Assignment 1 on Program Semantics

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Let us assume $e, e' \in \mathbf{AExp}$ and $x \in \mathbf{Var}$. The notation $e[x/e']$ denotes the result of replacing all occurrences of x in e by e'. For example: $(x+y)[x/(3*z)] = (3*z)+y$.

(a) Define $e[x/e']$ in a compositional way.

$$
\begin{aligned}
n[x/e'] &= n \\
x[y/e'] &= x \\
x[x/e'] &= e' \\
(e_1 + e_2)[x/e'] &= e_1[x/e'] + e_2[x/e'] \\
(e_1 - e_2)[x/e'] &= e_1[x/e'] - e_2[x/e'] \\
(e_1 * e_2)[x/e'] &= e_1[x/e'] * e_2[x/e']
\end{aligned}
$$

$\forall n \in \mathbf{Num}$, $\forall x, y \in \mathbf{Var}$ with $x \neq y$ and $\forall e_1, e_2, e' \in \mathbf{AExp}$.

(b) Prove the following *substitution lemma*: for all $e, e' \in \mathbf{AExp}$, $x \in \mathbf{Var}$, $\sigma \in \mathbf{State}$:

$$\mathcal{A}[\![e[x/e']]\!]\sigma = \mathcal{A}[\![e]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma]$$

Recall the compositional definition of $\mathcal{A}[\![\_]\!] : \mathbf{AExp} \to \mathbf{State} \to \mathbb{Z}$ as

$$
\begin{aligned}
\mathcal{A}[\![n]\!]\sigma &= n \\
\mathcal{A}[\![x]\!]\sigma &= \sigma(x) \\
\mathcal{A}[\![e_1 + e_2]\!]\sigma &= \mathcal{A}[\![e_1]\!]\sigma + \mathcal{A}[\![e_2]\!]\sigma \\
\mathcal{A}[\![e_1 - e_2]\!]\sigma &= \mathcal{A}[\![e_1]\!]\sigma - \mathcal{A}[\![e_2]\!]\sigma \\
\mathcal{A}[\![e_1 * e_2]\!]\sigma &= \mathcal{A}[\![e_1]\!]\sigma * \mathcal{A}[\![e_2]\!]\sigma
\end{aligned}
$$

$\forall n \in \mathbf{Num}$, $\forall x \in \mathbf{Var}$ and $\forall e_1, e_2 \in \mathbf{AExp}$. Recall also the definition of $\_[\_ \mapsto \_] : \mathbf{State} \to \mathbf{Var} \to \mathbf{AExp} \to \mathbf{State}$ as

$$\sigma[x \mapsto e](y) = \begin{cases} e & \text{if y = x} \\ \sigma(y) & \text{otherwise} \end{cases}$$

On the one hand, let $n \in \mathbf{Lit}$. As $\mathcal{A}[\![n[x/e']]\!]\sigma = \mathcal{A}[\![n]\!]\sigma = \mathcal{A}[\![n]\!]\sigma' = n$ $\forall \sigma, \sigma' \in \mathbf{State}$ and $\forall e' \in \mathbf{AExp}$, the property holds for literal expressions.

On the other hand, let $x \in \mathbf{Var}$. As $\mathcal{A}[\![x[y/e']]\!]\sigma = \mathcal{A}[\![x]\!]\sigma = \sigma(x)$ and $\sigma(x) = \sigma[y \mapsto e''](x)$ $\forall y \in \mathbf{Var}$ with $x \neq y$, $\forall \sigma \in \mathbf{State}$ and $\forall e', e'' \in \mathbf{AExp}$, the property also holds for variable expressions when we are not replacing the same variable.

We are left with the base element case of a variable expression which is being replaced. Let us develop each side of the property equation separately:

$$\mathcal{A}[\![x[x/e']]\!]\sigma = \mathcal{A}[\![e']\!]\sigma$$
$$\mathcal{A}[\![x]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] = \sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma](x) = \mathcal{A}[\![e']\!]\sigma$$

$\forall x \in \mathbf{Var}$, $\forall e' \in \mathbf{AExp}$ and $\forall \sigma \in \mathbf{State}$, so the property holds for all the base element cases.

Now, in order to proceed with the proof by structural induction, let $e_1, e_2 \in \mathbf{AExp}$. If both e_1 and e_2 satisfy the property, then

$$\begin{aligned} \mathcal{A}[\![(e_1 + e_2)[x/e']]\!]\sigma &= \mathcal{A}[\![e_1[x/e'] + e_2[x/e']]\!]\sigma \\ &= \mathcal{A}[\![e_1[x/e']]\!]\sigma + \mathcal{A}[\![e_2[x/e']]\!]\sigma \\ &= \mathcal{A}[\![e_1]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] + \mathcal{A}[\![e_2]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] \\ &= \mathcal{A}[\![(e_1 + e_2)]\!]\sigma[x \mapsto \mathcal{A}[\![e']\!]\sigma] \end{aligned}$$

$\forall \sigma \in \mathbf{State}$, $\forall e' \in \mathbf{AExp}$ and $\forall x \in \mathbf{Var}$.

The property can be proven in a similar way for the remaining composite elements (i.e. $-$ and $*$).