Math 32, Lecture #4

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Today's Lecture

- Why you should be a math major/minor
- Simulation and Probability
- Birthday Problem
- Conditional Probability

Why Should You Be a Math Major/Minor

Best Jobs of 2019*

1) Data Scientist** 	$114K	6) Medical Services Manager	$103K
2) Statistician 	$84K	7) Information Security Analyst 	$98K
3) University Professor	$76K	8) Mathematician 	$83K
4) Occupational Therapist	$73K	9) Operations Research Analyst 	$72K
5) Genetic Counselor	$62K	10) Actuary 	$97K

 Math/Statistics

 Computer Science

*http://www.careercast.com/

** Best job ***every year*** since 2016 (when Prof Sindi started this)

Computers & Probability = Sampling

- From Last Week: Probability of getting 3 Heads (out of 5 flips of a Fair Coin).
- We would need a way to **generate samples** of fair coin flips.
- We will map {H,T} to {1,0}, i.e., H = 1 and T = 0.
- We will then use a **random number generator** to create samples of 1's and 0's.
- As long as these samples satisfy P(1) = ½ and P(0) = ½, we can use them to calculate P(3 H).

Calculating P(3 H) using R

- Let us now get the computer to simulate a certain number of coin tosses in R, say flipping a coin 5 times.
- We can then simply count how many times we observe exactly 3 heads.
- In theory, we should get 5/16, the number we calculated by hand.

Calculating P(3 H) using R

- How do we flip a coin on a computer?!
- Computers are good at generating “pseudo” random numbers
- Let’s ask the computer to generate a random number in [0,1] (with no bias!)
- If the number is >1/2 we will call this heads!
- Otherwise, we will call this tails.

Calculating P(3H) Using R

1. Set-Up:
   – Decide how many “experiments” to run.
   – Create “counters”
2. For n = 1 to total experiments
   – Flip coin 5 times
   – Count number of Heads
   – Did we get 3 Heads?
     - Yes: Add to success counter!
     - No: Keep going
3. Output **Empirical Probability**
   – *Empirical:* outcome of a simulation.
   – **Law of Large Numbers:** # trials goes to infinity, *empirical* probability converges to the *theoretical* probability

coinFlip.R

coinflip.R

Source on Save

Run

```r
#How many times are we going to "flip 5 coins"?
numtrials      = 1000

#We need a variable to "count" the number of times our event (3 heads) comes up.
oureventcount = 0

######################
# Run the Experiment #
######################

for (j in seq(from=1,to=numtrials,by=1))
{
    # flip a coin 5 times (i.e., generates 5 numbers uniformly distributed between [0,1])
    x = runif(5)

    #Let's call the outcome heads if it is larger than 0.5
    #counts the number of heads
    numheads = sum(x>0.5)

    #if we have exactly 3 heads add 1 to oureventcount
    if (numheads == 3)
        oureventcount = oureventcount + 1
}

#output the probability we computed: number of times 3 heads/(total Trials)
empiricalProbability = oureventcount/numtrials;
cat(sprintf("Empirical Probability:= %f\n", empiricalProbability))
```

1. Set-Up

2. Run Experiment

3. Output
Empirical Probability

1:1 (Top Level)

Console

Observations

- We can use R to generate samples of **many different** random variables, not just coin flips.
- Examples:
  - Rolls of the dice
  - Samples that follow a “bell curve,” i.e., a Gaussian or normal distribution.
- Using these samples, we can use R to answer probability questions. Sometimes, we can use R to answer probability questions that **cannot be answered using pencil and paper.**

My Favorite Probability Example: The Birthday Problem

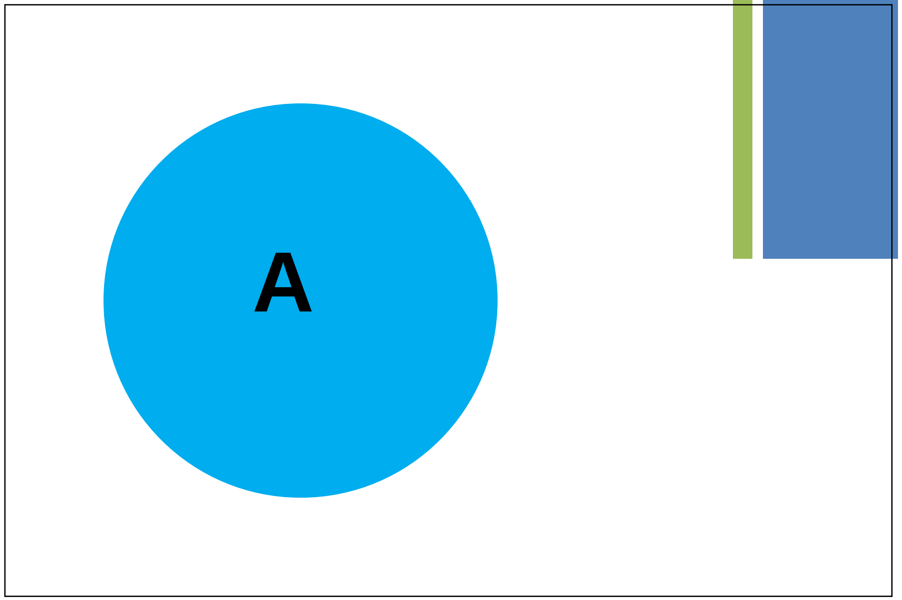
How many people have to be in the same room to guarantee two of them have the same birthday?

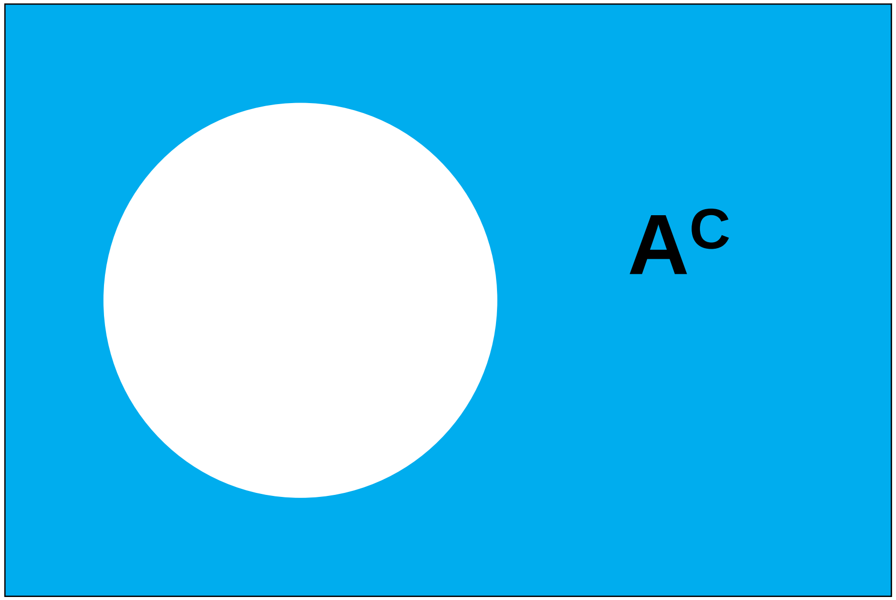
By the Pigeon Hole Principle we need 366 people (ignoring Leap Years) to make sure that 2 **must** have the same birthday.

But what about only a 50% chance that two of them have the same birthday?

What do you think?

$P(A) + P(A^c) = 1$





My Favorite Probability Example: The Birthday Problem

How many people have to be in the same room so that there's a **50% chance** that at least two of them have the same birthday?

Gut Answer: 366/2? NO!
Correct Answer: 23 (Wow!)

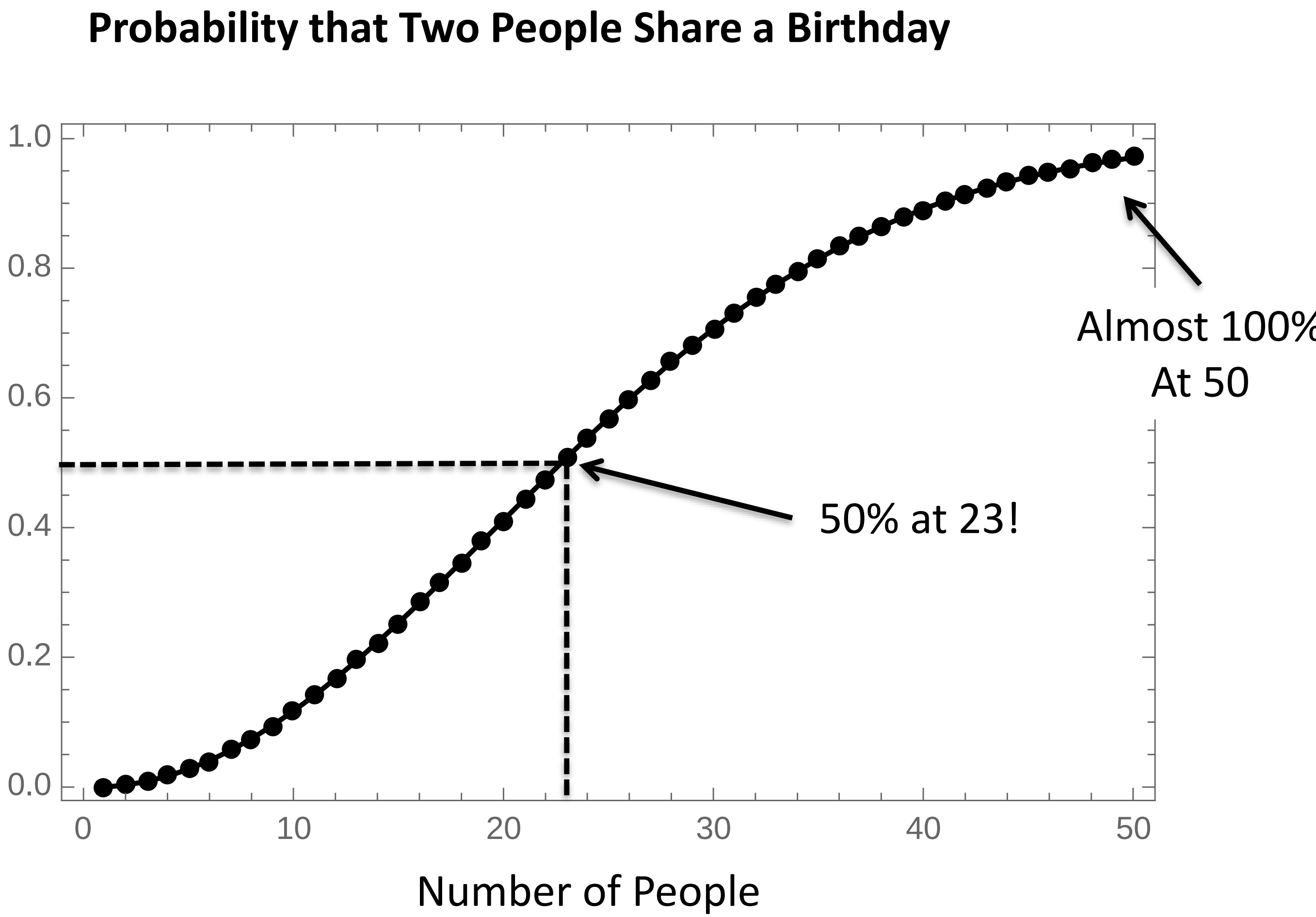
Prob(At least 2 Share a Birthday) = 1– Prob(Everyone has a different Birthday)

If we have 23 people and each of them have a different birthday then how could this happen?
Person 1 has 365/365 days they could be born on.
Person 2 has 364/365 days they could be born on.
Person 3 has 363/365 days they could be born on. Etc.

Prob(Everyone Has a Different Birthday)
$= (365/365)\times(364/365)\times\ldots\times(343/365)$
$=(1/365)^{23}\times(365!/342!)= 0.49270276$

Prob(At least 2 People Share a Birthday) = $1 - 0.49270276 = 0.507297$

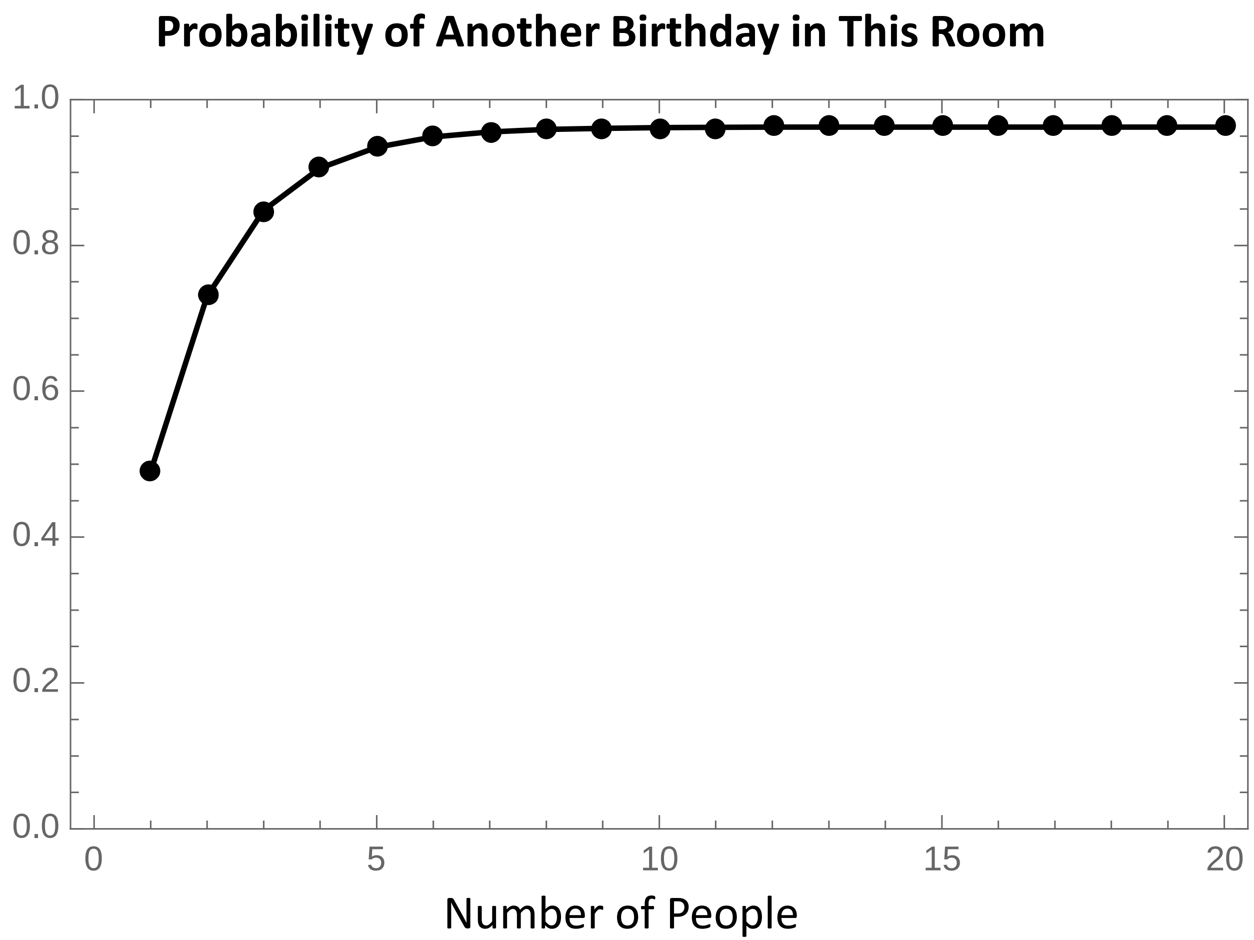
The Birthday Problem



Math helps us to make sense of problems where - at first glance - our intuition may mislead us.

A Twist on the Bday Problem

How many people **in this room** will I have to ask for their birthdays, before we find someone who has the same birthday?



Conditional Probability

What is the probability of A if we know B happens?

P(A | B) = Probability of A given B

Conditional Probability and Multiplication Rule

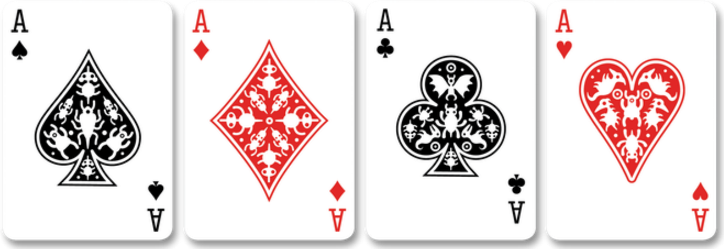
- Standard 52 cards deck: 4 suits, each suit contains 13 cards (Ace, 2-10, Jack, Queen, King), well shuffled.
- What is the probability that a player is dealt two cards, both are aces?
- We are sampling cards **without replacement.**
- P(Ace,Ace) = (4/52) x (3/51).
- Alternatively, **the Multiplication Rule**

For any two events A and B we have:

$$P(A \cap B) = P(A) \times P(B \mid A)$$

$$P(A \cap B) = P(B) \times P(A \mid B)$$

Usually, one of these is easier…



Conditional Probability and Multiplication Rule

- A = 1st Card is an Ace
  B = 2nd Card is an Ace
- From before: $P(A \cap B) = (4/52)x(3/51)$
- Multiplication Rule:
  $P(A \cap B) = P(A) \times P(B \mid A)$
  - $P(A) = 4/52$
  - $P(B \mid A) = 3/51$
- So our sampling w/o replacement is using conditional probability.
- Another way:

$$P(B \mid A) = P(A \cap B)/P(A)$$

Return to Independence

- A=player 1 gets two cards, both Aces
- B=player 2 gets two cards, both Aces
- Q: Are these independent?
  $P(A \cap B) = P(A) \times P(B)$
- $P(A) = (4/52)x(3/51)$
- $P(B) = (4/52)x(3/51)$
- But $P(A \cap B) = (4/52)x(3/51)x(2/50)x(1/49)$
- $\therefore$ Not independent

Independent $\neq$ Mutually Exclusive

- Events A and A^c are **mutually exclusive**, but they are NOT independent.
- Question: $P(A \cap A^c) = ?$
- Answer:
  - We always have: $P(A \cap A^c) = 0$.
  - We always have: $P(A) \times P(A^c) = P(A)(1 - P(A)) = P(A) - P(A)^2 \geq 0$
- Example:
  - A = Roll a 6,
  - A^c = We do not roll a 6 (i.e., we roll a 1,2,3,4 or 5)
  - $P(A \cap A^c) = P(\{6\} \cap \{1,2,3,4,5\}) = P(\emptyset) = 0$.

Conceptually, once A has happened, A^c is impossible; thus, they are completely dependent.

In Summary

- Study Math
- R is really fun for problem solving!
- Birthday problem is the best!
- Conditional probabilities