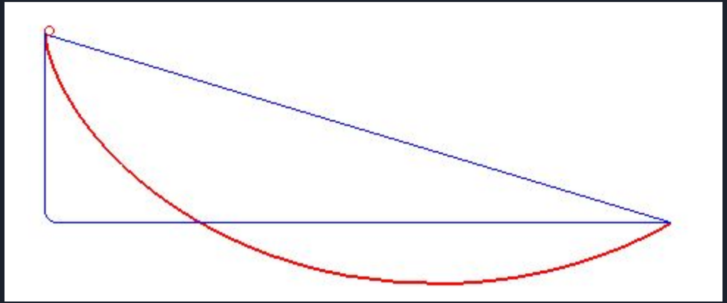
Optimal Curve Between Two Points

By Adrian Garcia

Introduction

Throughout history mathematicians continuously sought to challenge themselves with hypothetical problems.

Today we will be discussing one subsection of those hypothetical problems, namely, the Brachistochrone problem or “shortest time” problem.



Introduction

The Brachistochrone problem, introduced in 1696 by mathematician Johann Bernoulli, was given as follows:

- “Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time”

Although simple on the surface, its solution is anything but

Introduction

- Question:
    - Suppose we have an object at different starting points A that slide down a curve to an ending point B. What is the optimal curve that ensures the fastest travel time possible for each different starting point?

Model

For simplicity, let us define our variables as follows:

- Point A: $(0, y_A)$
- Point B: $(x_B, 0)$
    - y_A and x_B are both positive and nonzero
- An object of mass m starting at rest at point A
- Gravity $g \approx 9.8\ m/s^2$

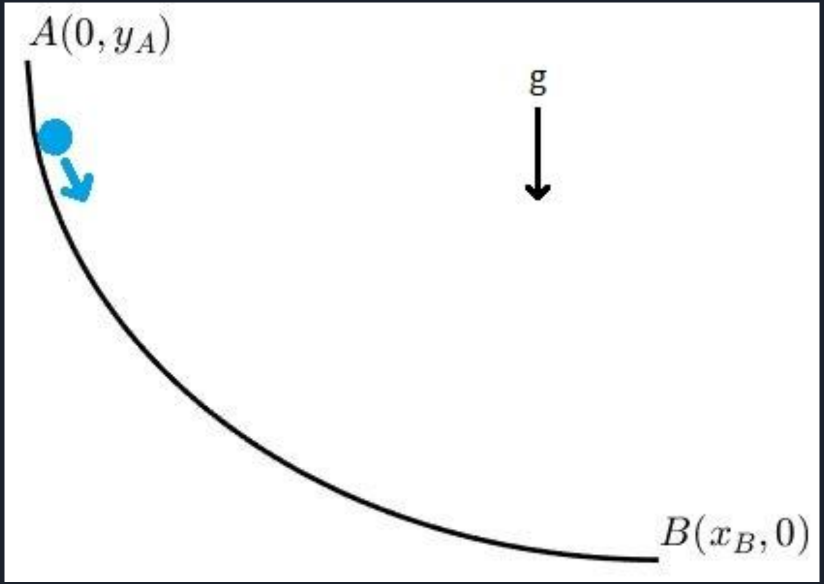


Figure 1: An object of mass m sliding down path y(x) from point A $(0,y_A)$ to point B $(x_B, 0)$

Model

In order to find the optimal curve that produces the fastest travel time we must then look into minimizing the integral

$$T = \int_A^B \frac{dS}{v}$$

where

- T: total travel time
- S: arc length of the optimal curve
- v: velocity of our object at any given point

Model

As our object is only under the influence of gravity, we may then assume the object's energy is conserved

- i.e. the sum of the potential and kinetic energy is the same at any point along the trajectory of the object

$$mgy_A = mgy(x) + \frac{1}{2}mv^2$$

$$v = \sqrt{2g(y_A - y(x))}$$

Similarly, by the Pythagorean theorem we know

$$dS = \sqrt{dx^2 + dy^2}$$

$$dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Model

Thus our integral then becomes

$$T(y(x)) = \int_0^{x_B} \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2g(y_A - y(x))}}\, dx$$

Now, to minimize this integral we must then consider the following. Suppose we have a minimizing function y(x) as well as a function ϵ(x) such that |ϵ(x) | <<1 and ϵ(x) =0 at both 0 and x_B. Then,

$$T(y(x)+\epsilon(x)) - T(y(x)) \propto \epsilon^2(x)$$

Model

As such, consider the following Taylor expansion of the integral near ϵ(x)=0

$$
\begin{aligned}
T(y+\epsilon)-T(y) &= \int_0^{x_B}\sqrt{\frac{1+[y'+\epsilon']^2}{2g(y_A-(y+\epsilon))}}\,dx-\int_0^{x_B}\sqrt{\frac{1+[y']^2}{2g(y_A-y)}}\,dx \\
&= \int_0^{x_B}\sqrt{\frac{1}{2g}}\left[\sqrt{1+[y']^2}+\frac{y'\epsilon'}{\sqrt{1+[y']^2}}+O(\epsilon^2)\right] \\
&\qquad\cdot\left[\frac{1}{\sqrt{y_A-y}}+\frac{\epsilon}{2(y_A-y)^{\frac{3}{2}}}+O(\epsilon^2)\right]dx-\int_0^{x_B}\sqrt{\frac{1+[y']^2}{2g(y_A-y)}}\,dx
\end{aligned}
$$

$$
T(y+\epsilon)-T(y) = \sqrt{\frac{1}{2g}}\int_0^{x_B}\frac{\sqrt{1+[y']^2}\epsilon}{2(y_A-y)^{\frac{3}{2}}}\,dx+\int_0^{x_B}\frac{y'\epsilon'}{\sqrt{(y_A-y)(1+[y']^2)}}\,dx+O(\epsilon^2)
$$

Model

As we have established $T(y(x)+\epsilon(x)) - T(y(x)) \propto \epsilon^2(x)$, we would then like the terms that are linear to ϵ(x) be 0, thus

$$\sqrt{\frac{1}{2g}}\int_0^{x_B} \frac{\sqrt{1+[y']^2}\epsilon}{2(y_A-y)^{\frac{3}{2}}}\,dx + \int_0^{x_B} \frac{y'\epsilon'}{\sqrt{(y_A-y)(1+[y']^2)}}\,dx = 0$$

Now using integration by parts on the second term of our integral we have,

$$\int_0^{x_B} \frac{y'\epsilon'}{\sqrt{(y_A-y)(1+[y']^2)}}\,dx = \left[\frac{y'}{\sqrt{(y_A-y)(1+[y']^2)}}\epsilon\right]_0^{x_B} - \int_0^{x_B} \epsilon\frac{d}{dx}\left[\frac{y'}{\sqrt{(y_A-y)(1+[y']^2)}}\right]dx$$

However, as ϵ(x)=0 at both boundaries 0 and x_B, we then have the subsequent integral

$$\int_0^{x_B} \frac{y'\epsilon'}{\sqrt{(y_A-y)(1+[y']^2)}}\,dx = -\int_0^{x_B} \epsilon\frac{d}{dx}\left[\frac{y'}{\sqrt{(y_A-y)(1+[y']^2)}}\right]dx$$

$$\implies \sqrt{\frac{1}{2g}}\int_0^{x_B} \epsilon\left[\frac{\sqrt{1+[y']^2}}{2(y_A-y)^{\frac{3}{2}}} - \frac{d}{dx}\left[\frac{y'}{\sqrt{(y_A-y)(1+[y']^2)}}\right]\right]dx = 0$$

Model

As this must be true for any function $\epsilon(x)$ the following must always be true

$$\frac{\sqrt{1+[y']^2}}{2(y_A-y)^{\frac{3}{2}}} = \frac{d}{dx}\left[\frac{y'}{\sqrt{(y_A-y)(1+[y']^2)}}\right]$$

Therefore to find the optimal curve we must then solve the above ODE subject to the boundary conditions $y(0) = y_A$ and $y(x_B) = 0$

Method

Following from the previous section, first note our time integral

$$T(y(x)) = \int_0^{x_B} \sqrt{\frac{1+\left(\frac{dy}{dx}\right)^2}{2g(y_A - y(x))}}\,dx$$

equates to a functional of the form

$$T(y(x)) = \int_0^{x_B} F(x, y(x), y'(x))$$

of which does not explicitly have the independent variable x; hence we may then use the Beltrami identity which states that if $\frac{\partial F}{\partial x} = 0$, then our ODE

$$\frac{\sqrt{1+[y']^2}}{2(y_A - y)^{\frac{3}{2}}} = \frac{d}{dx}\left[\frac{y'}{\sqrt{(y_A - y)(1+[y']^2)}}\right]$$

Equates to the Beltrami identity

$$F - y'\frac{\partial F}{\partial y'} = C$$

where C is a constant

Method

Thus

$$\sqrt{\frac{1+[y']^2}{2g(y_A-y)}}-\frac{[y']^2}{\sqrt{2g(y_A-y)(1+[y']^2)}}=C$$

$$C\sqrt{2g(y_A-y)(1+[y']^2)}=1$$

$$(y_A-y)(1+[y']^2)=C_1 \qquad \left(\because C_1=\frac{1}{2gC^2}\right)$$

$$(y_A-y)+[y']^2(y_A-y)=C_1$$

$$\sqrt{\frac{C_1-(y_A-y)}{y_A-y}}=y'=\frac{dy}{dx}$$

$$\sqrt{\frac{y_A-y}{C_1-(y_A-y)}}\,dy=dx$$

Method

After integrating both sides

$$x = \int \sqrt{\frac{y_A - y}{C_1 - (y_A - y)}}\, dy$$

and using trig substitution we have

$$y = y_A - C_1 \sin^2\left(\frac{\theta}{2}\right)$$

$$dy = -C_1 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta$$

$$\implies x = \int \sqrt{\frac{C_1 \sin^2\left(\frac{\theta}{2}\right)}{C_1 - C_1 \sin^2\left(\frac{\theta}{2}\right)}} - C_1 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta$$

Method

Subsequently

$$
\begin{aligned}
x &= \int \sqrt{\frac{C_1 \sin^2\left(\frac{\theta}{2}\right)}{C_1 - C_1 \sin^2\left(\frac{\theta}{2}\right)}} - C_1 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) d\theta \\
&= -C_1 \int \sin^2\left(\frac{\theta}{2}\right) d\theta \\
&= -\frac{C_1}{2} \int (1 - \cos\theta)\, d\theta \\
x &= -\frac{C_1}{2}(\theta - \sin\theta) + C_2
\end{aligned}
$$

where C_2 is a constant

Method

Thus we have the following parametric equations of our optimal curve

$$x = -\frac{C_1}{2}(\theta - \sin\theta) + C_2$$

$$y = y_A - \frac{C_1}{2}(1 - \cos\theta)$$

Now applying our starting boundary condition y(0) = y_A we have

$$\theta_A = 0 \implies x(0) = 0 = -\frac{C_1}{2}(0) + C_2 \implies C_2 = 0$$

Therefore

$$x = -\frac{C_1}{2}(\theta - \sin\theta)$$

$$y = y_A - \frac{C_1}{2}(1 - \cos\theta)$$

Method

As we have found our optimal curve, the subsequent travel time is given as follows

$$T = \sqrt{\frac{1}{2g}} \int_{\theta_A=0}^{\theta_B} \sqrt{\frac{1 + \left(\frac{dy/d\theta}{dx/d\theta}\right)^2}{(y_A - y)}} \frac{dx}{d\theta}\, d\theta$$

$$= \sqrt{\frac{1}{2g}} \int_{0}^{\theta_B} \sqrt{\frac{1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}}{\frac{C_1}{2}(1 - \cos\theta)}} \frac{-C_1}{2}(1 - \cos\theta)\, d\theta$$

$$= -\sqrt{\frac{C_1}{4g}} \int_{0}^{\theta_B} \sqrt{\frac{(1-\cos\theta)^2 + \sin^2\theta}{1 - \cos\theta}}\, d\theta$$

$$= -\sqrt{\frac{C_1}{2g}} \int_{0}^{\theta_B} \sqrt{\frac{1-\cos\theta}{1 - \cos\theta}}\, d\theta$$

$$T = -\sqrt{\frac{C_1}{2g}}\theta_B$$

Results

Disclaimer: It is significant to note that a final cosmetic transformation of our parametric equations was used so that the value of θ was increasing instead of decreasing as we plot from the first point to the next. To do this, we substitute -θ for θ in our parametric equations; hence

$$x = \frac{C_1}{2}(\theta - \sin\theta)$$
$$y = y_A - \frac{C_1}{2}(1 - \cos\theta)$$

Similarly

$$T = \sqrt{\frac{C_1}{2g}}\theta_B$$

Results

- Using an equation solver for a nonlinear system of equations, such as Mathematica's *FindRoot* function, we have C_1 = 1.14583 and θ_B = 2.41201 for the case A=(0,1). Subsequently

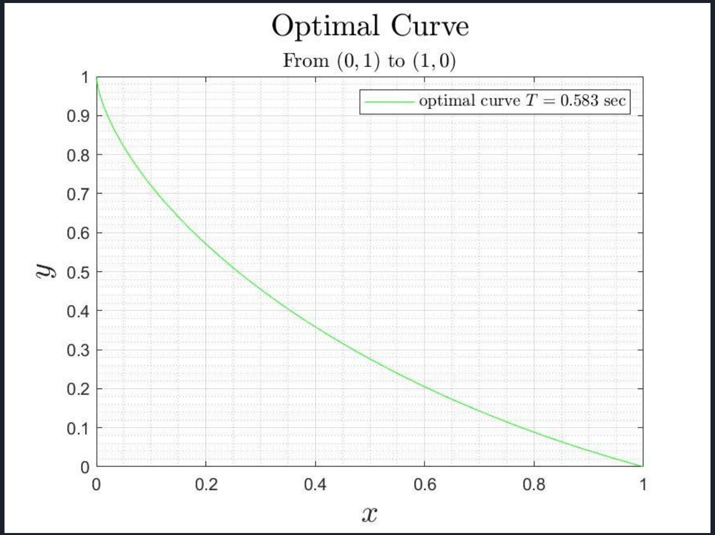


Figure 2: The optimal curve connecting point A (0,1) to point B (1, 0) with a total travel time of T = 0.583 seconds

Results

- Comparing the total travel time of the optimal curve to a select number of elementary functions, we see that we indeed found the most optimal curve from point A (0,1) to point B (1,0)

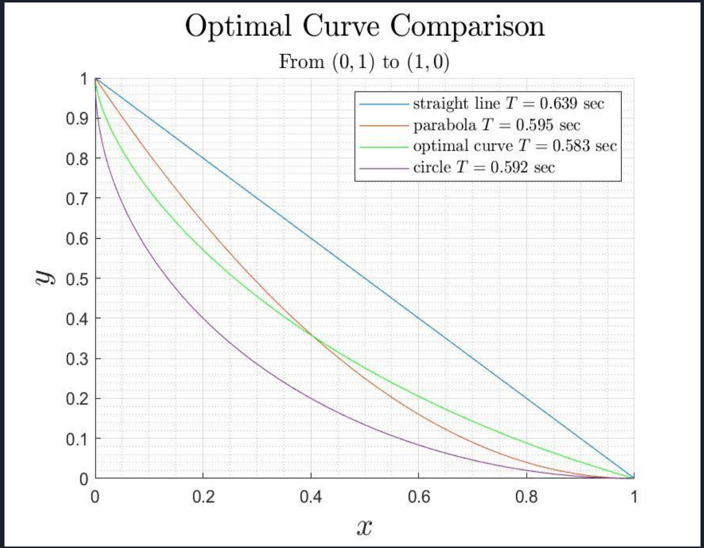


Figure 3: The total travel time of the optimal curve compared to a straight line, parabola, and circle from point A (0,1) to point B (1, 0)

Results

- Again using an equation solver for a nonlinear system of equations, such as Mathematica's *FindRoot* function, we have the following C_1, θ_B, and T values relative to a point A $(0, y_A)$

Point A	C_1	θ_B	T
(0, 0.5)	0.5172	3.50837	0.570
(0, 0.75)	0.762983	2.87995	0.568
(0, 1)	1.14583	2.41201	0.583
(0, 1.25)	1.70259	2.05822	0.607
(0, 1.5)	2.47222	1.78594	0.634

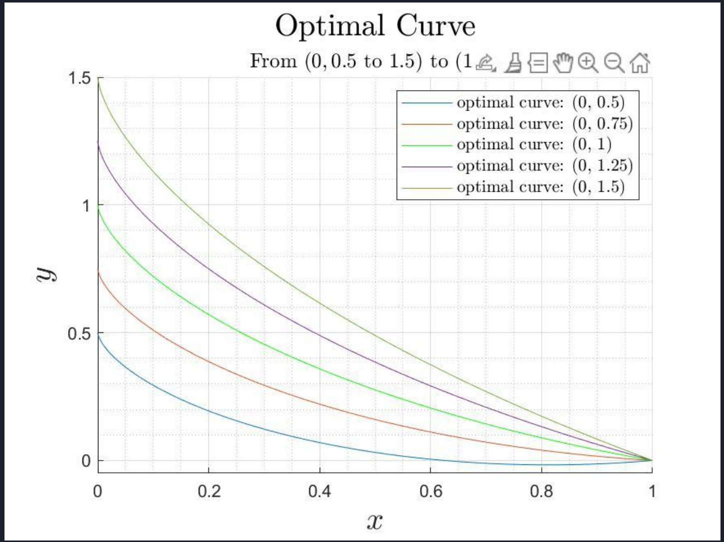


Figure 4: The optimal curve connecting various point A's to a fixed point B (1, 0)

Discussion

Although introduced as a hypothetical problem, the solution to the Brachistochrone problem does, in fact, have many real world applications as well as being a major factor into the creation of calculus of variation.

Discussion

- Caveats:
  - We did not consider any sort of friction or the object’s mass
  - In regards to a rollercoaster/ski ramp application, we did not check to see if the ride is safe for passengers
- Future work:
  - Answering these caveats either in conjunction or separately
  - Researching into why an object may start anywhere on our optimal curve and still end with the same time as any other starting point on the same curve