TIØ4317 Empirical and Quantitative Methods in Finance	Exercise 1
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1

(a)

Table 1: Bond Simple Returns

Year	Bond Value	Bond Simple Return
2006	36.90	NaN
2007	39.80	7.86
2008	42.40	6.53
2009	38.10	-10.14
2010	36.40	-4.46
2011	39.20	7.69
2012	44.60	13.78
2013	45.10	1.12

(b)

Table 2: Bond CC Returns

Year	Bond Value	Bond CC Return
2006	36.90	NaN
2007	39.80	7.57
2008	42.40	6.33
2009	38.10	-10.69
2010	36.40	-4.56
2011	39.20	7.41
2012	44.60	12.91
2013	45.10	1.11

(c)

Table 3: Bond Price in 2013 terms

Year	Bond Value	Bond Price in 2013 terms
2006	36.90	42.84
2007	39.80	45.25
2008	42.40	46.80
2009	38.10	41.15
2010	36.40	38.55
2011	39.20	40.66
2012	44.60	45.40
2013	45.10	45.10

(d)

Table 4: Bond Real Return

Year	Bond Value	Bond Real Return
2006	36.90	NaN
2007	39.80	5.61
2008	42.40	3.44
2009	38.10	-12.08
2010	36.40	-6.32
2011	39.20	5.47
2012	44.60	11.65
2013	45.10	-0.65

2

(a) Distribution of Y: Given X_i, Y_i follows a normal distribution: $Y_i|X_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$.

(b) Likelihood function: Given that $Y_i|X_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$, the conditional probability density function (pdf) of Y_i given X_i is:

$$f(y_i|x_i; \beta_0, \beta_1, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2}\right)$$

Since we have n independent observations $(Y_1, X_1), (Y_2, X_2), ..., (Y_n, X_n)$, the likelihood function $L(\beta_0, \beta_1, \sigma^2)$ is the product of the individual conditional pdfs:

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} f(y_i|x_i; \beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2} \right) \right]$$

This can be rewritten as:

$$L(\beta_0, \beta_1, \sigma^2) = \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \prod_{i=1}^{n} \exp\left(-\frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2} \right)$$

$$L(\beta_0, \beta_1, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(\sum_{i=1}^{n} \left[-\frac{(y_i - (\beta_0 + \beta_1 x_i))^2}{2\sigma^2} \right] \right)$$

$$L(\beta_0, \beta_1, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \right)$$

(c) MLE = OLS: To find the MLEs, we maximize the log-likelihood function $\ell(\beta_0, \beta_1, \sigma^2) = \ln(L(\beta_0, \beta_1, \sigma^2))$:

$$\ell(\beta_0, \beta_1, \sigma^2) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

To maximize ℓ with respect to β_0 and β_1, we take partial derivatives and set them to zero:

$$\frac{\partial \ell}{\partial \beta_0} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0 \implies \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\frac{\partial \ell}{\partial \beta_1} = \frac{1}{\sigma^2} \sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \implies \sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

These are the normal equations for MLE.

Now consider Ordinary Least Squares (OLS). OLS estimators minimize the sum of squared residuals (SSR):

$$SSR(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

To minimize SSR, we take partial derivatives with respect to β_0 and β_1 and set them to zero:

$$\frac{\partial SSR}{\partial \beta_0} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0 \implies \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\frac{\partial SSR}{\partial \beta_1} = -2 \sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0 \implies \sum_{i=1}^{n} x_i (y_i - \beta_0 - \beta_1 x_i) = 0$$

We observe that the normal equations derived from MLE are identical to the normal equations derived from OLS. Therefore, the MLEs for β_0 and β_1 are identical to the OLS estimators, i.e., $\hat{\beta}_{0,ML} = \hat{\beta}_{0,OLS}$ and $\hat{\beta}_{1,ML} = \hat{\beta}_{1,OLS}$.

3

PRF (Population Regression Function): Equation: $E(Y|X_i) = \beta_0 + \beta_1 X_i$. This equation describes the *true population relationship*. Notice $E(Y|X_i)$, the *expected* average Y for a given X, and β_0, β_1, the *true population parameters* in the equation.

SRF (Sample Regression Function): Equation: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$. This equation is our *sample-based estimate*. See $\hat{Y}_i$, the *predicted* Y for an observation in our sample, and $\hat{\beta}_0, \hat{\beta}_1$, the *estimated parameters* derived from our sample data in the equation.

Difference: The core difference is in the equations' components. PRF uses $E(Y|X_i)$ and true parameters (β_0, β_1) representing the *population's true relationship*. SRF uses $\hat{Y}_i$ and estimated parameters $(\hat{\beta}_0, \hat{\beta}_1)$ to *estimate* this relationship from a *sample*. Essentially, the SRF equation *estimates* the PRF equation using sample data.

4

Assumptions are made about the **unobservable error terms** (ϵ_i), which are part of the population regression model, not directly about the **estimated residuals** (e_i), which are their sample counterparts. We assume properties for ϵ_i (like zero mean, etc.) to ensure our statistical inferences about the population parameters are valid. Residuals are calculated from the sample data *after* estimation and are used to *check* if the assumptions about the error terms are reasonably plausible in our sample, but the core assumptions are about the unobservable error terms in the population model, not the observable residuals in a specific sample.

5

Linear regression assumes $E(Y|X_i) = \beta_0 + \beta_1 X_i$. This equation means the *average* value of Y for a given X_i is assumed to be a linear function of X_i.

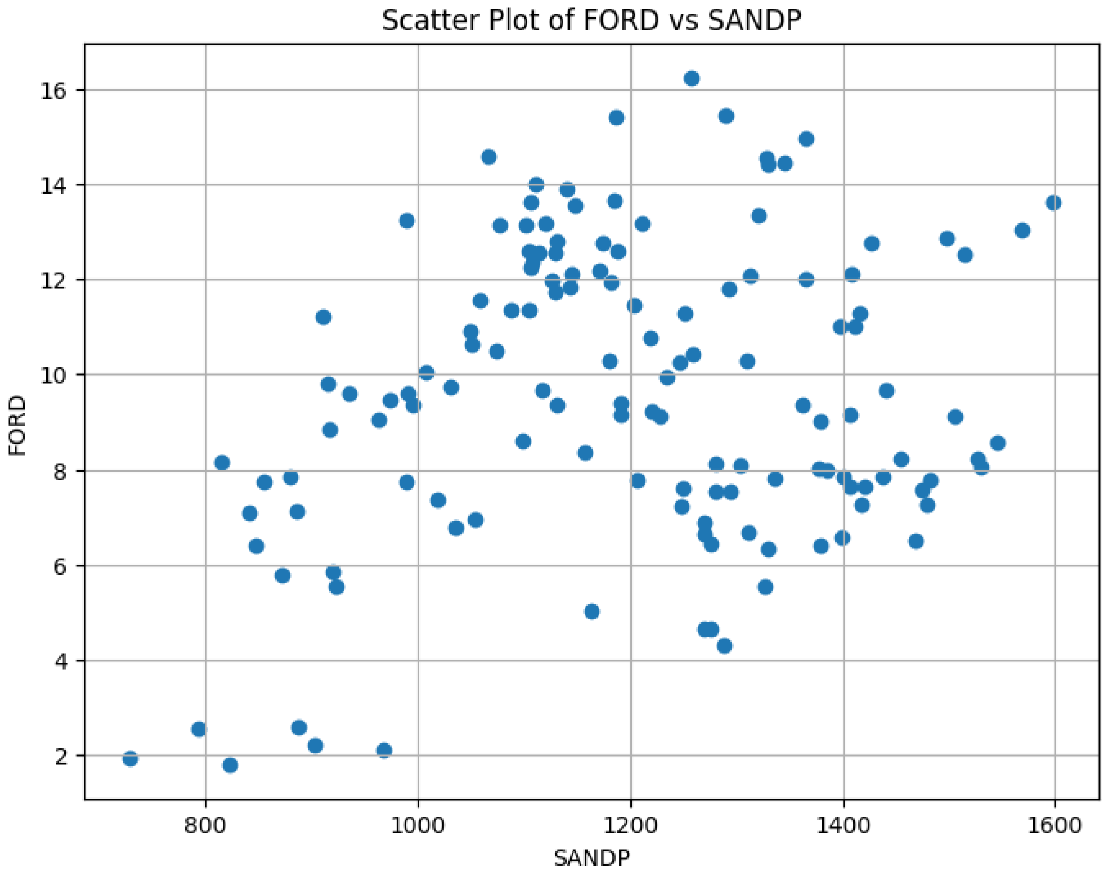
Estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ from $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$ estimate:

- β_1: the change in the *conditional mean* of Y for a one-unit change in X.
- β_0: the *conditional mean* of Y when X is zero.

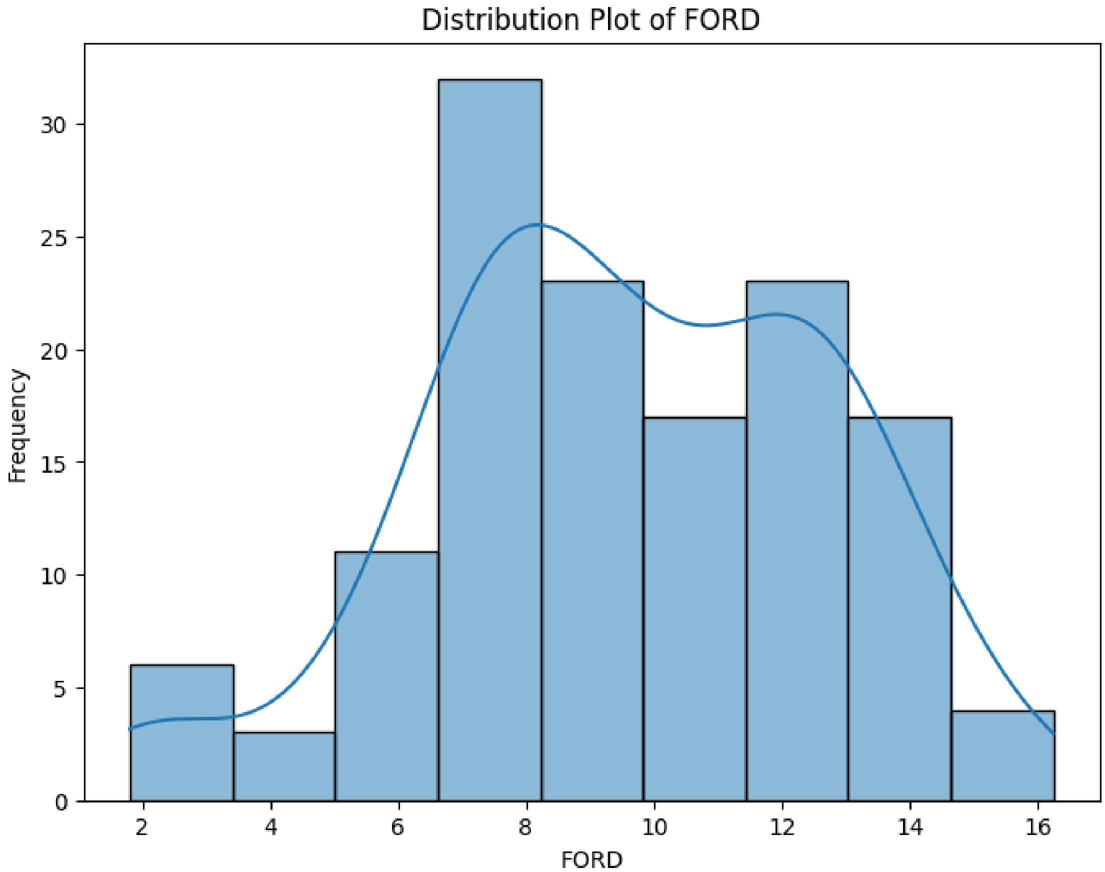
Meaning isn't guaranteed. If linearity is wrong or $X = 0$ is irrelevant, interpretations (especially of $\hat{\beta}_0$) can be meaningless.

6

(a)



Scatter Plot of FORD vs SANDP
FORD
SANDP
16
14
12
10
8
6
4
2
800
1000
1200
1400
1600



Based on the scatter plot, there seems to be a positive relationship between the S&P and FORD. As the S&P increases, FORD also increases.

(b)

Table 5: Summary Statistics

Mean	Median	Mode	Std Dev	Min	Max
9.60	9.42	9.35	3.12	1.81	16.24

Mean, Mode, and Median

The mean is the arithmetic average, calculated by summing all values and dividing by the number of values. It's important to note that the mean is sensitive to outliers, meaning extreme values can disproportionately influence it. The median, on the other hand, is the middle value in a dataset that has been sorted. This makes the median more robust to outliers as it's not pulled as much by extreme values. Finally, the mode is the most frequently occurring value within the dataset. There can be multiple modes or no mode at all. The most useful measure among these depends heavily on the data's distribution and the context of the analysis. For distributions that are roughly symmetrical, the mean, median, and mode tend to be similar. However, when dealing with skewed distributions, the median is often considered a more reliable measure of central tendency compared to the mean because of its resilience to outliers.

Arithmetic vs. Geometric Mean

When considering stock returns, it's important to distinguish between the arithmetic mean and the geometric mean. The arithmetic mean is simply the average of returns, but it can sometimes overestimate the actual growth of an investment over multiple periods, especially when volatility is present. This is because it doesn't fully account for the effects of compounding. In contrast, the geometric mean provides a more accurate picture of the compounded rate of return over multiple periods. It reflects the actual rate at which an investment grows over time, taking into account the compounding effect of returns. Therefore, for evaluating the long-term performance of stock investments, the geometric mean is generally considered a more useful metric as it provides a more realistic representation of investment growth.

(c)

Table 6: Regression Results

Parameter	Estimate	Std. Error
Alpha	0.01	0.01
Beta	2.07	0.28

Interpretation of Regression Results (FORD Stock):

- **Alpha (α):** The estimated alpha for FORD is approximately 0.01 (or 1%), with a standard error of 0.01. This positive alpha suggests that, over the period analyzed, FORD may have slightly outperformed what the market model would predict based on its risk (beta). However, given that the alpha estimate is of similar magnitude to its standard error, the statistical significance of this alpha might be weak, and we cannot draw a definitive conclusion based on this result alone.

- **Beta (β):** The estimated beta for FORD is approximately 2.07, with a standard error of 0.28. This beta is significantly greater than 1.0. It indicates that FORD stock is considerably more volatile than the market (SANDP). The beta estimate is statistically significant (much larger than its standard error), suggesting a robust relationship between FORD's returns and the market returns during this period.

(d)

P-value for Beta: 0.0000

There is statistically significant evidence that Beta () is different from zero. This suggests that there is a statistically significant relationship between FORD's return and the market (S&P) return. FORD's return is systematically related to market movements.

7

Type I Error means rejecting a true hypothesis. Like falsely claiming someone is guilty.

Type II Error means failing to reject a false hypothesis. Like letting a guilty person go free.

The **Significance Level (α):** means the chance of making a Type I error. If we set it at 0.05 it means there is a 5% risk of a false positive.

(a) Type I Error More Important: Testing a new drug for safety. A false positive (saying it's safe when it's not) could harm many.

(b) Type II Error More Important: Screening for a contagious disease. A false negative (missing a case) could lead to disease spread.

8

Hypotheses are tested about the **actual (population) coefficients** (β_0, β_1), not just our sample estimates $(\hat{\beta}_0, \hat{\beta}_1)$.

We use sample estimates to *infer* about the true population values. Hypothesis tests like testing if $\beta_1 = 0$ ask: "Is it likely the *true* slope in the population is zero, given our sample?".

Estimates vary sample to sample. We test claims about the fixed, population values, using our sample as evidence. We're interested in the underlying truth, not just describing our specific sample.