought of as the space of polynomials of degree $\leq k$.

Simplicial theory

- The maps $d^i : [n] \to [n+1], s^i : [n+1] \to [n]$ are defined as usual.
- In addition, we define the maps $a^i : [m] \to [n], k \mapsto k + i$.
- For any category $\mathcal{C}$ and any simplicial object $F : \Delta^{\mathrm{op}} \to \mathcal{C}$ we write
  - $d_i := F(d^i)$
  - $s_i := F(s^i)$
  - $a_i := F(a^i)$.
- Simplicial spaces are usually denoted by X, Y,...

Differential geometry

- Manifolds are typically denoted by M, N,..., and their dimensions, by m, n,...

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