The Fascinating Maths Behind Pouring Water Into Cups and Its Applications

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10 Pages

1 Introduction

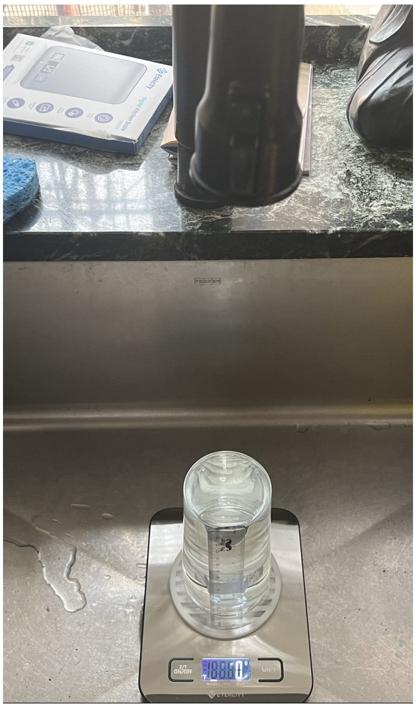
Pouring water into a cup is a task that is seen everyday in our lives. Whether it range from a waiter pouring water into a cup at a restaurant to simply pouring water for yourself at home, these applications can be seen in the theoretical world of maths. Given how the cup is cone-like, the rate at which the height of the shape taken by water in a cup increases is not consistent. As useless as this may seem, the maths behind pouring water into a truncated cone-like container has practical applications. In the concept of hydrology, truncated partial cones can be used to model the flow rate of rivers. Since the rate at which the water will rise is dependent on a certain rate of change in height, this can be reversed to find out the general change in volume over time and predict how quickly the container will fill as a result. As a result, flow rates can be obtained and hence can predict whether a flood were to occur based on measurements. The outcome of this experiment can also return a general rule describing how the rate at which water rises in a container with a specific geometric shape over time.

2 Investigation

To further understand this problem, data needs to be taken to have a real life representation of the volume of the cup to ensure the precision of calculations.

2.1 Methodology

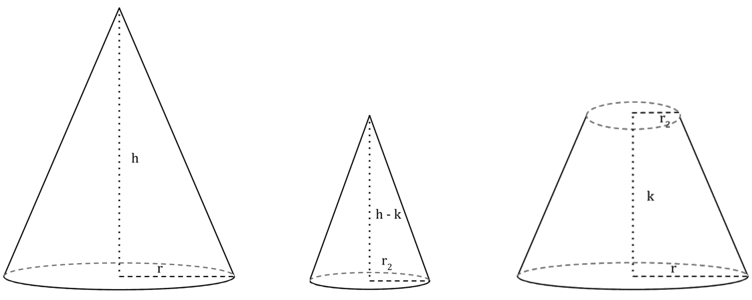
The method will consist of a cup of water with a ruler attached with tape put onto the cup of water to track the height over time, a tap that pours water into the cup, a scale to measure the mass of the water, and cameras to time the progress of the water pouring over time. Each and every item is required to obtain accurate measurements. A scale can be used to measure the mass of the water over time, where the concept of density can be used to obtain the volume of the water. To setup this experiment, the scale is to be put under the empty cup of water, where the other cup that will be used to pour will be suspended above the other cup. Cameras will be setup to serve as both a timer and a way to look at measurements over time, so that they can be referred to multiple times to measure the variables. Below is the setup of the experiment:



As simple as it is, all variables can be measured and allow for reliable data to be produced.

2.2 Assumptions

Several assumptions need to be made in this experiment, since the theoretical world differs greatly from the real. The relative geometry of the cup is assumed to be that of a partial cone, which consists of a larger cone being truncated by a smaller, though proportional cone. Observe the following figure, showing the assumed geometry of the cup:



This relationship of a partial cone can be used to get the measurements of the cup, then use the theory of angles of depression and elevation to find the entire height of the theoretical full cone. Since the cup in the setup is beaker-like, means that its radius will decrease once water is being poured into the cup, which is the opposite relationship to a traditional cup. Secondly, the rate at which the volume of the cup is assumed to be constant, which can be obtained by finding the amount of volume that the water takes up when being poured into the cup, divided over the time that it took to do so. Utilising this assumed constant rate means that calculations later on will be easier though retaining the logic of the instantaneous rate of change of the height increasing. Furthermore, the water is assumed to be pure and at room temperature, so that mass to volume measurements can be done by understanding that for every gram of water will entail 1 centimetre cubed of water. Since water in the real world splashes, water in the theoretical world is assumed to not have characteristic, so further variables would not have to be kept track of.

2.3 Variables

All of the variables that are accounted for in the experiment are in the following table below:

Symbol	Meaning	Description
V	Volume	Volume of the cup in cm^3
h	Height	The height of the entire theoretical cone in cm
k	Partial Height	The height of the partial cone in cm
t	Time	Time elapsed when pouring water in s
m	Mass	Mass of the water in g
r	Radius	Radius of the theoretical cone in cm
r_2	Radius 2	Radius of the partial cone in cm

2.4 Data Collection

Once having setup the experiment, data can be collected. All collected data is shown in the table below:

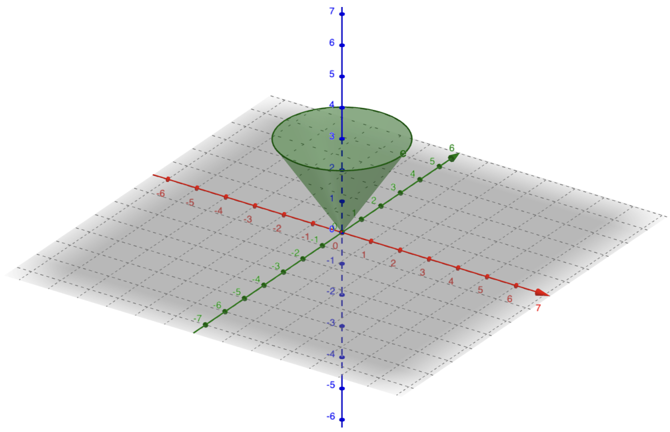
Time (t)	Partial Cone Height (k)	Mass of water (m)	Volume of water (v)
1	0.70	22	22
2	1.80	48	48
3	3.20	85	85
4	4.10	116	116
5	4.70	144	144
6	5.50	181	181
7	6.20	218	218
8	7.00	247	247
9	8.00	285	285
10	9.00	318	318
11	10.0	348	348
12	11.1	384	384
13	12.2	414	414
14	13.5	452	452

The data shows a general trend that the rate of change in k is increasing, with the rate at which volume changes is somewhat constant.

3 Theory and Calculations

3.1 Theory Behind the Cup's Volume and $\frac{dk}{dt}$

Interpreting the collected data requires a significant amount of mathematical theory, especially to obtain a general formula for $\frac{dk}{dt}$. First, the cone formula has to be derived, which on a fundamental is a series of cylinders with infinitesimal thickness. Observe the following diagram, representing a cone on a 3D plane.



To obtain the volume of this cone, the infinite sum of all of the theoretical cylinders needs to be obtained. Suppose that the radius of the cylinder be expressed as r_{cy}, to differentiate between the cylinder's and cone's radii. To find an expression for the area of a the theoretical cylinders, it can be expressed as the area of a circle times an infinitely small height, shown below:

$$V_{cylinder} = \pi r_{cy}^2 dz \tag{1}$$

dz represents the infinitely small height, which is with respect to the z axis. The z axis is representative of the height of the cone, which implies that the volume of the cone is the infinite sum of the cylinders from z coordinates 0 to h, which equation is shown below:

$$V_{cone} = \int_0^h \pi r_{cy}^2 dz \tag{2}$$

Before the integration of the volume of the cylinder, r_{cy} does not represent a constant value, which implies that usage of an expression for radius in terms of z, which can be done using the theory of the 3D plane. Given the linear nature of the slant of a cylinder, a linear proportion can be found to find a cylinder's radius based on height z. Suppose that a line passes through points A and B, with coordinates $(0, 0, 0)$ and $(0, r, h)$ respectively. Point B represents a point on the cone's circumference with A representing the origin and the very peak of the cone, where the line is the cone's slant. To find the gradient and hence equation of this line, the gradient formula can be used since the x coordinates for both points do not change. Given how the line passes through the origin, the z intercept will also be at the origin, and hence point-slope formula usage is unnecessary, and the gradient is multiplied by y to have a linear function:

$$z = \frac{\Delta z}{\Delta y} \cdot y = \frac{h}{r} y \tag{3}$$

Rearranging this formula to find an expression for y will lead to the following equation for r_{cy} in terms of r, h and z:

$$y = \frac{r}{h} z = r_{cy} \tag{4}$$

Once rearranged, the equation for r_{cy} can be substituted into the equation for the volume of a cone:

$$V_{cone} = \int_0^h \pi \left(\frac{r}{h} z\right)^2 dz \tag{5}$$

As both h and r do not change, $\frac{r}{h}$ is a constant and can be factored out of the integral along with π. Further integration can subsequently derive this the formula, using the steps shown below:

$$V_{cone} = \int_0^h \pi \frac{r^2}{h^2} z^2 dz$$

$$V_{cone} = \pi \frac{r^2}{h^2} \int_0^h z^2 dz$$

$$V_{cone} = \pi \frac{r^2}{h^2} \cdot \frac{z^3}{3} |_0^h$$

$$V_{cone} = \pi \frac{r^2}{h^2} \cdot \frac{h^3}{3}$$

$$V_{cone} = \frac{1}{3} \pi r^2 h \tag{6}$$

Now, the formula for the volume of a cone has been derived. Next, the formula for a partial cone, or in this case cup, can be obtained. Given how a partial cone consists of a cone being truncated by a part of itself, implies that the height of the smaller cone would be equivalent to $h - k$, where its radius would be r_2. Using this theory, the following relationship can be shown:

$$V_{cup} = V_{cone} - V_{cone2}$$

$$V_{cone} > V_{cone2}$$

$$V_{cup} = \frac{1}{3} \pi r^2 h - \frac{1}{3} \pi r_2^2 (h - k) \tag{7}$$

Now, this equation can be simplified:

$$V_{cup} = \frac{1}{3} \pi \left(r^2 h - r_2^2 (h - k) \right)$$

$$V_{cup} = \frac{1}{3} \pi \left(r^2 h - r_2^2 h + r_2^2 k \right)$$

$$V_{cup} = \frac{1}{3} \pi \left(h(r^2 - r_2^2) + r_2^2 k \right) \tag{8}$$

Given how the cup consists of a cone being truncated by a smaller proportional one, the proportion of their heights to radii should be equal to one another, hence implying the following relationship:

$$\frac{h}{r} = \frac{h - k}{r_2} \tag{9}$$

This relationship can be used to find an equation for h.

$$hr_2 = rh - rk$$

$$kr = h(r - r_2)$$

$$h = \frac{kr}{r - r_2} \tag{10}$$

Since an equation for h has been found, it can be inserted back into the equation:

$$V_{cup} = \frac{1}{3} \pi \left(\frac{kr}{r - r_2} (r - r_2)(r + r_2) + r_2^2 k \right)$$

$$V_{cup} = \frac{1}{3} \pi \left(kr(r + r_2) + r_2^2 k \right)$$

$$V_{cup} = \frac{1}{3}\pi k\left(r^2 + rr_2 + r_2^2\right) \tag{11}$$

Since the formula for the volume of a partial cone has been found, the general formula for the volume of the cup has also been found. Since no other volume variables are being used, suppose that the volume of the cup is V. There are several elements to understand prior to differentiating the volume formula with respect to time (t); of the variables, r_2 and k changing with time, where r is a constant.

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi kr^2 + \frac{1}{3}\pi krr_2 + \frac{1}{3}\pi kr_2^2\right)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2\frac{dk}{dt} + \frac{1}{3}\pi r\left(r_2\frac{dk}{dt} + k\frac{dr_2}{dt}\right) + \frac{1}{3}\pi\left(r_2^2\frac{dk}{dt} + 2kr_2\frac{dr_2}{dt}\right)$$

$$\frac{3}{\pi}\frac{dV}{dt} = r^2\frac{dk}{dt} + rr_2\frac{dk}{dt} + kr\frac{dr_2}{dt} + r_2^2\frac{dk}{dt} + 2kr_2\frac{dr_2}{dt} \tag{12}$$

Since the rate at which r_2 is decreasing over time is not known, an equation for $\frac{dr_2}{dt}$ has to be found in terms of other known variables. Using the proportion from the derivation of the cup volume formula, this is possible:

$$hr_2 = rh - rk$$

$$\frac{d}{dt}hr_2 = \frac{d}{dt}(rh - rk)$$

$$h\frac{dr_2}{dt} = -r\frac{dk}{dt}$$

$$\frac{dr_2}{dt} = -\frac{r}{h}\frac{dk}{dt} \tag{13}$$

With an equation for $\frac{dr_2}{dt}$ in terms of $\frac{dk}{dt}$, r and h, it can be substituted back into the differential equation:

$$\frac{3}{\pi}\frac{dV}{dt} = r^2\frac{dk}{dt} + rr_2\frac{dk}{dt} - kr\frac{r}{h}\frac{dk}{dt} + r_2^2\frac{dk}{dt} - 2kr_2\frac{r}{h}\frac{dk}{dt} \tag{14}$$

Subsequent factorisation of $\frac{dk}{dt}$ can be done to isolate it, which will find a general formula for $\frac{dk}{dt}$:

$$\frac{3}{\pi}\frac{dV}{dt} = \frac{dk}{dt}\left(r^2 + rr_2 - \frac{kr^2}{h} + r_2^2 - \frac{2krr_2}{h}\right)$$

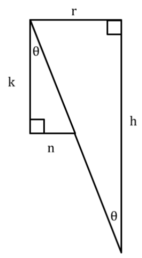
$$\frac{\frac{3}{\pi}\frac{dV}{dt}}{r^2 + rr_2 - \frac{kr^2}{h} + r_2^2 - \frac{2krr_2}{h}} = \frac{dk}{dt} \tag{15}$$

Simplifying this formula will eventually lead to a more elegant expression to find $\frac{dk}{dt}$:

$$\frac{dk}{dt} = \frac{3h}{\pi(hr^2 + hrr_2 - kr^2 + hr_2^2 - 2krr_2)}\frac{dV}{dt} \tag{16}$$

3.2 Theory Behind the Original Theoretical Cone

Since it is only the height of the entire partial cone (water cup) that is given, the height of the theoretical cone that the partial cone would be apart of has to be found independently. Using the theory of similar triangles and angles of elevation and depression, two measurements can be taken to find a ratio for the height of the cup (k) to the other leg (n) of a triangle. These two measurements can form a ratio and equated to the radius of the cone over the unknown height. The following figure represents this behaviour:



Once obtaining the measurements, an equation for the ratios is shown below:

$$\frac{k}{n} = \frac{h}{r} \tag{17}$$

Since variables k, n and r are known, an equation for h can be obtained:

$$h = \frac{kr}{n} \tag{18}$$

Calculations for constant h will allow for the height of the original theoretical cone to be known.

3.3 Measurements and Calculations

Since an equation for $\frac{dk}{dt}$ and h have been found, calculations can be conducted to apply all of the theoretical maths into the real world. In the real world, the height (k_{full}) and length (n) of the partial cone as well as the radius (r) of the cone are 14.5 cm, 1 cm and 3.5 cm respectively. Using these measurements, the height of the full cone, h, can be found:

$$h = \frac{14.5cm \cdot 3.5cm}{1cm} = 50.75cm$$

Referring back to the data table in 2.4, the amount of water that was poured into the cup totalled to 452 grams, and took approximately 14 seconds to do so. Assuming that $\frac{dV}{dt}$ is a constant, implies that it is representative of rate at which volume changed every second:

$$\frac{dV}{dt} = \frac{452g}{14s} \approx 32.286\ grams/s \approx 32.286\ cm^3/s$$

Given how there is a relationship between variables k and r_2 from the ratios of the cones, an equation for r_2 can be inserted into $\frac{dk}{dt}$. Refer back to equation set 10, where dividing the first equation by h can find r_2:

$$r_2 = \frac{hr - kr}{h}$$

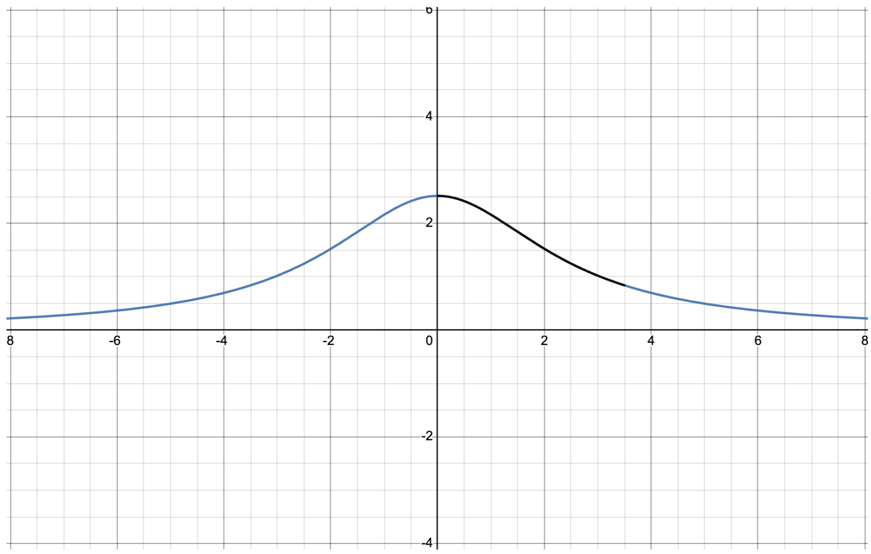
Obtaining the instantaneous rate of change in k can now be found with measurements obtained using only one measurement. For example, if $\frac{dk}{dt}$ is trying to be found if k is equivalent to $10cm$, measurements can be used to obtain it. Using the relationship for variable r_2 with k, r_2 is found:

$$r_2 = \frac{50.75 \cdot 3.5 - 10 \cdot 3.5}{50.75} \approx 2.81$$

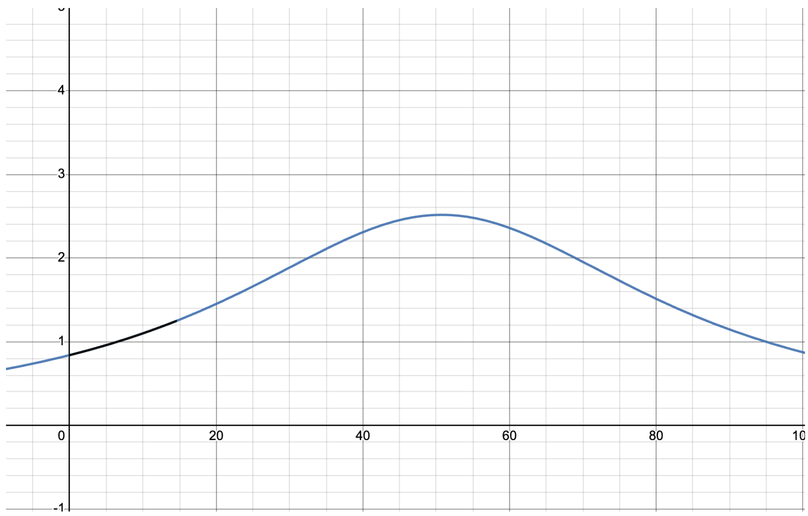
$$\frac{dk}{dt} = \frac{3(50.75)(32.286)}{\pi(50.75(3.5)^2 + 50.75(3.5)(2.81) - 10(3.5)^2 + 50.75(2.81)^2 - 2(10)(3.5)(2.81))}$$

$$\frac{dk}{dt}|_{k=10} \approx 1.3\ cm/s$$

Using these relationships however, a general curve that shows the relationship of $\frac{dk}{dt}$ with r_2 can be computed through Desmos, which is shown below:



The black segment of the graph represents the predicted $\frac{dk}{dt}$ within the domain of the water cup's radius, with the blue representing the actual graph in the theoretical world. The y axis represents $\frac{dk}{dt}$, with the x representing r_2. Since this is not with respect to time, the direction that r_2 moves relative to time is backwards. The relationship that is drawn based on the graph implies that the rate at which the cup height changes increases over time. Similarly with r_2, the relationship between k and $\frac{dk}{dt}$ is shown:



The general shape of the graph is very similar, but it is shifted to the right by a certain amount. Likewise with the previous graph, the black segment represents the cup's domain from 0 to 14.5 centimetres and the x axis representing k and y axis representing $\frac{dk}{dt}$. The maximum of the graph turns out to be 50.75, which is the height of the theoretical cone where in other words imply that the maximum rise in height in the entire theoretical cone occurs when $k = h$. These trends seen in each graph make sense when combining it with logic from the real world, implying the accuracy of theoretical calculations.

3.4 Cases With Cups and the Second Case

There are two separate cases for conical cup shapes: one where radius decreases as a result of height increase (refer to Sec. 2.2's partial cone), or one where radius increases with an increase in height (which shape is the upside-down iteration of the partial cone in 2.2). The graphs of $\frac{dk}{dt}$ relative to either r_2 or k will resemble one another, however will differ. Given the reverse relationship that cups of both cases have coupled with the theory of the first, it can be implied in the second case that the maximum for $\frac{dk}{dt}$ relative to r_2 is when r_2 is equivalent to the radius of the cup. Assume that the function for $\frac{dk}{dt}$ for cases 1 and 2 be K_{c1} and K_{c2} respectively. Suppose that the function that graphs $\frac{dk}{dt}$ relative to r_2 be $K_{c1}(r_2)$. Given how the graph of the first case's maximum value is at $r_2 = 0$, implies that this maximum has to be shifted to r. Since the cup shape still remains similar to the first case though with opposite behaviour, $K_{c2}(r_2) = K_{c1}(r_2 - r)$. A similar case is with a function for $\frac{dk}{dt}$ relative to k. Suppose this function is $K_{c1}(k)$. Since the maximum value of this function would be when $k = h$, the other case would display the opposite behaviour, reaching its maximum value when $k = 0$. Due to this, the function K_{c1} would be shifted to the left by h to account for this case, therefore implying $K_{c2}(k) = K_{c1}(k + h)$.

4 Conclusion

With the two separate cases, there are also numerous conclusions to be made. In the context of the first case of water cup shape, the general trend with the filling of the water cup had a trend that the rate at which the water rose increased over time. This stems from the inverse relationship that the rate at which water rises in the cone with the radius over time, where a decrease in radius r_2 will increase k. Such a relationship is mathematically shown in equation set 13 (located in Sec. 3.1), which shows how $\frac{dr_2}{dt}$ is equivalent to constant $\frac{r}{h}$ times $-\frac{dk}{dt}$, implying an non proportional inverse relationship. This is implied through the graphs, given how an increase in height implies a decrease in radius, hence increasing the rate at which the water rises in the cup. The results of the experiment coupled with the maths to back it up connect to the real world with this relationship, as it reflects how water flows and fills a container based on its shape. For any shape that decreases in diameter from the bottom to the top of the container, water will rise quicker as it increases in height, where the water will rise slower in the opposite case. In the aspect of the how the water flows, a detrimental change to diameter once increasing in height, will lead to water having less space to fill up the container in terms of width, which would push water molecules upwards to account for the lack of space, hence increasing the rate at which it rises over time. Both graphs from Sec. 3.3 shows how the $\frac{dk}{dt}$ will never be 0, given how liquids fill their respective container and will never have 0 change in height unless volume is not being added. These graphs also show how partial cones can still be used to predict rates at which water rises for the entire cone. Despite the various assumptions made, the general trend suggested by both the data and science imply how this relationship is still applied in the real world. In the context of measuring the rates at which river water would flow or somehow be poured into a conical container, variables h, k, r and r_2 would be known, where the average rate of increasing in the volume could proportionally show how quickly water would rise and potentially flood a place, and possibly find certain points in time that are optimal for evacuation or preparing for it.

5 Evaluation

5.1 General Evaluation

The experiment was subject to significant human error, since the there were several factors in the real world to account for. The cup has an inconsistent thickness throughout its height, which mean that measurements relating to the volume of the cup will be flawed, and will differ from the theoretical measurements, albeit the general trend is still there. The data table from Sec. 2.4 reflects how $\frac{dV}{dt}$ is not fully constant, since the rate at which volume changes in the cup cannot be fully controlled due to the possibly inconsistent pressure that is pushing the water. Furthermore, all data collection has to be filmed, since it would be difficult to instantaneously take measurements once per second. When reviewing the footage of the water being poured into the cup, there are several complications from the lighting and the angle that the camera is facing it, leading to imprecise measurements. The scale also is not perfect in the sense that it produces fully accurate measurements when mass is constantly changing, which means that it may be off. The water came from a tap, meaning that it is most likely pure and not at room temperature, leading to the risk of imprecise measurements. In the context of the real world, implementing these relationships could work for partial conical containers, however the imperfections of the real world entail that the rate at which the volume would be to increase is not a constant. To somewhat rectify this issue, the derivative of a volume function with respect to time in terms of other known variables could account for the inconsistent volume change in the real world.

5.2 Assessing Data Collected Versus Theoretical Measurements

To further assess the accuracy of the measurements, the percent error of the theoretical measurements relative to the real world measurements can be compared. Using the collected data in Sec. 2.4, this can happen. For example, using measurements from the moment when $t = 14$, k is equivalent to 13.5 cm, where the measured V is 452. The theoretical volume can be obtained by using k to find r_2, then use those values in the derived volume formula in Sec. 3.1:

$$r_{2_{theoretical}} = 3.5 - \frac{13.5 \cdot 3.5}{50.75} \approx 2.56896551724$$

$$V_{theoretical} = \frac{1}{3}\pi(13.5)(3.5^2 + 3.5(r_2) + r_2^2) \approx 393.5923437$$

$$\delta = \frac{|452 - 393.5923437|}{393.5923437} \approx 0.148396322324 \approx 14.8\%$$

Since there is a percent error, this implies that the real measurements are flawed. $\frac{dV}{dt}$ requires measurements from the real world, where with the flawed data in the data table will skew the theoretical measurements for $\frac{dk}{dt}$. Had there have been better measurements and more ideal cup shapes that resemble a cone better, calculations would almost be perfected.

6 Citations

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