Proof of Function Representation with Taylor Series

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We know that a power series is:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 \\ + a_5(x-a)^5 + ... + a_n(x-a)^n ...[\text{where } |x-a| < R]$$

0.1 Section (1)

Now: $f(a) = a_0$

$\therefore a_0 = \frac{f^{(0)}(a)}{0!}$ where $f^{(0)}(a) = f(a)$ and $0! = 1$

0.2 Section (2)

Taking the first derivative of f(x):

$f^{(1)}(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + 5a_5(x-a)^4 + ...$

$f^{(1)}(a) = a_1$.....(i)
$a_1 = \frac{f^{(1)}(a)}{1!}$.....[rearranging (i)]

0.3 Section (3)

Same way taking the second derivative of $f(x)$:

$f^{(2)}(x) = 2a_2 + 6a_3(x-a) + 12a_4(x-a)^2 + 20a_5(x-a)^3 + ...$

$f^{(2)}(a) = 2a_2$.....(ii)

$a_2 = \frac{f^{(2)}(a)}{2!}$....[rearranging (ii)]

0.4 Section (4)

Taking third derivative:

$f^{(3)}(x) = 6a_3 + 24a_4(x-a) + 60a_5(x-a)^2 + ...$

$f^{(3)}(a) = 6a_3$.....(iii)

$a_3 = \frac{f^{(3)}(a)}{3!}$....[rearranging (iii)]

0.5 Section (5)

Taking fourth derivative:

$f^{(4)}(x) = 24a_4 + 120a_5(x-a) + ...$

$f^{(4)}(a) = 24a_4.....(\text{iv})$

$a_4 = \frac{f^{(4)}(a)}{4!}....[\text{rearranging (iv)}]$

0.6 Section (6)

Taking fifth derivative:

$f^{(5)}(x) = 120a_5 + ...$

$f^{(5)}(a) = 120a_5.....(\text{iv})$

$a_5 = \frac{f^{(5)}(a)}{5!}....[\text{rearranaging (v)}]$

By plugging in values of a_0, a_1, a_2, a_3, a_4 and a_5 into $f(x)$ we get:

$$f(x) = \frac{f^{(0)}(a)(x-a)^0}{0!} + \frac{f^{(1)}(a)(x-a)^1}{1!} + \frac{f^{(2)}(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} + \frac{f^{(5)}(a)(x-a)^5}{5!} + ...$$

So the pattern is:

$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)(x-a)^n}{n!}$

And that's it. That's the Taylor series. [Q.E.D]