Solving The Goofy Problem On The Test

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1 Introduction To The Problem

Find the coordinates on curve $y = f(x) = (x-1)^2$ where the normal passes through the origin.

2 Interpreting The Problem

The problem involves finding the equation of the normal, which means that the derivative function has to be found, by differentiating y. This can be done without much difficulty:

$$\frac{dy}{dx} = 2(x-1) \cdot \frac{d}{dx}(x-1)$$
$$\frac{dy}{dx} = 2(x-1)$$
$$\frac{dy}{dx} = 2x - 2 \tag{1}$$

Given how the normal to a function is perpendicular to the tangent at the point, the formula for the tangent of a function can be slightly changed to account for the normal. In this case, variable a represents the x-coordinate of the solution:

$$y = f'(a)(x-a) + f(a)$$
$$y = -\frac{1}{f'(a)}(x-a) + f(a) \tag{2}$$

The following form is equivalent to point slope form of a linear function. Since the equation of the normal in the solution passes through the origin, this equation can be simplified:

$$y = -\frac{1}{f'(a)}(x-a) + f(a)$$

$$y = -\frac{1}{f'(x)}(x-0) + 0$$

$$y = -\frac{1}{f'(x)}x$$
$$y = -\frac{1}{2x-2}x \tag{3}$$

Since a is not known, x is now the input of $f'(x)$ where y can be substituted with $f(x)$:

$$y = -\frac{1}{2x-2}x$$
$$(x-1)^2 = -\frac{1}{2x-2}x \tag{4}$$

Now, a few steps can be taken to convert this algebraic equation into a cubic polynomial, to try and solve for roots:

$$(x-1)^2 = -\frac{1}{2x-2}x$$
$$(x-1)^2 = -\frac{1}{2(x-1)}x$$
$$2(x-1)^3 = -x$$
$$2(x^3-3x^2+3x-1) = -x$$
$$2x^3-6x^2+6x-2 = -x$$
$$2x^3-6x^2+7x-2 = 0 \tag{5}$$

This equation is now in a solvable form, where a formula can be used to find the roots.

3 Solving The Problem

Understand that for any cubic polynomial $y = ax^3+bx^2+cx+d$ the x-coordinate of a root is equivalent to:

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}+ \sqrt[3]{\left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}-\frac{b}{3a} \tag{6}$$

The bulkiness of the formula can be simplified into a more elegant form, since there are a few repetitions inside of it. The more simplified version of the cubic polynomial formula is shown below, with the definition of variables j and k:

$$k = \left(-\frac{b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)$$

$$j = \left(\frac{c}{3a} - \frac{b^2}{9a^2} \right)$$

$$x = \sqrt[3]{k + \sqrt{k^2 + j^3}} + \sqrt[3]{k - \sqrt{k^2 + j^3}} - \frac{b}{3a} \tag{7}$$

The proof for this is quite rigourous, and will not be covered here. Given how the polynomial from the problem cannot be solved with the rational root theorem nor with grouping of any sort, this formula has to be used. Additionally, the given formula only solves for one solution to the polynomial, where roots of unity (I'm not sure what those are) are needed to have a formula accounting for all three possible solutions. The equation itself does not have any rational roots, and actually only has one solution given the nature of the normal line and the shape of $f(x)$, meaning that this formula is the optimal way of solving for x. Using the following polynomial below:

$$2x^3 - 6x^2 + 7x - 2 = 0 \tag{8}$$

The coefficients a, b, c and constant d are used in the formula to find x:

$$k = -\frac{(-6)^3}{27\,(2)^3} + \frac{-6(7)}{6\,(2)^2} - \frac{-2}{2\,(2)}$$

$$j = \left(\frac{7}{3\,(2)} - \frac{(-6)^2}{9\,(2)^2} \right)$$

$$x = \sqrt[3]{k + \sqrt{k^2 + j^3}} + \sqrt[3]{k - \sqrt{k^2 + j^3}} - \frac{-6}{3\,(2)} \tag{9}$$

After some simplification, the solution in its exact form looks like this:

$$x = \sqrt[3]{\left(\frac{\sqrt{29} - 3\sqrt{3}}{12\sqrt{3}} \right)} - \sqrt[3]{\left(\frac{\sqrt{29} + 3\sqrt{3}}{12\sqrt{3}} \right)} + 1 \tag{10}$$

Once using a calculator to obtain an approximate value, the solution for the x-coordinate of the point on $f(x)$ is:

$$x \approx 0.410245487699 \tag{11}$$

Now that x (or in this case variable a) has been found, the y-coordinate is obtained by simply putting x in $f(x)$:

$$x \approx 0.410245487699$$

$$y \approx f(x)$$

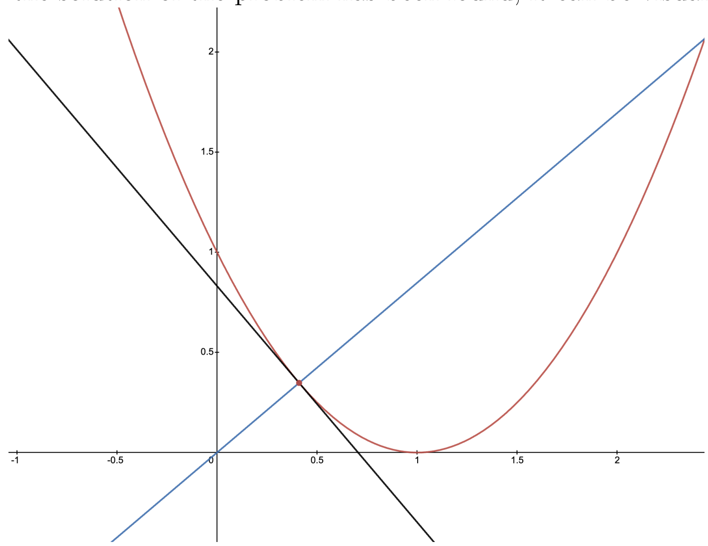
$$y \approx f(0.410245487699)$$

$$y \approx 0.34781038478 \tag{12}$$

As both x and y have been found, the coordinates of the curve $f(x)$ are approximately $(0.410245487699, 0.34781038478)$, where the normal at said points will pass through the origin.

4 Visualising The Solution

As the solution of the problem has been found, it can be visualised:



Using the solutions, the normal line passes through the origin, and is perpendicular to a tangent point on the curve, which by definition supports both conditions of the problem.