Direct product of Galois connections

Adrián Enríquez Ballester

February 22, 2022

Proposition (Direct product of Galois connections). *Let* $(L, \sqsubseteq), (L_1^{\#}, \sqsubseteq_1)$ *and* $(L_2^{\#}, \sqsubseteq_2)$ *be lattices such that* $(L, \alpha_1, \gamma_1, L_1^{\#})$ *and* $(L, \alpha_2, \gamma_2, L_2^{\#})$ *are Galois connections for some respective abstraction functions*

$$\alpha_1 : L \to L_1^{\#}$$
$$\alpha_2 : L \to L_2^{\#}$$

and concretization functions

$$\gamma_1 : L_1^{\#} \to L$$
$$\gamma_2 : L_2^{\#} \to L$$

The quadruple $(L, \alpha, \gamma, L_1^{\#} \times L_2^{\#})$ *is a Galois connection where the abstraction function* $\alpha : L \to L_1^{\#} \times L_2^{\#}$ *is defined as*

$$\alpha(l) = (\alpha_1(l), \alpha_2(l)) \quad \forall l \in L$$

and the concretization function $\gamma : L_1^{\#} \times L_2^{\#} \to L$ *is defined as*

$$\gamma(l_1, l_2) = \gamma_1(l_1) \sqcap \gamma_2(l_2) \quad \forall (l_1, l_2) \in L_1^{\#} \times L_2^{\#}$$

Proof. Recall the product partial order of the two lattices $(L_1^{\#}, \sqsubseteq_1)$ and $(L_2^{\#}, \sqsubseteq_2)$ defined as

$$\begin{aligned} (l_1, l_2) \sqsubseteq_\times (m_1, m_2) \iff & \; l_1 \sqsubseteq_1 m_1 \wedge l_2 \sqsubseteq_2 m_2 \\ & \forall l_1, m_1 \in L_1^{\#},\ \forall l_2, m_2 \in L_2^{\#} \end{aligned}$$

with which we already know that $(L_1^{\#} \times L_2^{\#}, \sqsubseteq_\times)$ is also a lattice.

Let us check that the required properties for α and γ to form a Galois connection are satisfied (i.e. both functions are monotonically increasing, $\gamma \circ \alpha \sqsupseteq id$ and $\alpha \circ \gamma \sqsubseteq_\times id$).

First, if $a \sqsubseteq b$ with $a, b \in L$, then

$$\alpha(a) = (\alpha_1(a), \alpha_2(a))$$
$$\alpha(b) = (\alpha_1(b), \alpha_2(b))$$

Due to the corresponding Galois connections for $L_1^\#$ and $L_2^\#$, the monotonicity of α_1 and α_2 implies that $\alpha_1(a) \sqsubseteq_1 \alpha_1(b)$ and $\alpha_2(a) \sqsubseteq_2 \alpha_2(b)$, so $\alpha(a) \sqsubseteq_\times \alpha(b)$. This proves that α itself is also monotonically increasing.

Second, if $(l_1, l_2) \sqsubseteq_\times (m_1, m_2)$ with $(l_1, l_2), (m_1, m_2) \in L_1^\# \times L_2^\#$, then

$$\gamma(l_1, l_2) = \gamma_1(l_1) \sqcap \gamma_2(l_2)$$
$$\gamma(m_1, m_2) = \gamma_1(m_1) \sqcap \gamma_2(m_2)$$

The product partial order requires that $l_1 \sqsubseteq_1 m_1$ and $l_2 \sqsubseteq_2 m_2$, so again, due to the Galois connections for $L_1^\#$ and $L_2^\#$, it must be the case that $\gamma_1(l_1) \sqsubseteq \gamma_1(m_1)$ and $\gamma_2(l_2) \sqsubseteq \gamma_2(m_2)$. These results together with the definition of $\sqcap$ yield to

$$\gamma_1(l_1) \sqcap \gamma_2(l_2) \sqsubseteq \gamma_1(l_1) \sqsubseteq \gamma_1(m_1)$$
$$\gamma_1(l_1) \sqcap \gamma_2(l_2) \sqsubseteq \gamma_2(l_2) \sqsubseteq \gamma_2(m_2)$$

As $\gamma_1(l_1) \sqcap \gamma_2(l_2)$ is a lower bound of $\{\gamma_1(m_1), \gamma_2(m_2)\}$, it must be smaller or equal than its greatest lower bound, so $\gamma_1(l_1) \sqcap \gamma_2(l_2) \sqsubseteq \gamma_1(m_1) \sqcap \gamma_2(m_2)$.

Once γ has been proved to be monotonically increasing, we are going to verify that$\gamma \circ \alpha \sqsupseteq id$. Let l be an element of the concrete domain L, so we have that

$$\begin{aligned}(\gamma \circ \alpha)(l) &= \gamma(\alpha(l)) \\ &= \gamma(\alpha_1(l), \alpha_2(l)) \\ &= \gamma_1(\alpha_1(l)) \sqcap \gamma_2(\alpha_2(l))\end{aligned}$$

The Galois connections for $L_1^\#$ and $L_2^\#$ already satisfy this property, so $l \sqsubseteq \gamma_1(\alpha_1(l))$ and $l \sqsubseteq \gamma_2(\alpha_2(l))$. As l is a lower bound of $\{\gamma_1(\alpha_1(l)), \gamma_2(\alpha_2(l))\}$, it must be smaller or equal than its greatest lower bound, so $l \sqsubseteq \gamma_1(\alpha_1(l)) \sqcap \gamma_2(\alpha_2(l))$.

Finally, for proving that $\alpha \circ \gamma \sqsubseteq_\times id$, let (l_1, l_2) be an element of the abstract domain $L_1^\# \times L_2^\#$. We have that

$$\begin{aligned}(\alpha \circ \gamma)(l_1, l_2) &= \alpha(\gamma(l_1, l_2)) \\ &= \alpha(\gamma_1(l_1) \sqcap \gamma_2(l_2)) \\ &= (\alpha_1(\gamma_1(l_1) \sqcap \gamma_2(l_2)), \alpha_2(\gamma_1(l_1) \sqcap \gamma_2(l_2)))\end{aligned}$$

Once more, the Galois connections for $L_1^\#$ and $L_2^\#$ already satisfy this property and also require α_1 and α_2 to be monotonically increasing. With all this and by knowing that $\gamma_1(l_1) \sqcap \gamma_2(l_2) \sqsubseteq \gamma_1(l_1)$ and $\gamma_1(l_1) \sqcap \gamma_2(l_2) \sqsubseteq \gamma_2(l_2)$ we have

$$\begin{aligned}
\alpha_1(\gamma_1(l_1) \sqcap \gamma_2(l_2)) \sqsubseteq_1 \alpha_1(\gamma_1(l_1)) \sqsubseteq_1 l_1 \\
\alpha_2(\gamma_1(l_1) \sqcap \gamma_2(l_2)) \sqsubseteq_2 \alpha_2(\gamma_2(l_1)) \sqsubseteq_2 l_2
\end{aligned}$$

thus $(\alpha_1(\gamma_1(l_1) \sqcap \gamma_2(l_2)), \alpha_2(\gamma_1(l_1) \sqcap \gamma_2(l_2))) \sqsubseteq_\times (l_1, l_2)$. $\square$