The Solution to Andrea's Problem

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1 Introduction of the Problem

Find the value of the expression below:

$$\lim_{x \to \int_0^\infty \sqrt{t}e^{-t}dt} \left(\sum_{n=0}^{\infty} \frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} \right)'' \tag{1}$$

2 Finding What x Approaches

To evaluate the limit, the value that x approaches needs to be found. Since the value of x is the improper integral, it will need to be evaluated by itself:

$$\int_0^\infty \sqrt{t}e^{-t}dt \tag{2}$$

Integration of this function will involve integration by parts, which formula is shown below:

$$\int uv = u \int v - \int \left(u' \int v \right) \tag{3}$$

In this case, using the order at which integration by parts is done (ILATE), $u = \sqrt{t}$ and $v = e^{-t}$:

$$\int \sqrt{t}e^{-t}dt = \sqrt{t} \int e^{-t}dt - \int \left(\frac{d}{dx}\sqrt{t} \int e^{-t}dt \right) dt \tag{4}$$

Now, further integration can happen in the function e^{-t}:

$$\int e^{-t}dt = \int e^{u}dt,\ u = -t$$

$$\frac{du}{dt} = -1$$

$$-du = dt$$

$$\int e^{-t}dt = -\int e^{u}du$$

$$\int e^{-t}dt = -e^{u}$$

$$\int e^{-t}dt = -e^{-t} + C$$

With integration of e^{-t} being completed, it can be substituted back into the equation.

$$\int \sqrt{t}e^{-t}dt = \sqrt{t} \cdot -e^{-t} - \int \left(\frac{d}{dx}\sqrt{t} \cdot -e^{-t} \right) dt$$

Before evaluating the other integral, the derivative of $\sqrt{t}$ has to be found, which can be easily found with the power rule in differentiation ($\frac{d}{dx}x^n = nx^{n-1}$):

$$\int \sqrt{t}e^{-t}dt = -\sqrt{t}e^{-t} + \int \frac{e^{-t}}{2\sqrt{t}}dt \tag{5}$$

Now, the integral of $\frac{e^{-t}}{2\sqrt{t}}$ can be found:

$$\int \frac{e^{-t}}{2\sqrt{t}}dt = \int \frac{e^{-u^2}}{2u}dt,\ u = \sqrt{t}$$

$$\frac{du}{dt} = \frac{1}{2\sqrt{t}}$$

$$dt = 2\sqrt{t}du$$

$$\int \frac{e^{-t}}{2\sqrt{t}}dt = \int \frac{e^{-u^2}}{2u} \cdot 2\sqrt{t}du$$

$$\int \frac{e^{-t}}{2\sqrt{t}}dt = \int e^{-u^2}du \tag{6}$$

Although integration of e^{-u^2} is possible, this will be done later. By converting the function into the following below, it can be immediately substituted as the erf function:

$$\int \frac{e^{-t}}{2\sqrt{t}}dt = \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int e^{-u^2}du \tag{7}$$

Using the rule below, the integral of e^{-u^2} with respect to u can be found:

$$erf\left(z\right) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}dt$$

$$\int \frac{e^{-t}}{2\sqrt{t}}dt = \frac{\sqrt{\pi}}{2}erf\left(u\right)$$

$$\int \frac{e^{-t}}{2\sqrt{t}}dt = \frac{\sqrt{\pi}}{2}erf\left(\sqrt{t}\right) + C$$

Now, inserting the evaluated integral back into the original equation, done below:

$$\int \sqrt{t}e^{-t}dt = -\sqrt{t}e^{-t} + \frac{\sqrt{\pi}}{2} erf\left(\sqrt{t}\right) + C \tag{8}$$

Since the original integral specifies a range between 0 and ∞, the limit of the integrated function (without the constant C) when t approaches ∞ can be found to obtain the value of the improper integral:

$$\begin{aligned}
\int_0^\infty \sqrt{t}e^{-t}dt &= \lim_{t\to\infty}\left(\sqrt{t}e^{-t} + \frac{\sqrt{\pi}}{2} erf\left(\sqrt{t}\right)\right)\\
\int_0^\infty \sqrt{t}e^{-t}dt &= -\lim_{t\to\infty}\frac{\sqrt{t}}{e^t} + \frac{\sqrt{\pi}}{2}\lim_{t\to\infty} erf\left(\sqrt{t}\right)\\
\int_0^\infty \sqrt{t}e^{-t}dt &= -\lim_{t\to\infty}\frac{\frac{d}{dt}\sqrt{t}}{\frac{d}{dt}e^t} + \frac{\sqrt{\pi}}{2}\lim_{t\to\infty} erf\left(\sqrt{t}\right)\\
\int_0^\infty \sqrt{t}e^{-t}dt &= -\lim_{t\to\infty}\frac{1}{2\sqrt{t}e^t} + \frac{\sqrt{\pi}}{2}\lim_{t\to\infty} erf\left(\sqrt{t}\right)\\
\int_0^\infty \sqrt{t}e^{-t}dt &= 0 + \frac{\sqrt{\pi}}{2}\cdot\lim_{t\to\infty} erf\left(\sqrt{t}\right)\\
\int_0^\infty \sqrt{t}e^{-t}dt &= \frac{\sqrt{\pi}}{2}\cdot\lim_{t\to\infty} erf\left(\sqrt{t}\right)
\end{aligned} \tag{9}$$

To find the limit of erf when x approaches ∞, $erf\left(\sqrt{t}\right)$ needs to be reverted back into its integral form, where $\sqrt{t}$ is the upper bound of the improper integral:

$$\lim_{t\to\infty} erf\left(\sqrt{t}\right) = \lim_{t\to\infty}\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx \tag{10}$$

Since $\sqrt{t}$ approaches ∞, it is used instead as the upper bound, shown below:

$$\lim_{t\to\infty} erf\left(\sqrt{t}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-x^2}dx \tag{11}$$

Given how the function e^{-x^2} is even, implies that that it is equivalent to half of the Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-x^2}dx \tag{12}$$

To evaluate this Integral, it has to be defined to a variable, in this case I:

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx \tag{13}$$

Variable I can be squared, which allows for a double integral to be used. Changing the variable in the second term does not change the value of the integral, where integrating with respect to different variables allows for flexibility.

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy$$

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dydx \tag{14}$$

To further evaluate this integral, more concepts need to be introduced. The function in the integral will create a 3D bell shape meaning that it has to be simplified in a way that can allow for integration with respect to one variable. The following below is the current function:

$$f(x, y) = e^{-\left(x^2+y^2\right)} \tag{15}$$

The equation for a circle can be implemented, which formula is below:

$$r^2 = x^2 + y^2 \tag{16}$$

Substituting r^2 for $x^2 + y^2$ changes the function to have only one input (r), shown below:

$$f(r) = e^{-r^2} \tag{17}$$

Given the nature at which the function f is graphed, an infinite sum of shells can be found to subsequently calculate the overall volume. To find the area of a shell in terms of r, it can be found by multiplying thickness (dr), circumference ($2\pi r$) and height ($f(r)$):

$$V = f(r) \cdot 2\pi r \cdot dr$$

$$V = e^{-r^2} 2\pi r dr \tag{18}$$

dr represents an infinitely small value. When the infinite sum of the volume function is taken, the area under the shape in the previous function ($f(x, y)$) can be found:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dydx = \int_{0}^{\infty} e^{-r^2} 2\pi r dr$$

Now, the volume function can be integrated:

$$\int_{0}^{\infty} e^{-r^2} 2\pi r dr = \pi \int_{0}^{\infty} e^{-r^2} 2r dr, u = r^2$$

$$\frac{du}{dr} = 2r$$

$$dr = \frac{du}{2r}$$

$$\int_{0}^{\infty} e^{-r^2} 2\pi r dr = \pi \int_{0}^{\infty} e^{-u} 2r \frac{du}{2r}$$

$$\int_0^\infty e^{-r^2} 2\pi r dr = \pi \int_0^\infty e^{-u} du$$

$$\int_0^\infty e^{-r^2} 2\pi r dr = \pi \cdot -e^{-r^2}|_0^\infty$$

$$\int_0^\infty e^{-r^2} 2\pi r dr = \pi \cdot \left(\lim_{r\to\infty} -e^{-r^2} + 1\right)$$

$$\int_0^\infty e^{-r^2} 2\pi r dr = \pi \tag{19}$$

With the evaluation of the integral complete, the value for I^2 has been calculated. To find the value of I, the square root of I^2 has to be taken. With a value of I, the Gaussian Integral has been found:

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi} \tag{20}$$

Returning back to the original expression, the value of the Gaussian Integral divided by 2 multiplied by $\frac{2}{\sqrt{\pi}}$ is equal to the following below:

$$\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx \tag{21}$$

Further evaluation and substitution is done to fully simplify:

$$\lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1 \tag{22}$$

Since this expression equates to 1, inserting this back into the expression fully evaluates the original integral and solving for the value that x approaches in the limit:

$$\int_0^\infty \sqrt{t} e^{-t} dt = \frac{\sqrt{\pi}}{2} \tag{23}$$

3 Using x to Find the Limit of the Infinite Sum

Now that the value of x has been found, the new expression is below:

$$\lim_{x\to\frac{\sqrt{\pi}}{2}} \left(\sum_{n=0}^\infty \frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)}\right)'' \tag{24}$$

Understand that for every derivative of function $f(x)$, the derivative of the sum of all of the terms is equal to if all of the terms were individually differentiated. Due to this rule, the following can be inferred:

$$\frac{d}{dx}\sum_{n=j}^{k} f(x) = \sum_{n=j}^{k} \frac{d}{dx} f(x) \tag{25}$$

The rule states that the function in the infinite sum can be differentiated and would not change the sum if evaluated. Using this, the second derivative of the function inside of the sum can be found with respect to x, where n would be considered as a constant:

$$\left(\sum_{n=0}^{\infty} \frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)}\right)'' = \sum_{n=0}^{\infty} \frac{d^2}{dx^2} \frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)}$$

$$\frac{d}{dx}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{1}{(4n+3)(2n+1)(4n+4)}\frac{d}{dx}x^{4n+4}$$

$$\frac{d}{dx}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{1}{(4n+3)(2n+1)(4n+4)}(4n+4)x^{4n+3}$$

$$\frac{d}{dx}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{1}{(4n+3)(2n+1)}x^{4n+3}$$

$$\frac{d^2}{dx^2}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{1}{(4n+3)(2n+1)}\frac{d}{dx}x^{4n+3}$$

$$\frac{d^2}{dx^2}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{1}{(4n+3)(2n+1)}(4n+3)x^{4n+2}$$

$$\frac{d^2}{dx^2}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{x^{4n+2}}{2n+1}$$

$$\frac{d^2}{dx^2}\frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} = \frac{x^{2(2n+1)}}{2n+1} \tag{26}$$

With the second derivative of the function inside of the infinite sum, the simplified expression is below:

$$\lim_{x\to\frac{\sqrt{\pi}}{2}} \sum_{n=0}^{\infty} \frac{x^{2(2n+1)}}{2n+1} \tag{27}$$

Without any external operations that may possibly change x, it can be inserted into the infinite sum:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{\sqrt{\pi}}{2}\right)^{2(2n+1)}}{2n+1}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{\pi}{4}\right)^{2n+1}}{2n+1} \tag{28}$$

The sum now resembles the Taylor Series approximation of the hyperbolic inverse tangent function, below:

$$arctanh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \tag{29}$$

Using this information, more simplification can be done to convert the infinite series into a simple function and solving the problem:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{\pi}{4}\right)^{2n+1}}{2n+1} = arctanh\left(\frac{\pi}{4}\right)$$

$$arctanh\left(\frac{\pi}{4}\right) \approx 1.05930617082 \tag{30}$$

Finally, the expression has been fully evaluated and the following has been proved:

$$\lim_{x \to \int_0^\infty \sqrt{t}e^{-t}dt} \left(\sum_{n=0}^{\infty} \frac{x^{4n+4}}{(4n+3)(2n+1)(4n+4)} \right)'' = arctanh\left(\frac{\pi}{4}\right) \approx 1.05930617082 \tag{31}$$

It's not as hard as it looks.