TIØ4140 Project Evaluation and Financing

Exercise 5: Brownian Motion and Itô's Lemma

Posted: Monday, February 10, 2025.
Deadline: Tuesday, February 18, 2025, 23:59.
Grading: Approved / Not approved.

N.B.:

- *To get "Approved", you should attempt to solve all mandatory tasks and have **70%** correct.*
- *Remember to write down the **main solution steps**!*

Overview of Tasks

- **Mandatory Tasks**:
  - Exercises from other sources
  - Problem from McDonald (2013)
  - Exam problem
- **Voluntary Tasks:**
  - Exam problem
  - Exercises from other sources

Mandatory Tasks

- ### Exercises from other sources

1. Assume we have a stock price that follows geometric Brownian motion. An expected return is 0.16, a volatility is 0.35. The current price is 38 EUR. Find:
   a. The probability that a European call option on the stock with an exercise price of 40 EUR and a maturity date in 6 months will be exercised?
   b. The probability that a European put option on the stock with the same exercise price and maturity to be exercised?

   Hint: For a stock price S_T following geometric Brownian motion, the natural logarithm of S_T at time T is normally distributed. The probability of the stock price is derived using the formula for the cumulative normal distribution: $\ln S_T \sim \phi\left[lnS_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$, and $P(S_T>S_0) = 1-N(Z)$.

2. Simulate 100 sample paths of the following stochastic processes:
   a) Brownian motion.
   b) Arithmetic Brownian motion.
   c) Ornstein-Uhlenbeck process.
   d) Geometric Brownian motion.

   For each case, draw a plot with the 100 sample paths and a line representing the expected value of the process.

3. Suppose that:
   3.1. S follows equation: $\frac{dS}{S} = (r - \delta_S)dt + \sigma_S dZ_S$ and Q follows equation $\frac{dQ}{Q} = (r - \delta_Q)dt + \sigma_Q dZ_Q$.
   Use Itô'sLemma to find the process followed by $\mathbf{S^2Q^{0.5}}$.

3.2. S follows equation: $\frac{dS}{S} = (r - \delta_S)dt + \sigma_S dZ_S$ and Q follows equation $\frac{dQ}{Q} = (r - \delta_Q)dt + \sigma_Q dZ_Q$.

Use Itô's Lemma to find the process followed by **ln(SQ).**

3.3. The processes for S_1 and S_2 are given by these two equations:

$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dZ_1$

$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dZ_2$

Note that the diffusions dZ_1 and dZ_2 are different.

3.3.1. Find **the expected return on Q, α_Q,** where Q follows the process:

$$dQ = \alpha_Q Q dt + Q(\eta_1 dZ_1 + \eta_2 dZ_2)$$

3.3.2. Show that, to avoid arbitrage:

$$\alpha_Q - r = \frac{\eta_1}{\sigma_1}(\alpha_1 - r) + \frac{\eta_2}{\sigma_2}(\alpha_2 - r)$$

(*Hint:* Consider the strategy of buying one unit of Q and shorting $Q\eta_1/S_1\sigma_1$ units of S_1 and $Q\eta_2/S_2\sigma_2$ units of S_2. Finance any net cost using risk-free bonds.)

- **Exam problem**

a. The stock price process is usually described by:

$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$ (*)

where S is the stock price, Δt the time increment, μ is the drift, σ is the volatility and ε is a random number from N(0,1). Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

$$\Delta S = \mu S \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

$$\Delta S = \mu \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

Why is the first process (*) a more appropriate model of stock price behavior than any of these three alternatives?

b. Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow a geometric Brownian motion? Explain your answer.

c. If S follows the geometric Brownian motion process, what is the process followed by $y=e^{\alpha S}$ where α is a constant?

Voluntary Tasks

- **Exam problem**

Let X and Y satisfy the following system of SDE's:

$$dX(t) = \alpha X(t)dt + Y(t)dB(t), \quad X(0) = x_0$$

$$dY(t) = \alpha Y(t)dt + \sigma X(t)dB(t), \quad Y(0) = y_0$$

a. Calculate the expected values E[$X(t)$] and E[$Y(t)$].

b. Use Itô's lemma to compute $dR(t)$ where $R(t) = X^2(t) + Y^2(t)$

c. Show that for $\sigma = -1$ and some arbitrary α the process R(t) becomes deterministic.

d. For arbitrary α and $\sigma = -1$, calculate the expected values $E[X(t)]$, $E[Y(t)]$ and the covariance $Cov(X(t),Y(t))$.
Hint: $Cov(X(t),Y(t)) = E[X(t)Y(t)] - E[X(t)]E[Y(t)]$.

- ## Exercises from other sources

1. Suppose that a stock price is currently $20 and that a call option with an exercise price of $25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:
   a. Stock price increases steadily from $20 to $35 during the life of the option.
   b. Stock prices oscillates wildly, ending up at $35.

   Which scenario would make the synthetically created option more expensive? Explain your answer.

2. An exchange rate is currently 0.8000. The volatility of the exchange rate is quoted as 12% and interest rates in the two countries are the same. Using the lognormal assumption, estimate the probability that the exchange rate in three months will be (a) less than 0.7000, (b) between 0.7000 and 0.7500, (c) between 0.7500 and 0.8000, (d) between 0.8000 and 0.8500, (e) between 0.8500 and 0.9000, and (f) greater than 0.9000. Based on the volatility smile usually observed in the market for exchange rates, which of these estimates would you expect to be too low, and which would you expect to be too high?