# Using Fermat's Principle of Least Time to Prove the Law of Reflection and Refraction

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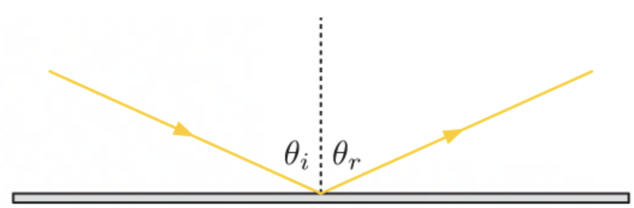
April 30, 2023

## 1 Fermat's Principle of Least Time

**Sometime** around 1650, Pierre de Fermat developed the principle of least time explain the behaviour of light: *Out of all possible paths that it might take to get from one point to another, light takes the path which requires the shortest time.*

## 2 Deriving the Law of Reflection

**To** understand the general behaviour of the law of reflection, it can be visualised in the diagram below:




Assuming that all measurements are different, two separate triangles can be created to represent the traversal of light, and its subsequent reflection once hitting the mirror. Let triangle $A$ represent the triangle with $\theta_i$, where triangle $B$ contains $\theta_r$. The entire width of the triangle can be expressed with a distance of $k$, where $x$ can be variable representative of the bottom leg (on the mirror) of triangle $A$ on the mirror. Given this, the bottom leg of $B$ is $k - x$, since $k$ is the full length of the mirror, where $x + (k - x)$ equates to $k$. The other legs of triangles $A$ and $B$ can be expressed as constants, $a$ and $b$ respectively. Now, the path length for light traversal prior and after reflection can be found, which can be added to obtain a function for the overall path length.

The Pythagorean Theorem can be used to do this, which is shown below:

$$a^2 + b^2 = c^2$$

To find the path length prior to reflection:

$$p_i = \sqrt{a^2 + x^2}$$

Now, to find the path length after reflection:

$$p_r = \sqrt{b^2 + (k - x)^2}$$

With individual formulae for distances, a general formula to find the total path length can be derived, defined as function $D$:

$$D = p_r + p_i$$

$$D = \sqrt{b^2 + (k - x)^2} + \sqrt{a^2 + x^2}$$

Since light takes the shortest possible path to get from one point to another, implies that $D$ will have to be a minimum point to satisfy this principle. In Differential Calculus, a minimum point implies a stationary point, where the following relationship is below:

$$\frac{dD}{dx} = 0$$

Given this information, equation $D$ can be differentiated with respect to $x$.

$$D = \sqrt{b^2 + (k - x)^2} + \sqrt{a^2 + x^2}$$

$$\frac{dD}{dx} = \frac{1}{2\sqrt{b^2 + (k - x)^2}} \cdot 2(k - x) \cdot -1 + \frac{1}{2\sqrt{a^2 + x^2}} \cdot 2x$$

$$\frac{dD}{dx} = \frac{-2(k - x)}{2\sqrt{b^2 + (k - x)^2}} + \frac{2x}{2\sqrt{a^2 + x^2}}$$

$$\frac{dD}{dx} = \frac{-(k - x)}{\sqrt{b^2 - (k - x)^2}} + \frac{x}{\sqrt{a^2 + x^2}}$$

Since $\frac{dD}{dx} = 0$, the left side of the equation is 0, where further manipulation can be done:

$$0 = \frac{-(k - x)}{\sqrt{b^2 - (k - x)^2}} + \frac{x}{\sqrt{a^2 + x^2}}$$

$$\frac{(k - x)}{\sqrt{b^2 - (k - x)^2}} = \frac{x}{\sqrt{a^2 + x^2}}$$

With the following equation, the definition of $sin$ can be leveraged from referring back to both triangles $A$ and $B$. The definition of $sin$ states that for any right triangle, the sine of an angle is equivalent to the side opposite to it divided by the length of the hypotenuse. Since both triangles are right, this definition can be applied directly into the equation, using the relationships below:

$$sin\theta_i = \frac{x}{\sqrt{a^2 + x^2}}$$

$$sin\theta_r = \frac{k - x}{\sqrt{b^2 - (k - x)^2}}$$

Now, substituting both $sin\theta_r$ and $sin\theta_i$ into the equation gives the following relationship:

$$sin\theta_i = sin\theta_r$$

$$\theta_i = \theta_r$$

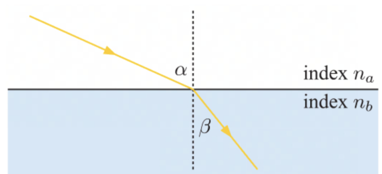
Since both angle of incidence and reflection are equal, it shows that when light hits a mirror, the angle of incidence ($\theta_i$) is equal to the angle of reflection ($\theta_r$).

# 3 Deriving the Law of Refraction

**Deriving** the law of Refraction can leverage the same function $D$:

$$D = \sqrt{b^2 + (k-x)^2} + \sqrt{a^2 + x^2}$$

The following diagram represents the behaviour of refraction:




Refraction occurs when light travels from a substance with index of refraction $n_a$ to another with index of refraction $n_b$, and are related. To find this relation, the following formulae can be used:

$$v = \frac{d}{t}$$

$$v = \frac{c}{n}$$

The first formula is a general formula for velocity, where $d$ represents displacement and $t$, time in seconds. The second formula represents a formula to find the speed of light within a substance, where c is the speed of light and $n$ is the index of refraction of said substance. Since refraction occurs with the changing of substances, $v$ will be different for indices $n_a$ and $n_b$. Since the formula from the proof for the law of reflection solves for distance, the theory from the first formula can be used to obtain a function for time:

$$v = \frac{d}{t}$$

$$t = \frac{d}{v}$$

Since $v$ is theoretically known as well as the path lengths, a function for time can be found:

$$D = \sqrt{b^2 + (k-x)^2} + \sqrt{a^2 + x^2}$$

$$\frac{D}{v} = \frac{1}{v_b} \cdot \sqrt{b^2 + (k-x)^2} + \frac{1}{v_a} \cdot \sqrt{a^2 + x^2}$$

$$t = \frac{1}{v_b} \cdot \sqrt{b^2 + (k-x)^2} + \frac{1}{v_a} \cdot \sqrt{a^2 + x^2}$$

According to the principle of least time, $\frac{dt}{dx} = 0$ given how the shortest time to traverse a path implies the shortest path taken. Next, differentiating the equation can get to the desired result of the proof:

$$\frac{dt}{dx} = \frac{1}{v_b}\frac{d}{dx}\sqrt{b^2 + (k-x)^2} + \frac{1}{v_a}\frac{d}{dx}\sqrt{a^2 + x^2}$$

$$\frac{dt}{dx} = \frac{1}{v_b} \cdot \frac{-(k-x)}{\sqrt{b^2 - (k-x)^2}} + \frac{1}{v_a} \cdot \frac{x}{\sqrt{a^2 + x^2}}$$

$$0 = \frac{1}{v_b} \cdot \frac{-(k-x)}{\sqrt{b^2 - (k-x)^2}} + \frac{1}{v_a} \cdot \frac{x}{\sqrt{a^2 + x^2}}$$

$$\frac{1}{v_b} \cdot \frac{(k-x)}{\sqrt{b^2 - (k-x)^2}} = \frac{1}{v_a} \cdot \frac{x}{\sqrt{a^2 + x^2}}$$

By using the previously derived relationships from Sec 2., the equation can be simplified:

$$\frac{1}{v_b} sin\beta = \frac{1}{v_a} sin\alpha$$

Now, referring back to the second formula for velocity within a substance, it can be rearranged to be re-substituted into the equation:

$$v = \frac{c}{n}$$

$$\frac{1}{v} = \frac{n}{c}$$

Using this relationship, the equation is further simplified. Understand that $c$ represents the speed of light, which is a constant and can hence be cancelled out of the equation:

$$\frac{n_b}{c} sin\beta = \frac{n_a}{c} sin\alpha$$

$$n_b sin\beta = n_a sin\alpha$$

Now that the equation has been fully simplified, the angle of incidence $\alpha$ and angle of refraction $\beta$ are shown to be related by $n_b sin\beta = n_a sin\alpha$.