Experimentally Measuring the Gravitational Constant, G

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The gravitational constant, G, is a highly important universal physical constant. We provide an experimental way to measure it using a torsion pendulum with a reflecting mirror, a laser, heavy masses, and a laser tracker. This process involves moving heavy masses close to the torsion pendulum, causing it to twist towards a new equilibrium. This motion is tracked by bouncing a laser of the center of the pendulum and using a laser tracker. A gravitational constant of $G = (6.6 \pm 0.6) * 10^{-11} \frac{m^3}{kgs^2}$ was calculated using a weighted average, by errors, method and a value of $G = (6.58 \pm 0.03) * 10^{-11} \frac{m^3}{kgs^2}$ was calculated by using a method of spread between experiment values. Nearly all uncertainty in this experiment came from the uncertainty in the equilibrium displacement angle of the pendulum.

I. INTRODUCTION AND HISTORY

The gravitational constant has been postulated and indirectly measured since 1680. Is is among one of the earliest discovered fundamental constants. It was first estimated by Issac Newton, who did so by estimating the density of the Earth. As it was he who discovered the the gravitational force between two masses is inversely proportional with the distance between the two masses squared; with this estimate he could use the estimated mass of the Earth and its measured attractive, or gravitational, force to estimate G. The gravitational constant, as it is now viewed through Einstein's Theory of General Relativity, relates energy, or mass, to the curvature of space-time. For many reason, the gravitational constant has applications in the fields of cosmology, astronomy, and geophysics.

The first direct measurement of the gravitational constant was motivated by trying to find the mass of the Earth. This experiment was preformed by Henry Cavendish, who used an experimental set-up extremely similar to the one we used: using a torsion pendulum and tracking its displacement from the gravitational force between the masses on the torsion pendulum and other relatively large masses. [1]

Interestingly enough, the gravitational constant remains one of the least accurately known fundamental constants. The CODATA recommended value for the gravitational constant is currently $G = (6.67430 \pm 0.00015) * 10^{-11} \frac{m^3}{kgs^2}$. In addition, the different experiments for G, specifically the ones considered in the CODATA recommended value, are not in agreement with each other. Measuring the gravitational constant is hard: the gravitational force itself is an extremely weak fundamental force. In addition, accurately measuring the gravitational constant requires measuring many values and geometries. [2]

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II. THEORY

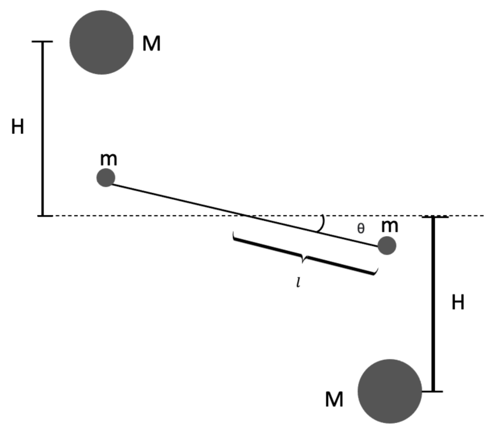

FIG. 1: A simplified version of the laboratory set-up used: looking down from the torsion pendulum.

Looking at Figure 1, an expression for the G, the gravitational constant, can be derived by considering when the torsion pendulum reaches equilibrium. At this point, the torque from the twisted torsion pendulum and the gravitational forces are equal.

$$\sum \tau_g = \tau_P \tag{1}$$

It is important to note that small angle approximations are used in the simplification of the geometries shown in Figure 1 and for future simplification of Equation 1. This is justified as the estimated displacement angle, θ, was about 0.009 radians.

Hooke's Law describes that the torque from a torsion pendulum is proportional to the angle it is displaced: $\tau_P = \kappa\theta$. The magnitude of the torque from the gravitational force, using cross products, is $\tau_g = 2\frac{GMm}{(H - lsin(\theta))^2} lsin(\frac{\pi}{2} + \theta)$. Simplifying these two torques, through the small angle approximation and trigonometric identities, the following expression for G, the gravitational constant, is derived:

$$G = \frac{\kappa\theta(H - l\theta)^2}{2Mml} \tag{2}$$

where κ is a constant determined my the torsion pendulum's restoring strength.

κ is not directly measured. The system reached an equilibrium because energy was dissipated: through many forms such as air resistance and heating of the pendulum wire. The equation for this dampened simple harmonic motion is described below.

$$\ddot{\theta} + C\dot{\theta} + \kappa\theta = 0 \tag{3}$$

Where C is a factor relating to the dampening of the torsion pendulum.

The solution to this equation is of the form:

$$\theta = Ae^{-\alpha t}\cos\omega t \tag{4}$$

Through derivation and substitutions of this solution κ is found using the α and ω, which can be estimated from the tracked motion of the pendulum.

$$\kappa = I\omega^2 + I\alpha^2 \tag{5}$$

The moment of inertia for the pendulum is estimated using just only the masses on each end of the pendulum: $I = 2ml$. The expression for the G simplified using these equations.

$$G = \frac{l(\omega^2 + \alpha^2)(H - l\theta_0)^2\theta_0}{M} \tag{6}$$

Where ω is the angular frequency of the pendulum's motion and α is its characteristic dampening frequency. This is the equation we will use for calculating the gravitational constant.

All values of G and intermediate values are calculated using the best value of its parameters. The error in each of these calculations is based of the following equation:

$$\sigma_V = \sqrt{\sum_i \left(\frac{\partial V}{\partial p_i}\right)^2 \sigma_{p_i}^2} \tag{7}$$

where V is any value and p_i is each parameters.

To get a weighted average value for G, each experimentally calculated value of G is weighted by its error in accordance with the following equation:

$$G = \frac{\sum_i \frac{G}{\sigma_G^2}}{\sum_i \frac{1}{\sigma_G^2}} \tag{8}$$

With an uncertainty according to the following equation:

$$\sigma_G = \left(\sqrt{\sum_i \frac{1}{\sigma_G^2}}\right)^{-1} \tag{9}$$

A best value for G is also calculated using the spread in each experimentally calculated best value. It's best value is the mean of each experimentally calculated value with an uncertainty calculated using the equation for the typical standard error of means.

III. APPARATUS AND PROCEDURE

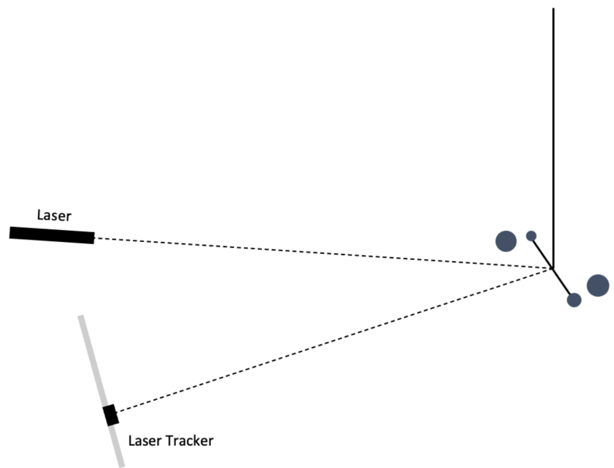

FIG. 2: A visual of the torsion pendulum, laser, and laser tracker.

The experiment used a setup like the one shown in Figure 2. For this experiment, the large masses were switched between different sides of the torsion pendulum. Therefore the masses would come back to two different equilibrium, whose displacement angle could be measured. While the pendulum returned to a new equilibrium we collected data of the motion by tracking the laser which bounced off of a mirror in the center of the torsion pendulum. This experiment required extreme precaution as any push on the table or airflow from opening the door would disturb the torsion pendulum and ruin our experiment, taking hours for the pendulum to revert back to rest once again.

The distance from the laser tracker to the torsion pendulum was measured. The average distance between the two displacement points was also measured; this was a large source of error as the laser tracker, used in marking these points, was easily disturbed by trying to take this measurement. Using these measurements, the angle at which the torsion pendulum had been displaced, θ, could be calculated, approximating with a small angle approximation. The diameter of the large mass and the width of the torsion pendulum were measured, to calculate H. Uncertainty from these two values came from uncertainties in measurements using a tape ruler. The mass of the large balls, or masses, was measured using a mass scale. The total length of the bar between the masses in the torsion pendulum was given as $10cm$.

Using the laser tracked path; a regression of the form: $\theta = Ae - tcos(t+) + c$, could be completed to calculate α and ω, with errors in the estimates used as uncertainties. Figure 3, below, shows an example of the regression-fitted data and the laser-tracked data.

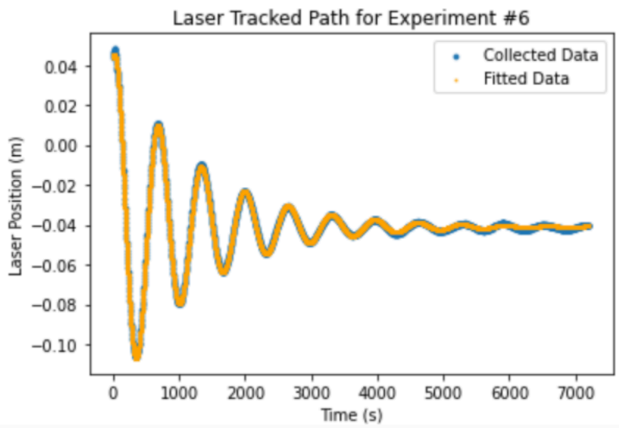

FIG. 3: The fitted and laser-tracked data for Experiment 6.

IV. OBSERVATIONS AND ANALYSIS

The experiment was successfully completed 7 times; with the calculated gravitational constants and uncertainties shown in Figure 4.

As expected, none of these values were outliers in accordance with Chauvenet's Criterion. This data was used to calculate a weighted average of these values in accordance with Equation 6 as explained in Theory. A spread calculated value of these is also calculated as explained in the Theory section. The uncertainty in the spread calculated value was notably much smaller than the weighted average value uncertainty. This is shown as the uncertainty in each value is much larger than the spread between the best values for each calculated gravitational constant. When calculating the spread value for the gravitational constant, it became clear that this spread should should be corrected as they are t-statistics. This t-statistic correction in the uncertainty was 9%.

To give more insight on this experiment, specifically what our largest sources of error were, the proportion of error coming from our estimated α, ω, θ, M, and H were all calculated. The largest average proportion of error came from the error in the θ, accounting for about 95% of the uncertainty.

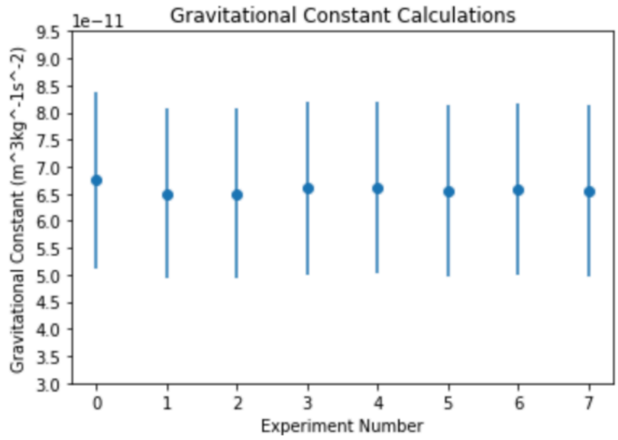

FIG. 4: The calculated values and uncertainties of the gravitational constant.

V. CONCLUSIONS AND OUTLOOK

The weighted average value, weighted by errors, for the gravitational constant is $G = (6.6 \pm 0.6) * 10^{-11} \frac{m^3}{kgs^2}$. The spread calculated value for the gravitational constant is $G = (6.58 \pm 0.03) * 10^{-11} \frac{m^3}{kgs^2}$, which was the same value whether accounting for a t-statistic correction or not. Using Chauvenet's Criterion, there were no outliers in the data. These experimental values are reasonably in agreement with the CODATA recommended value of $G = (6.67430 \pm 0.00015) * 10^{-11} \frac{m^3}{kgs^2}$. A reason for why the spread calculated value strays from this recommended value is discussed below.

As seen from Figure 4, the uncertainty in each calculated G is much larger than the spread in these calculations; however, this can be explained. Calculating the proportion of error coming from each of our measurement values, it is found that on average nearly 95% of the error comes from the uncertainty in θ. This makes sense as out measurements for the average laser displacements were so large, $8.20 \pm .9cm$, about one part in ten, similar to the uncertainty proportion of the weighted average calculated value of the gravitational constant. However, this uncertainty can be explained. It was noticed on one of the experiments, that the laser tracker was partially blocked when marking the equilibrium laser point, changing its equilibrium position. Had this been noticed earlier and avoided, some of the points would not have been so far from the others, creating a large uncertainty in these laser displacements, as there was clear clustering beside some outliers. This also could have been avoided by calculating an average laser displacement from the electronically collected data by converting the changes in the final and initial voltages to positions, as these readings were more accurate than plotting the final laser positions manually with a pencil. In this way, the error in each experiment was overestimated, explaining the disagreement between the uncertainties in the weighted average and spread calculated gravitational constants.

In addition, using this average displacement angle value held parameter in each experiment constant, except for the regressed values of α and ω. This put more error in each experiment rather than more spread between each experiment. Systematically lowering the spread between experiments also explains why the spread calculated value does not include the CODATA recommended value.

Acknowledgments

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[1] J. Wu et al., Annalen der Physik **531**, 1900013 (2019).
[2] T. Quinn, Nature **408**, 919 (2000).