The smooth Oka principle

Adrian Clough

incl. of pt. CW-compl.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ 1 \arrow[r] & \cancel{X/A} = C_f \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ 1 & C_f \end{tikzcd}$$

Mapping cones behave well w.r.t (co)homology

$$\begin{tikzcd} \cdots \arrow[r] & \tilde{H}_n(A) \arrow[r] & \tilde{H}_n(X) \arrow[r] & \tilde{H}_n(\overset{C_f}{\cancel{X/A}}) \arrow[dlll, out=-10, in=170] \\ \tilde{H}_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

58' Dieter Puppe : C_f some sort of "homotopical cofibre / pushout"

$D: K \longrightarrow TSpc$

Homotopy (co)limits , denoted holim D, hocolim D

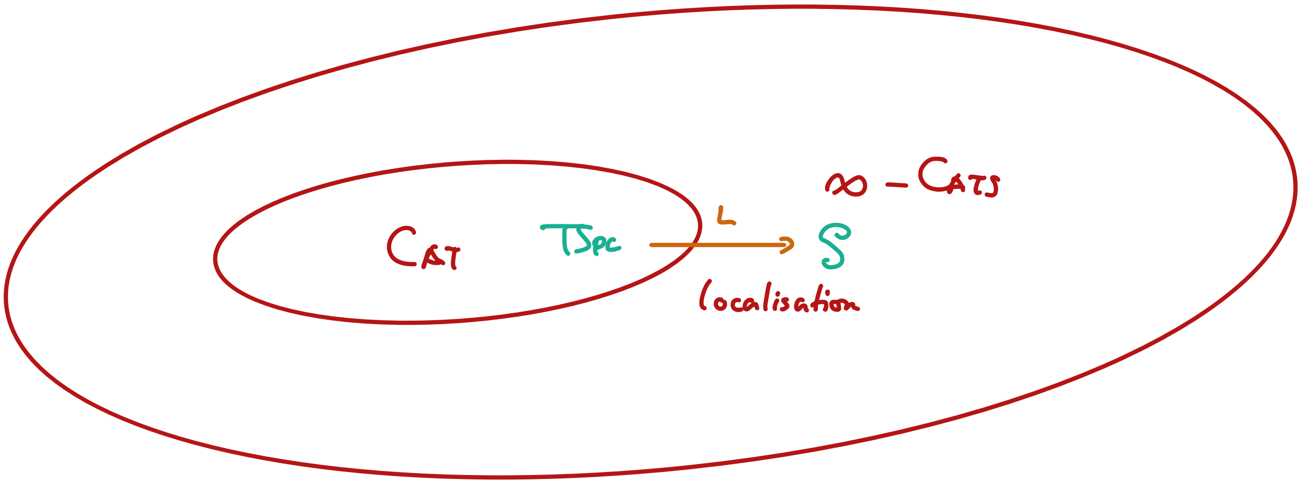
Vogt, Boardman, Kan, Segal, May, ...	via	Bar resolution	equivalent
Verdier, Quillen, ...	via	derived functors	

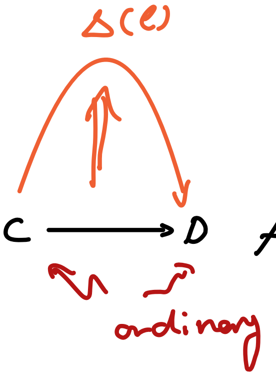
Behave well w.r.t. $\pi_\bullet$, $H_\bullet$, $H^\bullet$, etc.

Until early 00's holim D, hocolim D simply denoted any construction weakly equivalent to above.

∞-categories

The theory of ∞-categories is a faithful generalisation of the theory of ordinary categories.





$f: C \longrightarrow D$ functor of ~~∞-~~categories.

$W \subseteq \mathrm{Mor}\, C$

Definition: Consider functor $D: K \longrightarrow C$ & localisation $L: C \longrightarrow C[W^{-1}]$ then $X \in C$ is homotopy colimit of D if $LX = \mathrm{colim}\, L \circ D$.

Obs. H_*, H^*, π_*, etc. naturally functors out of S (satisfying exactness properties)

Similarly for

- A CW complex
- X topological space

asserted that $\underline{\mathrm{TSpc}}(A,X)$ is correct mapping space.

↖ CO-topology

$L\,\underline{\mathrm{TSpc}}(A,X) \xrightarrow{\simeq} \mathcal{S}(LA, LX)$ Continuous Oka principle

$$\begin{tikzcd}
\mathrm{TSpc} \arrow[r, shift left, "- \times A"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathrm{TSpc} \arrow[l, shift left, "\underline{\mathrm{TSpc}}(A, -)"] \arrow[d, "w.e"] \\
\mathcal{S} \arrow[r, shift left, "- \times LA"] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, shift left, "\mathcal{S}(LA, -)"]
\end{tikzcd}$$

Cartesian model structure

all obj. fibrant

$$\begin{tikzcd}
\hat{\Delta} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{TSpc} \arrow[l, shift left] \\
\Delta \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

Given smooth manifold M, many ways to extract homotopy type

- LM { underlying top. sp.
- smooth total singular complex

My thesis: $\mathrm{Mfd} \longrightarrow \mathrm{Sh}(\mathrm{Mfd}) =: \mathrm{Diff}^\infty \xrightarrow{\pi_!} S$

Shape

Differentiable sheaves

$\pi_!$ computes underlying h.t.

$$D : \overset{\overset{\Delta,\square}{\shortparallel}}{A} \longrightarrow \mathrm{Diff}^\infty$$

$$\operatorname{colim} \mathrm{Diff}^\infty(D(-), X) = \pi_! X$$

Thm. (Smooth Oka principle) [Berwick-Evans, Boavida de Brito, Pavlov / C.]

$A, X \in \mathrm{Diff}^\infty$ $\qquad \pi_! \underline{\mathrm{Diff}}(A, X) \xrightarrow{\sim} S(\pi_! A, \pi_! X) \qquad (*)$

Manifolds (Hausdorff, paracompact)

<u>Definition</u>: $A \in \mathrm{Diff}^\infty$ satisfies the <u>smooth Oka principle</u> if $(*)$ holds for A.

<u>Proposition</u> (C.) $\underline{\mathrm{Hom}}(\overset{\overset{\Delta,\square}{\shortparallel}}{A}{}^{op}, S)$ admits cofibrantly generated model structure.

Idea: Transfer model structure

$$\begin{tikzcd} \underline{\mathrm{Hom}}(A^{op}, S) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Diff}^\infty \arrow[l, shift left] \end{tikzcd}$$

as for topological spaces

Homotopical calculi on ∞-categories

Slogan: Model structures & other homotopical calculi serve to compute homotopy colimits.

Let C ∞-category, $W \subseteq \operatorname{Mor} C$, $L : C \longrightarrow C[W^{-1}]$

Definition: $c \in C$ is right proper if $C_{/c}[W^{-1}] \xrightarrow{\simeq} C[W^{-1}]_{/c}$ Lc

Definition: $c' \longrightarrow c$ in C is sharp if $\forall$

$$\begin{tikzcd}
a' \arrow[r, "\sim"'] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & b' \arrow[r] \arrow[d] & c' \arrow[d] \\
a \arrow[r, "\sim"] & b \arrow[r] & c
\end{tikzcd}$$

Proposition: c', c right proper, $f : c' \longrightarrow c$ sharp, then any pullback along $f : c' \longrightarrow c$ homotopy pullback

Proof:

$$\begin{tikzcd}
C_{/c'} \arrow[r, shift left, "f_!"] \arrow[d] \arrow[r, phantom, "\perp"] & C_{/c} \arrow[l, shift left, "f^*"] \arrow[d] \\
C[W^{-1}]_{/c'} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & C[W^{-1}]_{/c} \arrow[l, shift left, "f^*"]
\end{tikzcd}$$

Proposition: If C is equipped with a fibration structure ($\sim$ fibrations in model structure) then

- c fibrant $\Rightarrow$ c right proper
- $c' \longrightarrow c$ fibration $\Rightarrow$ $c' \longrightarrow c$ sharp

Until early 00's model categories were one of few ways to access ∞-categories.

$$A \overset{\mathbb{R}\text{-eq.}}{\sim} A'$$

Oka princ. ⟺ Oka princ.

Want model structure on Diff^∞ s.t.:

~~a)~~ manifolds $\mathbb{R}$-eq to cofibrant objects

b) all objects fibrant

c) Cartesian closed

1) $\Delta \longrightarrow \mathrm{Diff}_\infty$ ordinary simplices

~~b)~~ ~~c)~~

2) $\square \longrightarrow \mathrm{Diff}_\infty$ ordinary cubes

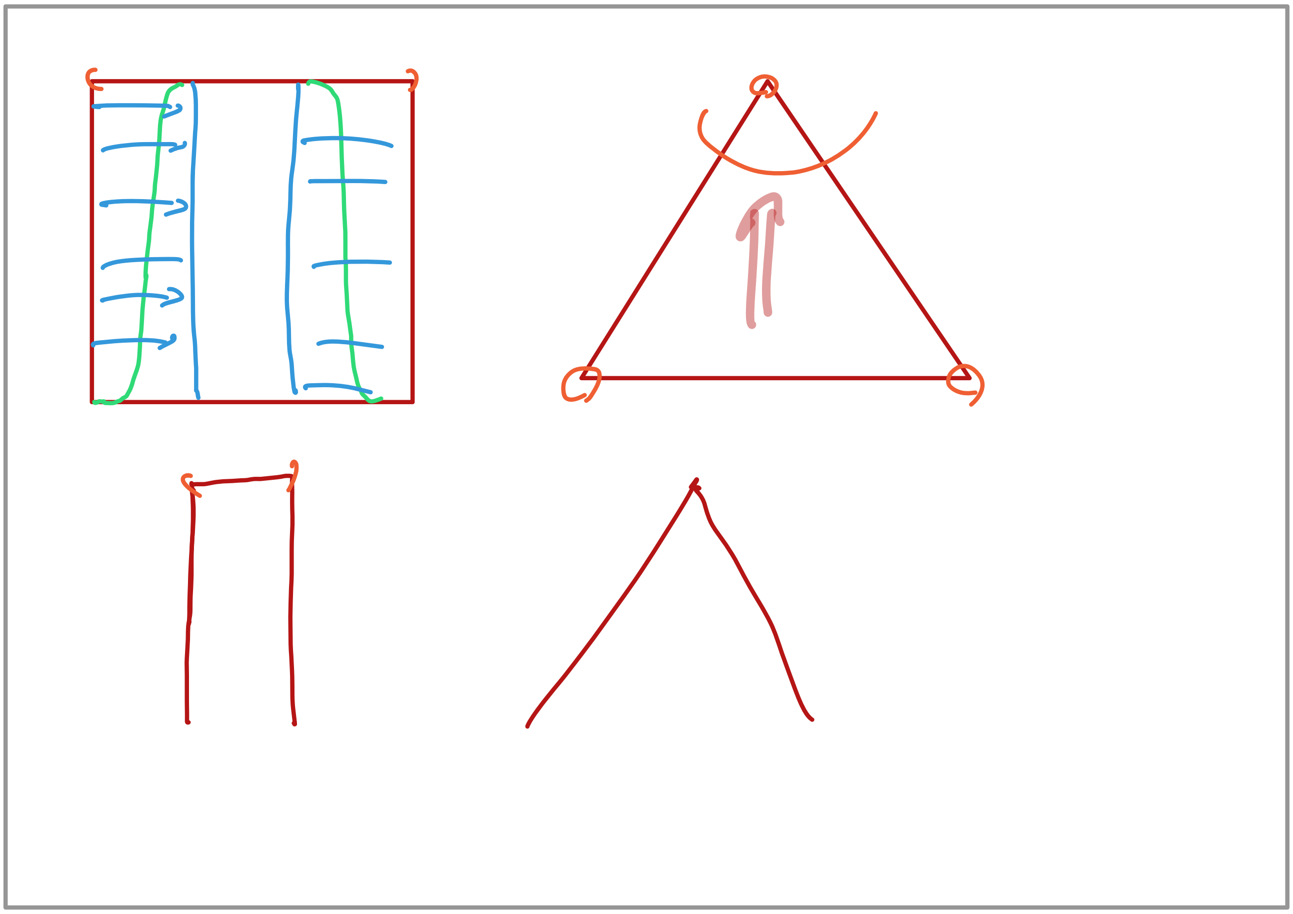
~~b)~~ c) ✓

c) $\Delta \longrightarrow \mathrm{Diff}_\infty$ Kihara's simplices

b) ✓ ~~c)~~ a)

Theorem (Kihara) Any (paracompact Hausdorff) manifold $\mathbb{R}^1$-equivalent to a Kihara complex.

$$\Delta^n_{Kih} \setminus Sk^{n-1}\Delta^n_{Kih} = \Delta^n \setminus Sk^{n-1}\Delta^n$$



Assume A, S, D satisfy smooth Oka principle. Consider $S \longrightarrow D$, $f : S \to A$

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & D \arrow[d] \\
A \arrow[r] & A \cup_f D
\end{tikzcd}$$

$A \cup_f D$ sat. smooth Oka principle $\Leftrightarrow$ $\forall\, X \in \mathrm{Diff}^\infty$:

$$\begin{tikzcd}
\overset{\mathcal{S}(\pi_! S,\, \pi_! X)}{\pi_! \, \underline{\mathrm{Diff}}^\infty(S, X)} & \pi_! \, \underline{\mathrm{Diff}}^\infty(D, X) \arrow[l] \\
\pi_! \, \underline{\mathrm{Diff}}^\infty(A, X) \arrow[u] & \pi_! \, \underline{\mathrm{Diff}}^\infty(A \cup_f D, X) \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \simeq \mathcal{S}(A \cup_f D, X)$$

h.t. pullback.

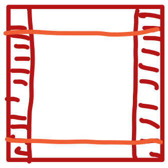
Strategy:

Show $\forall\, X \in \mathrm{Diff}^\infty$: $\overset{D}{\underline{\mathrm{Diff}}^\infty(\Delta^n_{k,h}, X)} \longrightarrow \overset{S}{\underline{\mathrm{Diff}}^\infty(\partial\Delta^n_{k,h}, X)}$ sharp.

$\Rightarrow$ Any complex built from $\Delta^n_{k,h}$ satisfies smooth Oka principle.

Definition: $X \twoheadrightarrow Y$ is a squishy fibration if all

$$\begin{tikzcd}
\sqcap^n_k \arrow[r] \arrow[d] & X \arrow[d] \\
\square^n \arrow[r] & Y
\end{tikzcd}$$



admit a lift.

Thm (C.) Squishy fibrations + shape equivalences form fibration structure.

Thm (C.) $\forall\, X \in \mathrm{Diff}^\infty$: $\underline{\mathrm{Diff}}^\infty(\Delta^n_{k,h}, X) \longrightarrow \underline{\mathrm{Diff}}^\infty(\partial\Delta^n_{k,h}, X)$

is a squishy fibration.

$$\overbrace{A \quad \text{compact} \quad X}^{\text{manifolds}}$$

Fréchet manifold

$$\underline{\mathrm{Diff}}^\infty(A,X)$$

$$\begin{tikzcd}
\circ \arrow[r] \arrow[d] & \underline{\mathrm{Diff}}^\infty(\Delta^2_{\mathrm{kih}}, X) \arrow[d] \\
\square' \arrow[r] & \underline{\mathrm{Diff}}^\infty(\partial\Delta^2_{\mathrm{kih}}, X)
\end{tikzcd}$$