Complete the following tasks. You need to show work for full credit. For situations that start with double integrals, the workload expectation is a correct setup and numerical answer for a probability (or an expression for a marginal distribution); you may use software such as Wolfram Alpha to perform the integration. Some answers have been provided.

Assemble your work into one PDF document and upload the PDF back into our CatCourses page.

1. A hobbyist family is making PPE (personal protective equipment) to donate to local health care workers. Let

   - $T_1 \sim U(1, 4)$ be the amount of time (in hours) to 3D print a face mask and
   - $T_2$ be an exponentially distributed random variable with an average of 3 hours to represent the time (in hours) to cut out and sew a suit.

   Describe the distribution of time to complete construction of one suit-and-mask outfit by computing the mean, median, and standard deviation of the sum $T_1+T_2$ assuming independence between $T_1$ and $T_2$.[1]

2. Let us try using the following bivariate function to describe the presence of sneeze particles from a person's mouth (distance units are in feet).[2]

   $$f(x, y) = kye^{-xy}, \quad x > 0, \quad 0 < y < 5$$

   (a) Find the value of $k$ so that $f$ is a probability density function.

   (b) Mindful of social distancing, compute the proportion of sneeze particles that exceed $x = 6$ feet in horizontal distance.

   (c) Compute the standard deviation of $Y$.

3. Your math teacher intentionally misinterprets the definition of N99 masks to bring you this challenge. If the diameters of saliva particles are uniformly distributed between 5 and 21 micrometers, use Chebyshev's Inequality to compute how many particles are needed so that the average diameter of the saliva particles is within 3.2 micrometers of the true population mean with at least 99 percent probability?[3]

4. Our story begins at the Pearson Cooking Academy where many new students are tasked with cooking eggs on a rectangular pan. An analyst at Cal Kulas Consulting recommends the following function form to model the locations of the eggs

   $$f(x) = k(4x^2 + y^2), \quad -2 \leq X \leq 2, \quad -1 \leq Y \leq 1$$

   Compute $\text{Var}(X)$, $\text{Var}(Y)$, the correlation between $X$ and $Y$, and describe the correlation with a complete sentence.[4]

[1] Hint: there is only one input variable—time—so there is no need for double integrals. This was an exam question during the Spring 2020 semester.

[2] Optional reading: In case anyone is interested in the physics setup I assumed, the $x$ axis is positive away from the person, and the $y$ axis is positive downward toward the ground. This was an exam question during the Spring 2020 semester.

[3] Source: `https://www.envirosafetyproducts.com/resources/dust-masks-whats-the-difference.html` This was an example question during the Spring 2020 semester

[4] Hint: leave intermediate calculations in terms of $k$. This was an exam question during the Summer 2020 session.

5. Alas, not every egg is cracked onto the pan well.[5] Among experienced chefs, let us assume that the number of broken yolks in preparation for one order can be modeled with $X \sim Pois(\lambda)$. Among novice students in a cooking academy, let us assume that the number of broken yolks can be similarly modeled with $Y = 3X + 2$. In today's lecture, a sample of experienced chefs demonstrate how to prepare a recipe, and a student tracked the number of broken yolks:

| **Chef** | Aang | Appa | Iroh | Katara | Sokka | Toph | Zuko |
|---|---|---|---|---|---|---|---|
| **Broken Yolks** | 1 | 3 | 0 | 0 | 2 | 0 | 2 |

   (a) What is the probability that an experienced chef will break at least one yolk?
   (b) What is the probability that a novice chef will break at most two yolks?
   (c) Build a range-rule-of-thumb interval $(\mu - 2\sigma, \mu + 2\sigma)$ for $Y$.

6. Today, the student chefs are preparing the "Tortilla de Merced", a traditional Spanish egg dish with an infusion of spicy hot potato chips whose mascot is a cheetah. Let us model the radius (in inches) of the dish with $R \sim U(3, 5)$. Use Chebyshev's Inequality to compute how many dishes need to be measured so that their average radius is within 2 percent error with at least 90 percent probability.[6]

7. Let $X$ be a continuous random variable with probability density function:

$$f_X(x) = \begin{cases} \frac{3}{4}x\,(2-x) & 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine the cumulative distribution function $F_X$.
   (b) Let $Y = \sqrt{X}$. Determine the cumulative distribution function $F_Y$.
   (c) Determine the probability density of $Y$.

8. Let $X$ have an exponential distribution with parameter $\lambda = 1/2$.

   (a) Determine the cumulative distribution function of $Y = \frac{1}{2}X$.
   (b) Determine the probability distribution function of $Y$.
   (c) What kind of distribution does $Y$ have?

9. A Poisson distribution with rate parameter $\lambda$ has probability mass function

$$f(k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

   Show that all probabilities do indeed add up to 100 percent. (Hint: this is a quick proof. You do not have to do a proof within a proof.)

[5] This was an exam question during the Summer 2020 session.
[6] This was an exam question during the Summer 2020 session.

Some answers

1. sample statistics
   - mean: 5.5 hours
   - median: 4.5794 hours
   - standard deviation: 3.1225 hours
2. (a) 0.2
   (b) 0.0333
   (c) 1.4434 feet
3. $n > 209$ particles (coincidentally Merced's area code!)
4. (a) $\mathrm{Var}(X) = \dfrac{4768k}{45} \approx 105.956k$ or $\dfrac{648448}{135} \approx 4803.3185$
   (b) $\mathrm{Var}(Y) = \dfrac{712k}{45} \approx 15.8222k$ or $\dfrac{96832}{135} \approx 717.2741$
   (c) The random variables $X$ and $Y$ are uncorrelated
5. (a) 0.6811
   (b) 0.0929
   (c) $(-0.9856, 11.8428)$
6. $n > 521$
7. (a)
   (b) $F_Y(y) = \begin{cases} \dfrac{3}{4}\left(y^4 - \dfrac{1}{3}y^6\right) & 0 \leq y \leq \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$
   (c)
8. (a)
   (b)
   (c) $Y \sim Exp(1)$