Constraints selection for the Hamiltonian graph problem

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The set of constraints stated in the assignment description are not minimal. Some of them follow from the others, so a subset can be choosen while keeping the problem requirements. This document explains the constraints selection and proves it to be correct.

The constraints, as specified in the assignment, are:

1. For each $i \in \{1..n\}$, v_i appears in the sequence.
2. For each $i \in \{1..n\}$, v_i does not appear in two different positions of the sequence.
3. For each $j \in \{1..n\}$, the j-th element of the sequence contains a vertex.
4. For each $i, j \in \{1..n\}$ such that $i \neq j$, v_i and v_j do not appear in the same position of the sequence.
5. For each $i, j \in \{1..n\}$ such that $i \neq j$: if v_i and v_j are not adjacent in the graph, then they do not appear together in the sequence.

Let p_{ij} be the propositional variable meaning that 'vertex i is in the path position j'. The first four constraints can be translated into the following propositional formulas:

1. For each $i \in \{1..n\}$, we have $p_{i1} \vee p_{i2} \vee ... \vee p_{in}$.
2. For each $i, j \in \{1..n\}$, we have $p_{ij} \Rightarrow \neg p_{i1} \wedge \neg p_{i2} \wedge ... \wedge \neg p_{i(j-1)} \wedge \neg p_{i(j+1)} \wedge ... \wedge \neg p_{in}$.
3. For each $j \in \{1..n\}$, we have $p_{1j} \vee p_{2j} \vee ... \vee p_{nj}$.
4. For each $i, j, k \in \{1..n\}$ such that $i \neq j$, we have $\neg(p_{ik} \wedge p_{jk})$.

We are going to prove that 2 and 3 are logical consequences of 1 and 4, thus an equivalent smaller set of constraints is $\{1, 4, 5\}$.

Lemma. $\{1, 4\} \models 2$ *and* $\{1, 4\} \models 3$.

Proof. Let n be the number of nodes in the graph and let α be an assignment that satisfies $\{1, 4\}$. Let $S = \{p_{ij} : \alpha(p_{ij})\}$ be the set of variables assigned to *true* by α.

On the one hand, 1 implies that $|S| \geq n$, because at least one p_{ij} must be *true* for each $i \in \{1..n\}$. On the other hand, 4 implies that $|S| \leq n$, because for every pair $p_{ij}, p_{kj} \in S$, i must be the same as k in order to not contradict 4, so both elements are the same. This leaves us with the result $|S| = n$.

If we suppose $\neg 2$, then there exists a pair $p_{ij}, p_{ik} \in S$ with $j \neq k$ and, together with 1, this would fall into the case $|S| > n$, which is not compatible with 4. As $1 \wedge 4 \wedge \neg 2$ is unsatisfiable, we have that $\{1, 4\} \models 2$.

For the last part, if we suppose $\neg 3$, then there exists some $j \in \{1..n\}$ such that $p_{ij} \notin S \ \forall i \in \{1..n\}$ and, together with 4, this time it falls into the case where $|S| < n$, which is not compatible with 1. This ends the proof with the result $\{1, 4\} \models 3$. $\square$