TIØ4140 Project Evaluation and Financing	**Exercise 5**
16.02.2025	**Scribe: Adrian Fagerland**

Chapter 26: The Black-Scholes-Merton Equation

Remembering

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Calculating the required partial derivatives

1. $\frac{\partial V}{\partial t}$:

   We use the product rule, and chain rule. $\frac{\partial V}{\partial t} = Se^{-\delta(T-t)}\left(\delta N(d_1) + \frac{\partial N(d_1)}{\partial t}\right)$.

   Note that $\frac{\partial N(d_1)}{\partial t} = N'(d_1)\frac{\partial d_1}{\partial t}$. $N'(d_1)$ is just the pdf of the standard normal distribution, with d_1 as an argument.

   After calculating $\frac{\partial d_1}{\partial t}$, we get that

$$\frac{\partial V}{\partial t} = \delta Se^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}N'(d_1)\left[-\frac{\ln(S/K)}{2\sigma(T-t)^{3/2}} + \frac{r-\delta+\sigma^2/2}{-\sigma\sqrt{T-t}}\right]$$

$$\frac{\partial V}{\partial t} = \delta Se^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}N'(d_1)\left[\frac{-\sigma^2(T-t)d_1 - (r-\delta+\sigma^2/2)(T-t)}{\sigma(T-t)\sqrt{T-t}}\right]$$

$$\frac{\partial V}{\partial t} = Se^{-\delta(T-t)}\left(\delta N(d_1) - \frac{N'(d_1)}{\sigma\sqrt{T-t}}\left[(r-\delta+\frac{1}{2}\sigma^2) + d_1\sigma\frac{1}{\sqrt{T-t}}\right]\right)$$

2. $\frac{\partial V}{\partial S}$:

   Using the chain rule, we have:

$$\frac{\partial V}{\partial S} = e^{-\delta(T-t)}N(d_1) + Se^{-\delta(T-t)}N'(d_1)\frac{\partial d_1}{\partial S}$$

   Since $\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$, the result is,

$$\frac{\partial V}{\partial S} = e^{-\delta(T-t)}\left[N(d_1) + \frac{N'(d_1)}{\sigma\sqrt{T-t}}\right]$$

3. $\frac{\partial^2 V}{\partial S^2}$:

$$\frac{\partial^2 V}{\partial S^2} = e^{-\delta(T-t)} N'(d_1) \frac{\partial d_1}{\partial S} + e^{-\delta(T-t)} \left[\frac{N''(d_1) \frac{\partial d_1}{\partial S} \sigma\sqrt{T-t} - N'(d_1)\sigma\sqrt{T-t}\frac{\partial d_1}{\partial S}}{\sigma^2 (T-t)} \right]$$

After simplifying by applying $\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$, it reduces to

$$\frac{\partial^2 V}{\partial S^2} = \frac{e^{-\delta(T-t)}}{S\sigma\sqrt{T-t}} \left[N'(d_1) \left(\frac{-d_1}{S\sigma\sqrt{T-t}} \right) \right]$$
$$\frac{\partial^2 V}{\partial S^2} = -\frac{e^{-\delta(T-t)} N'(d_1) d_1}{S^2 \sigma^2 (T-t)}$$

Plugging these values into the Black-Scholes equation, we get $0 = 0$.

Hence, $V(S,t) = Se^{-\delta(T-t)} N(d_1)$ satisfies the Black-Scholes equation.

Exercises from other sources 1

(a)

$f = e^{-r(T-t)} E^*[\ln(S_T)]$, where E^* denotes the expectation under the risk-neutral measure.

Under the risk-neutral measure, the stock price follows a geometric Brownian motion:

$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma W^*_{(T-t)}}$, where $W^*_{(T-t)}$ is a standard Brownian motion under the risk-neutral measure, and thus is normally distributed ~$N(0, T-t)$.

Taking the natural logarithm of both sides:

$$\ln(S_T) = \ln(S_t) + (r - \frac{1}{2}\sigma^2)(T-t) + \sigma W^*_{(T-t)}$$

Calculating $E^*[\ln(S_T)]$:

$$E^*(\ln S_T) = \ln S + (r - \frac{\sigma^2}{2})(T-t)$$

Substituting into the pricing formula:

$$f = e^{-r(T-t)}[\ln(S_t) + (r - \frac{1}{2}\sigma^2)(T-t)]$$

(b)

1. $\frac{\partial f}{\partial t}$:

$$\frac{\partial f}{\partial t} = re^{-r(T-t)}[\ln(S) + (r - \frac{1}{2}\sigma^2)(T-t)] + e^{-r(T-t)}[-(r - \frac{\sigma^2}{2})]$$

$$\frac{\partial f}{\partial t} = rf - e^{-r(T-t)}(r - \frac{1}{2}\sigma^2)$$

2. $\frac{\partial f}{\partial S}$:

$$\frac{\partial f}{\partial S} = e^{-r(T-t)}\frac{1}{S}$$

3. $\frac{\partial^2 f}{\partial S^2}$:

$$\frac{\partial^2 f}{\partial S^2} = -e^{-r(T-t)}\frac{1}{S^2}$$

Substituting into equation:

$$rf - e^{-r(T-t)}(r - \frac{1}{2}\sigma^2) + \frac{\sigma^2}{2}S^2(-e^{-r(T-t)}\frac{1}{S^2}) + rSe^{-r(T-t)}\frac{1}{S} - rf = 0$$

Simplifying leads to:

$$-e^{-r(T-t)}(r - \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 e^{-r(T-t)} + re^{-r(T-t)} = 0$$

hence $0 = 0$

Thus, the price f does satisfy the equation.

Exercises from other sources 2

The code is in the attached notebook.

a

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Annualized Volatility for AAPL: 0.2383

Last Closing Price: 248.01

b

Asian Call Option Price: 3.7913

Asian Call Option Price STD: 5.8150

c

	Price	STD
100	4.094162	5.931863
1000	3.659383	5.689137
10000	3.769529	5.758022
20000	3.792481	5.729376
50000	3.767705	5.749505
100000	3.817293	5.800743