Discrete Mathematics: HW9

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5.1 Mathematical Induction

8 Prove that $2 - 2\dot{7} + 2\dot{7}^2 - \cdots + 2(-7)^n = \frac{(1-(-7)^{n+1})}{4}$ whenever n is a nonnegative integer.

$P(n) := 2 - 2\dot{7} + 2\dot{7}^2 - 2\dot{7}^3 + \cdots + 2\dot{(} - 7)^n = \frac{(1-(-7)^{n+1})}{4}$

For $n = 0$, $\frac{(1-(-7)^{0+1})}{4} = 2$

For $n = k + 1$, $\frac{(1-(-7)^{k+1})}{4} + 2(-7)^{k+1} = \frac{(1-(-7)^{k+2})}{4}$

16 Prove that for every positive integer n, $1\dot{2}\dot{3} + 2\dot{3}\dot{4} + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

$P(n) : 1\dot{2}\dot{3} + 2\dot{3}\dot{4} + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

$1\dot{2}\dot{3} + 2\dot{3}\dot{4} + \cdots + n(n+1)(n+2) = \sum_{i=1}^{n} i(i+1)(i+2)$

For $n = 1, 1\dot{2}\dot{3} = 6, \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1\dot{2}\dot{3}\dot{4}}{4} = 6$

$\sum_{i=1}^{k+1} i(i+1)(i+2) = 1\dot{2}\dot{3}+2\dot{3}\dot{4}+\cdots+k(k+1)(k+2)+(k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$

$\therefore P(n) : 1\dot{2}\dot{3} + 2\dot{3}\dot{4} + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

20 Prove that $3^n < n!$ if n is an integer greater than 6.

$P(n) : 3^n < n!$

For $n = 7, 3^7 = 2187$ and $7! = 5040$

$3^k < k!, k > 6$

$P(k+1) = 3^{k+1} = (k+1)!$

5.2 Strong Induction and Well-Ordering

6 a. Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

Objective is to determine the amounts of postage can be formed using just 3 cents and 10 cents stamps.

The amounts of postages that be formed using just 3 - cent and 10 - cent stamps are $3, 6, 9, 10, 12, 13, 15, 16$ and all values greater than or equal to 18

b. Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

$19 = 3 + 3 + 3 + 10$ is true

c. Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

$19 = 3 + 3 + 3 + 10$ is true

$23 = 10 + 10 + 3$

12 Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$, and so on. [Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even note that $(k + 1)/2$ is an integer.]

32 Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

5.3 Recursive Definitions and Structural Induction

4 Find $f(2), f(3), f(4)$, and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1, 2, \ldots$

c. $f(n+1) = f(n)^2 + f(n-1)^3$

$f(2) = f(1)^2 + f(0)^3$

$1^2 + 1^3 = 2$

$f(3) = f(2)^2 + f(1)^3$

$2^2 + 1^3 = 5$

$f(4) = f(3)^2 + f(2)^3$

$5^2 + 2^3 = 33$

$f(5) = f(4)^2 + f(3)^3$

$33^2 + 5^3 = 1214$

d. $f(n+1) = \frac{f(n)}{f(n-1)}$

$f(2) = 1$

$f(3) = 1$

$f(4) = 1$

$f(5) = 1$

8 Give a recursive definition of the sequence $\{a_n\}, n = 1, 2, 3, \ldots$ if

c. $a_n = 10^n$

$a_n = n(n+1) = n^2 + n$

$a_{n+1} = n^2 + 3n + 2 = a_n + 2(n+1)$

d. $a_n = 5$

$a_{n+1} = (n+1)^2 = a_n + 2n + 1$

14 Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

$f_{k+1}f_{k-1} - f_k^2 = (-1)^k$

$f_{(k+1)+1}f_{(k+1)-1} - f_{k+1}^2 = f_{k+2}f_k - f_{k+1}^2$

$= (-1)^{k+1} \therefore P(k+1)$ is true.

26 Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis Step: $(0, 0) \in S$

Recursive Step: If $(a, b) \in S$, then $(a+2, b+3) \in S$ and $(a+3, b+2) \in S$.

a. List the elements of S produced by the first five applications of the recursive definition.

$(2, 3), (3, 2) \in S$

$(4, 6), (5, 5), (6, 4) \in S$

$(6, 9), (7, 8), (8, 7), (9, 6) \in S$

$(8, 12), (9, 11), (10, 10), (11, 9), (12, 8) \in S$

$(10, 15), (11, 14), (12, 13), (13, 12), (14, 11), (15, 10) \in S$ b. Use strong induction on the number of applications of the recursive step of the definition to show that $5 | a + b$ when $(a, b) \in S$.

$n = 0, a_0 = 0, b_0 = 0, 5 | 0 + 0$ and $(0, 0) \in S, (a_0, b_0) \in S$

$(a_0 + 2) + (b_0 + 3) = 0 + 2 + 0 + 3 = 5$

$(a_0 + 3, b_0 + 2), (a_0 + 2, b_0 + 3) \in S$

c. Use structural induction to show that $5 | a + b$ when $(a, b) \in S$.

$(a, b) \in S, 5 | (a + b)$

$(a+2) + (b+3) = 5(m+1), (a+3) + (b+2) = 5(m+1)$

$(a+2, b+3), (a+3, b+2) \in S$