TIØ4140 Project Evaluation and Financing	**Exercise 7**
15.03.2025	**Scribe: Adrian Fagerland**

Problems from McDonald (2013)

16.7

The formula for Delta-Gamma-Theta approximation:

$$\begin{aligned} C(S_t, t) \approx C(S, 0) &+ \Delta(S, 0) \times [S_t - S] \\ &+ \frac{1}{2} \times \Gamma(S, 0) \times [S_t - S]^2 \\ &+ \theta(S, 0) \times t \end{aligned}$$

First, we can compute d_1 and d_2:

$$d_1 = \frac{\ln(\frac{S}{K}) + (r - \delta + \frac{\sigma^2}{2}) \times T}{\sigma \times \sqrt{T}} = \frac{\ln(\frac{40}{40}) + (0.08 - 0 + \frac{0.3^2}{2}) \times 0.49315}{0.3 \times \sqrt{0.49315}} = 0.2926$$

$$d_2 = d_1 - \sigma \times \sqrt{T} = 0.2926 - 0.3 \times \sqrt{0.49315} = 0.08193$$

Now using the Black-Scholes formula we compute the Greeks at Day 0.

$$\Delta = e^{-\delta \times T} \times N(d_1) = e^{-0 \times 0.49315} \times N(0.2926) = 0.6151$$

$$\Gamma = \frac{N'(d_1)}{S \times \sigma \times \sqrt{T}} = \frac{\frac{1}{\sqrt{2\pi}} e^{-0.2926^2/2}}{40 \times 0.3 \times \sqrt{0.49315}} = 0.4536$$

$$\theta = \frac{-S \times N'(d_1) \times \sigma}{2\sqrt{T}} - r \times K \times e^{-rT} \times N(d_2)$$

$$= \frac{-40 \times N'(0.2926) \times 0.3}{2\sqrt{0.49315}} - 0.08 \times 40 \times e^{-0.08 \times 0.49315} \times N(0.08193) = -0.0134 \text{ per day}$$

Initial Call Premium:

$$C(S, 0) = S \times e^{-\delta \times T} \times N(d_1) - K \times e^{-r \times T} \times N(d_2) = \$4.122$$

	1		5		25	
	Actual	Predicted	Actual	Predicted	Actual	Predicted
Stock Price						
36.00	2.0365	2.0108	1.9921	1.9571	1.7660	1.6884
36.25	2.1424	2.1207	2.0971	2.0669	1.8663	1.7982
36.50	2.2515	2.2333	2.2053	2.1796	1.9701	1.9108
36.75	2.3637	2.3488	2.3168	2.2951	2.0773	2.0264
37.00	2.4792	2.4672	2.4315	2.4134	2.1880	2.1447
37.25	2.5979	2.5883	2.5495	2.5346	2.3020	2.2659
37.50	2.7198	2.7123	2.6707	2.6586	2.4195	2.3899
37.75	2.8449	2.8392	2.7951	2.7854	2.5403	2.5167
38.00	2.9730	2.9689	2.9226	2.9151	2.6645	2.6464
38.25	3.1044	3.1014	3.0534	3.0476	2.7921	2.7789
38.50	3.2387	3.2367	3.1872	3.1830	2.9231	2.9142
38.75	3.3762	3.3749	3.3241	3.3211	3.0573	3.0524
39.00	3.5167	3.5159	3.4642	3.4622	3.1948	3.1934
39.25	3.6602	3.6598	3.6072	3.6060	3.3355	3.3373
39.50	3.8067	3.8064	3.7533	3.7527	3.4794	3.4840
39.75	3.9560	3.9560	3.9023	3.9022	3.6265	3.6335
40.00	4.1083	4.1083	4.0542	4.0546	3.7767	3.7858
40.25	4.2634	4.2635	4.2090	4.2098	3.9300	3.9410
40.50	4.4213	4.4215	4.3667	4.3678	4.0863	4.0991
40.75	4.5820	4.5824	4.5271	4.5286	4.2456	4.2599
41.00	4.7454	4.7461	4.6903	4.6923	4.4078	4.4236
41.25	4.9114	4.9126	4.8562	4.8589	4.5729	4.5901
41.50	5.0800	5.0820	5.0247	5.0282	4.7408	4.7595
41.75	5.2513	5.2542	5.1958	5.2004	4.9115	4.9317
42.00	5.4250	5.4292	5.3695	5.3755	5.0848	5.1067
42.25	5.6012	5.6071	5.5456	5.5533	5.2609	5.2846
42.50	5.7798	5.7878	5.7242	5.7340	5.4395	5.4653
42.75	5.9607	5.9713	5.9052	5.9176	5.6206	5.6488
43.00	6.1440	6.1577	6.0885	6.1039	5.8043	5.8352
43.25	6.3295	6.3469	6.2741	6.2931	5.9903	6.0244
43.50	6.5173	6.5389	6.4619	6.4852	6.1787	6.2165
43.75	6.7071	6.7338	6.6519	6.6801	6.3694	6.4113
44.00	6.8991	6.9315	6.8440	6.8778	6.5623	6.6090

The approximation is most accurate when the change in the underlying stock price and time are both small. As these increase, the approximation becomes less accurate.

17.11

a.

We are given: $S_0 = \$40$, $\sigma = 0.30$, $\delta = 0$, $K = \$40$, $r = 0.08$, and $T = 2$ years. We'll use the Black-Scholes formula to determine the price of the call.

$d_1 = \frac{\ln(\frac{S_0}{K}) + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = \frac{\ln(\frac{40}{40}) + (0.08 - 0 + \frac{0.3^2}{2})2}{0.3\sqrt{2}} = 0.49497$

$d_2 = d_1 - \sigma\sqrt{T} = 0.49497 - 0.3\sqrt{2} = 0.0707$

Now, the Call option price is

$C = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) = 40 \times e^{-0 \times 2} \times N(0.49497) - 40 \times e^{-0.08 \times 2} \times N(0.0707) = 40 \times 0.6897 - 40 \times e^{-0.16} \times 0.5282 = \$9.864$

b.

This compound call gives the right to buy the 2-year call option (from part a) in 1 year for \$2. We will exercise the call option at time $t = 1$ when the value of the option at that time is larger than \$2.

We must obtain the price S_1 such that, when entered to the black-scholes formula, will give us a call option price of \$2.

After calculating in Python: The stock price S1 at which the compound call is exercised: 31.72

So, we will want to execute the compound option when $S_1 > \$31.72$.

c.

We are pricing a call on a call. We have an option that gives the right after $t = 1$ to buy a call for $x = \$2$ and with expiration date $T = 2$ years. We already computed $S_1^* = \$35.89$, at which price, we are going to start paying for the call. The price for the compound call is found by using Geske's Formula.

$C_c = S_0 e^{-\delta T} M(a_1, b_1; \rho) - K e^{-rT} M(a_2, b_2; \rho) - e^{-rt} x N(a_2)$

Where: $a_1 = \frac{\ln(\frac{S_0}{S_1^*}) + (r - \delta + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} = \frac{\ln(\frac{40}{35.89}) + (0.08 - 0 + \frac{0.3^2}{2})1}{0.3\sqrt{1}} = 0.76844$ $a_2 = a_1 - \sigma\sqrt{t} = 0.76844 - 0.3 = 0.46844$

$b_1 = \frac{\ln(\frac{S_0}{K}) + (r - \delta + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = \frac{\ln(\frac{40}{40}) + (0.08 - 0 + \frac{0.3^2}{2})2}{0.3\sqrt{2}} = 0.49497$ $b_2 = b_1 - \sigma\sqrt{T} = 0.0707$

$\rho = \sqrt{\frac{t}{T}} = \sqrt{\frac{1}{2}} = 0.7071$

$M(a, b, \rho)$ is the cumulative bivariate normal distribution.

After calculating in Python: Price of compound call option is: 7.8419

d.

Here, we have a put on a call. We have an option that gives the right after $t = 1$ to sell a call option for $x = \$2$. We want to find the price of the compound put. The option with expiration at $T = 2$. Geske's formula for a compound put is:

$P_c = Ke^{-rT}M(-a_2, b_2; -\rho) - S_0e^{-\delta T}M(-a_1, b_1; -\rho) + e^{-rt}xN(-a_2)$

Where a_1, a_2, b_1, b_2, ρ are the same as on part c.

After calculating in Python: Compound Put Price: 0.0783

21.13

Since the cash flows grow at the same rate as the risk-free rate during the initial 10 years, we can recognize that these cash flows are equivalent to the initial cash flow of \$1. The cash flow after year 10 remains constant. We want to find the present value formula for the project, we denote the time in which we invest t^*. The value of the cash flows at time t^* is:

$$PV_{t^*} = \sum_{i=1}^{10} \frac{1 * (1.05)^{i-1}}{(1.05)^i} + \frac{1}{0.05} * \frac{(1.05)^9}{(1.05)^{10}} = 28.5714$$

So at the moment of starting the project (t^*), the present value of benefits is \$28.5714. Therefore, the net present value at time zero is:

$$NPV(t^*) = \frac{PV_{t^*} - I}{(1+r)^{t^*}}$$

We should invest as soon as $PV \geq I$, therefore, the best is to invest immediately at $t = 0$.

The value of investing at time $t = 0$ is: $NPV(0) = PV_0 - 20 = 8.5714$. The value of the option to invest is simply the NPV of the project, as it is the most optimal to start as soon as possible.

21.14

The land's value is the option to extract the oil plus the residual value discounted back, like a perpetual American call option.

The option value is $V(S) = AS^\beta$ when $S < S^*$. First, we need β from the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r = 0$. Calculating with Python gives $\beta = 1.512430$ (the positive root).

Next, the optimal exercise threshold is $S^* = \frac{\beta(X-R)}{\beta-1}$. With $X - R = 12.60$ and $\beta - 1 = 0.512430$, we get $S^* = 37.19$. So, extraction happens when the oil price hits 37.19.

Then, we find A using the value-matching condition $A(S^*)^\beta = S^- X + R$. The payoff is $S^* - X + R = 24.59$, and $(S^*)^\beta = 237.21$, so $A = 0.103657$.

Finally, the land's value at $S = 15$ is $V(S) = AS^\beta$. With $S^\beta = 60.08$, we compute $V(15) = 6.23$. Since $15 < 37.19$, we don't extract now, and 6.23 is the land's value.

Exam problem

Since the cash flows grow at the same rate as the risk-free rate during the initial 10 years, we can recognize that these cash flows are equivalent to the initial cash flow of \$1. The cash flow after year 10 remains constant. We want to find the present value formula for the project, we denote the time in which we invest t^*. The value of the cash flows at time t^* is:

$$PV_{t^*} = \sum_{i=1}^{10} \frac{1 * (1.05)^{i-1}}{(1.05)^i} + \frac{1}{0.05} * \frac{(1.05)^9}{(1.05)^{10}} = 28.5714$$

So at the moment of starting the project (t^*), the present value of benefits is \$28.5714. Therefore, the net present value at time zero is:

$$NPV(t^*) = \frac{PV_{t^*} - I}{(1+r)^{t^*}}$$

We should invest as soon as $PV \geq I$, therefore, the best is to invest immediately at $t = 0$.

The value of investing at time $t = 0$ is: $NPV(0) = PV_0 - 20 = 8.5714$. The value of the option to invest is simply the NPV of the project, as it is the most optimal to start as soon as possible.

a)

The price of a European gap call option can be computed using a variation of the Black-Scholes formula.

Given that there is no dividend, we set $q = 0$. Substituting the provided values into the Black-Scholes formula:

$$d_1 = \frac{\ln(100/100) + (0.02 + 0.8^2/2) \cdot 2}{0.8\sqrt{2}} \approx 0.6010$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.6010 - 0.8\sqrt{2} \approx -0.5303$$

$N(d_1) \approx 0.7261$ and $N(d_2) \approx 0.2979$.

Substituting these into the gap call option price formula:

$$C = 100 \cdot 0.7261 - 150e^{-0.02 \cdot 2} \cdot 0.2979 \approx 29.6705$$

Therefore, the price of one gap call option is approximately 29.67.

b)

The Black-Scholes formula for a plain vanilla call option is:

$$C_{vanilla} = S_0 e^{-qT} N(d_{1,vanilla}) - Ke^{-rT} N(d_{2,vanilla})$$

where

$$d_{1,vanilla} = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_{2,vanilla} = d_{1,vanilla} - \sigma\sqrt{T}$$

The critical difference is that the gap call option uses the trigger price, L, in the calculation of d_1 and d_2, while the vanilla option uses the strike price, K. This leads to different values for $N(d_1)$ and $N(d_2)$, and therefore a different option price.

The gap call option price can be expressed as:

$$C_{gap} = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where d_1 and d_2 are calculated using L (the trigger), as shown in part a. The vanilla call option, in contrast, would use K in its d_1 and d_2 calculations. The core formula structure is the same; it is the inputs to the d_1 and d_2 terms that create the difference.