Discrete Mathematics: HW7

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3.2 The Growth of Functions

2 Determine whether each of these functions is $O(x^2)$

b. $f(x) = x^2 + 1000$

$f(n^2 + 1)$

$\therefore f(x) = x^2 + 1000$ is $O(x^2)$

d. $f(x) = \frac{x^4}{2}$

$f(x) = \frac{x^4}{2}$

$f(x) = \frac{1}{2}(x^2 \cdot x^2) \qquad f(x) \geq \frac{1}{2}(x \cdot x),\ x^2 \geq x$

$f(x) = \frac{1}{2}x^2$

$\therefore f(x) = \frac{x^4}{2}$ is not $O(x^2)$

f. $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$

$f(x) \leq x \cdot (x+1),\ x - 1 < \lfloor x \rfloor \leq x,\ x \leq \lceil x \rceil < x + 1$

$f(x) = x^2 + x$

$f(x) \leq x^2 + x^2,\ x^2 \geq x$

$f(x) = 2x^2,\ x \geq 1$

$\therefore f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$ is $O(x^2)$

6 Show that $\frac{(x^3+2x)}{2x+1}$ is $O(x^2)$.

$f(x) = (\frac{1}{2}x^2 - \frac{1}{4}x + \frac{9}{8}) - \frac{\frac{9}{8}}{2x+1}$

$|f(x)| = |(\frac{1}{2}x^2 - \frac{1}{4}x + \frac{9}{8}) - \frac{\frac{9}{8}}{2x+1}|$

$|f(x)| \leq |\frac{1}{2}x^2 - \frac{1}{4}x + \frac{9}{8}$

$|f(x)| \leq |\frac{1}{2}x^2 + \frac{9}{8}|$

$|f(x)| = \frac{1}{2}x^2 + \frac{9}{8}$

$|f(x)| < \frac{1}{2}x^2 + x^2$

$|f(x)| = \frac{3}{2}|x^2|$

$\therefore$ it is $O(x^2)$

8 Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.

a. $f(x) = 2x^2 + x^3\ log\ x$

$log\ x \leq x$

$x^3\ log\ x \leq x^3 \cdot x$

$2x^2 + x^3\ log\ x \leq 2x^4 + x^4 = 3x^4$

$\therefore f(x) \leq 3x^4,\ x > 1$

d. $f(x) = \frac{(x^4 + 5\ log\ x)}{(x^4+1)}$

$f(x) = \frac{x^3}{(x^4+1)} + \frac{5\ log\ x}{(x^4+1)}$

$\frac{x^3}{x^4+1} < \frac{x^3}{x^4}$ and $\frac{5\ log\ x}{x^4+1} < \frac{5x}{x^4+1}$

$\therefore f(x) = \frac{x^3}{x^4+1} + \frac{5\ log\ x}{x^4+1} < x^{-1} + 1 < x$

12 Show that $x\ log\ x$ is $O(x^2)$ but that x^2 is not $O(x\ log\ x)$.

$x\ log\ x \leq x(x) = x^2$

$x\ log\ x \leq x^2,\ x > e$
$x\ log\ x \in O(x^2)$
$x \leq C\ log\ x$
$\frac{x}{log\ x} \leq C,\ x > k$
$x^2 \notin O(x\ log\ x)$

18 Let k be a positive integer. Show that $1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$.
$|1^k + 2^k + \cdots + n^k| = 1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots + n^k$ (n times)
$|1^k + 2^k + \cdots + n^k| \leq n \cdot n^k = n^{k+1} = |n^{k+1}|$
$|1^k + 2^k + \cdots + n^k| \leq 1 \cdot |n^{k+1}|$
$\therefore 1^k + 2^k + \cdots + n^k$ is $O(n^{k+1})$

20 Determine whether each of the functions $log(n+1)$ and $log(n^2+1)$ is $O(log\ n)$.
$log(n+1) < log(2n-1)$
$log(n+1) < log(n^2)$
$n^2 > 2n - 1 \Leftrightarrow (n-1)^2 > 0$
$log(n+1) = 2log\ n$
$\therefore log(n+1)$ is $O(log\ n)$
$log(n^2+1) < log(2n^2-1)$
$log(n^2+1) < log(n^4)$
$log(n^2+1) = 4log\ n$
$log(n^2+1) \leq Olog\ n$
$\therefore log(n^2+1)$ is $O(log\ n)$

22 Arrange the function $(1.5)^n,\ n^{100},\ (log\ n)^3,\ \sqrt{n}\ log\ n,\ 10^n,\ (n!)^2,\ and\ n^{99} + n^{98}$ in a list so that each function is big-O of the next function.
$(log\ n)^3,\ \sqrt{n}\ log\ n,\ n^{99} + n^{98},\ n^{100},\ (1.5)^n,\ 10^n,\ (n!)^2$

3.3 Complexity of Algorithms

2 Give a big-O estimate for the number additions used in this segment of an algorithm
$t := 0$
for $i := 1$ to n
  for $j := 1$ to n
    $t := t + i + j$
$\therefore O(n^2)$

4 Give a big-O estimate for the number of operations, where an operation is an addition or a multiplication, used in this segment of an algorithm (ignoring comparisions used to test the conditions in the while loop). $i := 1$
$t := 0$
while $i \leq n$
  $t := t + i$
  $i := 2i$
$\therefore O(log\ n)$

8 Given a real number x and a positive integer k, determine the number of multiplications used to find x^{2^k} starting with x and successively squaring (to find x^2, x^4, and so on). Is this a more efficient way to find x^{2^k} than by multiplying x by itself the appropriate number of times?
$x^{2^k} = \frac{x^2 \cdot x^2 \cdot x^2 \cdot \cdots \cdot x^2}{k\ times}$

18 How much time does an algorithm take to solve a problem of size n if this algorithm uses $2n^2 + 2^n$ operations, each requiring 10^{-9} seconds, with these values of n?
a. 10
$[2(10)^2 + 2^{10}]x10^{-9} = [2(100) + 1024]x10^{-9}$
$= (200 + 1024)x10^{-9}$
$= 1224x10^{-9}$

$\therefore 1.224x10^{-6}$ sec

b. 20

$[2(20)^2 + 2^{20}]x10^{-9} = [2(400) + 1048576]x10^{-9}$
$= (800 + 1048576)x10^{-9}$
$= 1049376x10^{-9}$
$\therefore 1.05x10^{-3}$ sec

c. 50

$[2(50)^2 + 2^{50}]x10^{-9} = [2(2500) + 1125899906842624]x10^{-9}$
$= (5000 + 1125899906842624)x10^{-9}$
$= 1125899906847624x10^{-9}$
$\therefore 1.13x10^{6}$ sec

d. 100

$[2(100)^2 + 2^{100}]x10^{-9} = [2(10000) + 1.267x10^{30}]x10^{-9}$
$= (20000 + 1.267x10^{30})x10^{-9}$
$= 1.267x10^{30}x10^{-9}$
$\therefore 1.27x10^{21}$ sec

20 What is the effect in the time required to solve a problem when you double the size of the input from n to $2n$, assuming that the number of milliseconds the algorithm uses to solve the problem with input size n is each of these function? [Express your answer in the simplest form possible, either as a ratio or a difference. Your answer may be a function of n or a constant].

a. $loglogn$

$f(n) = log(log\ 2n)$
$f(2n) = log(log\ 2n) = log(log\ n + log\ 2)$

c. $100n$

$f(n) = 100n$
$f(2n) = 100(2n) = 2(100n)$

e. n^2

$f(n) = n^2$
$f(2n) = (2n)^2 = 4n^2$

g. 2^n

$f(n) = 2^n$
$f(2n) = 2^{2n} = (2^n)^2$

22 Determine the least number of comparisons, or best-case performance,

a. required to find the maximum or a sequence of n integers, using Algorithm 1 of section 3.1
b. used to locate an element in a list of n terms using a linear search.
c. used to locate an element in a list of n terms using a binary search.