# The Solution to the Hamburger Problem

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## 1 Introduction

A friend of mine presented an intimidating function that would be difficult to integrate. Whilst eating a hamburger, I decided to integrate that function:

$$f(x) = \frac{e^x}{x^2 - 1} \tag{1}$$

As intimidating as it looks, the process is not complicated as it involves simple algebra.

## 2 The Solution

To integrate the $f$, the easiest course of action would be to split the denominator into partial fractions, since it is a difference of squares (a quadratic in the form $x^2 - b^2$). Since the numerator is not a polynomial, $e^x$ will be factored out and excluded to enable for the fraction to be decomposed into two separate more convenient terms.

$$\int f(x)dx = \int e^x \left(\frac{1}{x^2 - 1}\right) dx \tag{2}$$

$$\int f(x)dx = \int e^x \left(\frac{A}{x+1} + \frac{B}{x-1}\right) dx \tag{3}$$

The fraction has been decomposed into two parts, where values $A$ and $B$ need to be solved for.

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{1}{x^2 - 1} \tag{4}$$

Solving for $A$:

$$A(x+1) = 1 \tag{5}$$

$$A(1+1) = 1 \tag{6}$$

$$2A = 1 \tag{7}$$

$$A = \frac{1}{2} \tag{8}$$

Next, solving for B requires a similar method:

$$B(x-1) = 1 \tag{9}$$

$$B(-1-1) = 1 \tag{10}$$

$$-2B = 1 \tag{11}$$

$$B = -\frac{1}{2} \tag{12}$$

With values for both $A$ and $B$, the integral can properly be decomposed.

$$\int f(x)dx = \int e^x \left( \frac{1}{2(x-1)} + \frac{1}{2(x+1)} \right) dx \tag{13}$$

$$\int f(x)dx = \frac{1}{2} \int \left( \frac{e^x}{x-1} - \frac{e^x}{x+1} \right) dx \tag{14}$$

$$\int f(x)dx = \frac{1}{2} \left( \int \frac{e^x}{x-1} dx - \int \frac{e^x}{x+1} dx \right) \tag{15}$$

Now, two smaller and easier integrals will require solving. Both integrals will follow a similar process. Let's divide the process into determining them separately. Let's first focus on the right function:

$$\int \frac{e^x}{x+1} dx \tag{16}$$

To assist with solving the integral, we can equate a variable $u$ to $x+1$ to remove the inconvenience of the denominator.

$$\text{Let } u = x+1 \tag{17}$$

For those who aren't familiar with U-substitution, I'll add these few extra steps as U-Substitution isn't always as merciful with composite functions. Linear functions are a very easy example of when to implement U-substitution, as their derivative with respect to $x$ would be a constant, which makes them immediately applicable into an integral.

$$\frac{du}{dx} = \frac{d}{dx}(x+1) \tag{18}$$

$$\frac{du}{dx} = 1 \tag{19}$$

$$dx = du \tag{20}$$

Now that $dx$ has been solved in terms of $u$, the integral is follows a more elegant format:

$$\int \frac{e^x}{u} du \tag{21}$$

Since there is still a term containing $x$ $(e^x)$, this can be solved pretty easily by understanding that $u = x + 1$ which means that $x = u - 1$:

$$\int \frac{e^{u-1}}{u} du \tag{22}$$

The term $e^{u-1}$ can be converted into $\frac{1}{e} \cdot e^u$ to increase the simplicity of the integral, and also enable for the pesky constant term to be factored out entirely:

$$\frac{1}{e} \int \frac{e^u}{u} du \tag{23}$$

This integral can now go two ways: the dark path of integration by-parts, or the very appealing alternative route. The following is formula for the formula for the Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty} \left( \frac{f^{(n)}(0)}{n!} x^n \right) \tag{24}$$

As $e^x$ is famously easy when it comes to the Maclaurin Series, the best course of action is to use it. The next step of solving the integral will involve substituting the Maclaurin Series for $e^x$ and factoring out the $\frac{1}{u}$ term, which will be worked with in the subsequent step. Since $\frac{d}{dx} e^x = e^x$ and $e^0 = 1$, the Maclaurin Series for $e^x$ is:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{25}$$

This can now be inserted into the integral:

$$\frac{1}{e} \int \frac{1}{u} \sum_{n=0}^{\infty} \frac{u^n}{n!} du \tag{26}$$

$$\frac{1}{e} \int \frac{1}{u} \left( 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots \right) du \tag{27}$$

$$\frac{1}{e} \int \left( \frac{1}{u} + 1 + \frac{u}{2!} + \frac{u}{3!} + \dots \right) du \tag{28}$$

$$\frac{1}{e} \left( ln|u| + u + \frac{u^2}{2 \cdot 2!} + \frac{u^3}{3 \cdot 3!} + \dots \right) \tag{29}$$

$$\frac{1}{e} \left( ln|u| + \sum_{n=1}^{\infty} \frac{u^n}{n \cdot n!} \right) \tag{30}$$

Now that the integral has been solved, the last part is to resubstitute $x+1$ back into $u$ and insert the useless constant, $C$:

$$\int \frac{e^x}{x+1} dx = \frac{1}{e} \left( ln|x+1| + \sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot n!} \right) + C \tag{31}$$

The next part of the entire process is to integrate the left integral. This process is very similar, instead involving the U-Substitution of $u = x - 1$ which will yield a slightly different antiderivative.

$$\int \frac{e^x}{x-1} dx \tag{32}$$

$$\text{Let } u = x - 1 \tag{33}$$

$$\int \frac{e^{u+1}}{u} dx \tag{34}$$

$$e \int \frac{e^u}{u} dx \tag{35}$$

This paper already went through the integral of $\frac{e^u}{u}$, so I won't bore you with more steps.

$$e \int \frac{e^u}{u} dx = e \left( ln|u| + \sum_{n=1}^{\infty} \frac{(u)^n}{n \cdot n!} \right) + C \tag{36}$$

$$\int \frac{e^x}{x-1} dx = e \left( ln|x-1| + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot n!} \right) + C \tag{37}$$

These smaller solved integrals can now be inserted into the main expression:

$$\int f(x) dx = \frac{1}{2} \left( e \left( ln|x-1| + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot n!} \right) - \frac{1}{e} \left( ln|x+1| + \sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot n!} \right) \right) + C \tag{38}$$

The main integral has finally been solved. Not so arbitrary afterall, is it Neil?