Formula for $x!$ and its Relations to the Harmonic Numbers

Adrian Fang

May 31, 2023

1 Generalising $x!$ For All $\mathbb{R}$

Suppose that the definition of $n!$ is the product of all of the numbers from 1 to n. To generalise this as an equation, $n!$ can be written as:

$$x! = x \cdot (x-1)! \tag{1}$$

Given how a property of logarithms allow for products to become sums of logs, the natural logarithm of $x!$ can be taken:

$$x! = \sum_{k=1}^{n} ln\,(k) \tag{2}$$

Suppose that the natural log of the factorial is associated with a function, in this case called M:

$$M(x) = \sum_{k=1}^{x} ln\,(k) \tag{3}$$

In this case, this general formula for the natural log of a factorial can be raised as a power of e to obtain a formula for $n!$, yet will only account for integer values. Suppose that a translated iteration of this function exists, which can be altered to account for two separate sums rather than 1, where x is less than n:

$$M(x+n) = \sum_{k=1}^{x+n} ln\,(k) \tag{4}$$

$$M(x+n) = \sum_{k=1}^{x} ln\,(k) + \sum_{k=(x+1)}^{n} ln\,(k) \tag{5}$$

$$M(x+n) = \sum_{k=1}^{x} ln\,(k) + \sum_{k=1}^{n} ln\,(k+x) \tag{6}$$

Now, this function can be shown to be recursive, where for function $M(x)$, if it was to be shifted by any value, the end result would contain $M(x)$:

$$M(x+n) = M(x) + \sum_{k=1}^{n} ln\,(x+k) \tag{7}$$

As function M would diverge, the limit of M as a random variable N approaches infinity can be taken, replacing x.

$$\lim_{N\to\infty} M(N+n) = \lim_{N\to\infty} \left(M(N) + \sum_{k=1}^{n} ln\,(N+k) \right) \tag{8}$$

To simplify this equation further, the behaviour of the logarithmic factorial must be understood. Given how both $ln(x)$ and $x!$ diverge, a generalisation of $ln(x!)$ can be made. Since $ln(x)$ diverges very slowly, $ln(x!)$ will also diverge slowly, which can be shown by taking its second derivative:

$$\frac{d}{dx} ln\,(x!) = \frac{1}{x!} \cdot \frac{d}{dx} x! \tag{9}$$

$$\frac{d^2}{dx^2} ln\,(x!) = \frac{d}{dx} \left(\frac{1}{x!} \cdot \frac{d}{dx} x! \right) \tag{10}$$

$$\frac{d^2}{dx^2} ln\,(x!) = -\left(\frac{1}{x!}\right)^2 \left(\frac{d}{dx} x!\right)^2 + \frac{1}{x!} \frac{d^2}{dx^2} x! \tag{11}$$

Now, taking the limit as x approaches infinity:

$$\lim_{x\to\infty} \frac{d^2}{dx^2} ln\,(x!) = \lim_{x\to\infty} \left(-\left(\frac{1}{x!}\right)^2 \left(\frac{d}{dx} x!\right)^2 + \frac{1}{x!} \frac{d^2}{dx^2} x! \right) \tag{12}$$

It is implied that $x!$ diverges faster than its derivative, let alone its second as well, meaning that the second derivative of $ln(x!)$ will eventually converge:

$$\lim_{x\to\infty} \frac{d^2}{dx^2} ln\,(x!) = 0 \tag{13}$$

Due to this, the logarithmic factorial slowly approaches that of a line; implying that for $M(N) - M(N-1) \approx ln(N)$ and $ln(N) \approx ln(N+1) \approx ln(N+2) \approx ln(N+k)$. As this statement goes for all $\mathbb{Z}$, it can be generalised as $ln(N) \approx ln(N+k)$. Hence, the finite sum $ln(N+k)$ can be written as $ln(N)$, where the summation would no longer be necessary. As N approaches infinity, the statement fundamentally becomes true, which means that the approximation becomes a true statement.

$$\lim_{N\to\infty} M(N+n) = \lim_{N\to\infty} \left(M(N) + \sum_{k=1}^{n} ln\,(N) \right) \tag{14}$$

$$\lim_{N\to\infty} M(N+n) = \lim_{N\to\infty} \left(M(N) + n \cdot ln(N) \right) \tag{15}$$

Now, the n can be replaced with x to represent an extended definition of the recursive formula:

$$\lim_{N\to\infty} M(N+x) = \lim_{N\to\infty} \left(M(N) + xln(N)\right) \tag{16}$$

To generalise this and find $M(x)$ for all $\mathbb{R}$, $M(N+x)$ can be replaced with the recursive formula, where $M(x)$ can be isolated:

$$\lim_{N\to\infty} \left(M(x) + \sum_{k=1}^{N} ln\left(k+x\right)\right) = \lim_{N\to\infty} \left(M(N) + xln(N)\right) \tag{17}$$

$$M(x) = \lim_{N\to\infty} \left(\sum_{k=1}^{N} ln\left(k\right) - \sum_{k=1}^{N} ln\left(k+x\right) + xln(N)\right) \tag{18}$$

$$M(x) = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \left(ln\left(k\right) - ln\left(k+x\right)\right) + xln(N)\right) \tag{19}$$

$$M(x) = \lim_{N\to\infty} \left(\sum_{k=1}^{N} ln\left(\frac{k}{k+x}\right) + xln(N)\right) \tag{20}$$

With a generalised formula for $M(x)$ that encompasses all real numbers, it can be raised as a power of e to find $x!$, since $M(x) = ln(x!)$:

$$ln(x!) = \lim_{N\to\infty} \left(\sum_{k=1}^{N} ln\left(\frac{k}{k+x}\right) + xln(N)\right) \tag{21}$$

$$x! = \lim_{N\to\infty} exp\left(\sum_{k=1}^{N} ln\left(\frac{k}{k+x}\right) + xln(N)\right) \tag{22}$$

$$x! = \lim_{N\to\infty} N^x \prod_{k=1}^{N} \frac{k}{k+x} \tag{23}$$

Now, a generalised formula for $x!$ has been derived, which extends to non-integer values.

2 Generalising the Harmonic Numbers

The fundamental definition of a harmonic number is defined as the sum of all of the reciprocal integers from 1 to some value x:

$$H_x = \sum_{k=1}^{x} \frac{1}{k} \tag{24}$$

Since H_x slowly diverges, an extended definition of the harmonic numbers can be written:

$$\lim_{N \to \infty} (H_N - H_{N+x}) = 0 \tag{25}$$

As N would approach infinity, when being subtracted by H_{N+x}, the result would be 0. By definition, the following can be inferred:

$$H_x = \lim_{N \to \infty} (H_N - (H_{N+x} - H_N)) \tag{26}$$

Expanding this out as sums can now be done, where the generalisation for the harmonic numbers for all $\mathbb{R}$ has been derived:

$$H_x = \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \left(\sum_{k=1}^{N+x} \frac{1}{k} - \sum_{k=1}^{x} \frac{1}{k} \right) \right) \tag{27}$$

$$H_x = \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \left(\sum_{k=1}^{x} \frac{1}{k} + \sum_{k=(x+1)}^{N} \frac{1}{k} - \sum_{k=1}^{x} \frac{1}{k} \right) \right) \tag{28}$$

$$H_x = \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{N} \frac{1}{k+x} \right) \tag{29}$$

$$H_x = \lim_{N \to \infty} \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+x} \right) \tag{30}$$

Now, the x harmonic number can be taken, and has been extended to all real numbers.

3 Relating the Harmonic Numbers to Factorials

With generalised formulae for both the Factorial and Harmonic Numbers, there happens to be a strange relationship between both of them. By differentiating and applying a vertical translation upwards by γ, the Euler Mascheroni Constant, it will equate to the harmonic numbers. Assume that γ is known, with its definition below:

$$\gamma = \lim_{N \to \infty} \left(\sum_{k=1}^{N} \frac{1}{k} - ln(N) \right) \approx 0.577 \tag{31}$$

First, differentiating the logarithmic factorial is to be done, which can be done by taking the derivative of $M(x)$:

$$\frac{d}{dx} M(x) = \lim_{N \to \infty} \frac{d}{dx} \left(\sum_{k=1}^{N} ln \left(\frac{k}{k+x} \right) + xln(N) \right) \tag{32}$$

$$\frac{d}{dx}M(x) = \lim_{N\to\infty}\left(\frac{d}{dx}\sum_{k=1}^{N} ln\left(\frac{k}{k+x}\right) + \frac{d}{dx}xln(N)\right) \tag{33}$$

$$\frac{d}{dx}M(x) = \lim_{N\to\infty}\left(\frac{d}{dx}\left(\sum_{k=1}^{N} ln\left(k\right) - \sum_{k=1}^{N} ln\left(k+x\right)\right) + ln(N)\right) \tag{34}$$

$$\frac{d}{dx}M(x) = \lim_{N\to\infty}\left(-\sum_{k=1}^{N}\frac{1}{k+x} + ln(N)\right) \tag{35}$$

Once fully differentiated, γ is used to vertically shift $M'(x)$:

$$\frac{d}{dx}M(x) + \gamma = \lim_{N\to\infty}\left(-\sum_{k=1}^{N}\frac{1}{k+x} + ln(N) + \sum_{k=1}^{N}\frac{1}{k} - ln(N)\right) \tag{36}$$

$$\frac{d}{dx}M(x) + \gamma = \lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{k} - \frac{1}{k+x}\right) \tag{37}$$

$$\frac{d}{dx}ln\left(x!\right) + \gamma = H_x \tag{38}$$

As a result, the derivative of the Logarithmic Factorial shifted upwards by γ is equivalent to the Harmonic Numbers.