AdaBoost

the original paper

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Simplicity vs. fitting to data

1.2	2.8	8.0	3.3	5.0	4.5	7.4	5.6	3.8	6.6	6.1	1.7
-	-	+	-	-	-	+	+	-	+	+	-

What is your classifier?

Simplicity vs. fitting to data

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-	-	+	-	-	-	+	+	-	+	+	-

What is your classifier?

$$h(x) = \begin{cases} +1 & \text{if } x \geq 5.3 \\ -1 & \text{otherwise} \end{cases}$$

It was so simple!

Simplicity vs. fitting to data

Let's change only two labels:

1.2	2.8	8.0	3.3	5.0	4.5	7.4	5.6	3.8	6.6	6.1	1.7
-	-	+	-	-	-	+	+	+	-	+	-

What is your classifier?

Simplicity vs. fitting to data

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-	-	+	-	-	-	+	+	+	-	+	-

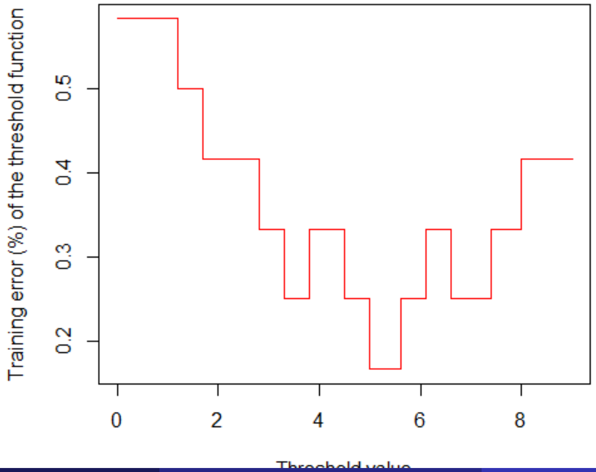
What is your classifier?

$$h(x) = \begin{cases} -1 & \text{if } x < 3.5 \\ +1 & \text{if } 3.5 \leq x < 4 \\ -1 & \text{if } 4 \leq x < 5.5 \\ +1 & \text{if } 5.5 \leq x < 6.5 \\ -1 & \text{if } 6.5 \leq x < 7 \\ +1 & \text{otherwise} \end{cases}$$

It's hard to find a single decision stump

Simplicity vs. fitting to data

Training error of a single decision stump for the last training set



PAC learning definitions I

- Let X be a set called the *instance space*.
- A *concept* c over X is a subset $c \subseteq X$ of the instance space.
- A *concept class* C over X is a set of concepts over X, typically with an associated representation.
- A *target concept* may be any concept $c^* \in C$.
- A *learning algorithm* (called also *learner*) tries to infer an unknown concept (called *hypothesis*), chosen from a known concept class.
- A set of random variables is *independent and identically distributed (i.i.d.)* if each random variable has the same *identical* probability distribution as the others and all are mutually *independent*.
- The instances the learner receives from a distribution D, are independently and identically distributed (i.i.d.).

PAC learning definitions II

- After observing a sequence S of i.i.d. *training examples* of the target concept c^*, the learner L outputs the hypothesis h (its estimate of c^*) from the set H of possible hypotheses.
- The *true error* of h with respect to c^* and D is the probability that h will misclassify an instance drawn randomly according to D:
  $err(h) \equiv \underset{x \in D}{\mathbf{Pr}}[c^*(x) \neq h(x)] = \underset{(x,y) \sim D}{\mathbf{Pr}}[h(x) \neq y]$
- The *training error* is the fraction of training examples misclassified by the hypothesis h:
  $\widehat{err}(h) \equiv \underset{x \in S}{\mathbf{Pr}}[c^*(x) \neq h(x)] =$
  $= \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}[c^*(x_i) \neq h(x_i)] = \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}[h(x_i) \neq y_i]$

Weak learnability

A concept class C is *PAC-learnable* or *strongly learnable* by L if for any D over X, and for any $c \in C$, and for any positive value of ε, and for any positive value of δ, the learner L will output in a polynomial time in $\frac{1}{\varepsilon}$, $\frac{1}{\delta}$, $size(x \in X)$, $size(c)$, a hypothesis h such that $\mathbf{Pr}[err(h) > \varepsilon] \leq \delta$

An arbitrarily small true error means nearly perfect generalization and that is usually unrealistic!

A concept class C is *weakly learnable* by L if for any D over X, and for any $c \in C$, and for any positive value of δ, and for some fixed positive value of $\varepsilon = \frac{1}{2} - \gamma$ with $\gamma > 0$, the learner L will output in a polynomial time in $\frac{1}{\varepsilon}$, $\frac{1}{\delta}$, $size(x \in X)$, $size(c)$, a hypothesis h such that $\mathbf{Pr}[err(h) > \varepsilon] \leq \delta$

Weak $\equiv$ Strong

Hypothesis Boosting Problem - Michael Kearns, December 1988

> *"is it the case that any target class that is weakly learnable is in fact strongly learnable? Note that we pose the question in a representation-independent setting in that the hypothesis class used by the strong learning algorithm need not be the same as that used by the weak learning algorithm."*

Proof - Robert Schapire, June 1990

> *"Theorem: A concept class C is weakly learnable if and only if it is strongly learnable"*

Boosting introduction

Let's consider a simple classifier, the decision stump:
$h(\mathbf{x}; \theta) = \text{sign}(w_1 x_k - w_0), \mathbf{x} \in \mathbb{R}^d, \theta = \{k, w_1, w_0\}$
Note it acts on only one component of $\mathbf{x}$

Voted combination of m classifiers, built iteratively in m rounds:
$h_m(\mathbf{x}) = \alpha_1 h(\mathbf{x}; \theta_1) + \ldots + \alpha_m h(\mathbf{x}; \theta_m)$
where the votes α_i are non-negative (a higher vote means a better classifier)

Let's consider this loss function:
$L(y, f(\mathbf{x})) = \exp(-yf(\mathbf{x}))$
with the value $1/e$ (small) when classifies correctly,
or the value e (large) when classifies incorrectly

$$\min \sum_{i=1}^{n} L(y_i, h_m(\mathbf{x}_i)) = \min \sum_{i=1}^{n} \exp(-y_i h_m(\mathbf{x}_i)) =$$
$$\min \sum_{i=1}^{n} \exp(-y_i(h_{m-1}(\mathbf{x}_i) + \alpha_m h(\mathbf{x}_i;\theta_m))) =$$
$$\min \sum_{i=1}^{n} \exp(-y_i h_{m-1}(\mathbf{x}_i))\exp(-y_i\alpha_m h(\mathbf{x}_i;\theta_m)))=$$

At round m, the expression $\exp(-y_i h_{m-1}(\mathbf{x}_i))$ under min is fixed (it's already constructed), let's denote it by W_i^{m-1} so:

$$= \min \sum_{i=1}^{n} W_i^{m-1}\exp(-y_i\alpha_m h(\mathbf{x}_i;\theta_m))) =$$
$$= \min\{\sum_{y_i=h(\mathbf{x}_i;\theta_m)} W_i^{m-1}\exp(-\alpha_m) + \sum_{y_i\neq h(\mathbf{x}_i;\theta_m)} W_i^{m-1}\exp(\alpha_m)\} =$$
$$= \min\{\exp(-\alpha_m)\sum_{y_i=h(\mathbf{x}_i;\theta_m)} W_i^{m-1} + \exp(\alpha_m)\sum_{y_i\neq h(\mathbf{x}_i;\theta_m)} W_i^{m-1}\} =$$
$$= \min\{\exp(-\alpha_m)\sum_{i=1}^{n} W_i^{m-1}\mathbf{1}(y_i = h(\mathbf{x}_i;\theta_m))+$$
$$+\exp(\alpha_m)\sum_{i=1}^{n} W_i^{m-1}\mathbf{1}(y_i \neq h(\mathbf{x}_i;\theta_m))\} =$$
$$= \min\{\exp(-\alpha_m)(\sum_{i=1}^{n} W_i^{m-1} - \sum_{i=1}^{n} W_i^{m-1}\mathbf{1}(y_i \neq h(\mathbf{x}_i;\theta_m))+$$
$$+\exp(\alpha_m)\sum_{i=1}^{n} W_i^{m-1}\mathbf{1}(y_i \neq h(\mathbf{x}_i;\theta_m))\} =$$
$$= \min\{\exp(-\alpha_m)\sum_{i=1}^{n} W_i^{m-1}-$$
$$-\exp(-\alpha_m)\sum_{i=1}^{n} W_i^{m-1}\mathbf{1}(y_i \neq h(\mathbf{x}_i;\theta_m))+$$
$$+\exp(\alpha_m)\sum_{i=1}^{n} W_i^{m-1}\mathbf{1}(y_i \neq h(\mathbf{x}_i;\theta_m))\}$$

But this min is on α_m, so:

$\frac{\partial \sum_{i=1}^{n} L(y_i, h_m(\mathbf{x}_i))}{\partial \alpha_m} = 0$, i.e.

$$-\exp(-\alpha_m) \sum_{i=1}^{n} W_i^{m-1} +$$
$$+\exp(-\alpha_m) \sum_{i=1}^{n} W_i^{m-1} \mathbf{1}(y_i \neq h(\mathbf{x}_i; \theta_m)) +$$
$$+\exp(\alpha_m) \sum_{i=1}^{n} W_i^{m-1} \mathbf{1}(y_i \neq h(\mathbf{x}_i; \theta_m)) = 0$$

Results:

$$\exp(-\alpha_m) \frac{\sum_{i=1}^{n} W_i^{m-1} \mathbf{1}(y_i \neq h(\mathbf{x}_i; \theta_m))}{\sum_{i=1}^{n} W_i^{m-1}} +$$
$$+\exp(\alpha_m) \frac{\sum_{i=1}^{n} W_i^{m-1} \mathbf{1}(y_i \neq h(\mathbf{x}_i; \theta_m))}{\sum_{i=1}^{n} W_i^{m-1}} = \exp(-\alpha_m)$$

$$\frac{\sum_{i=1}^{n} W_i^{m-1} \mathbf{1}(y_i \neq h(\mathbf{x}_i; \theta_m))}{\sum_{i=1}^{n} W_i^{m-1}} = \frac{\exp(-\alpha_m)}{\exp(-\alpha_m) + \exp(\alpha_m)}$$

The left term is the *weighted* training error at round m, so:

$$\varepsilon_m = \frac{\exp(-\alpha_m)}{\exp(-\alpha_m)+\exp(\alpha_m)} = \frac{1}{1+\exp(2\alpha_m)}$$

We obtain the formula of the vote at round m:

$$\alpha_m = \frac{1}{2}\ln(\frac{1-\varepsilon_m}{\varepsilon_m})$$

We have denoted above the weights by:

$$W_i^m = \exp(-y_i h_m(\mathbf{x}_i)) = exp(-y_i(h_{m-1}(\mathbf{x}_i) + \alpha_m h(\mathbf{x}_i;\theta_m))) =$$
$$= W_i^{m-1}\exp(-y_i\alpha_m h(\mathbf{x}_i;\theta_m))$$

Results a recursive update rule of the weights at round m based on the value at the previous round:

$$W_i^m = W_i^{m-1}\exp(-y_i\alpha_m h(\mathbf{x}_i;\theta_m))$$

At every round, these weights can be normalized such that: $\sum_{i=1}^n W_i^m = 1$

AdaBoost Algorithm - Freund & Schapire, 1995

Input:

a sequence of n labeled examples $\{(\mathbf{x}_i, y_i), i = 1, \ldots, n\}$,
a number T of iterations,
and a weak learner *WeakLearn* that at round m will produce a hypothesis $h(\mathbf{x}; \theta_m)$

Initialize:

$W_i^1 = \frac{1}{n}, i = 1, \ldots, n$, because we don't have any information about weights at the first round.

Iterate from m = 1 to T:

Call *WeakLearn* to produce the hypothesis $h(\mathbf{x}; \theta_m)$ (*AdaBoost has no direct control on the hypothesis*)

Calculate the weighted training error: $\varepsilon_m = \frac{\sum_{i=1}^n W_i^m \mathbf{1}(y_i \neq h(\mathbf{x}_i; \theta_m))}{\sum_{i=1}^n W_i^m}$

Calculate the vote: $\alpha_m = \frac{1}{2} \ln(\frac{1-\varepsilon_m}{\varepsilon_m})$

Update the weights: $W_i^m \leftarrow W_i^m \exp(-y_i \alpha_m h(\mathbf{x}_i; \theta_m))/Z_m$ and normalize their values (division by Z_m).

Output the final hypothesis: $\text{sign}(h_T(\mathbf{x}))$

From the normalized weights update formula, we have:

$$W_i^T = W_i^1 \frac{\exp(-\sum_{m=1}^T y_i \alpha_m h(\mathbf{x}_i;\theta_m))}{\prod_{m=1}^T Z_m} = W_i^1 \frac{\exp(-y_i \sum_{m=1}^T \alpha_m h(\mathbf{x}_i;\theta_m))}{\prod_{m=1}^T Z_m}$$

But $W_i^1 = \frac{1}{n}$ and $\sum_{m=1}^T \alpha_m h(\mathbf{x}_i;\theta_m) = h_T(\mathbf{x}_i)$, so:
$W_i^T = \frac{1}{n} \frac{\exp(-y_i h_T(\mathbf{x}_i))}{\prod_{m=1}^T Z_m}$

The training error of the final classifier is:

$\widehat{err}(h_T) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}[h_T(\mathbf{x_i}) \neq y_i] = \frac{1}{n} \sum_{i=1}^n \begin{cases} 1, & \text{if } y_i \neq h_T(\mathbf{x_i}) \\ 0, & \text{otherwise} \end{cases} =$

$= \frac{1}{n} \sum_{i=1}^n \begin{cases} 1, & \text{if } y_i h_T(\mathbf{x_i}) < 0. \\ 0, & \text{otherwise} \end{cases}$

But $\exp(-y_i h_T(\mathbf{x_i})) > 1$ if $y_i h_T(\mathbf{x_i}) < 0$, so:
$\widehat{err}(h_T) < \frac{1}{n} \sum_{i=1}^n \exp(-y_i h_T(\mathbf{x}_i)) = \frac{1}{n} n \prod_{m=1}^T Z_m \sum_{i=1}^n W_i^T$
From $\sum_{i=1}^n W_i^T = 1$ we obtain the following upper bound for the training error of the final classifier:

$\widehat{err}(h_T) < \prod_{m=1}^{T} Z_m$

$$W_i^{m+1} = W_i^m \exp(-y_i \alpha_m h(\mathbf{x}_i; \theta_m))/Z_m =$$
$$\frac{W_i^m}{Z_m} \begin{cases} \sqrt{\frac{1-\varepsilon_m}{\varepsilon_m}}, & \text{if } y_i h_T(\mathbf{x_i}) < 0. \\ \sqrt{\frac{\varepsilon_m}{1-\varepsilon_m}}, & \text{otherwise} \end{cases} \text{ because } \alpha_m = \frac{1}{2} \ln(\frac{1-\varepsilon_m}{\varepsilon_m})$$
$$1 = \sum_{i=i}^{n} W_i^{m+1} = \sqrt{\frac{\varepsilon_m}{1-\varepsilon_m}} \sum_{y_i = h_m(\mathbf{x}_i)} \frac{W_i^m}{Z_m} + \sqrt{\frac{1-\varepsilon_m}{\varepsilon_m}} \sum_{y_i \neq h_m(\mathbf{x}_i)} \frac{W_i^m}{Z_m} =$$
$$= \frac{1}{Z_m}(\sqrt{\frac{\varepsilon_m}{1-\varepsilon_m}}(1-\varepsilon_m) + \sqrt{\frac{1-\varepsilon_m}{\varepsilon_m}}\varepsilon_m) = \frac{2}{Z_m}\sqrt{\varepsilon_m(1-\varepsilon_m)}, \text{ so:}$$
$$Z_m = 2\sqrt{\varepsilon_m(1-\varepsilon_m)}$$

We denote by *edge* $\gamma_m = \frac{1}{2} - \varepsilon_m$, i.e. $\varepsilon_m = \frac{1-2\gamma_m}{2}$, so:

$$Z_m = \sqrt{1 - 4\gamma_m^2}$$

From the inequality $\sqrt{1-x} \leq \exp(-x/2)$, we obtain:
$$Z_m = \sqrt{1-4\gamma_m^2} \leq exp(-2\gamma_m^2)$$

That means:
$\widehat{err}(h_T) < \prod_{m=1}^{T} Z_m \leq \exp(-2\sum_{m=1}^{T} \gamma^2)$

Now, turning back to our weak learning assumption, we have: $\gamma_m \geq \gamma$ for $m = 1, \ldots, T$, so have completely proved that:

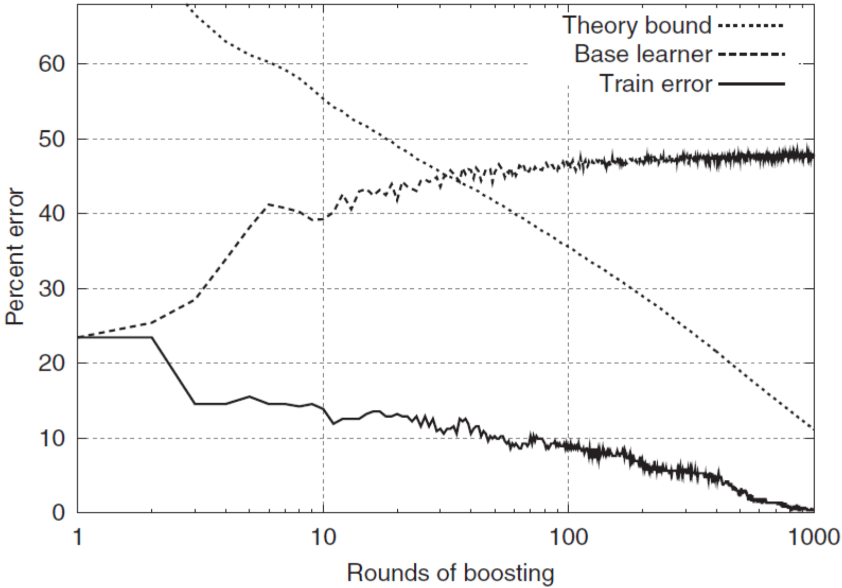
Theorem

$\widehat{err}(h_T) < \exp(-2T\gamma^2)$

The training error of the final classifier constructed with AdaBoost is exponentially converging to 0 for any γ with the number of rounds.

In reality, the convergence is even better, as it results from the following figure:

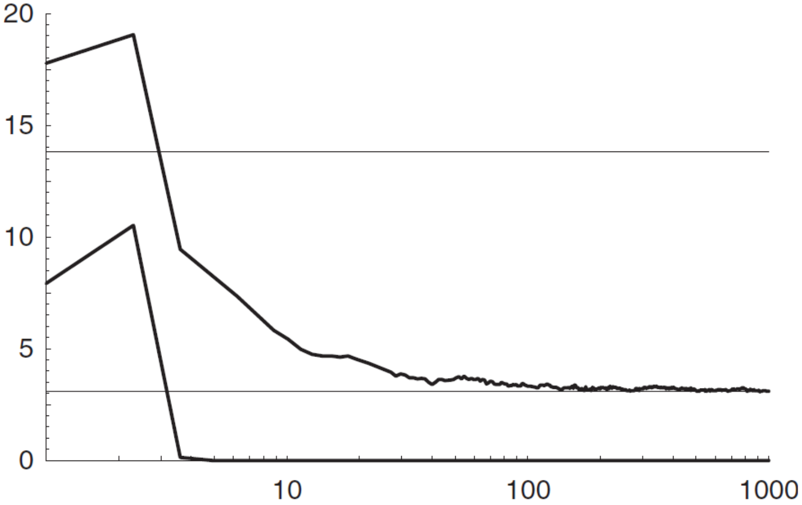
[Schapire & Freund's book, p. 57]



No overfitting

Test error and training error

[Schapire & Freund's book, p. 16]



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