LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MAY 12, 2025

NETWORKED MINDS:
OPINION DYNAMICS AND COLLECTIVE INTELLIGENCE IN SOCIAL NETWORKS

RELAXING THE ASSUMPTIONS OF THE CJT

Adrian Haret
a.haret@lmu.de

The Condorcet Jury Theorem showed us that groups can be wise.

The Condorcet Jury Theorem showed us that groups can be wise. But what happens if its assumptions are not satisfied?

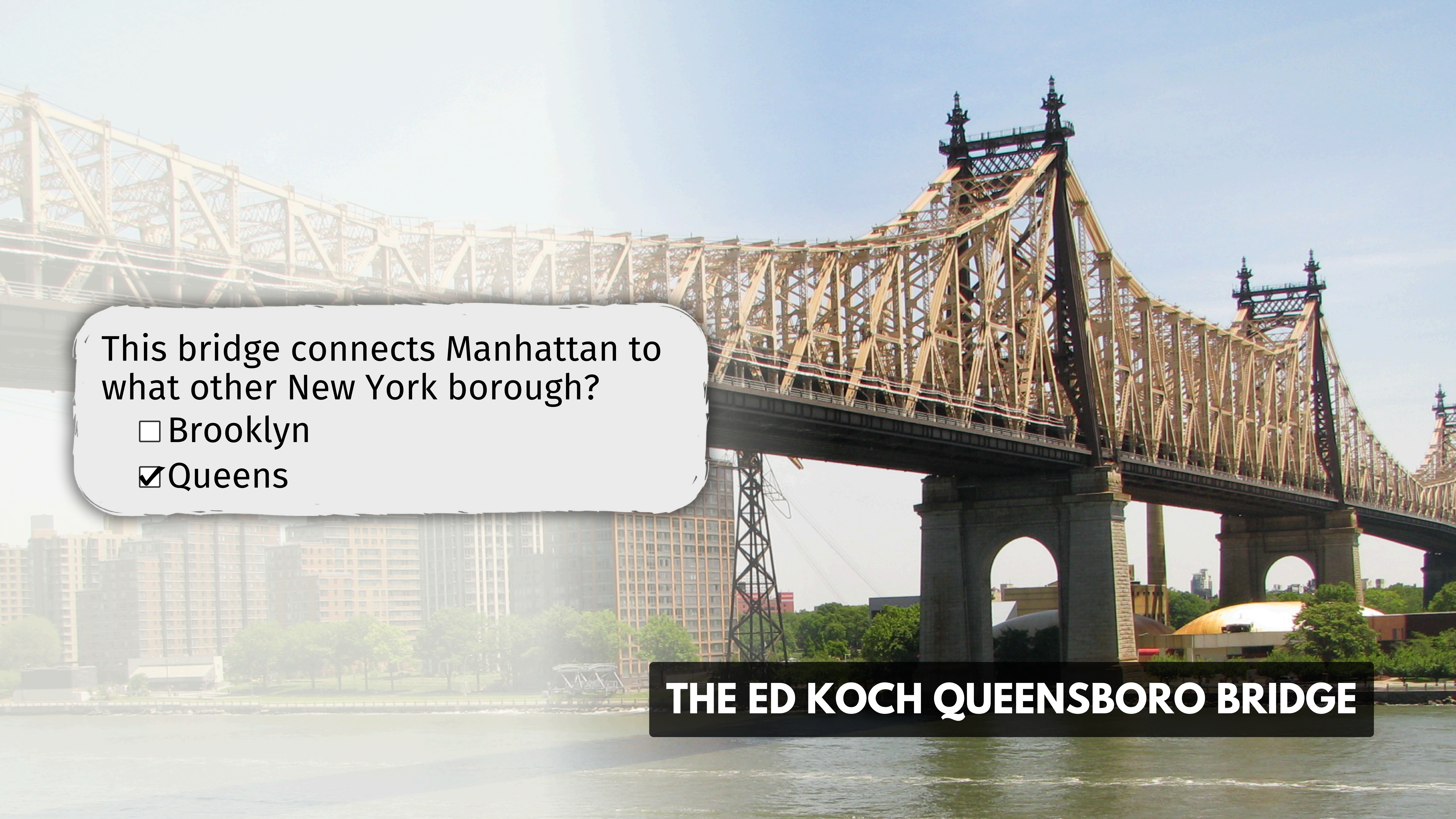
We’ve assumed competence is better than random, i.e., above ½.

We've assumed competence is better than random, i.e., above ½. What would be a reason for competence to be below ½?

Time for a pop quiz.

This bridge connects Manhattan to what other New York borough?

- ☐ Brooklyn
- ☐ Queens



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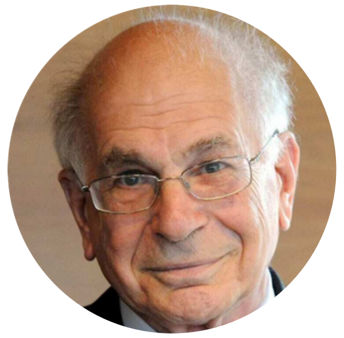
THE ED KOCH QUEENSBORO BRIDGE



DANIEL KAHNEMAN

You thought it was Brooklyn, didn’t you?

Humans have biases!

Like the *availability bias*.

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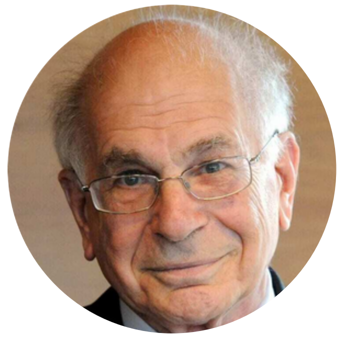

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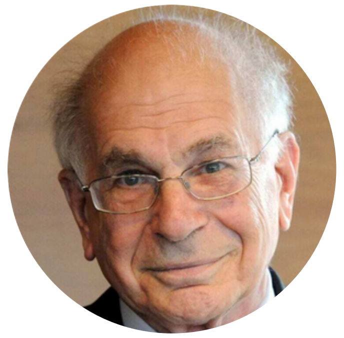

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HÉLÈNE LANDEMORE

Yeah let’s not exaggerate.

Landemore, H. (2013). *Democratic Reason: Politics, Collective Intelligence, and the Rule of the Many*. Princeton University Press.

Also, what does this competence p even mean?

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Also, what does this competence p even mean? Does it make sense to rate voters in this way? Especially if predicting rare, or unique, events…

GLENN W. BRIER

Sure! There’s an entire line of research dedicated to this.

Check out *scoring rules*.

For instance, the Brier score.

Even so, is it realistic to assume that competence is better than random?

PHILIP E. TETLOCK

Some people seem to be good at it.

Superforecasters!

Tetlock, P. E., & Gardner, D. (2016). *Superforecasting: The Art and Science of Prediction*. Random House.

What happens if voters have different competences?

GROUP ACCURACY WITH DIFFERENT COMPETENCES

Take three voters with competences p_1, p_2, p_3.

The probability of a correct majority decision is:

$$\begin{aligned}
\Pr\Big[S_n > 1\Big] &= \Pr\Big[S_n = 2 \text{ or } S_n = 3\Big] \\
&= p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3 + p_1 p_2 p_3 \\
&= p_1 p_2 + p_2 p_3 + p_1 p_3 - 2 p_1 p_2 p_3.
\end{aligned}$$

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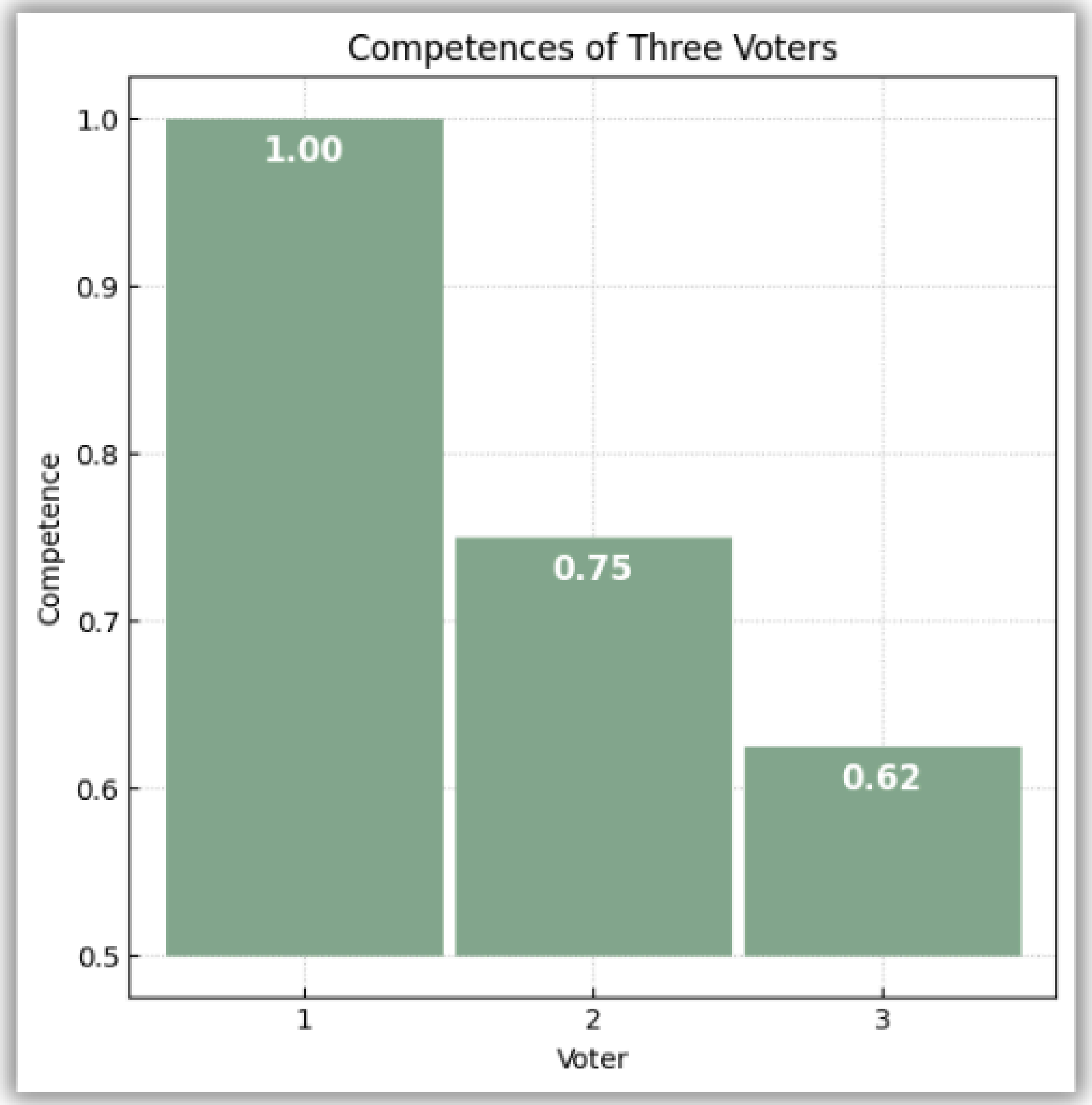
For n voters, n odd, with competences p_1, ..., p_n, the probability of a correct majority decision is:

$$\Pr\Big[S_n > {}^{n}\!/_{2}\Big] = \sum_{C \subseteq N, |C| > {}^{n}\!/_{2}} \left(\prod_{i \in C} p_i \cdot \prod_{N \setminus C} (1 - p_i) \right).$$

With different competences, things unravel a bit.

GROUP NO LONGER BETTER THAN ITS MEMBERS

Take three voters with competences $p_1 = 1$, $p_2 = {}^3\!/_4$ and $p_3 = {}^5\!/_8$.

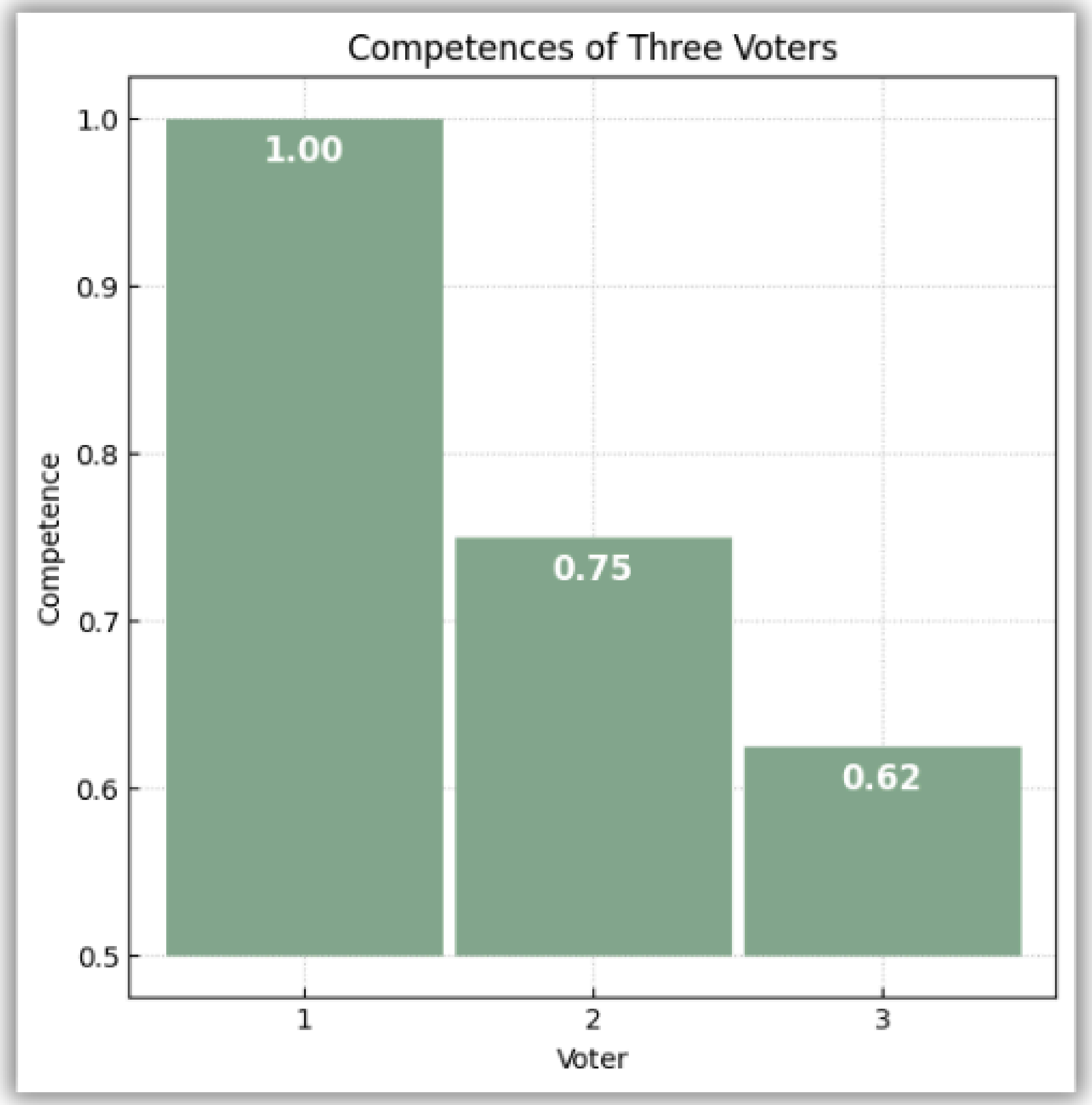


GROUP NO LONGER BETTER THAN ITS MEMBERS

Take three voters with competences $p_1 = 1$, $p_2 = 3/4$ and $p_3 = 5/8$.

The probability of a correct majority decision is:

$$\begin{aligned}
\Pr\left[S_3 > 1\right] &= \Pr\left[S_3 = 2 \text{ or } S_3 = 3\right] \\
&= p_1 p_2 (1 - p_3) + p_1 (1 - p_2) p_3 + (1 - p_1) p_2 p_3 + p_1 p_2 p_3 \\
&= 1 \cdot \frac{3}{4} \cdot \left(1 - \frac{5}{8}\right) + 1 \cdot \left(1 - \frac{3}{4}\right) \cdot \frac{5}{8} + (1 - 1) \cdot \frac{3}{4} \cdot \frac{5}{8} + 1 \cdot \frac{3}{4} \cdot \frac{5}{8} \\
&= 0.90625 \\
&< p_1.
\end{aligned}$$



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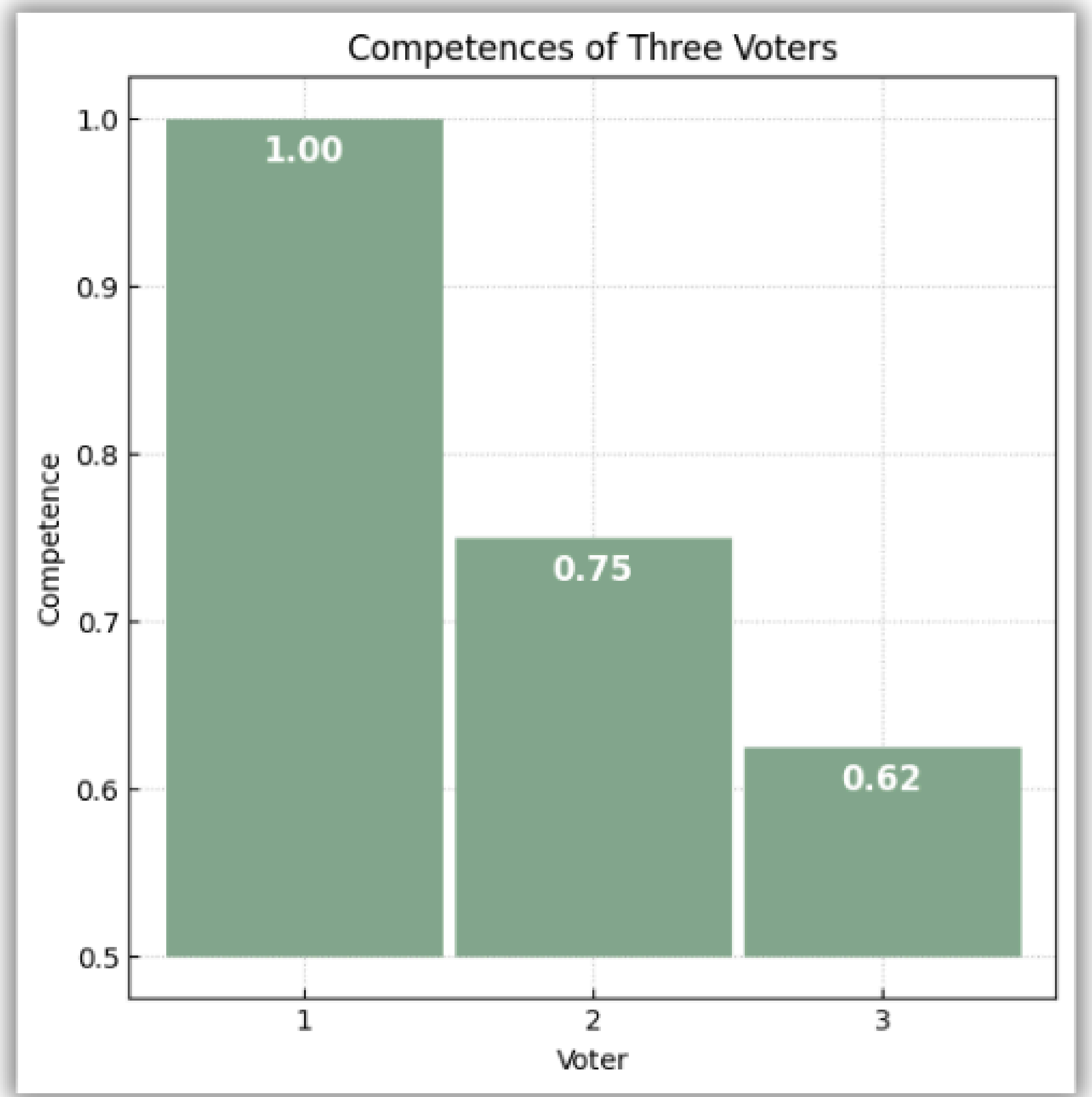
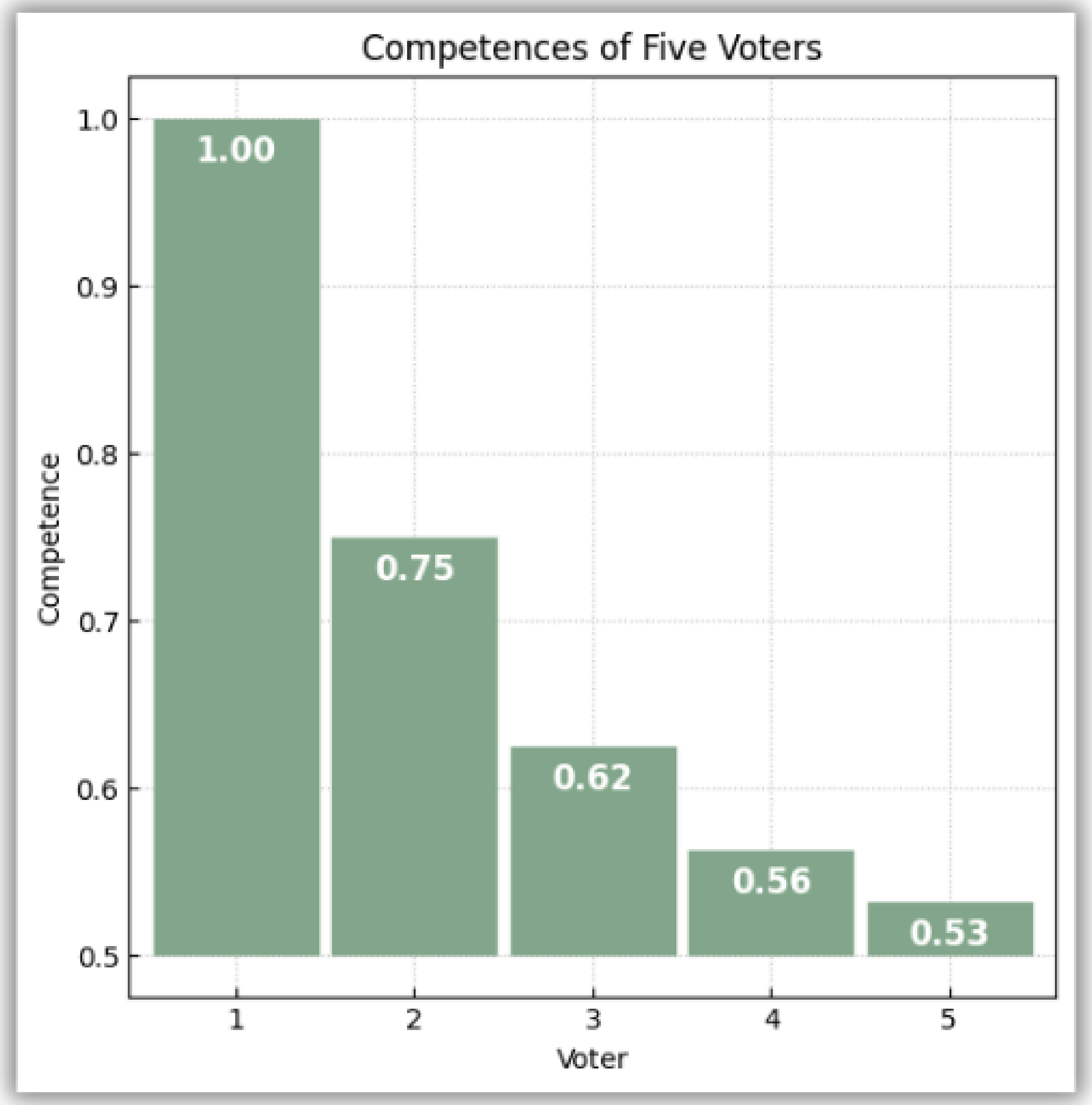
Now we have an expert (i.e., voter 1) who is actually better than the majority vote.



LARGER GROUPS NO LONGER GUARANTEED BETTER

Suppose we add two voters with competences $p_4 = {}^{9}/_{16}$ and $p_5 = {}^{17}/_{32}$ to the previous group.



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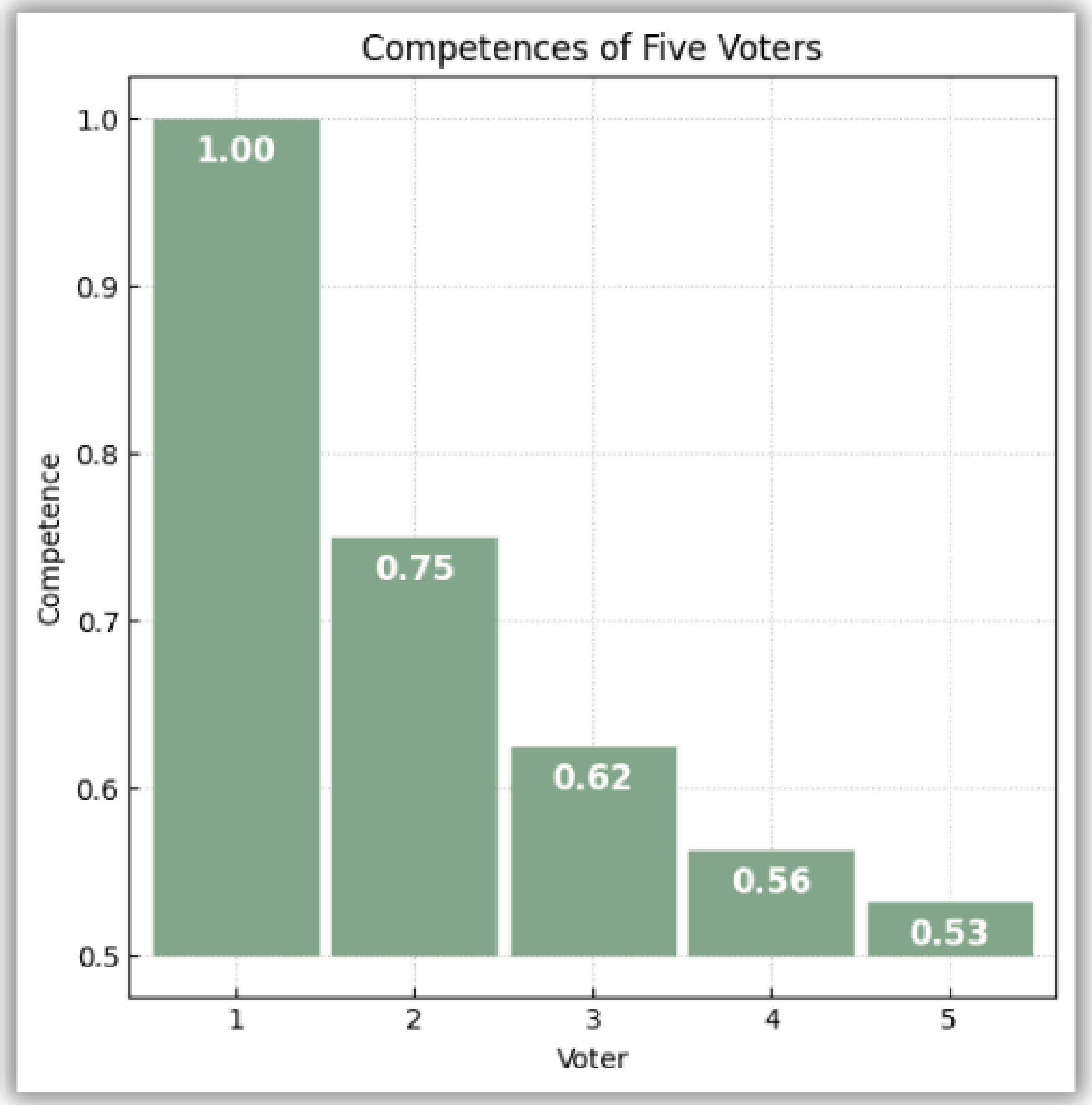
Suppose we add two voters with competences $p_4 = {}^{9}/_{16}$ and $p_5 = {}^{17}/_{32}$ to the previous group.

We now have a group of five voters whose competences are $\boldsymbol{p} = \left(1, {}^{3}/_{4}, {}^{5}/_{8}, {}^{15}/_{16}, {}^{16}/_{32}\right)$.



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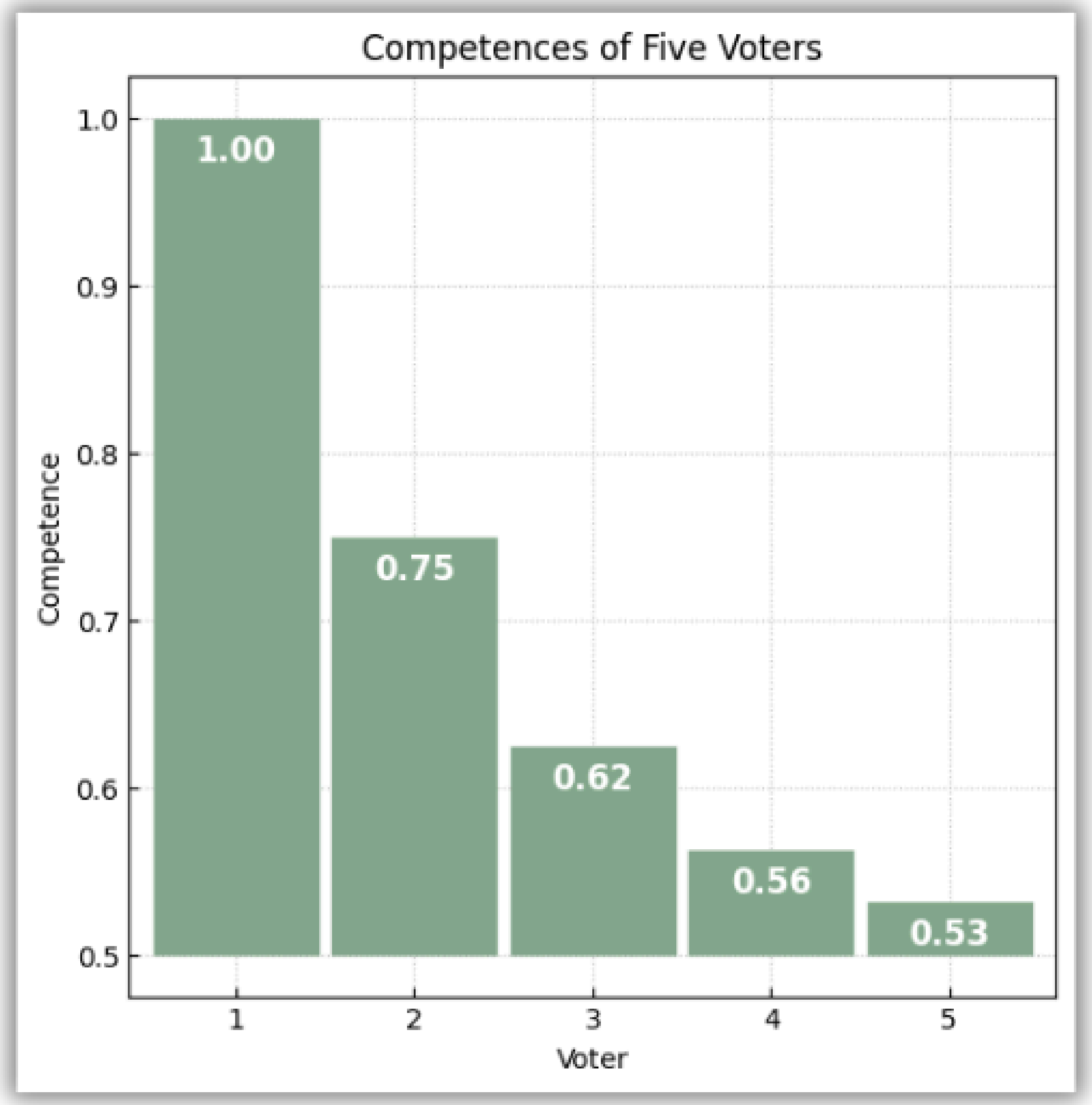

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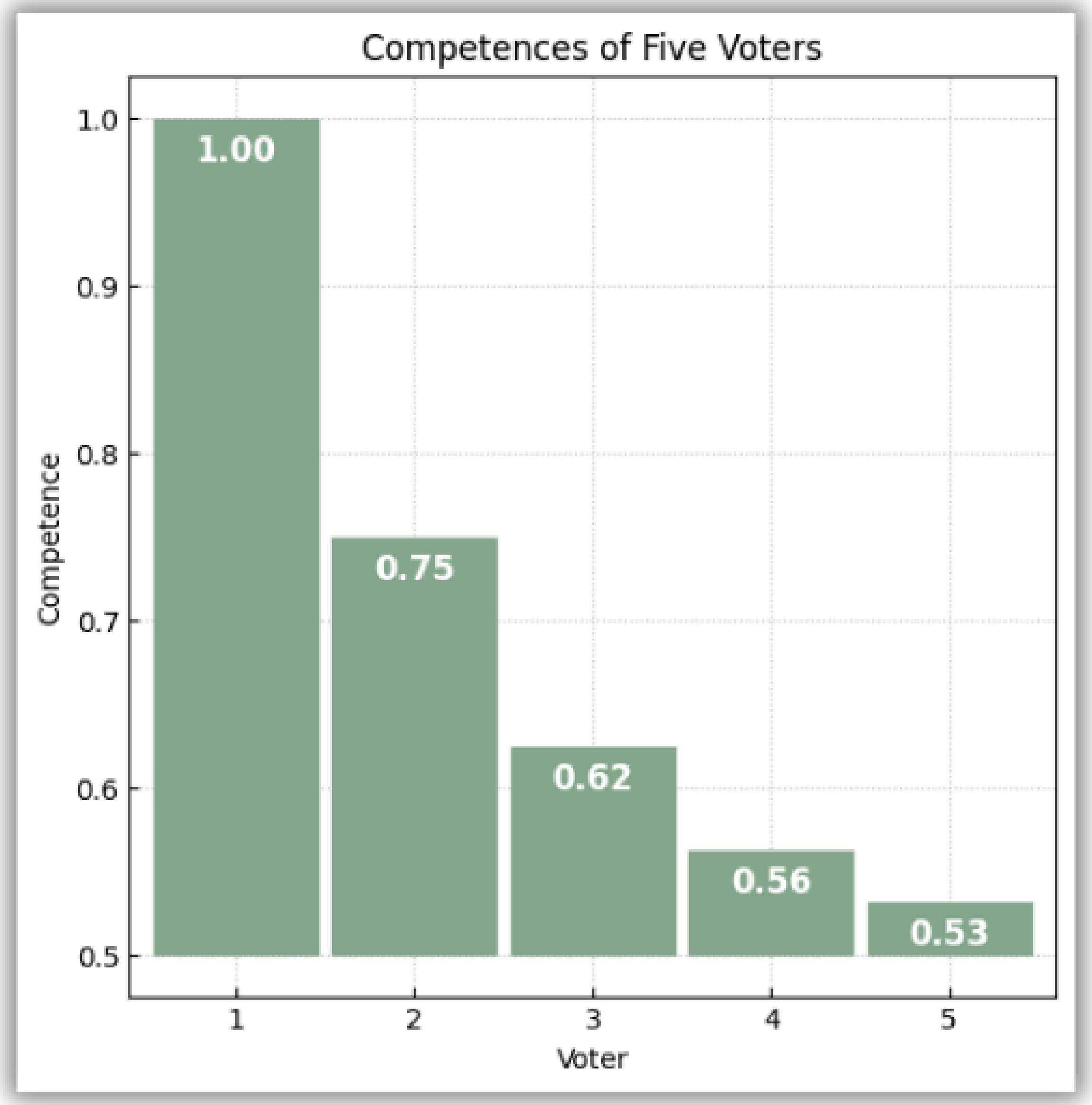
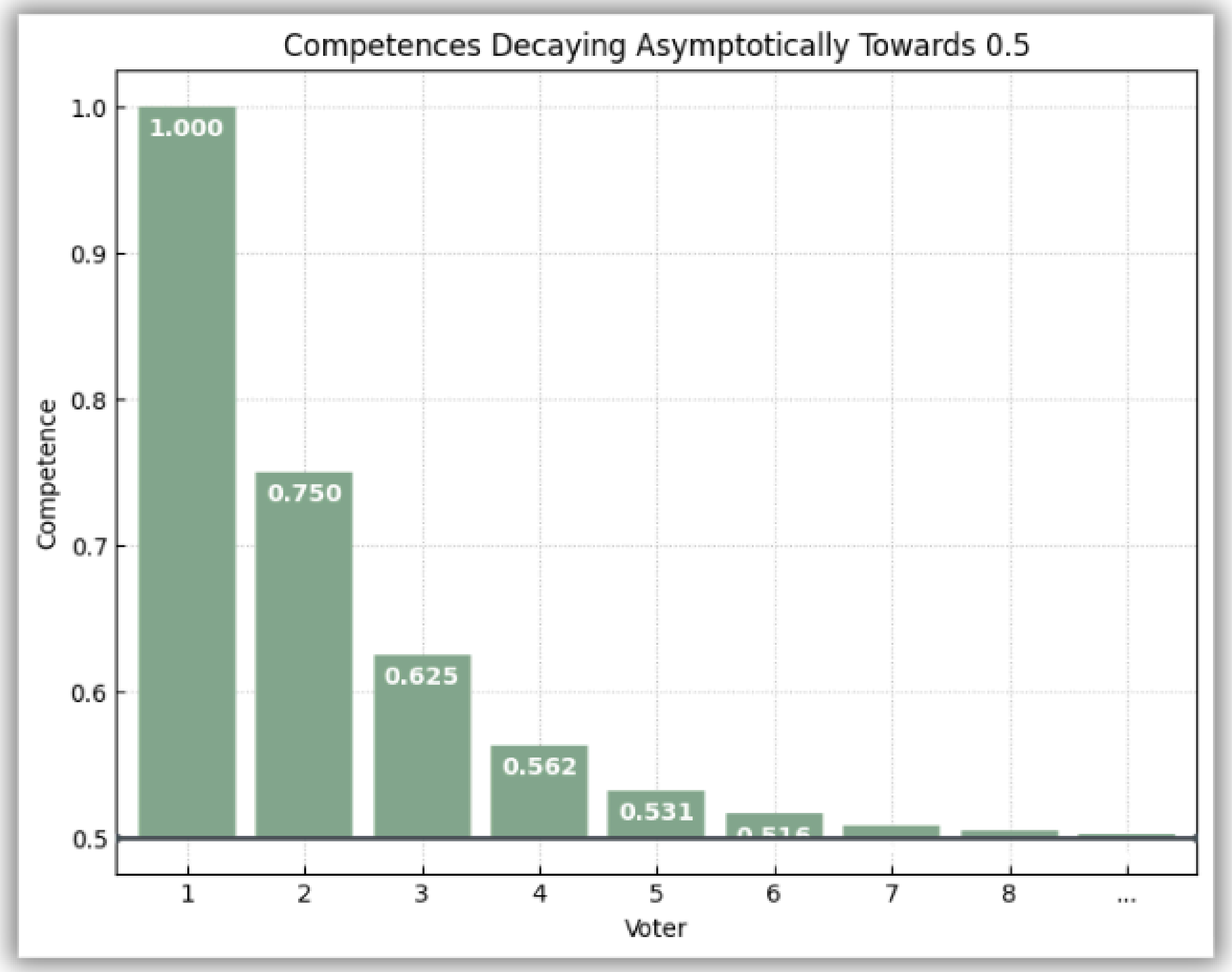
Adding voters 4 and 5 made the group less accurate than before!



GROUP ACCURACY IN THE LIMIT NO LONGER PERFECT

Take n voters with competences:

$$p_1 = \frac{1}{2} + \frac{1}{2}, \quad p_2 = \frac{1}{2} + \frac{1}{2^2}, \quad \ldots, \quad p_n = \frac{1}{2} + \frac{1}{2^n}.$$



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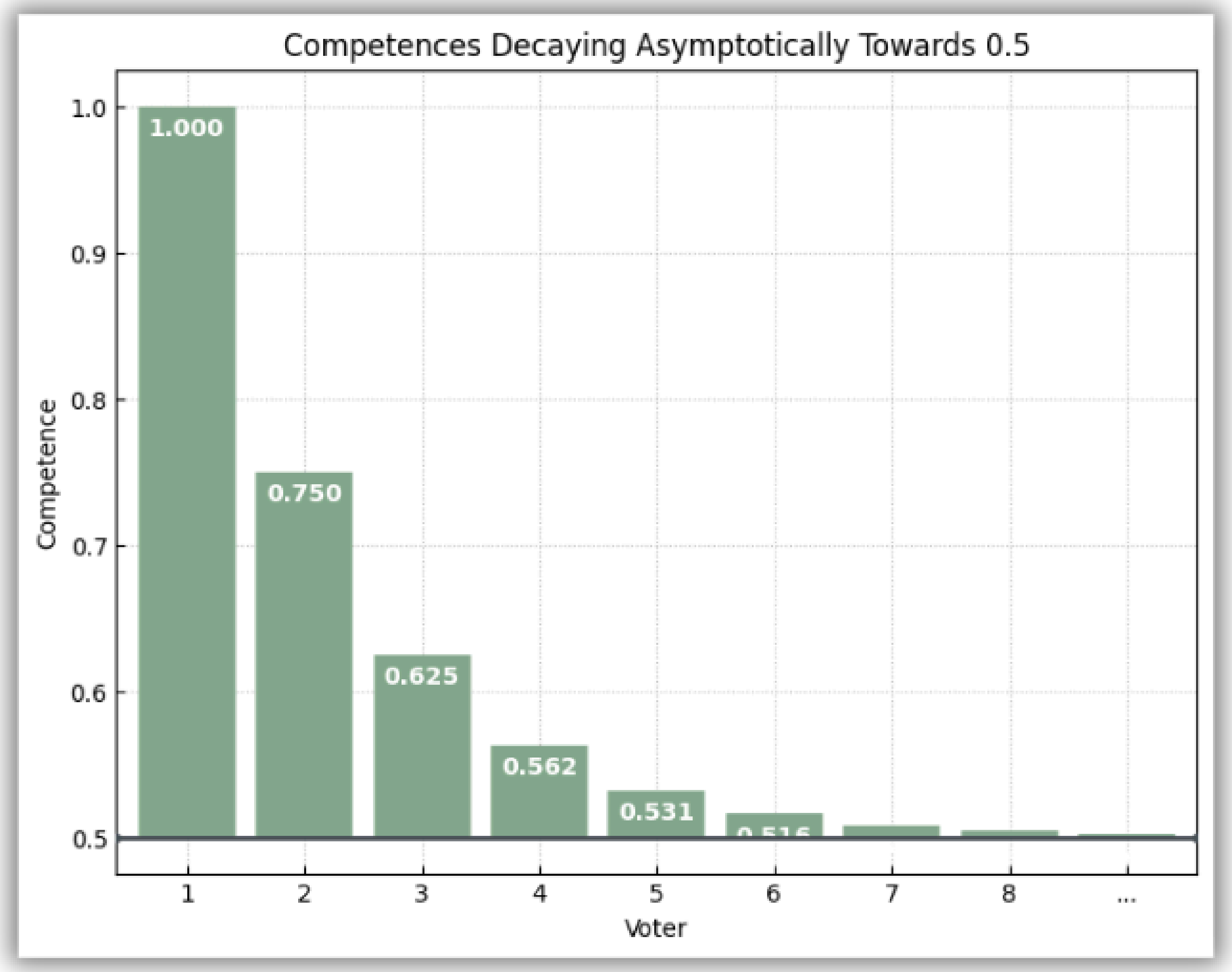
The probability of a correct majority decision, as n grows, is:

$$\lim_{n \to \infty} \Pr\left[S_n > {}^{n}\!/_{2}\right] = \frac{1}{2}.$$



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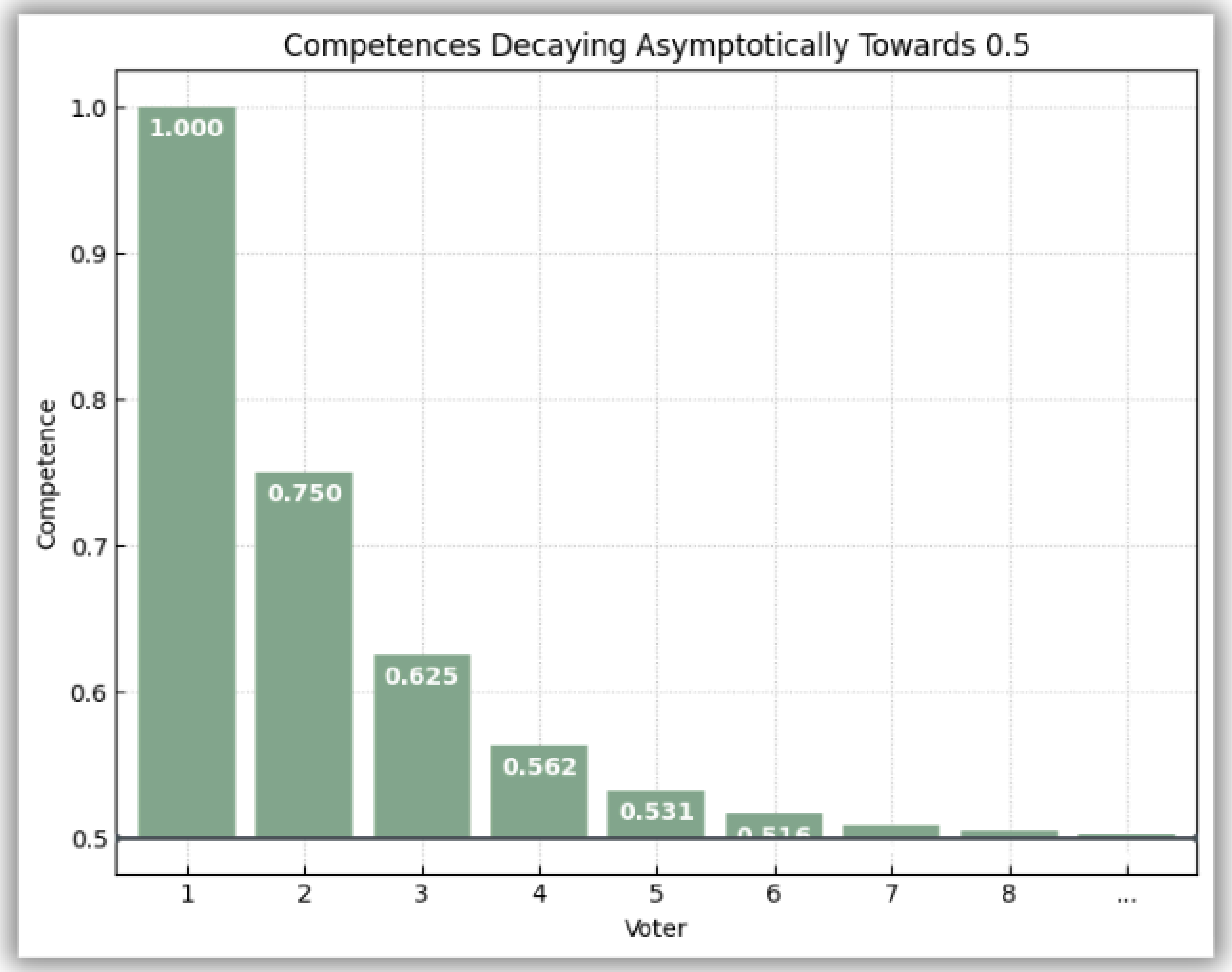

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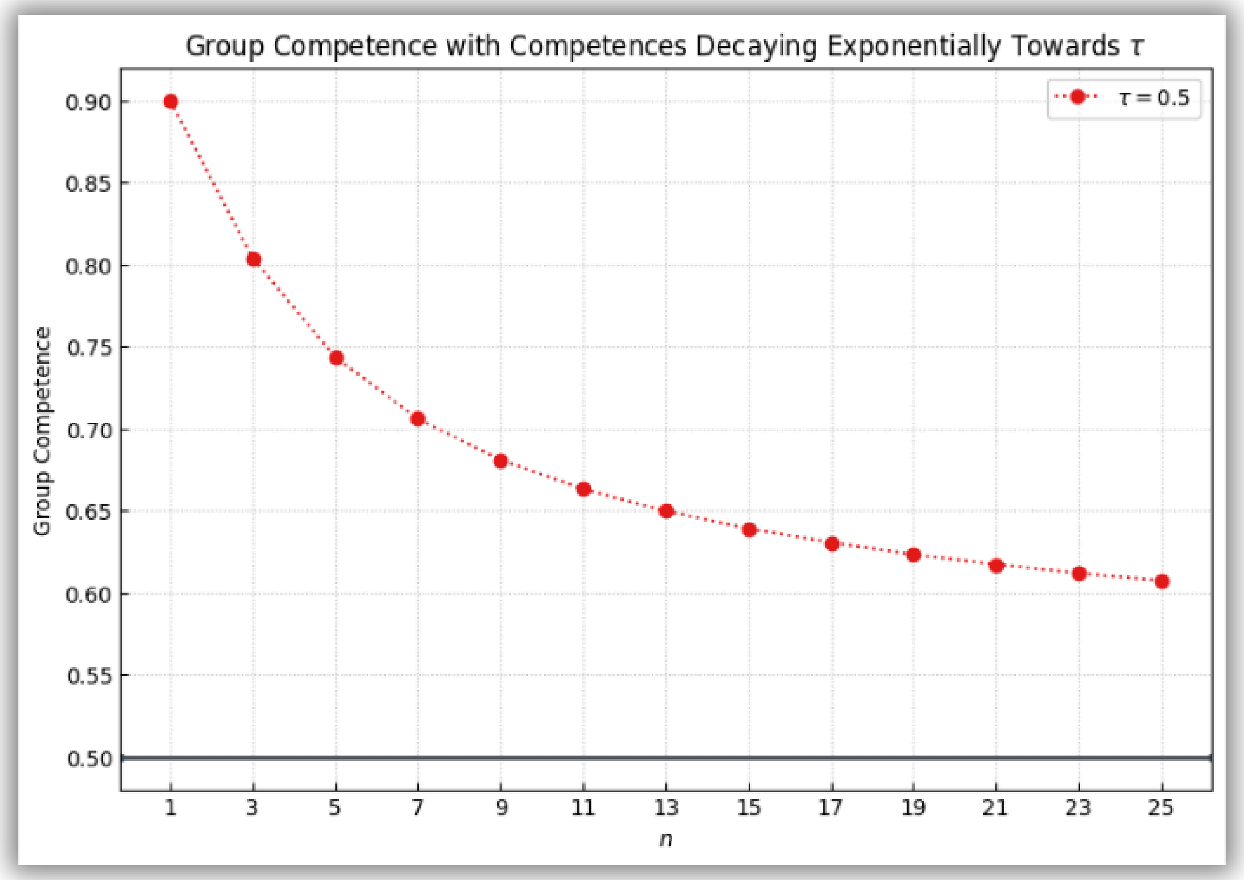

The problem, here, is that competences cna arbitrarily close to ½.

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COMPETENCES BOUNDED ABOVE ½

THEOREM (PAROUSH, 1998)

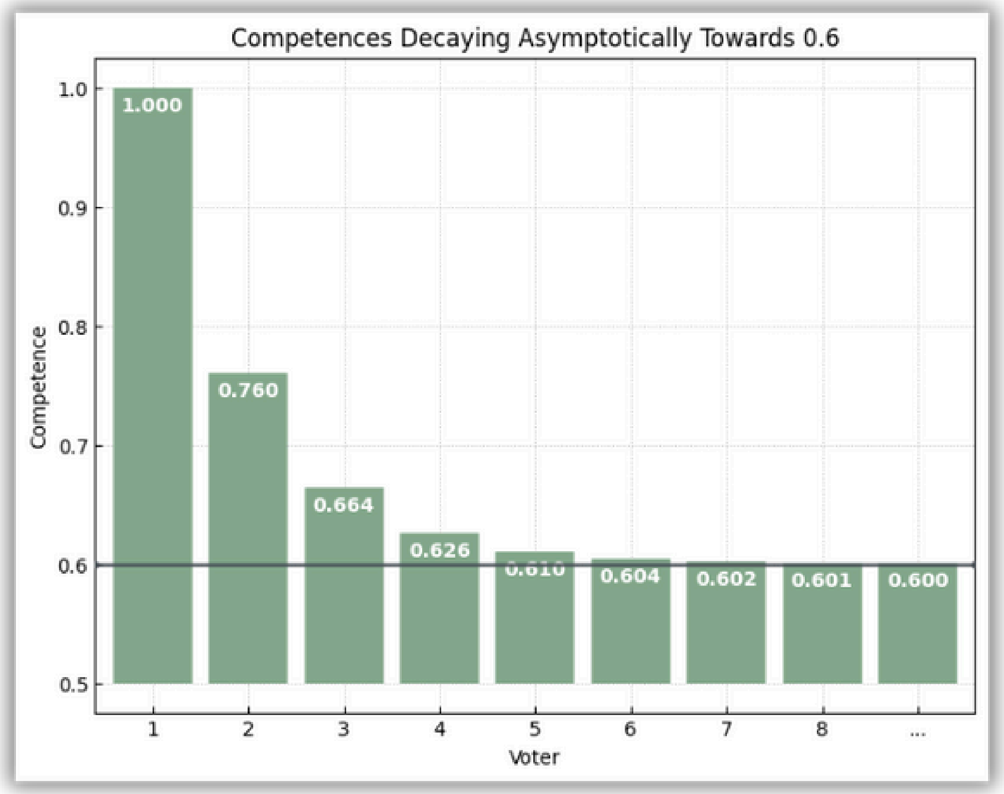
For an odd number n of voters with competences $p_1, \ldots, p_n$ who vote independently of each other, then, if $p_i > {}^1\!/_2 + \varepsilon$, for every voter i and some $\varepsilon > 0$, it holds that:

$$\lim_{n \to \infty} \Pr\left[S_n > {}^n\!/_2\right] = 1.$$

Paroush, J. (1998). Stay away from fair coins: A Condorcet jury theorem. *Social Choice and Welfare*, 15(1), 15–20.

COMPETENCES DECAYING EXPONENTIALLY TO ½ + ε

Suppose competences decay towards 0.6, which implies that ε = 0.1.



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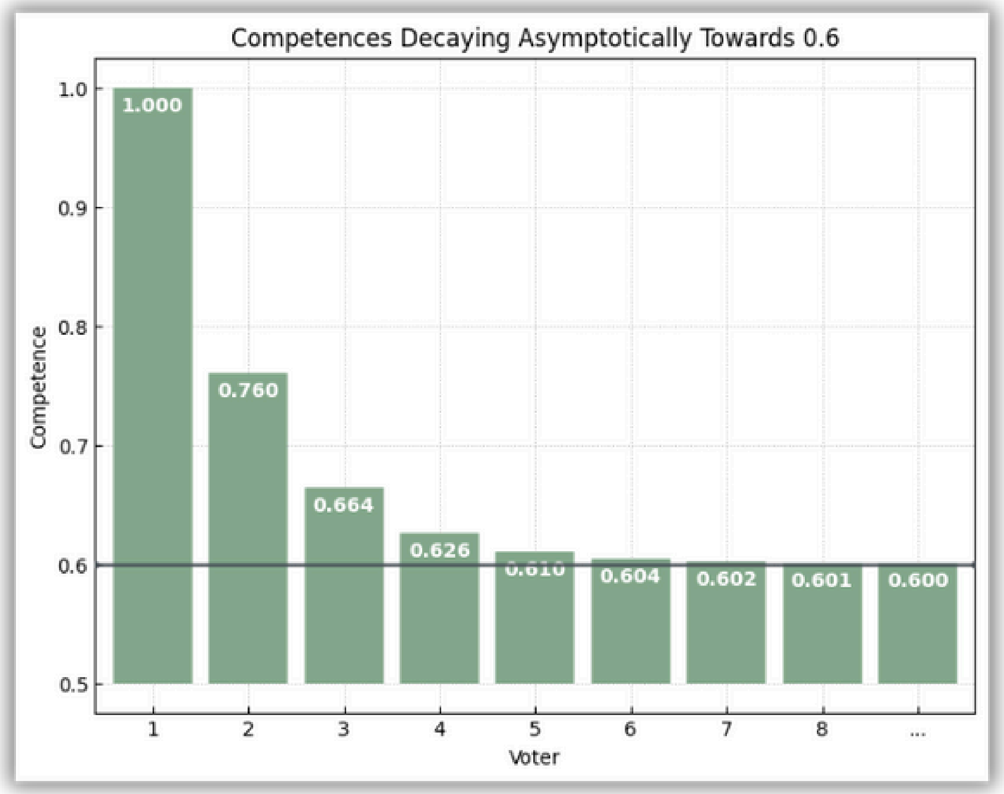
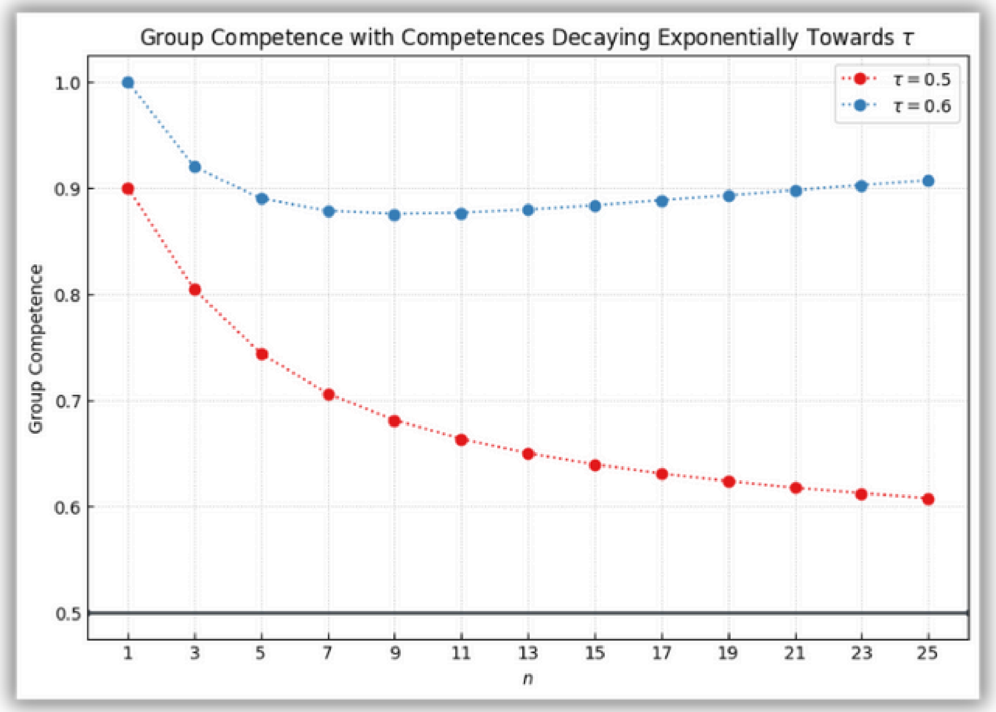
Now group competence goes to 1, again.





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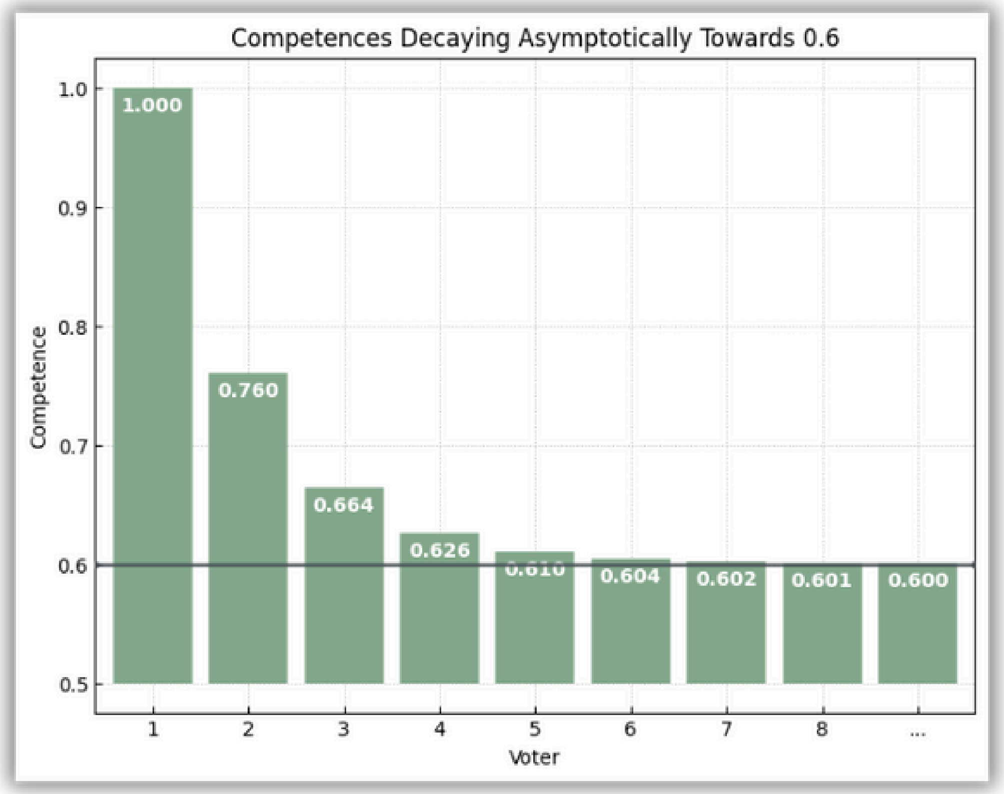
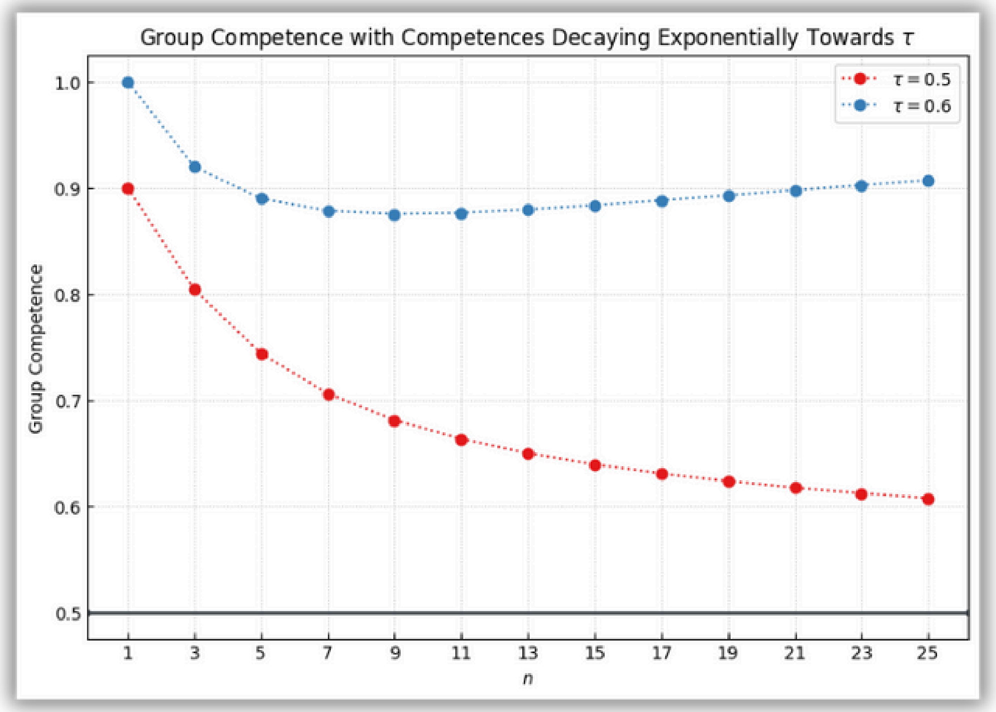
Eventually…





How do we prove this?

COMPETENCES BOUNDED ABOVE ½: PROOF

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PROOF

Take $\boldsymbol{p} = \Big(p_1, \ldots, p_n\Big)$ to be the vector of competences of n voters.

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Note, first, that if we improve the competence of one voter, then the probability of a correct majority decision increases.

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Formally, suppose we replace some p_i in $\boldsymbol{p}$ with $p_i' > p_i$, while keeping all other competences the same. We say the resulting vector $\boldsymbol{p}'$ is an *improvement* of $\boldsymbol{p}$. If S_n' is the sum of the votes determined by $\boldsymbol{p}'$, we have that:

$$\Pr\left[S_n' > n/2\right] > \Pr\left[S_n > n/2\right]$$

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Formally, suppose we replace some p_i in $\boldsymbol{p}$ with $p'_i > p_i$, while keeping all other competences the same. We say the resulting vector $\boldsymbol{p}'$ is an *improvement* of $\boldsymbol{p}$. If S'_n is the sum of the votes determined by $\boldsymbol{p}'$, we have that:

$$\Pr\left[S'_n > n/2\right] > \Pr\left[S_n > n/2\right]$$

Note, now, that we can get from $\boldsymbol{p}^* = \left(1/2 + \varepsilon, \ldots, 1/2 + \varepsilon\right)$ to any $\boldsymbol{p} = \left(p_1, \ldots, p_n\right)$ by a series of improvements.

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But we already know, from the Condorcet Jury Theorem, that the group competence of $\boldsymbol{p}^*$ approaches 1 asymptotically. So the group competence of $\boldsymbol{p}$ does the same.

This still requires, though, that competences are above ½. Is this essential?

COMPETENCES NORMALLY DISTRIBUTED

THEOREM (GROFMAN, 1978)

For an odd number n of voters with accuracies normally distributed with a mean of $p > 1/2$ and a variance of $p(1-p)/n$, then:

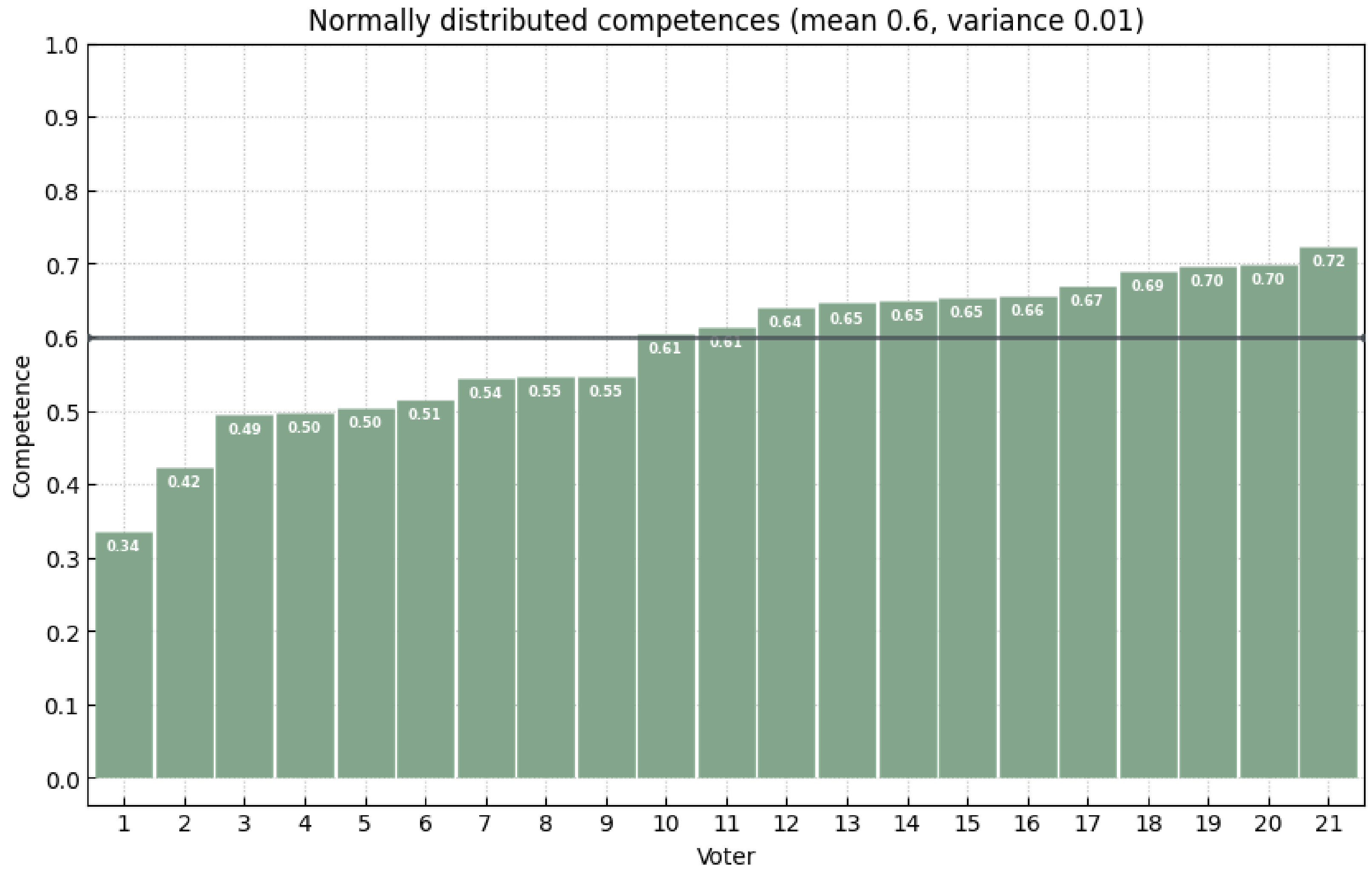
$$\lim_{n \to \infty} \Pr\left[S_n > n/2\right] = 1.$$

Grofman, B. (1978). Judgmental competence of individuals and groups in a dichotomous choice situation: Is a majority of heads better than one? *Journal of Mathematical Sociology*, 47–60.

COMPETENCES NORMALLY DISTRIBUTED: ILLUSTRATION

With normally distributed competences, most values for p_i cluster around the mean p.

The variance σ^2 describes how much values jump deviate from the mean.



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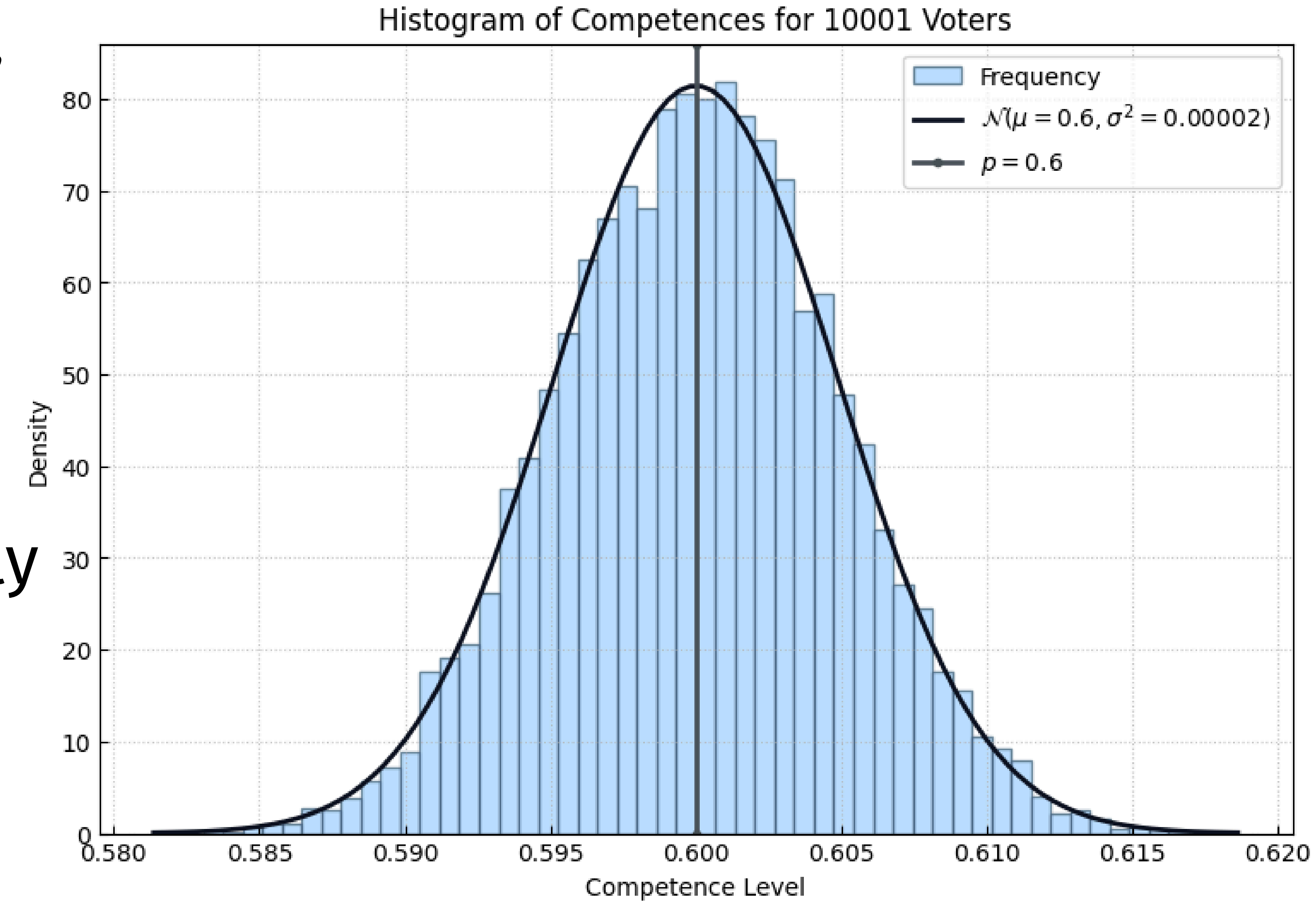

COMPETENCES NORMALLY DISTRIBUTED: ILLUSTRATION

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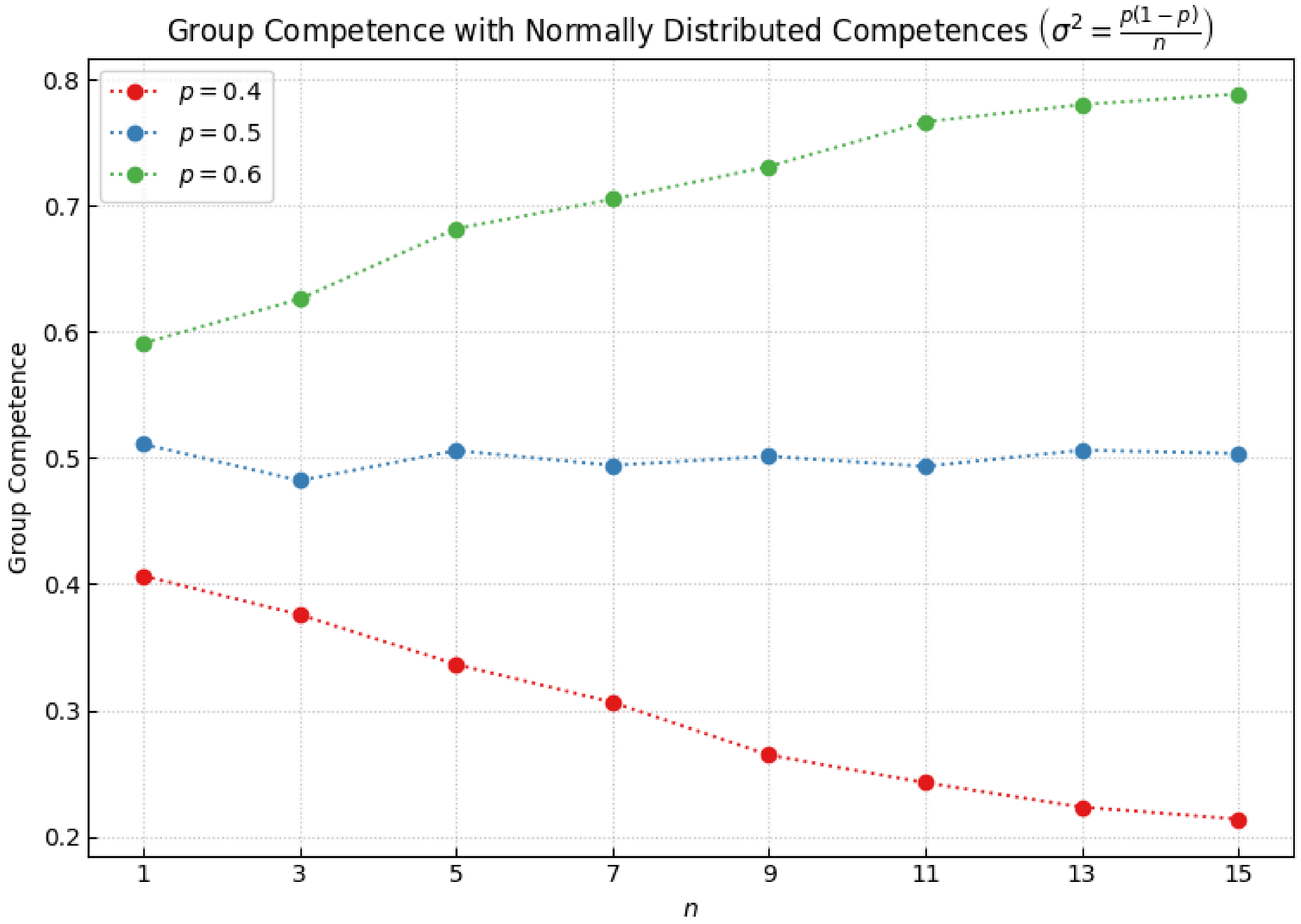
Normal distributions are more commonly represented using histograms.

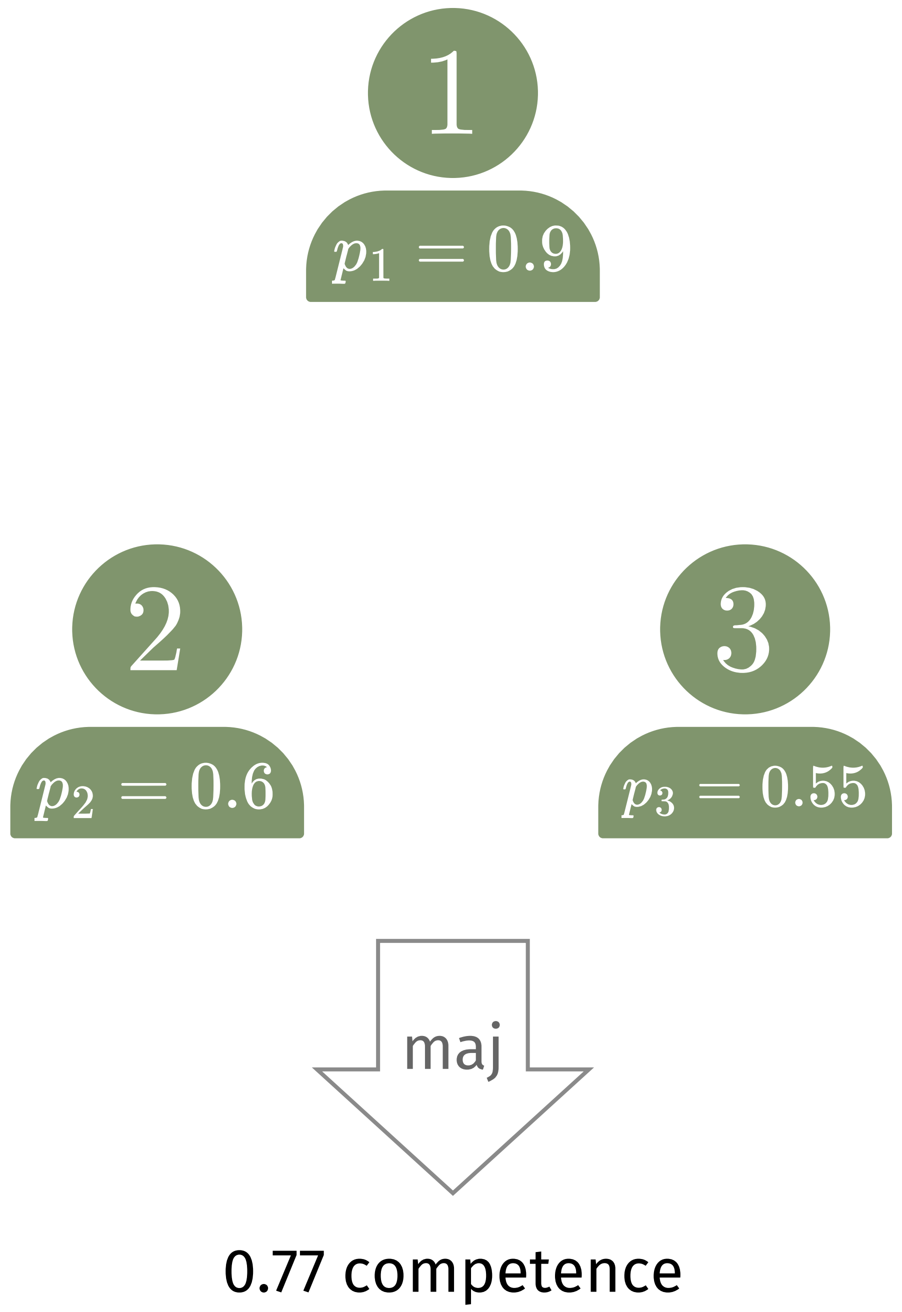
And the mean determines group competence.



What about independence?

SOCIAL INFLUENCE

Majority opinion with independent voters has a competence of 0.77.



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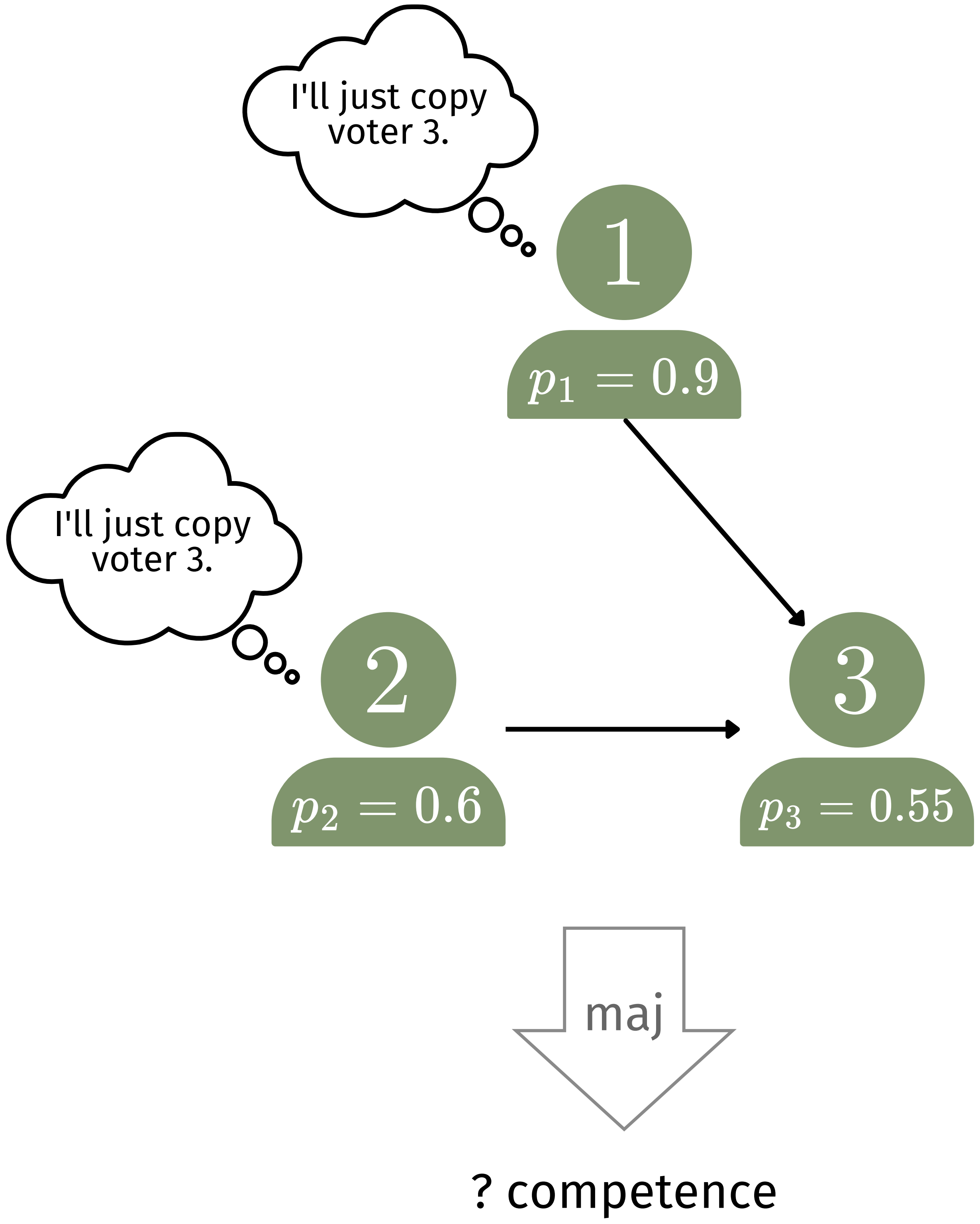
Suppose voters 1 and 2 copy voter 3.



SOCIAL INFLUENCE

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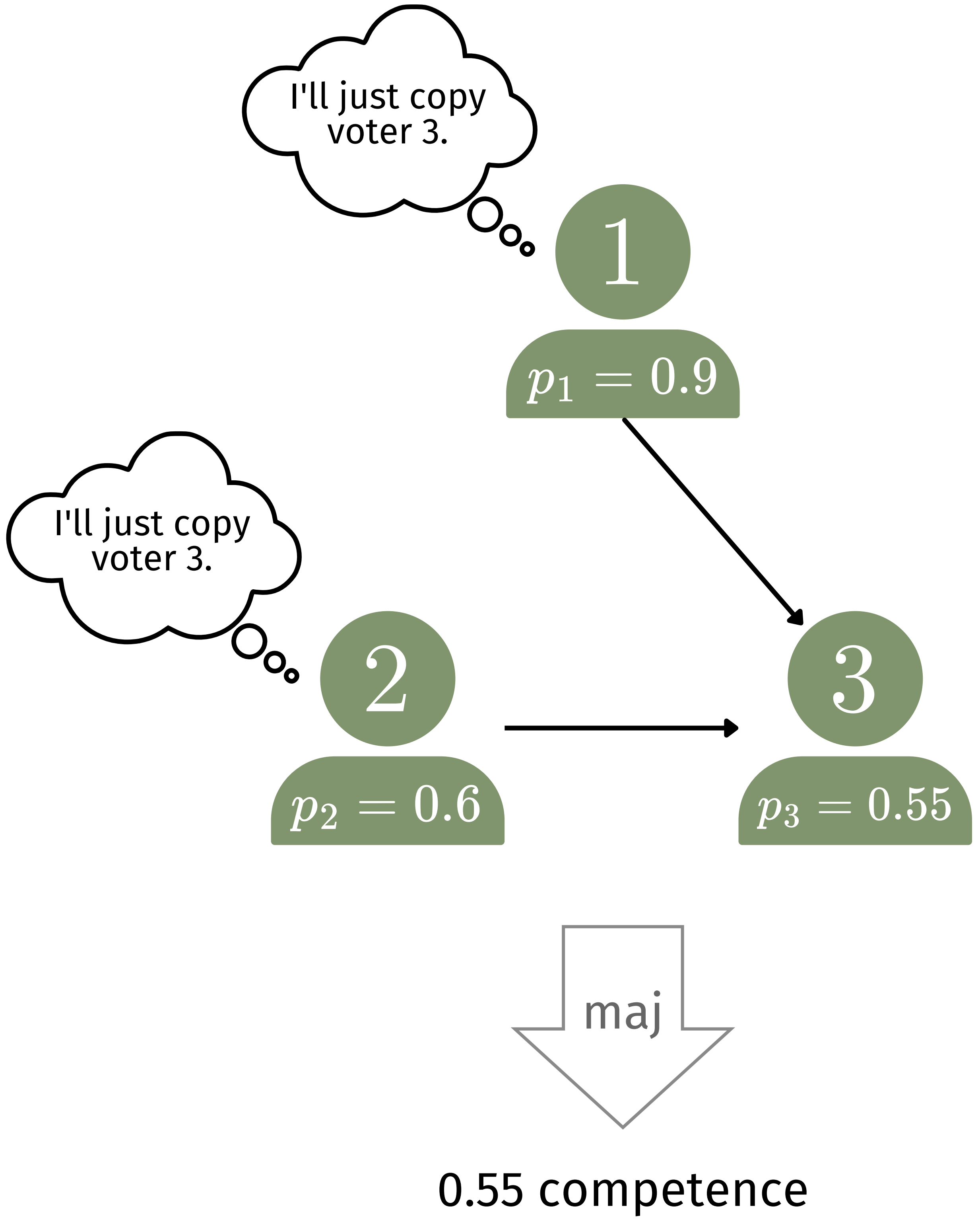
Suppose voters 1 and 2 copy voter 3.

Now the group is only as good as voter 3!



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