Proof of Euler's Formula

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We have to prove that:

$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Let $f(\theta) = e^{i\theta}$ According to Maclaurin series we know that:

$f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)x^n}{n!}$ where $f^{(0)}(a) = f(a)$ and $0! = 1$

Also:

$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ i^6 = -1, i^7 = -i,\ i^8 = 1$

$$
\begin{aligned}
f^{(0)}(0) &= 1\\
f^{(1)}(0) &= i\\
f^{(2)}(0) &= -1\\
f^{(3)}(0) &= -i\\
f^{(4)}(0) &= 1\\
f^{(5)}(0) &= i\\
f^{(6)}(0) &= -1\\
f^{(7)}(0) &= -i\\
f^{(8)}(0) &= 1
\end{aligned}
$$

$$e^{i\theta} = \frac{1}{0!} + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + ...$$

Now to find Maclaurin series representation of:

$g(\theta) = \cos(\theta)$

$$\begin{aligned}
g^{(0)}(0) &= \cos(0) = 1 \\
g^{(1)}(0) &= -\sin(0) = 0 \\
g^{(2)}(0) &= -\cos(0) = -1 \\
g^{(3)}(0) &= \sin(0) = 0 \\
g^{(4)}(0) &= \cos(0) = 1 \\
g^{(5)}(0) &= -\sin(0) = 0 \\
g^{(6)}(0) &= -\cos(0) = -1 \\
g^{(7)}(0) &= \sin(0) = 0 \\
g^{(8)}(0) &= \cos(0) = 1
\end{aligned}$$

$$\cos(\theta) = \frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + ...$$

Maclaurin series representation of:

$h(\theta) = \sin(x)$

$$h^{(0)}(0) = \sin(0) = 0$$
$$h^{(1)}(0) = \cos(0) = 1$$
$$h^{(2)}(0) = -\sin(0) = 0$$
$$h^{(3)}(0) = -\cos(0) = -1$$
$$h^{(4)}(0) = \sin(0) = 0$$

$$h^{(5)}(0) = \cos(0) = 1$$
$$h^{(6)}(0) = -\sin(0) = 0$$
$$h^{(7)}(0) = -\cos(0) = -1$$
$$h^{(8)}(0) = \sin(0) = 0$$

$$i\sin(\theta) = \frac{i\theta}{1!} - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + ...$$

Now:

$$\begin{aligned}
\cos(\theta) + i\sin(\theta) &= \frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \frac{i\theta}{1!} - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + ... \\
&= \frac{1}{0!} + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \frac{\theta^8}{8!} + ... \\
&= e^{i\theta}
\end{aligned}$$

[Q.E.D.]

0.1 Another way to prove Euler's formula

Another way to prove Euler's formula is that:
$\cos(\theta) + i\sin(\theta) = e^{i\theta}$

Now rearranging that equation we get:

$$1 = \frac{e^{i\theta}}{\cos(\theta) + i\sin(\theta)}$$

$$\begin{aligned}
0 &= \frac{d}{d\theta}\left(\frac{e^{i\theta}}{\cos(\theta) + i\sin(\theta)}\right) \text{.....[Taking der. of both sides]} \\
0 &= \frac{ie^{i\theta}\cos(\theta) - e^{i\theta}\sin(\theta) + e^{i\theta}\sin(\theta) - ie^{i\theta}\cos(\theta)}{\cos^2(\theta) + 2i\cos(\theta)\sin(\theta) - \sin^2(\theta)} \\
0 &= \frac{0}{\cos^2(\theta) + 2i\cos(\theta)\sin(\theta) - \sin^2(\theta)} \\
0 &= 0
\end{aligned}$$

$$L.H.S = R.H.S$$

[Q.E.D.]