Discrete Mathematics: HW3

Adrian Darian

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2.5 Cardinality of Sets

2 Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a. the integers greater than 10

$X_n = 10 + n$

$X_1 = 10 + n = 11$

$X_2 = 10 + 2 = 12$

$\therefore$ the set greater than 10 are a countable infinite

b. the odd negative integers

$X_n = -(2n - 1)$

$X_1 = -(2(1) - 1)$

$\therefore$ the set of negative odd integers are a countable infinite

c. the integers with absolute value less than $1,000,000$

$-999,999 \Leftrightarrow 999,999$

$\therefore$ the set is finite

d. the real numbers between 0 and 2

the integers between 0 and 2 are uncountable

4 Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a. integers not divisible by 3

$$f(n) = \begin{cases} 3n + 1, \text{ if } n \text{ is even} \\ 3n + 2, \text{ if } n \text{ is odd} \end{cases}$$

$\therefore$ this set is not countable

b. integers divisible by 5 but not by 7

$0 \to 5$

$1 \to -5$

$2 \to 10$

$3 \to -10$

$4 \to 15$

$5 \to -15$

$\vdots$

$\therefore$ the set is countable

c. the real numbers with decimal representations consisting of all $1s$

$\overline{1} = 0.111\cdots$

$A = \{b.\overline{1} | b \in []\}$

$[] \to A$

$\therefore$ the set is countable

d. the real numbers with decimal representations of all $1s$ or $9s$

$M = \{$ real numbers with decimal representations of all 1s or 9s$\}$

$\begin{cases} 1 \text{ if } (i-1)th = 9 \\ 9 \text{ if } (i-1)th = 1 \end{cases}$
$\therefore$ the set is countable

12 Show that if A and B are sets and $A \subset B$ then $|A| \leq |B|$.
$x \in A$ since $A \subset B$
$\therefore x \in B$
which means $A \leq B$ when $A \subset B$

16 Show that a subset of a countable set is also countable.
X is a countable set and Y is a subset to X
$x_1, x_2, x_3, \cdots, x_n, \cdots$
$\therefore$ Y is countable if we list out all the elements of Y in the sequence

20 Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.
$x \in X, y \in Y, z \in Z$
$|X| = |Y|$ and $|Y| = |Z|$
$\therefore f : X \to Y$ and $g : Y \to Z$ are bijections
$y \in Y$ such $g(y) = z$ and $x \in X$ such $f(x) = y$
$g \circ f(x) = g(f(x)) = g(y) = z$
$z \in Z, g \circ f(x) = z, x \in X$
$g \circ f : X \to Z$
$x = y \Rightarrow f(x) = f(y)$
$y = z \Rightarrow f(y) = f(z)$
$g \circ (f(x)) = g \circ (f(y)) = g \circ (f(z))$
$g \circ f : X \to Z$ is one-to-one
$g \circ f : X \to Z$ is a bijection
$\therefore |X = |Y$ and $|Y = |Z|$, then $|X| = |Z|$

2.6 Matrices

14 The nxn matrix $A = [a_{ij}]$ is called a diagonal matrix if $a_{ij} = 0$ when $i \neq j$. Show that the product of two nxn diagonal matrices is again a diagonal matrix. Give a simple rule for determining this product.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$AB = [0, \cdots, 0, a_{ii}, 0, \cdots, 0]\dot{[}0, \cdots, 0, b_{jj}, 0, \cdots, 0]$ if $j \neq i$
$0\dot{0} + \cdots + a_{ii}\dot{0} + \cdots + 0\dot{b}_{jj} + 0\dot{0} = 0$
$\therefore a_{ii}$ and b_{ij} do not multiply so $AB = 0$ and AB is then a diagonal

22 Let A be a matrix. Show that the matrix AA^t is symmetric.
$(AA^t)^t = (A^t)^t A^t$
$(AB)^t = B^t A^t$
$(A^t)^t = A$
$(AA^t)^t = AA^t$
$\therefore AA^t$ is a symmetric matrix

28 Find the Boolean product of A and B, where $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$A[]B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} [] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A[]B = \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \vee 0 \end{bmatrix}$$
$$A[]B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

30 Let A be a zero–one matrix. Show that

a. $A \vee A = A$

$A \vee A = [0] \vee [0]$

$A \vee A = [0 \vee 0]$

$A \vee A = [0]$

$\therefore A \vee A = A$

b. $A \wedge A = A$

$A \wedge A = [0] \wedge [0]$

$A \wedge A = [0 \wedge 0]$

$A \wedge A = [0]$

$\therefore A \wedge A = A$

3.1 Algorithms

6 Describe an algorithm that takes a input as a list of n integers and finds the number of negative integers in the list.

```
P = 0
if X = 0
    return 0
if N = 0
    return 1
for i = 0 to |N|
    P = PxX
    if n < 0
        return 1/P
    return P
```

10 Devise an algorithm to compute x^n, where x is a real number and n is an integer.

```
for i = 0 to N
    if N < 0
        k = k * 1/X
    P = P * X
```

36 Use the bubble sort to sort d, f, k, m, a, b, showing the lists obtained at each step.

```
bubblesort(a_1, ..., a_n : real numbers with n ≥ 2)
    for i = 1 to n − 1
        for j = 1 to n − i
            if a_j > a_{j+1} then swap a_j and a_{j+1}
```

40 Use the insertion sort to sort the list in Exercise 36, showing the lists obtained at each step.

```
insertionsort(a_1, ..., a_n : real numbers with n ≥ 2)
    for j = 2 to n
        i = 1
        while a_j > a_i
            i = i + 1
        m = a_j
        for k = 0 to j − i − 1
            a_{j−k} = a_{j−k−1}
        a_i = m
```

52 Use the greedy algorithm to make change using quarters, dimes, nickels, and pennies for

a. 87 cents.
   3 quarters, 1 dime, 2 pennies
b. 49 cents.
   1 quarter, 2 dimes, 4 pennies
c. 99 cents.
   3 quarters, 2 dimes, 4 pennies
d. 33 cents.   1 quarter, 1 nickel, 3 pennies

54 Use the greedy algorithm to make change using quarters, dimes, and pennies (but no nickels) for each of the amounts given in Exercise 52. For which of these amounts does the greedy algorithm use the fewest coins of these denominations possible?

a. 87 cents.
   3 quarters, 1 dime, 2 pennies
b. 49 cents.
   1 quarter, 2 dimes, 4 pennies
c. 99 cents.
   3 quarters, 2 dimes, 4 pennies
d. 33 cents.   1 quarter, 8 pennies

56 Show that if there were a coin worth 12 cents, the greedy algorithm using quarters, 12-cent coins, dimes, nickels, and pennies would not always produce change using the fewest coins possible.

   15 cents = 12 cent coin, 3 pennies
   better solution would be to use a nickel and a dime