LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

COOPERATION: HOW TO MODEL IT, HOW TO FOSTER IT, AND HOW IT MIGHT HAVE EMERGED

TOMORROW & TOMORROW & TOMORROW

ITERATED GAMES

Adrian Haret
a.haret@uva.nl

December 7-14, 2023

How did the Prisoner's Dilemma come about?

MERRIL FLOOD

Melvin and I came up with the idea behind the Prisoner's Dilemma in the 50's, while working for the RAND corporation.

MELVIN DRESHER

●●●

Poundstone, W. (1993). *Prisoner's Dilemma: John Von Neumann, Game Theory and the Puzzle of the Bomb*. Anchor Books.

MERRIL FLOOD

Melvin and I came up with the idea behind the Prisoner's Dilemma in the 50's, while working for the RAND corporation.

MELVIN DRESHER

Amidst concern about nuclear war and political instability, we devised a little game to show that the Nash equilibrium sometimes makes strange predictions.

MERRIL FLOOD

Poundstone, W. (1993). *Prisoner's Dilemma: John Von Neumann, Game Theory and the Puzzle of the Bomb*. Anchor Books.

MERRIL FLOOD

Melvin and I came up with the idea behind the Prisoner's Dilemma in the 50's, while working for the RAND corporation.

MELVIN DRESHER

Amidst concern about nuclear war and political instability, we devised a little game to show that the Nash equilibrium sometimes makes strange predictions.

MERRIL FLOOD

We made two of our friends, AA and JW, play the game over 100 times, and recorded their reactions.

MELVIN DRESHER

Poundstone, W. (1993). *Prisoner's Dilemma: John Von Neumann, Game Theory and the Puzzle of the Bomb*. Anchor Books.

MERRIL FLOOD

Melvin and I came up with the idea behind the Prisoner's Dilemma in the 50's, while working for the RAND corporation.

MELVIN DRESHER

Amidst concern about nuclear war and political instability, we devised a little game to show that the Nash equilibrium sometimes makes strange predictions.

MERRIL FLOOD

We made two of our friends, AA and JW, play the game over 100 times, and recorded their reactions.

MELVIN DRESHER

For all the confusion, mutual cooperation occurred 60 out of the 100 trials.

Poundstone, W. (1993). *Prisoner's Dilemma: John Von Neumann, Game Theory and the Puzzle of the Bomb*. Anchor Books.

Game	AA	JW	AA's comments	JW's comments
1	D	C	JW will play [D]—sure win. Hence if I play [C]—I lose.	Hope he's bright.
2	D	C	What is he doing?!!	He isn't but maybe he'll wise up.
3	D	D	Trying mixed?	Okay, dope.
4	D	D	Has he settled on [D]?	Okay, dope.
5	C	D	Perverse!	It isn't the best of all possible worlds.
6	D	C	I'm sticking to [D] since he will mix for at least 4 more times.	Oh ho! Guess I'll have to give him another chance.
7	D	C		Cagey, ain't he? Well . . .
8	D	D		In time he could learn, but not in ten moves so:
9	D	D	If I mix occasionally, he will switch—but why will he ever switch from [D]?	
10	D	D	Prediction. He will stick with [D] until I change from [D]. I feel like DuPont.	I can guarantee myself a gain of 5, and guarantee that Player AA breaks

even (at best). On the other hand, with nominal assistance from AA, I can transfer the guarantee of 5 to Player AA and make 10 for myself, too. This means I have control of the game to a large extent, so Player AA had better appreciate this and get on the bandwagon.

With small amounts of money at stake, I would (as above) try (by using [C]) to coax AA into mutually profitable actions. With large amounts at stake I would play [D] until AA displayed some initiative and a willingness to invest in his own future. One play of [C] by AA would change me from [D] to [C], where I would remain until bitten.

On the last play, it would be conservative for me to switch to [D], but I wouldn't do so if the evidence suggested that AA was a nice stable personality

Game	AA	JW	AA's comments	JW's comments
				and not in critical need of just a little extra cash.
11	D	C		Probably learned by now.
12	C	C		I'll be damned! But I'll try again.
13	C	C		That's better.
14	C	C		Ha!
15	C	C		(bliss)
16	D	C		
17	C	D		The stinker.
18	C	D		He's crazy. I'll teach him the hard way.
19	D	D	I'm completely confused. Is he trying to convey information to me?	Let him suffer.
20	D	D		
21	D	C		Maybe he'll be a good boy now.
22	C	C		Always takes time to learn.

Poundstone, W. (1993). *Prisoner's Dilemma: John Von Neumann, Game Theory and the Puzzle of the Bomb*. Anchor Books.

Are AA and JW irrational?

MERRIL FLOOD

What do you say to that, John?!

JOHN NASH

MERRIL FLOOD
What do you say to that, John?!

JOHN NASH
You know, playing the Prisoner's Dilemma one time is not the same as playing it 100 times.

Playing it over and over again is like playing a different, multi-round game.

In the one-shot game there's no room for things like loyalty, trust, threats, or revenge.

But in the iterated version, these things can be relevant!

This gives us the first way out of the pessimistic outlook of the Prisoner's Dilemma.

This gives us the first way out of the pessimistic outlook of the Prisoner's Dilemma.

Does the equilibrium change if the game is played repeatedly?

So far we’ve been assuming that players make moves simultaneously, in ignorance of the other players’ actions.

So far we've been assuming that players make moves simultaneously, in ignorance of the other players' actions.

But, of course, some games are played over rounds.

GAMES IN EXTENSIVE FORM

In *perfect-information extensive-form games*, players take turns deploying their actions.

And are aware of actions taken at previous rounds: perfect memory!

Player 1 takes an action

...out of their action set: {*a*, *a'*}

Player 2 follows up

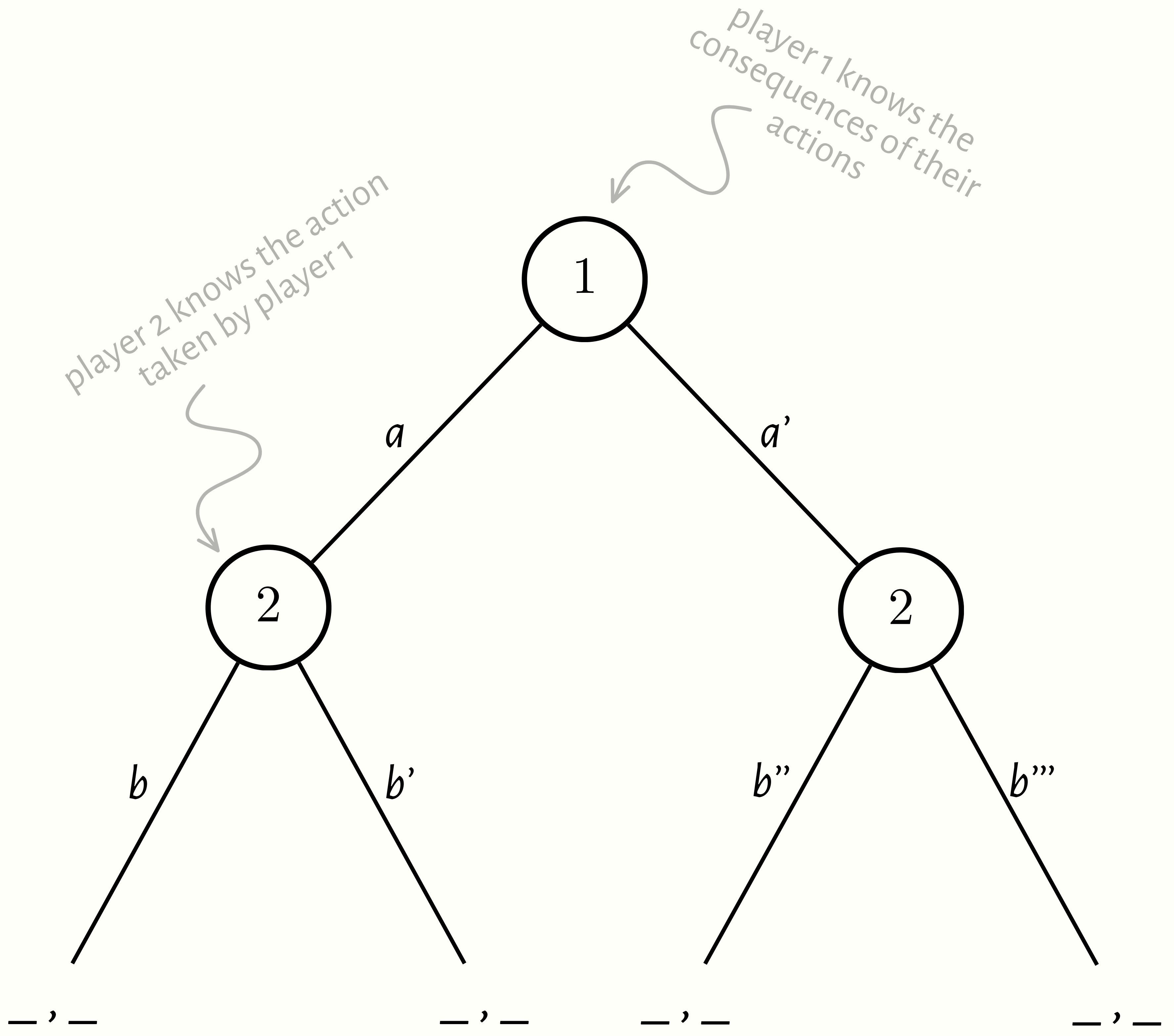
... knowing the action player 1 has taken

Every player receives a payoff

... specific to the branch taken

The whole game tree is known

... to all players



Extensive-form games with perfect information are modeled as *trees*, where non-terminal nodes (called *choice nodes*) correspond to *players*.

Extensive-form games with perfect information are modeled as *trees*, where non-terminal nodes (called *choice nodes*) correspond to *players*.

At every one of its choice nodes, an agent has some *actions* available.

Extensive-form games with perfect information are modeled as *trees*, where non-terminal nodes (called *choice nodes*) correspond to *players*.

At every one of its choice nodes, an agent has some *actions* available.

Each edge is labeled with the action taken by the parent agent at that node.

Extensive-form games with perfect information are modeled as *trees*, where non-terminal nodes (called *choice nodes*) correspond to *players*.

At every one of its choice nodes, an agent has some *actions* available.

Each edge is labeled with the action taken by the parent agent at that node.

Terminal nodes are labeled with the *utilities* of the players for the combination of actions that led to that particular outcome.

Extensive-form games with perfect information are modeled as *trees*, where non-terminal nodes (called *choice nodes*) correspond to *players*.

At every one of its choice nodes, an agent has some *actions* available.

Each edge is labeled with the action taken by the parent agent at that node.

Terminal nodes are labeled with the *utilities* of the players for the combination of actions that led to that particular outcome.

A *strategy* for an agent is a combination of actions, one for each node corresponding to that agent.

The Ultimatum Game ⚖️

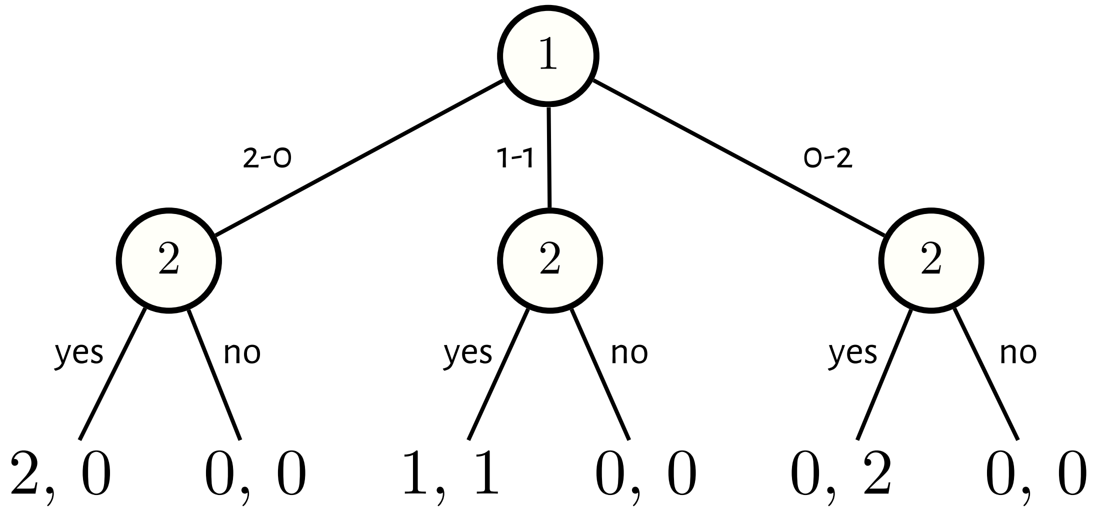
Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs



strictly dominant strategies

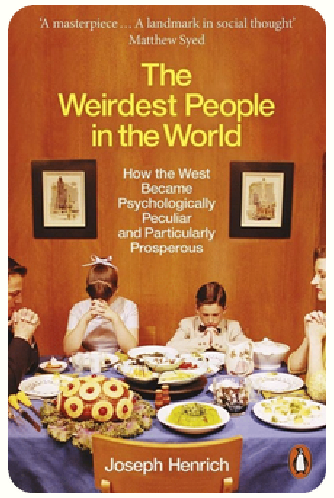
Pareto optimal strategy profiles

pure Nash equilibria

mixed Nash equilibria

JOE HENRICH

There are interesting cultural differences in the offers people from different cultures accept and reject when playing The Ultimatum Game.



Henrich, J. (2020). *The WEIRDest People in the World: How the West Became Psychologically Peculiar and Particularly Prosperous*. Farrar, Straus and Giroux.

Players

$N = \{1, 2\}$

Strategies of player 1

{2-0, 1-1, 0-2}

Strategies of player 2

(yes, yes, yes), (yes, yes, no), (yes, no, yes), (no, yes, yes),
(yes, no, no), (no, yes, no), (no, no, yes), (no, no, no)

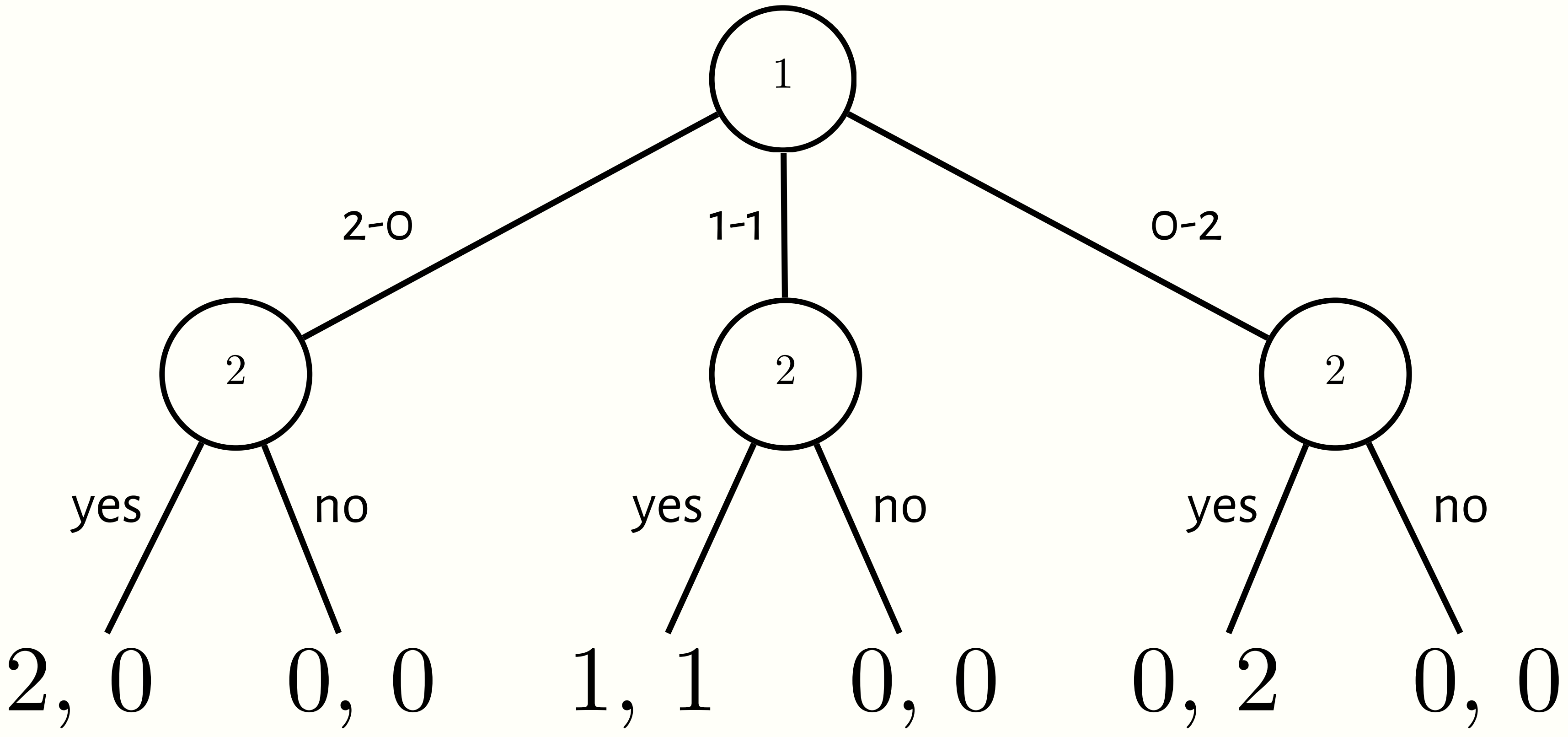
Strategy profiles

(2-0, (yes, yes, yes)), (2-0, (yes, yes, no)), ...

Payoffs (aka utilities)

$u_1\big(\text{1-1, (yes, no, yes)}\big) = 0$

$\dots$



Note that there is a subtlety in the definition of strategies.

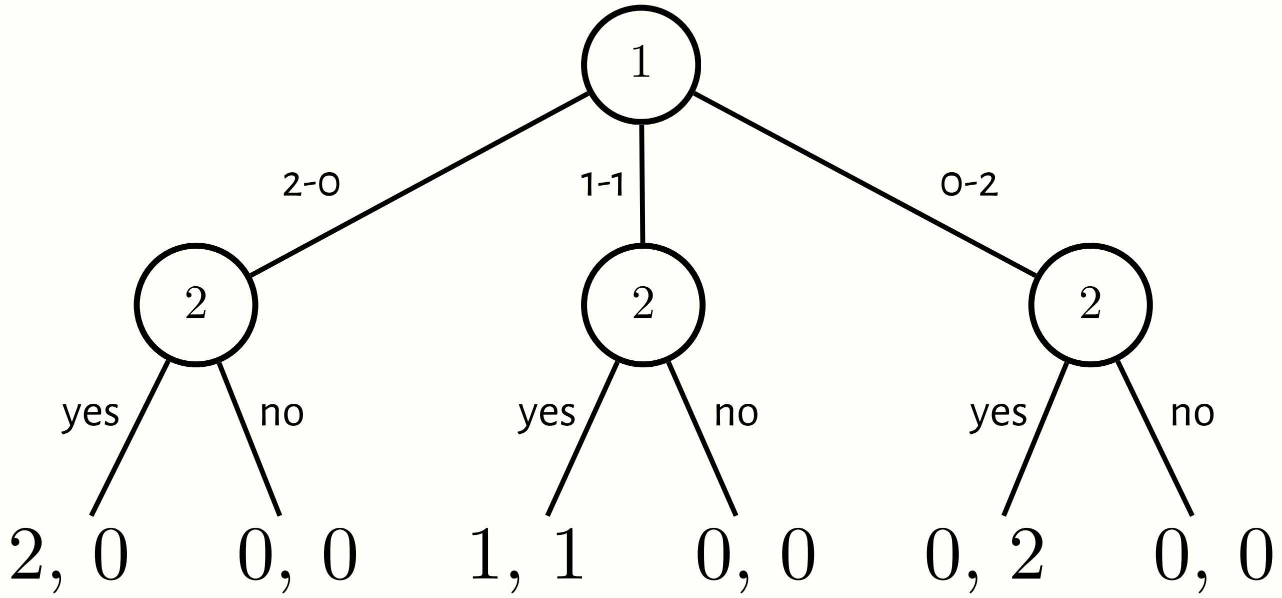
The strategies of each player need to be defined at every choice node of that player.

Even if there is no way to reach that node, given the other choice nodes.

To reason our way through a perfect-information game in extensive form, we just turn it into a *normal-form game*.

To reason our way through a perfect-information game in extensive form, we just turn it into a *normal-form game*.

Yes, we can always do it.



	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

Nash equilibria and everything else is computed with respect to the induced normal-form game.

The Ultimatum Game

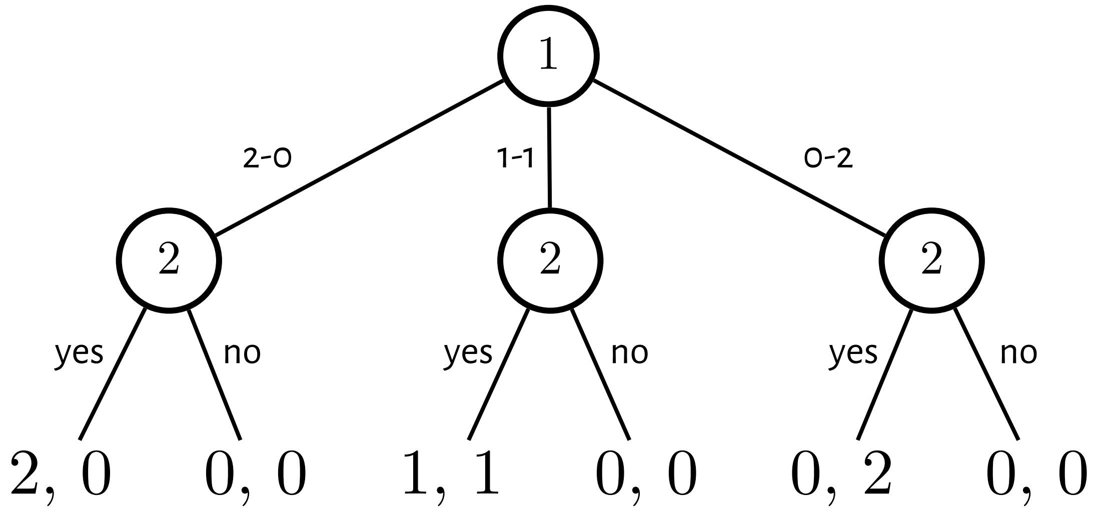
Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs



strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

The Ultimatum Game

Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

The Ultimatum Game

Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

strictly dominant strategies

none

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

The Ultimatum Game

Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

strictly dominant strategies

none

Pareto optimal strategy profiles

everything except (0, 0)

pure Nash equilibria

?

mixed Nash equilibria

?

The Ultimatum Game

Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	**2, 0**	**2, 0**	**2, 0**	**2, 0**	0, 0	0, 0	**0, 0**	**0, 0**
(1-1)	1, 1	1, 1	0, 0	0, 0	**1, 1**	**1, 1**	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	**0, 2**	0, 0

strictly dominant strategies

none

Pareto optimal strategy profiles

everything except (0, 0)

pure Nash equilibria

see above

mixed Nash equilibria

?

The Ultimatum Game

Player 1 has two euros, which it has to divide between themselves and player 2.

Player 1 makes an offer, which player 2 can *accept* or *reject*.

If player 2 accepts, money is divided according to player 1's offer.

If player 2 rejects, no one gets anything.

payoffs

	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	2, 0	2, 0	2, 0	2, 0	0, 0	0, 0	0, 0	0, 0
(1-1)	1, 1	1, 1	0, 0	0, 0	1, 1	1, 1	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0

strictly dominant strategies

none

Pareto optimal strategy profiles

everything except (0, 0)

pure Nash equilibria

see above

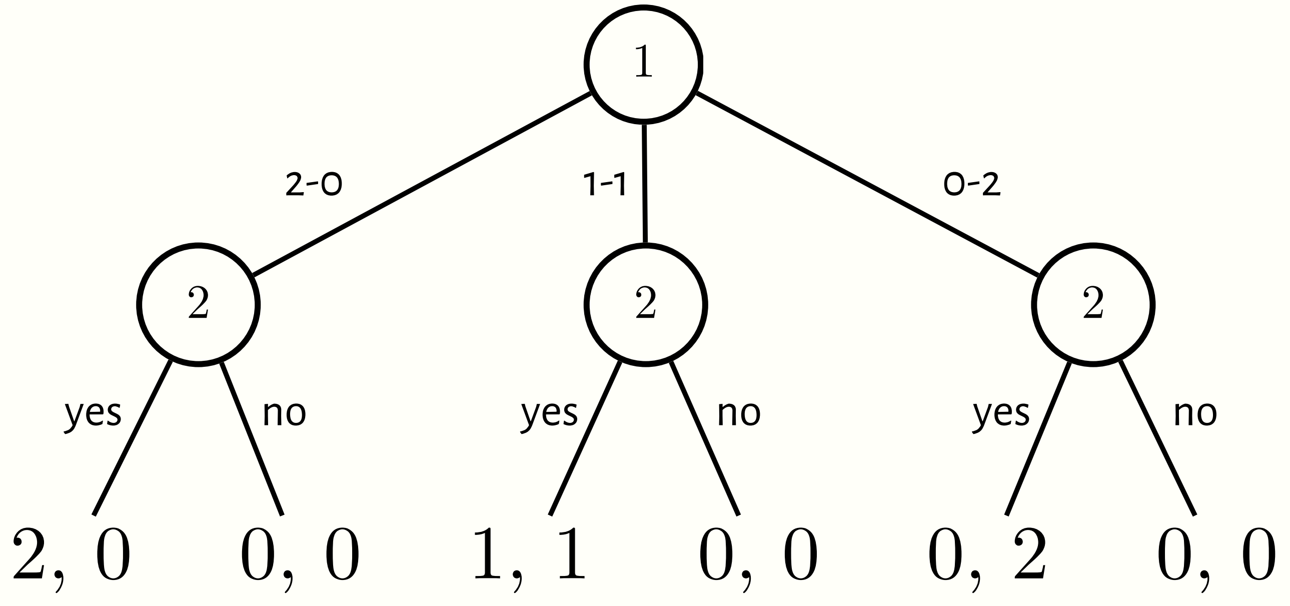
mixed Nash equilibria

too lazy to figure out

What makes (2-0, nnn) a Nash equilibrium depends crucially on what Player 2 does at *all* nodes: including 'irrelevant' ones.

Think: why does Player 1 not want to deviate?

Because Player 2 always says *no*, so there's no point!



	yyy	yyn	yny	ynn	nyy	nyn	nny	nnn
(2-0)	**2, 0**	**2, 0**	**2, 0**	**2, 0**	0, 0	0, 0	**0, 0**	**0, 0**
(1-1)	1, 1	1, 1	0, 0	0, 0	**1, 1**	**1, 1**	0, 0	0, 0
(0-2)	0, 2	0, 0	0, 2	0, 0	0, 2	0, 0	**0, 2**	0, 0

Games in extensive form afford a refinement of Nash equilibria: *subgame perfect equilibria*.

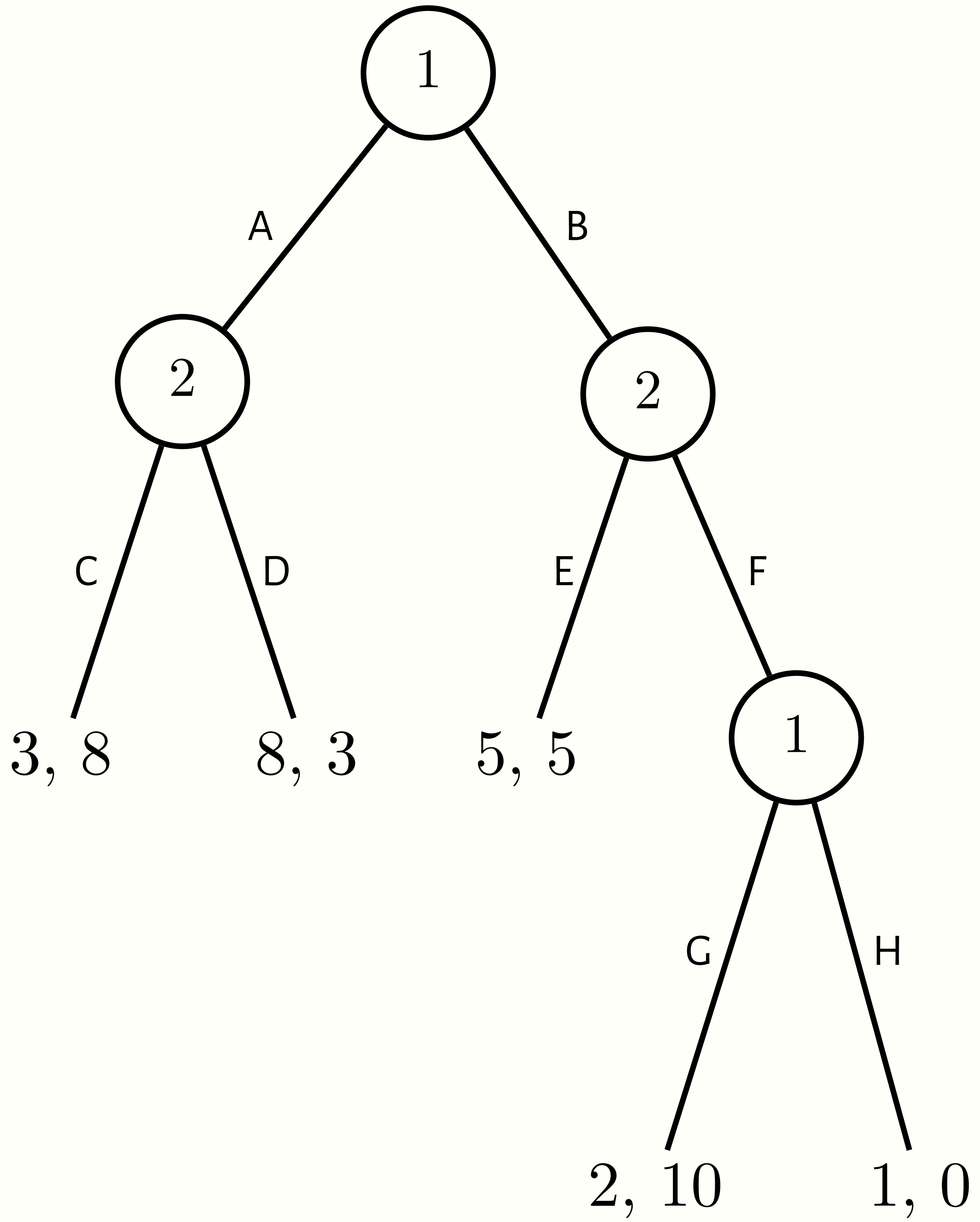
These involve playing a Nash equilibrium at every node of the game.

A subgame perfect equilibrium can be found by *backward induction*.

A subgame perfect equilibrium can be found by *backward induction*.

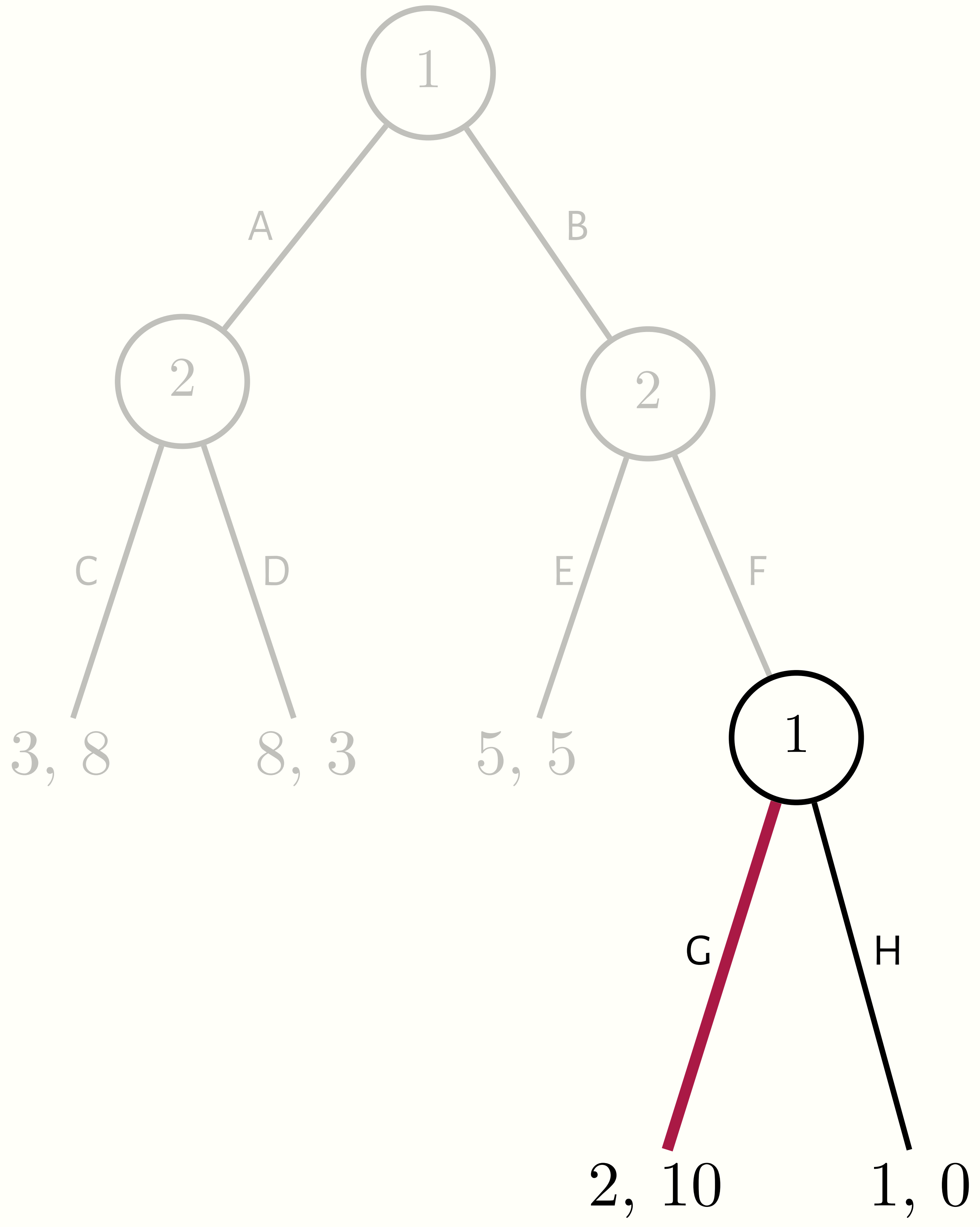
We reason backwards, from the end stages of a game, by finding the optimal action at every intermediate step.

Backward Induction: An Example



Backward Induction: An Example

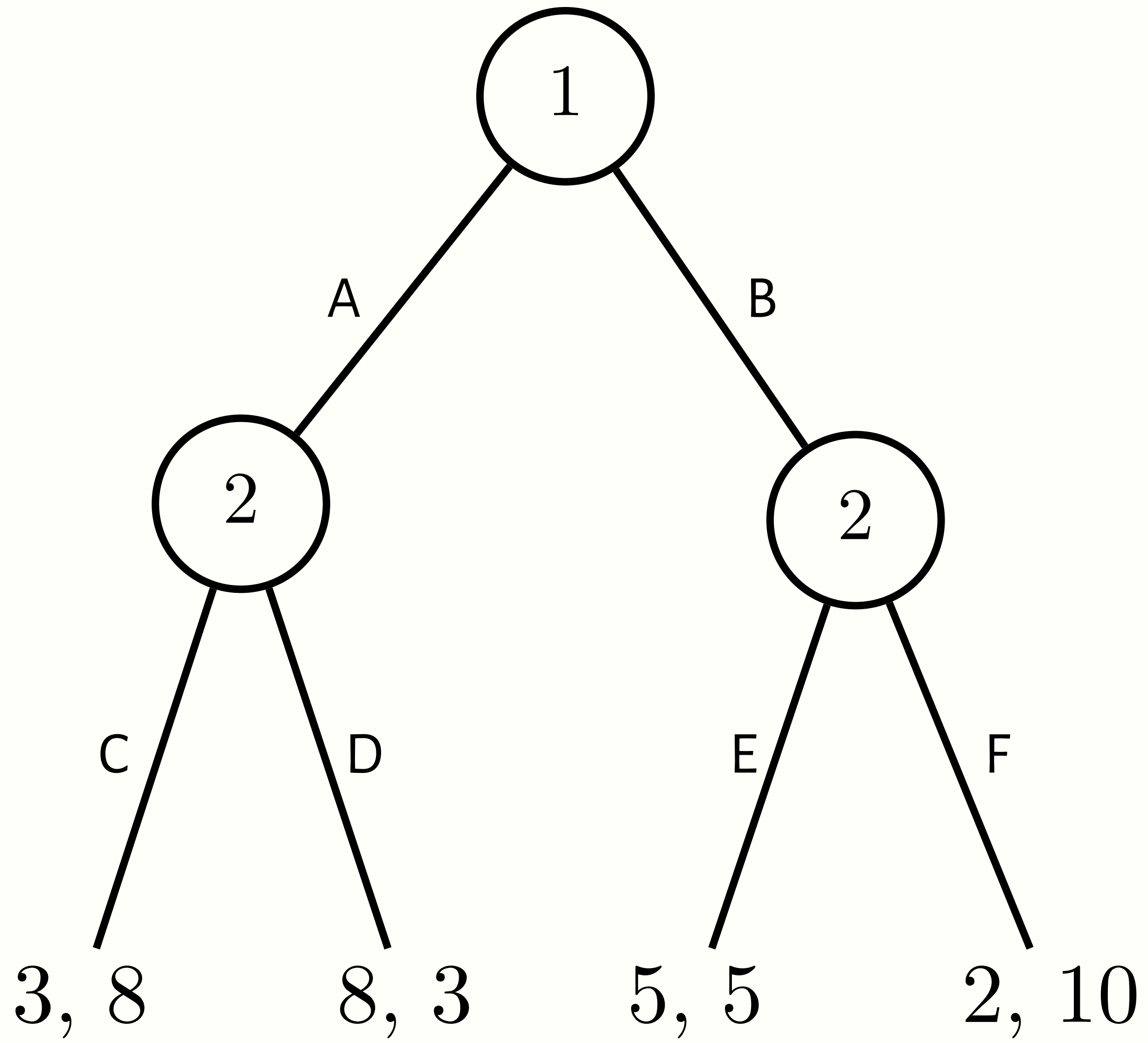
Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).



Backward Induction: An Example

Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

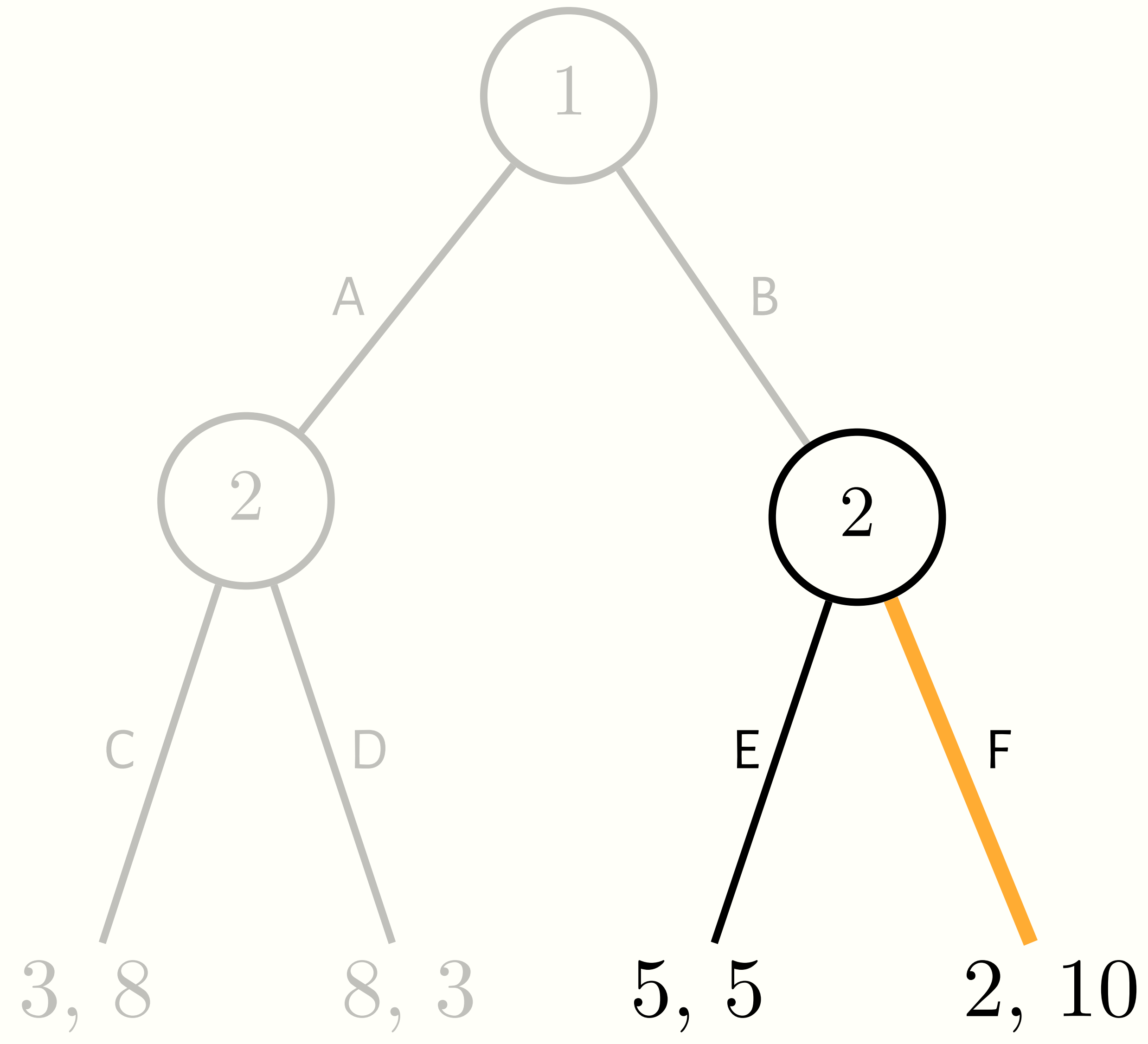


Backward Induction: An Example

Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

We infer that player 2 chooses F here.



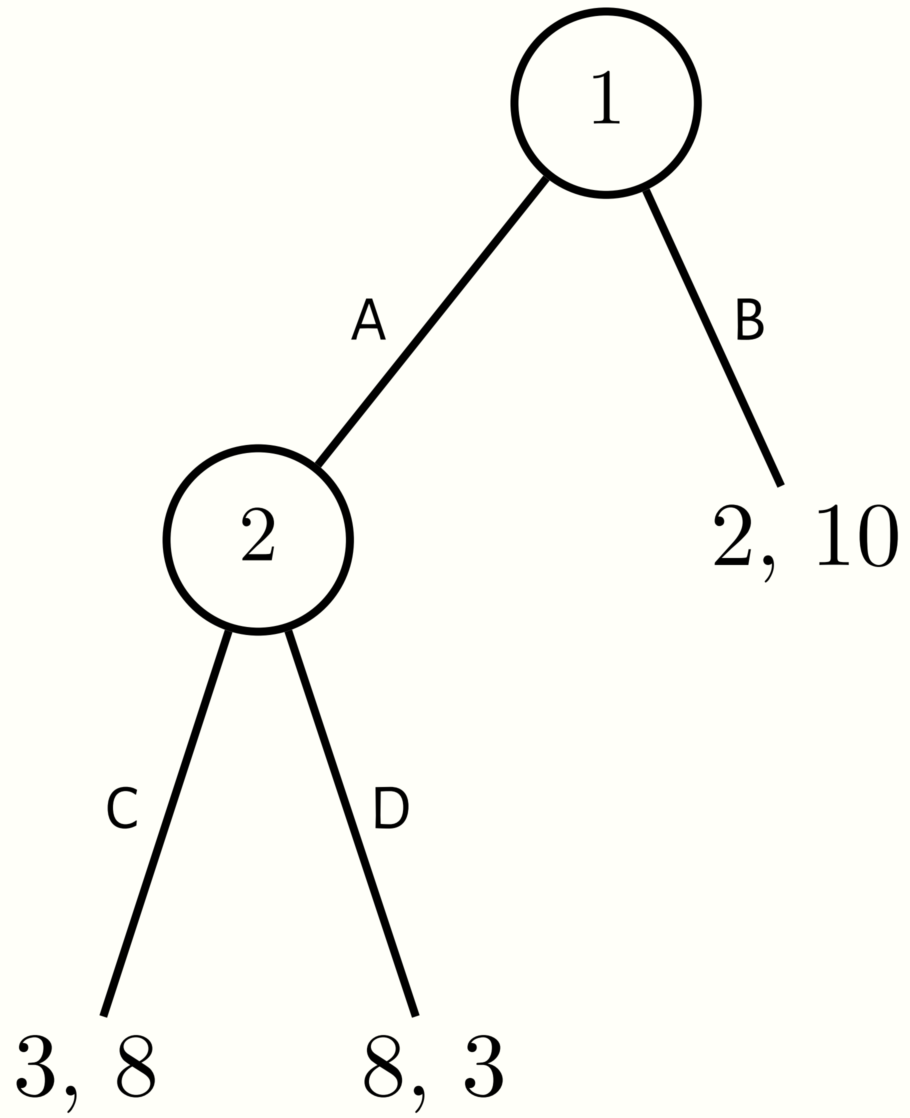
Backward Induction: An Example

Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

We infer that player 2 chooses F here.

Which further means that player 1 sees a payoff of 2 if they go down this path.



Backward Induction: An Example

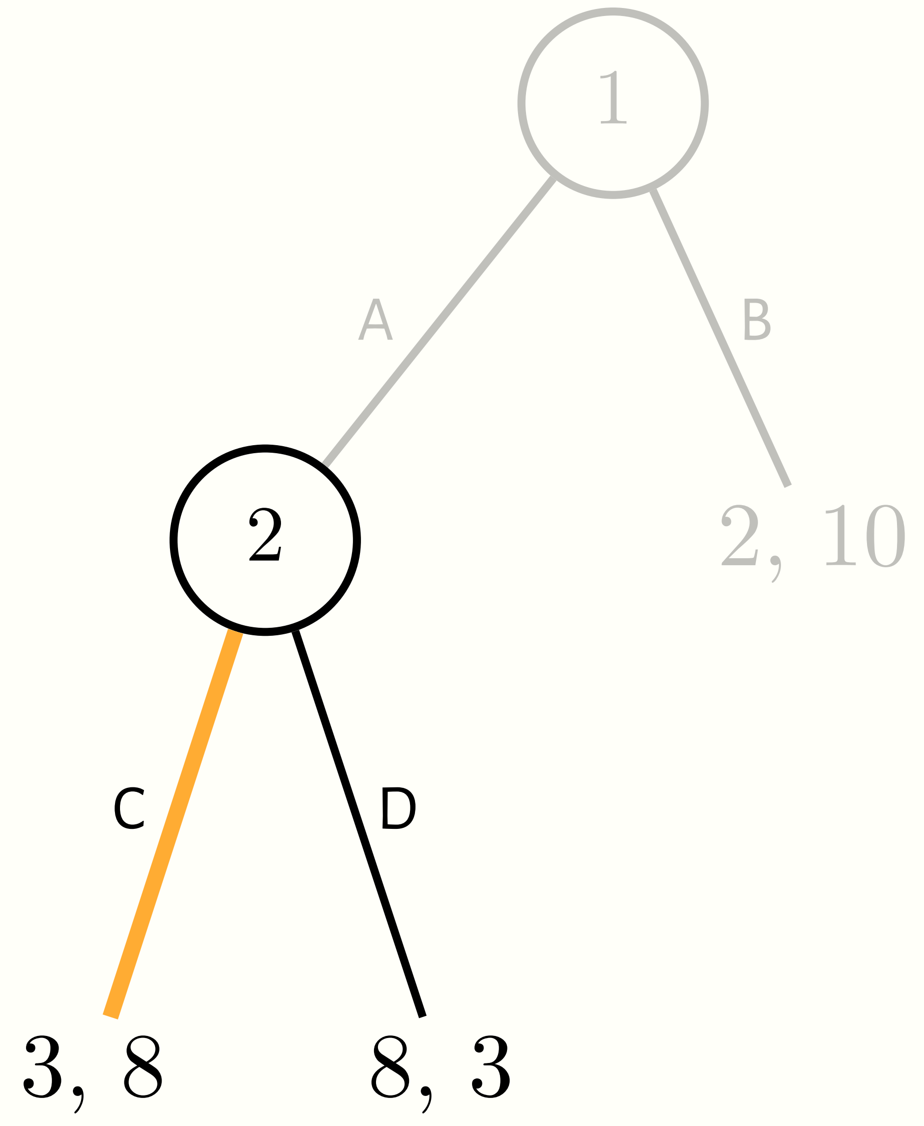
Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

We infer that player 2 chooses F here.

Which further means that player 1 sees a payoff of 2 if they go down this path.

On the other branch player 2 chooses C.



Backward Induction: An Example

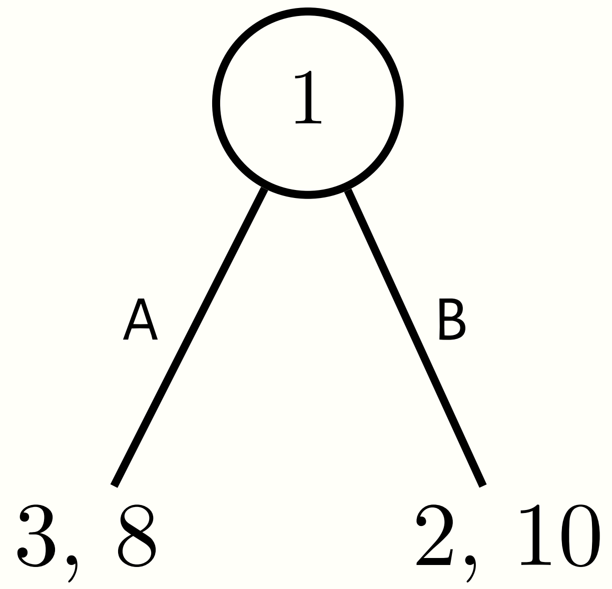
Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

We infer that player 2 chooses F here.

Which further means that player 1 sees a payoff of 2 if they go down this path.

On the other branch player 2 chooses C.



Backward Induction: An Example

Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

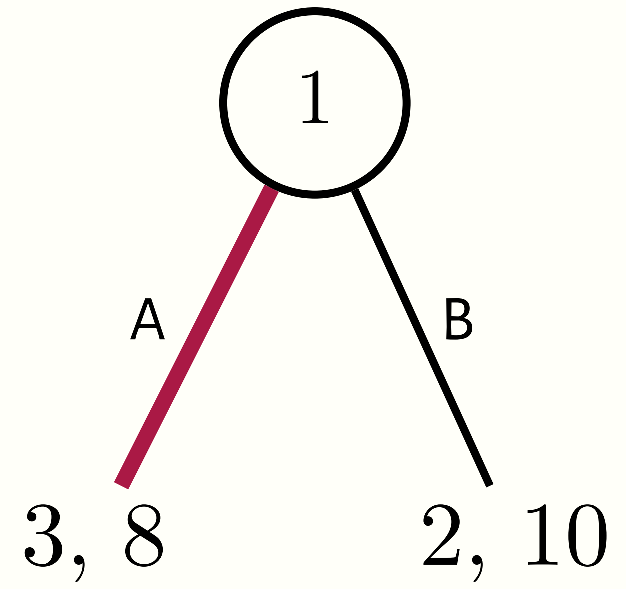
Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

We infer that player 2 chooses F here.

Which further means that player 1 sees a payoff of 2 if they go down this path.

On the other branch player 2 chooses C.

Which means player 1 chooses A.



Backward Induction: An Example

Faced with a choice only between G and H, player 1 surely chooses G, leading to a payoff of (2, 10).

Player 2 takes that into account when making their own decision one step earlier, i.e., they know that choosing F leads to a payoff of (2, 10).

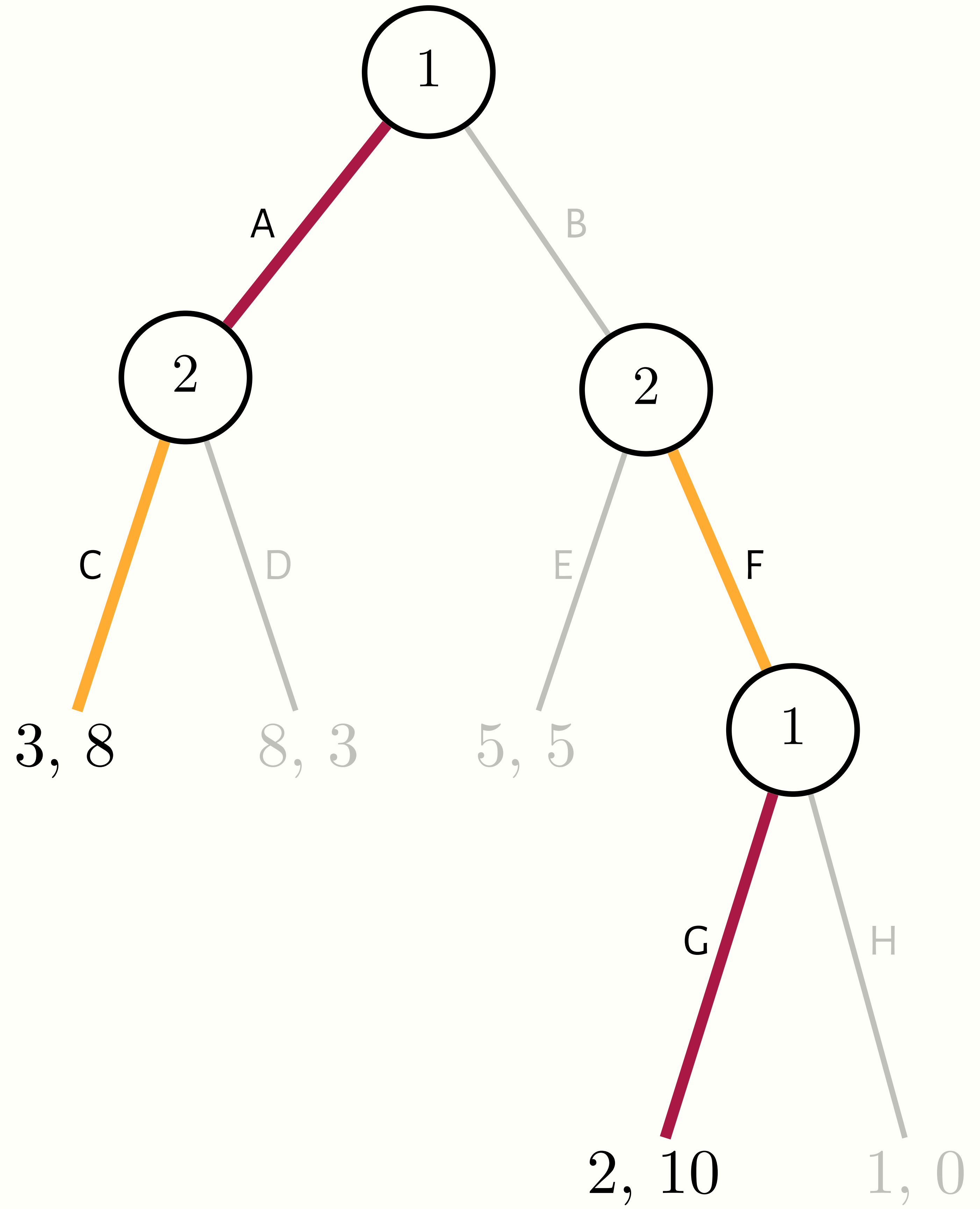
We infer that player 2 chooses F here.

Which further means that player 1 sees a payoff of 2 if they go down this path.

On the other branch player 2 chooses C.

Which means player 1 chooses A.

After which we can just read off the subgame-perfect equilibrium: ((A, G), (C, F)).



Backward induction is well-defined and terminates, if the game tree is finite.

So what have we shown?

THEOREM (SELTEN, 1965)

Every finite extensive-form game has at least one subgame-perfect equilibrium.

Selten, R. (1965). Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetraegheit. *Zeitschrift fuer die Gesamte Staatswissenschaft*, 121(2):301–324.

THEOREM (ZERMELO, 1913)

Every finite extensive-form game has at least one pure Nash Equilibrium.

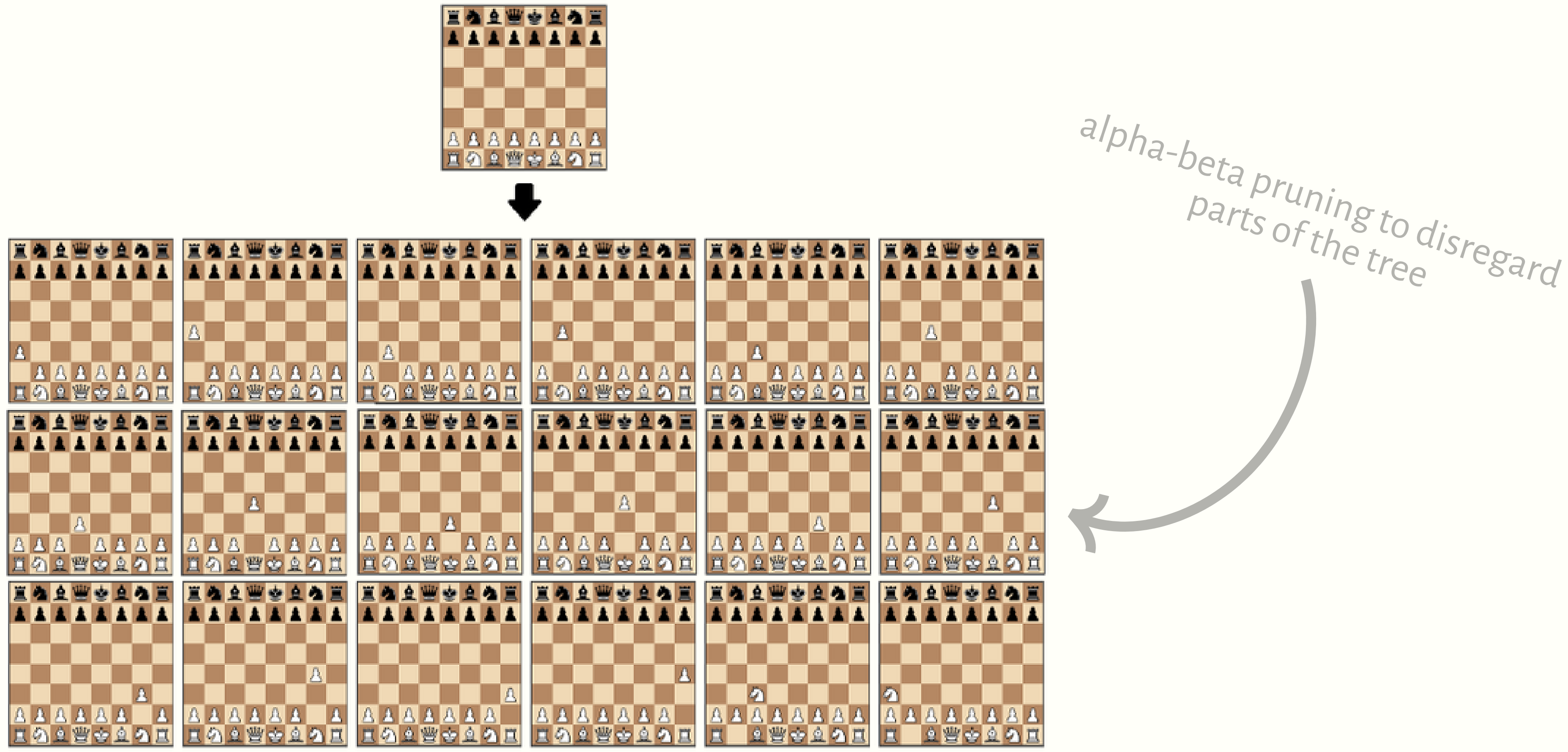
Zermelo, E. (1913). Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. *Proceedings of the 5th International Congress of Mathematicians*.

ERNST ZERMELO

I arrived at these ideas while thinking about whether chess is determined, i.e., whether either white or black has a winning strategy, or can force a draw.

Which is true if we can bound the length of a game.

At the same time, the game tree of chess is too large to actually survey the strategies, let alone represent it explicitly.



With extensive-form games, we can go even a bit more general.

With extensive-form games, we can go even a bit more general.

In perfect-information games players know what actions were played at previous rounds.

And thus, what nodes they are in.

With extensive-form games, we can go even a bit more general.

In perfect-information games players know what actions were played at previous rounds.

And thus, what nodes they are in.

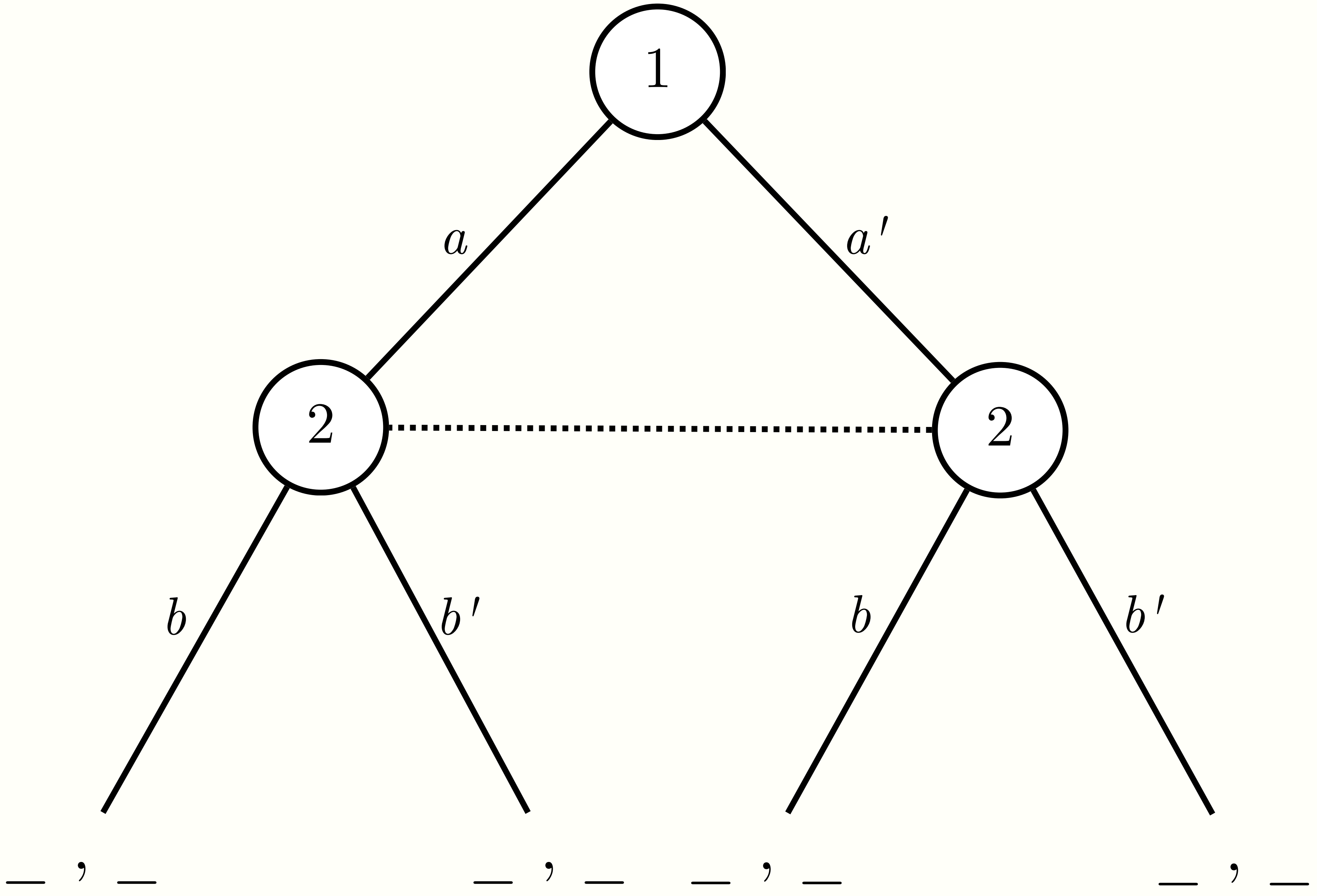
But in many other situations, players have only partial knowledge.

Enter extensive-form games with *imperfect* information.

We represent an agent's uncertainty over what choice node they're at by an *information set*.

Adding Uncertainty: A Dashed Line

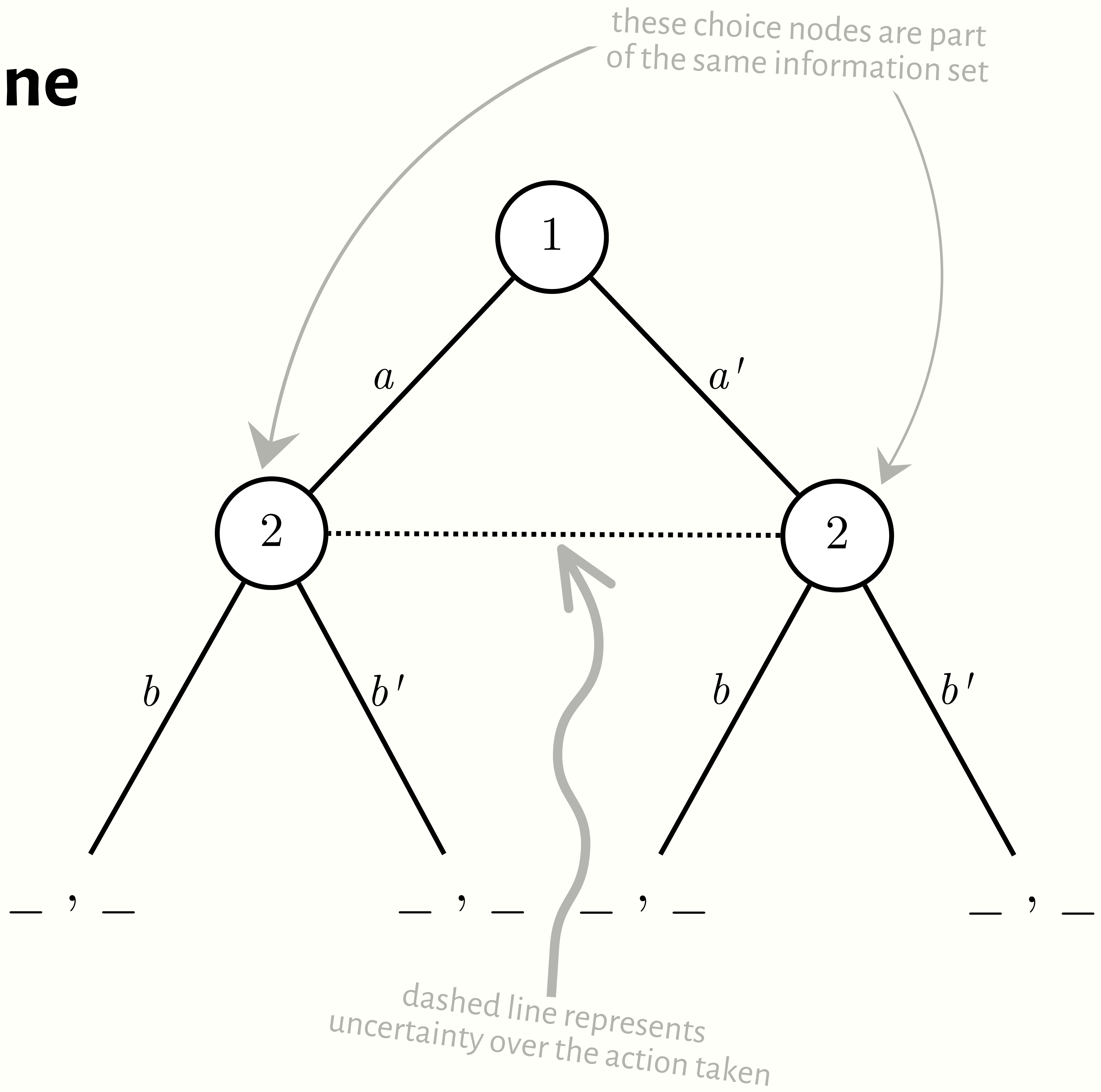
Player 1 takes takes an action: *a* or *a*’.



Adding Uncertainty: A Dashed Line

Player 1 takes takes an action: *a* or *a'*.

Player 2 follows up, *not* knowing what action Player 1 has actually taken: the two nodes connected by a dashed line are in the same information set.



Adding Uncertainty: A Dashed Line

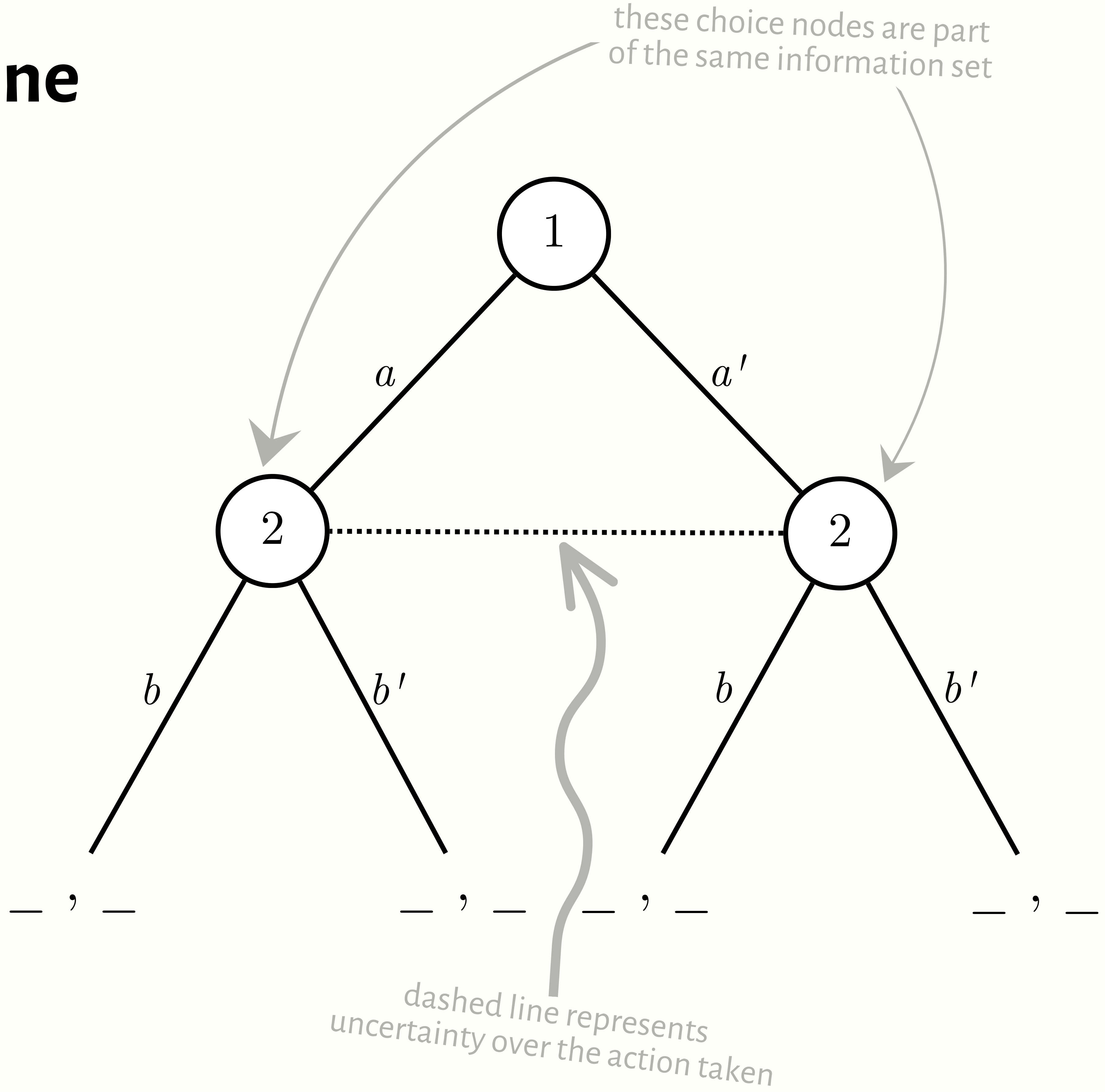
Player 1 takes takes an action: *a* or *a'*.

Player 2 follows up, *not* knowing what action Player 1 has actually taken: the two nodes connected by a dashed line are in the same information set.

Payoffs are specific to the branch taken.

Players know the actions available to all players, and the payoffs corresponding to each sequence of actions, i.e., the structure of the game.

But *do not know* which node from a particular information set they're in.



Intuitively, an agent cannot distinguish between the actions in one of their information sets.

Like their perfect-information counterparts, extensive-form games with perfect information are modeled as *trees*.

Like their perfect-information counterparts, extensive-form games with perfect information are modeled as *trees*.

The main difference is that every agent's choice nodes are partitioned into *information sets*.

Like their perfect-information counterparts, extensive-form games with perfect information are modeled as *trees*.

The main difference is that every agent's choice nodes are partitioned into *information sets*.

With the added proviso that the actions available at every information set are the *same* for all actions in that set.

Like their perfect-information counterparts, extensive-form games with perfect information are modeled as *trees*.

The main difference is that every agent's choice nodes are partitioned into *information sets*.

With the added proviso that the actions available at every information set are the *same* for all actions in that set.

A *strategy* for an agent is a combination of actions, one for each information set corresponding to that agent.

rather than for each choice node

Players
$N = \{1, 2\}$

Information sets of Player 1
{1a}, {1b, 1c}

Information sets of Player 2
{2}

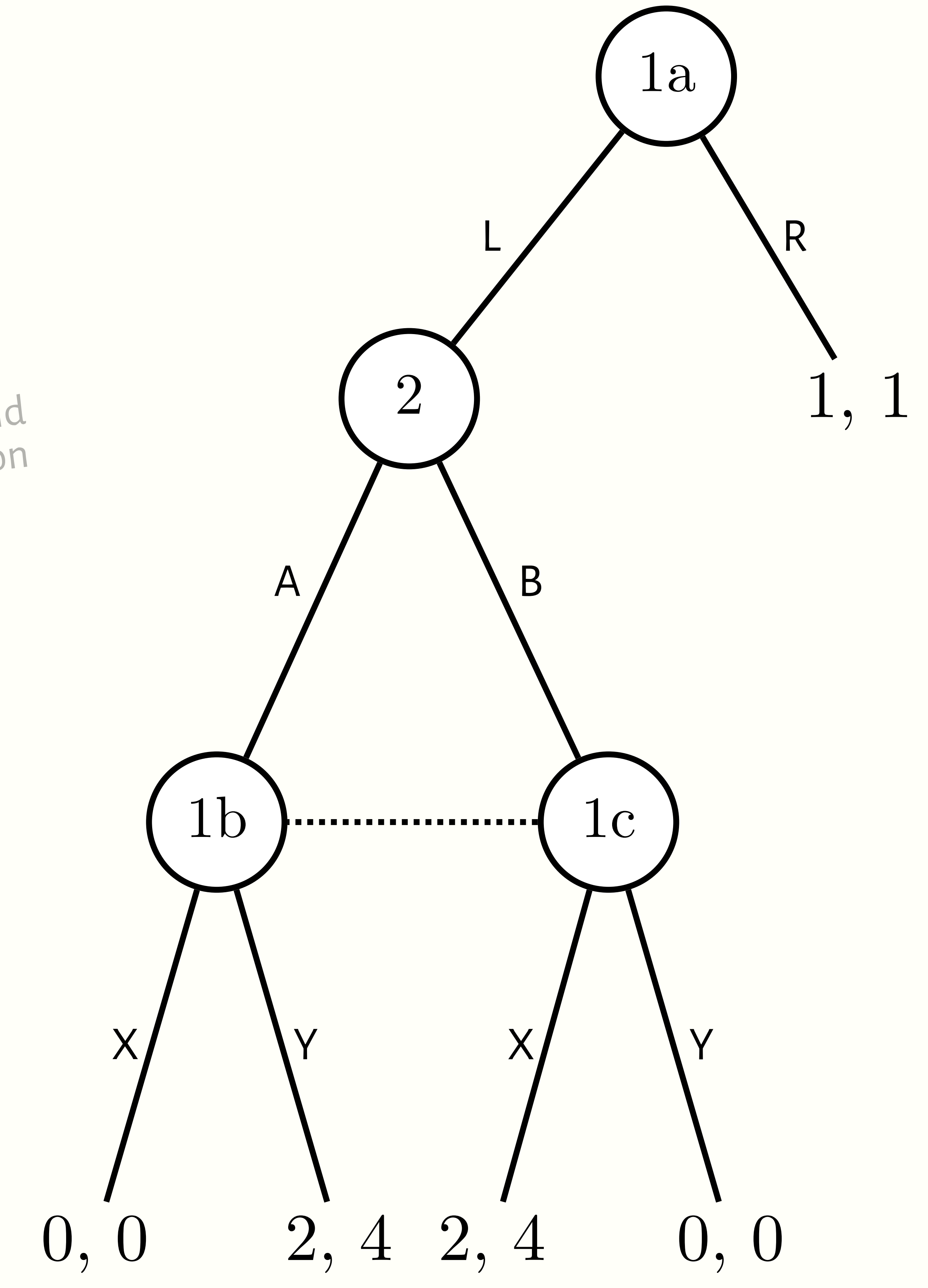
not (L, X, X), (L, X, Y), ..., which would be the case with perfect information

Strategies of player 1
(L, X), (L, Y), (R, X), (R, Y)

Strategies of player 2
A, B

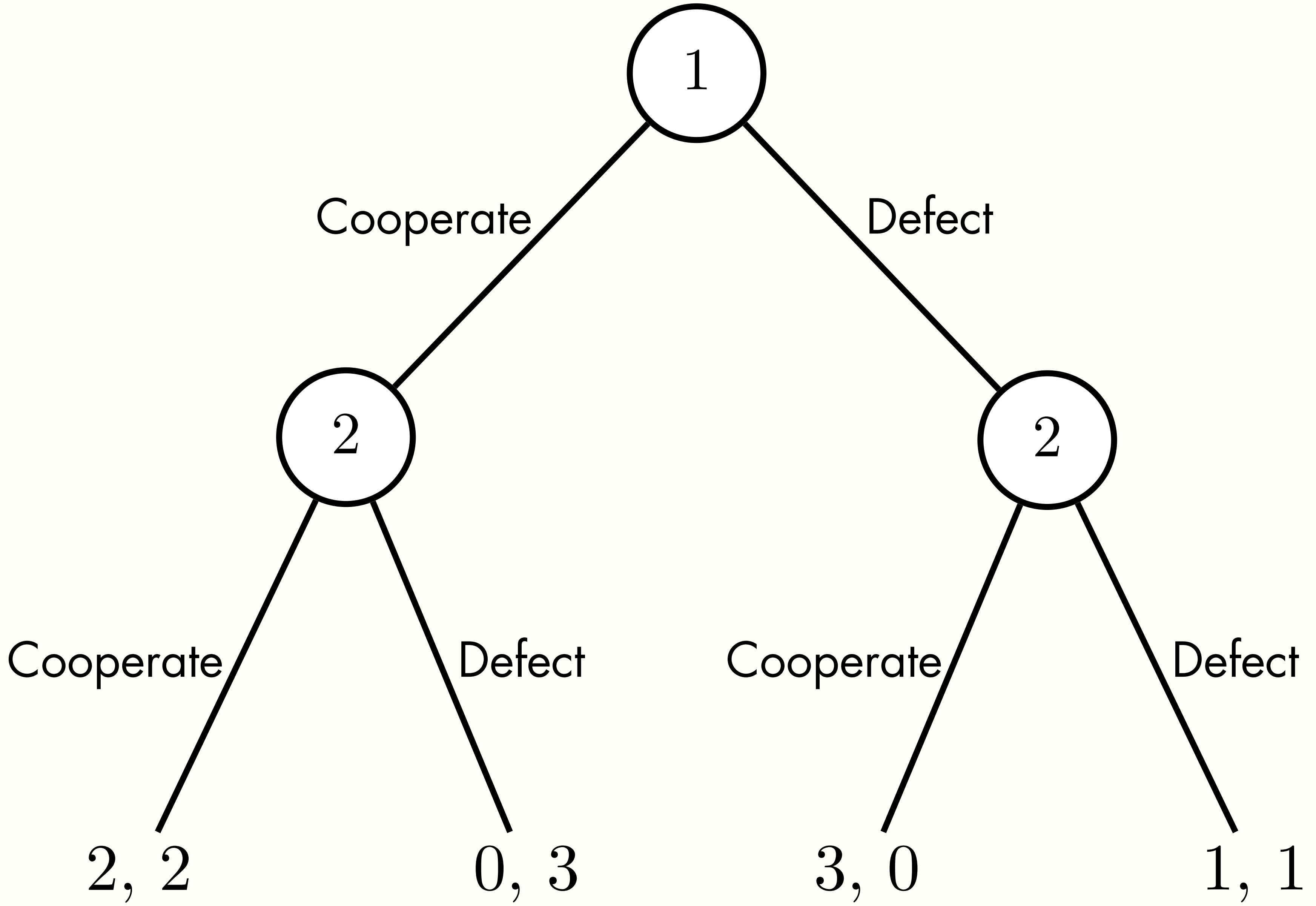
Strategy profiles
((L, X), A), ((L, X), B), ...

Payoffs (aka utilities)
you can figure this out

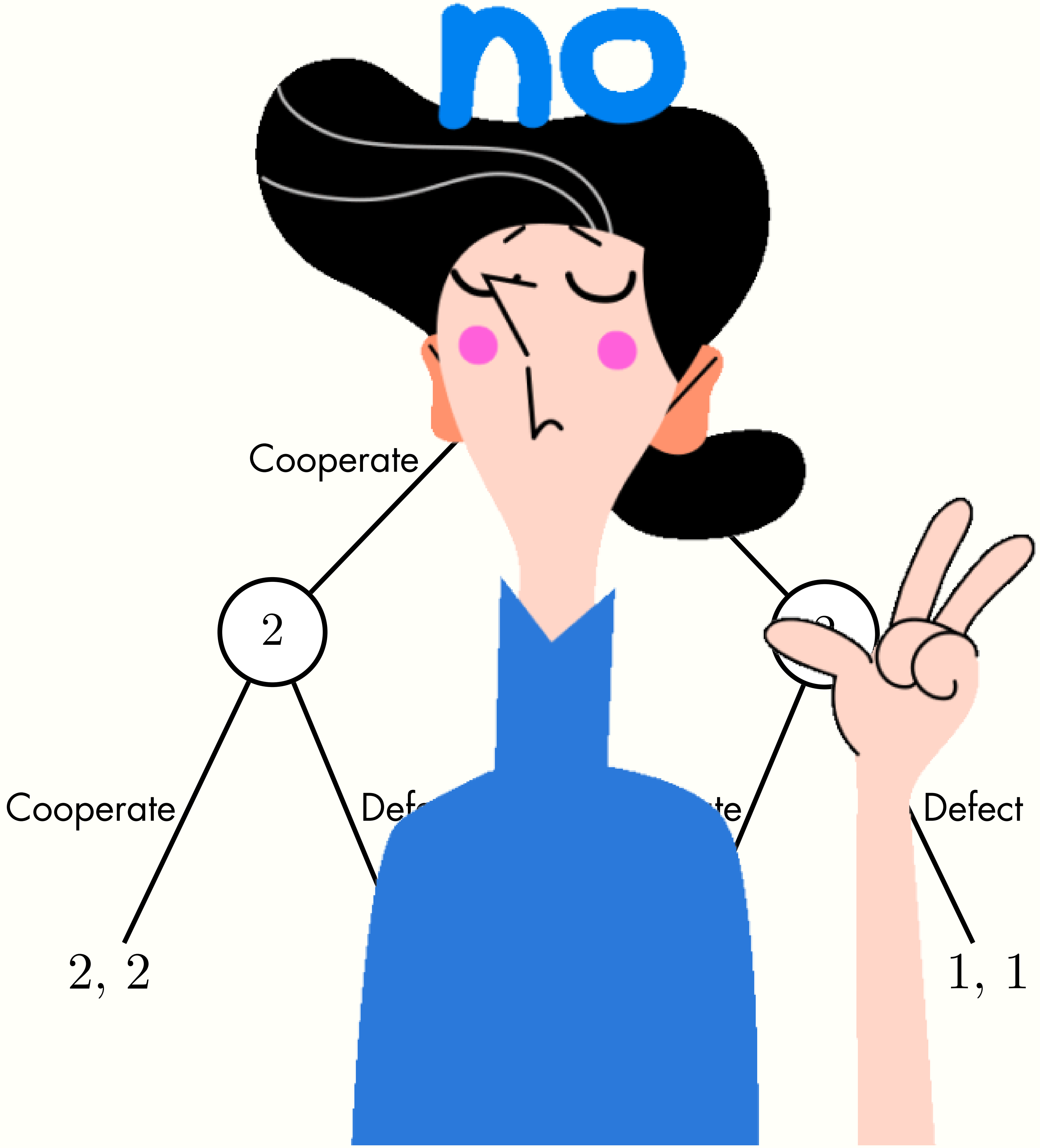


Now we can finally get back to the Prisoner's Dilemma!

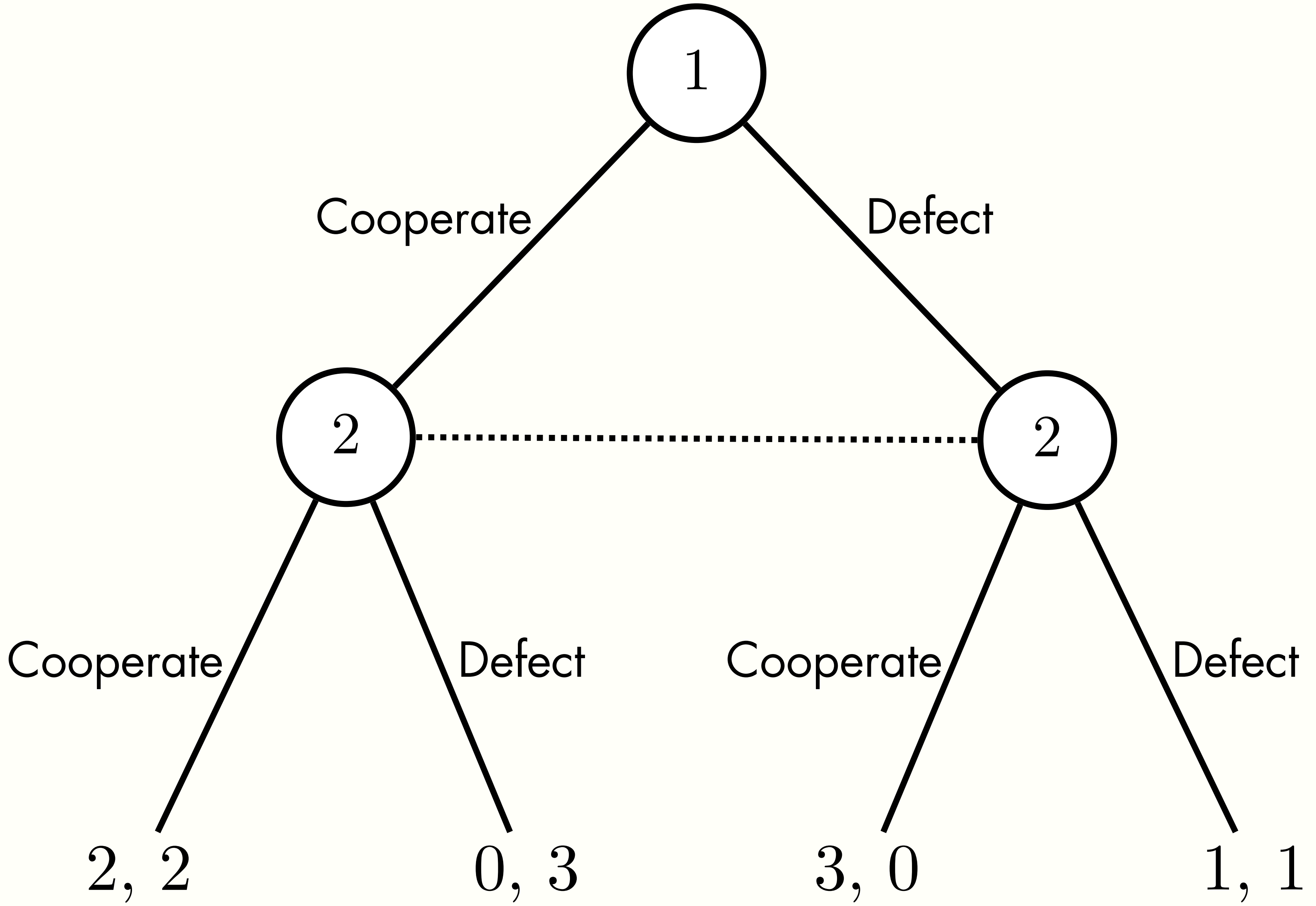
	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1



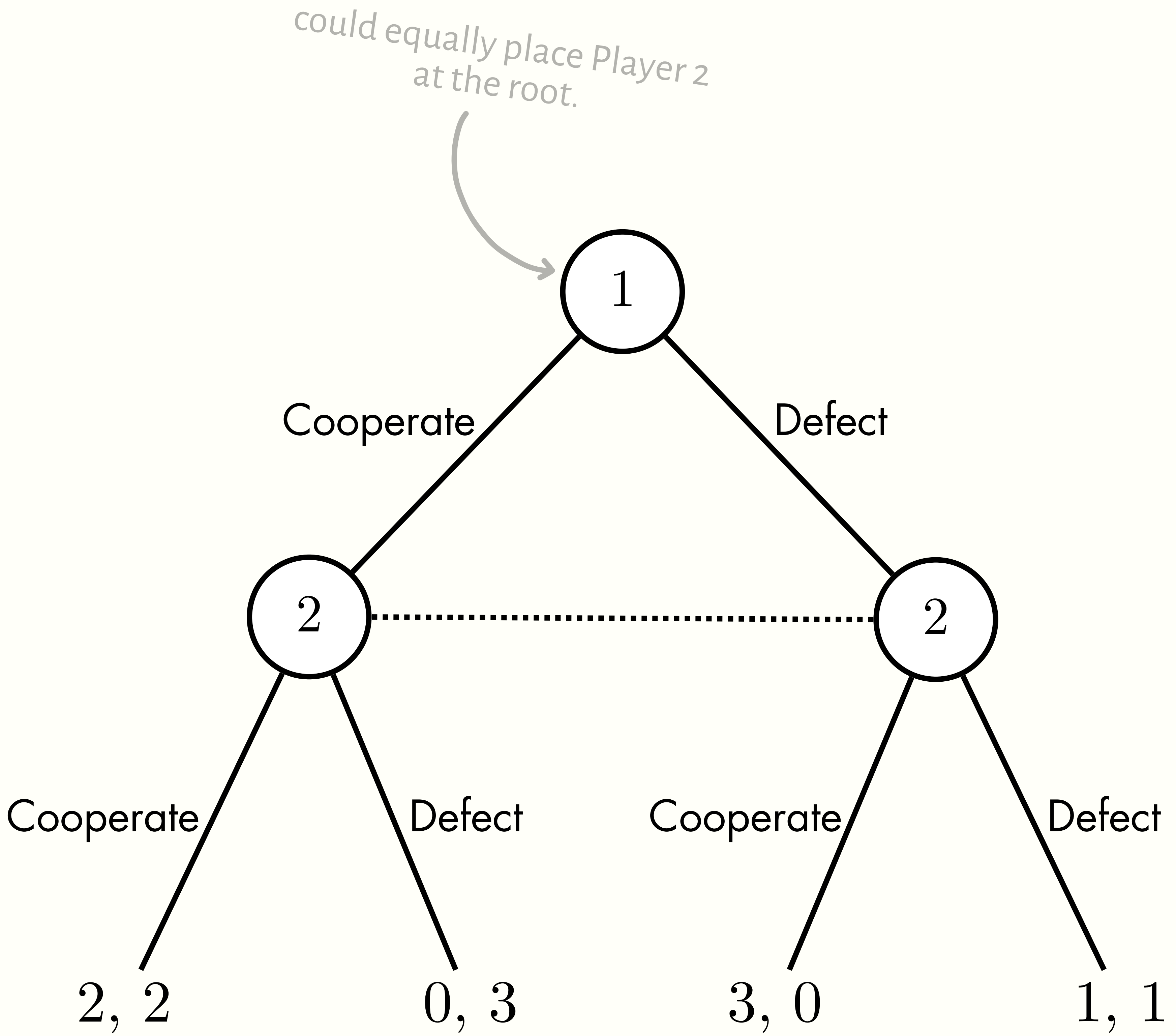
	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1



	Cooperate	Defect
Cooperate	$2,\ 2$	$0,\ 3$
Defect	$3,\ 0$	$1,\ 1$



	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1



Note that we can’t model the Prisoner’s Dilemma as an extensive-form game with *perfect* information.

Because, well, players don’t have perfect information.

But we *can* model it as a game of imperfect information.

Not only that, but now we can even model the iterated Prisoner's Dilemma!

A finite number of rounds.

A finite number of rounds.

Like, say, two.

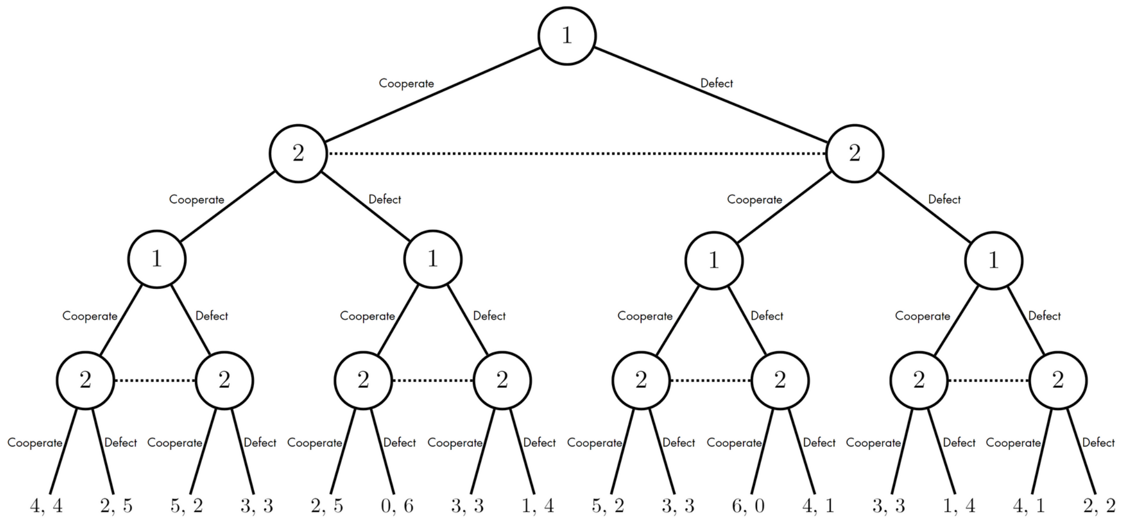
Iterated Prisoner's Dilemma

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs



strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

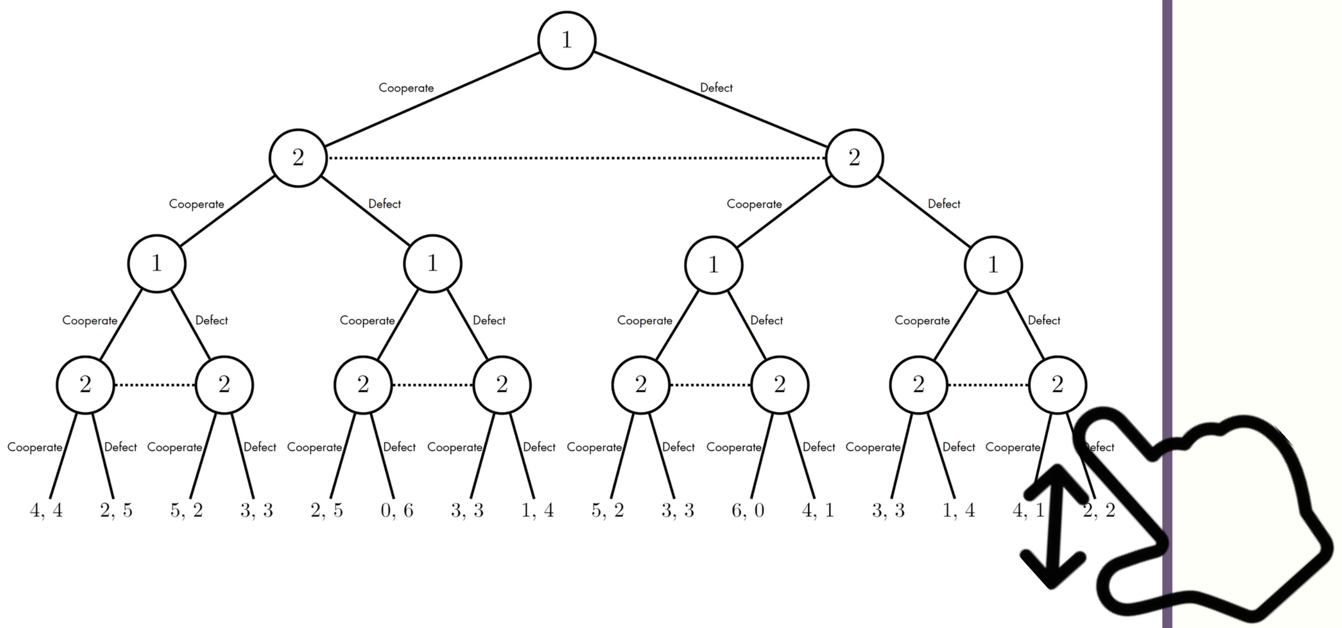
Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs



strictly dominant strategies

?

Pareto optimal strategy profiles

?

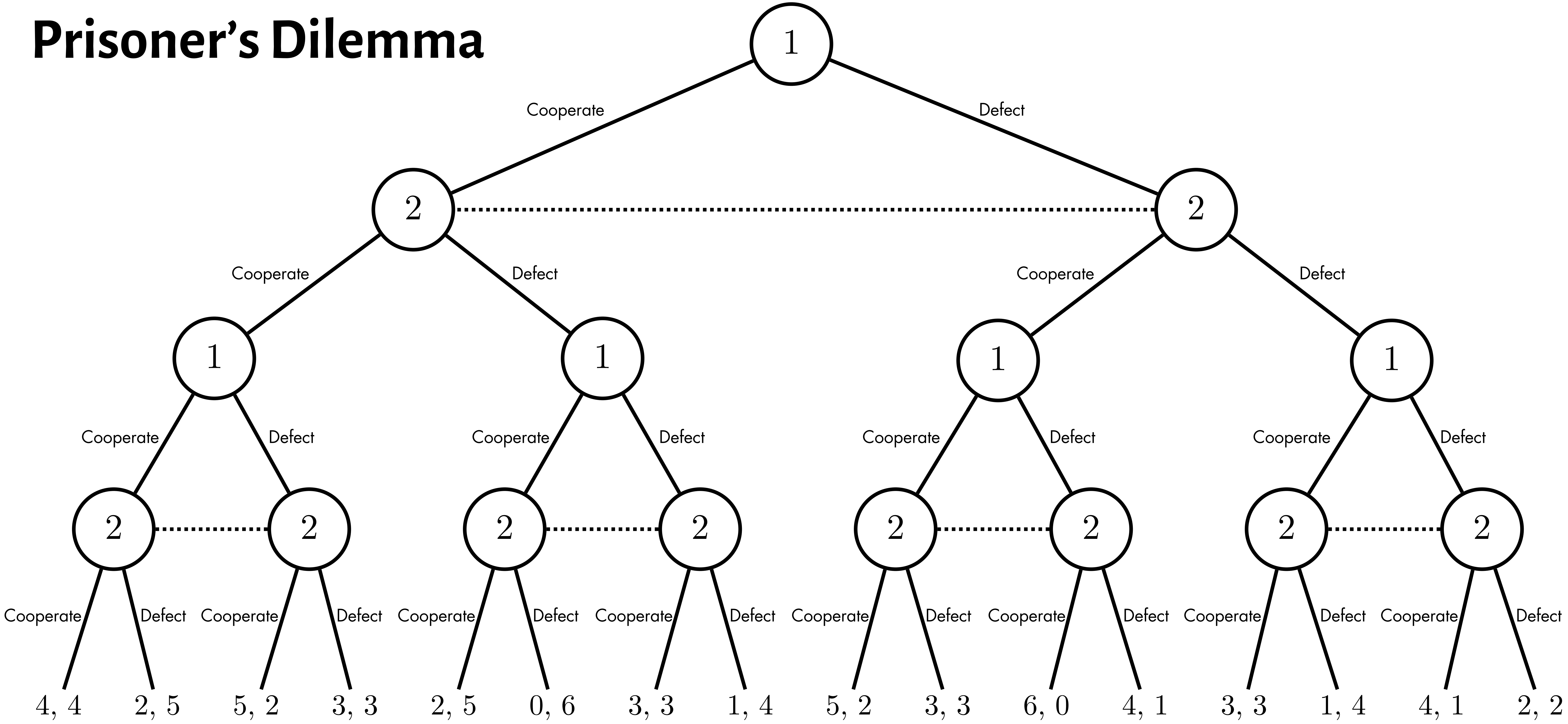
pure Nash equilibria

?

mixed Nash equilibria

?

Two Rounds of the Prisoner's Dilemma



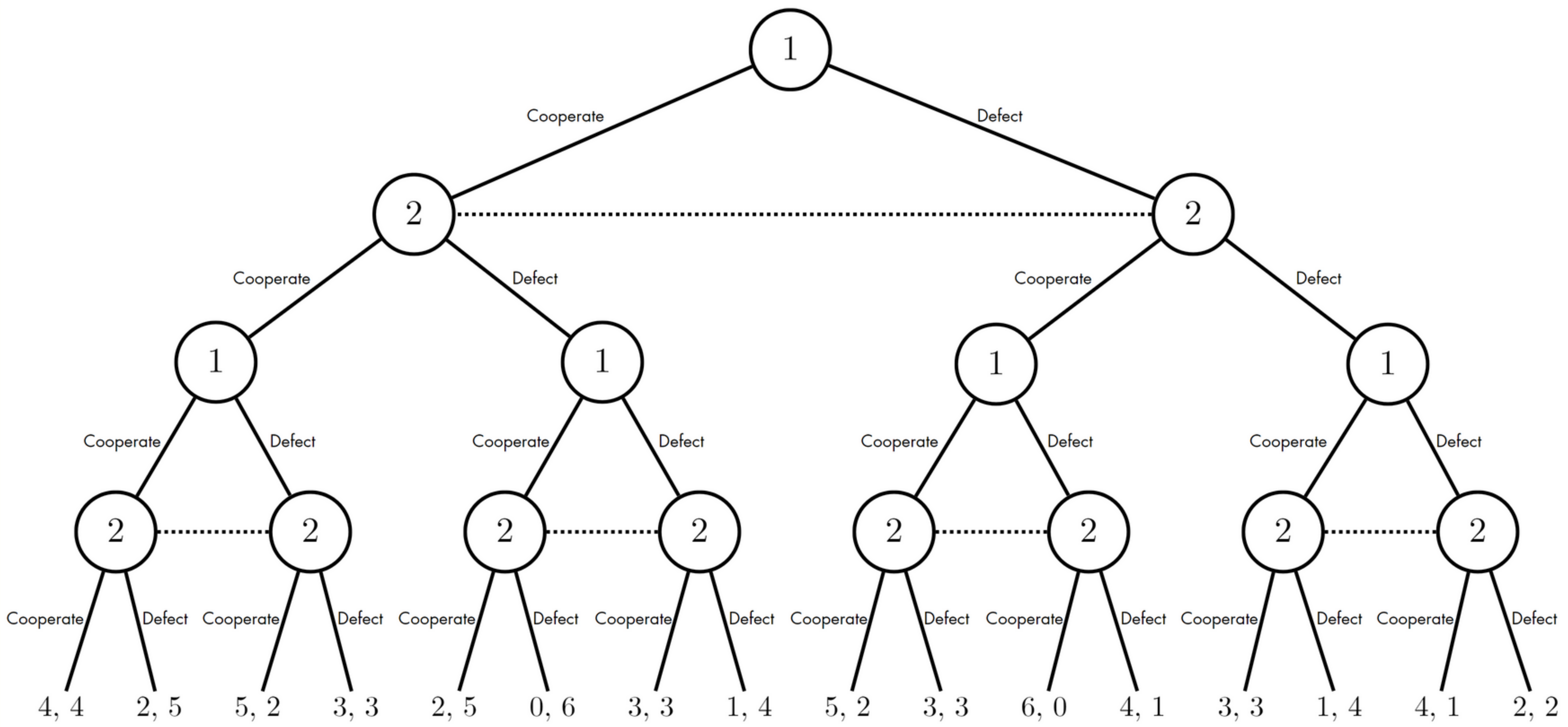
Players
$N = \{1, 2\}$

Strategies of Player 1
(C, C), (C, D), (D, C), (D, D)

Strategies of Player 2
(C, C), (C, D), (D, C), (D, D)

Strategy profiles
((C, C), (C, C)), ((C, C), (C, D)), ...

Payoffs (aka utilities)
hopefully clear



Straightforward to get a table now.

Iterated Prisoner’s Dilemma 

2 iterations

Two players play the Prisoner’s Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	$2+2, 2+2$	$2+0, 2+3$	$0+2, 3+2$	$0+0, 3+3$
C, D	$2+3, 2+0$	$2+1, 2+1$	$0+3, 3+0$	$0+1, 3+1$
D, C	$3+2, 0+2$	$3+0, 0+3$	$1+2, 1+2$	$1+0, 1+3$
D, D	$3+3, 0+0$	$3+1, 0+1$	$1+3, 1+0$	$1+1, 1+1$

strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

?

Pareto optimal strategy profiles

?

pure Nash equilibria

?

mixed Nash equilibria

?

In general, every game of imperfect information corresponds to a normal-form game, and vice-versa.

Thus, Nash equilibria and everything else are defined as for normal-form games.

So how do we analyze the 2-round Prisoner's Dilemma?

Iterated Prisoner's Dilemma

2 iterations

Two players play the Prisoner's Dilemma over $k = 2$ rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

?

Pareto optimal strategy profiles

pure Nash equilibria

mixed Nash equilibria

Iterated Prisoner’s Dilemma 

2 iterations

Two players play the Prisoner’s Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

((D, D), (D, D))

Pareto optimal strategy profiles

?

pure Nash equilibria

mixed Nash equilibria

Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

((D, D), (D, D))

Pareto optimal strategy profiles

see above

pure Nash equilibria

?

mixed Nash equilibria

Iterated Prisoner’s Dilemma 

2 iterations

Two players play the Prisoner’s Dilemma over k = 2 rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

((D, D), (D, D))

Pareto optimal strategy profiles

see previous

pure Nash equilibria

((D, D), (D, D))

mixed Nash equilibria

?

Iterated Prisoner's Dilemma 

2 iterations

Two players play the Prisoner's Dilemma over $k = 2$ rounds.

The final payoffs are the sum of the payoffs from each round.

payoffs

	C, C	C, D	D, C	D, D
C, C	4, 4	2, 5	2, 5	0, 6
C, D	5, 2	3, 3	3, 3	1, 4
D, C	5, 2	3, 3	3, 3	1, 4
D, D	6, 0	4, 1	4, 1	2, 2

strictly dominant strategies

((D, D), (D, D))

Pareto optimal strategy profiles

see previous

pure Nash equilibria

((D, D), (D, D))

mixed Nash equilibria

none

Again, the only Nash equilibrium is to always defect, for both players.

Again, the only Nash equilibrium is to always defect, for both players.

Note that we'd get the same equilibrium by backward induction.

Again, the only Nash equilibrium is to always defect, for both players.

Note that we'd get the same equilibrium by backward induction.

Note, as well, that we'd get the same conclusion for $k > 2$ rounds.

Well that was pointless.

ROBERT AUMANN

What if the game is played for an infinite number of times?

As in, we don't have a fixed number *k* of rounds at which the game ends.

Iterated Prisoner's Dilemma

infinitely iterated

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times

The final payoffs are the sum of the payoffs from each round.

Players

$N = \{1, 2\}$

Strategies of Player 1

(C, C, ...), (C, D, ...), ...

Strategies of Player 2

(C, C, ...), (C, D, ...), ...

Payoffs (aka utilities)

In general, infinite sums.

For instance, if both players always cooperate, payoffs are infinite series: (2, 2, ...), and the final payoff is:

$$2 + 2 + \cdots = \infty$$

ROBERT AUMANN

What if the game is played for an infinite number of times?

As in, we don't have a fixed number *k* of rounds at which the game ends.

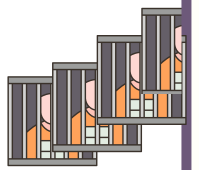
What we can add is some uncertainty about the future.

Expressed as a *discount factor* δ, with $0 < \delta < 1$.

At every new round, the payoffs are multiplied by δ.

For $\delta = 0.8$, we have that \$100 today is worth $0.8 \cdot \$100 = \$80$ tomorrow, and $0.8 \cdot \$80 = \$64$ in two days.

Iterated Prisoner’s Dilemma



infinitely iterated, with discount factor, $0 < \delta < 1$

Two players play the regular Prisoner’s Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

1/2

Players

$N = \{1, 2\}$

Strategies of Player 1

(C, C, ...), (C, D, ...), ...

Strategies of Player 2

(C, C, ...), (C, D, ...), ...

Payoffs (aka utilities)

In general, infinite sums.

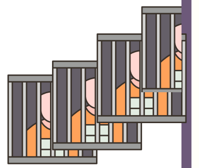
For instance, if both players always cooperate, payoffs are infinite series: $(2, 2\delta, 2\delta^2, \ldots)$, and the final payoff is:

$$2 + 2\delta + 2\delta^2 + \ldots$$

In general, for infinite sums we can use the following identity, for $0 < x < 1$:

$$1 + x + x^2 \cdots = \frac{1}{1-x}$$

Iterated Prisoner's Dilemma



infinitely iterated, with discount factor, $0 < \delta < 1$

Two players play the regular Prisoner's Dilemma:

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

but an infinite number of times

The final payoffs are the sum of the payoffs from each round, taking into account the discount factor δ.

1/2

Players

$N = \{1, 2\}$

Strategies of Player 1

(C, C, ...), (C, D, ...), ...

Strategies of Player 2

(C, C, ...), (C, D, ...), ...

Payoffs (aka utilities)

In general, infinite sums.

For instance, if both players always cooperate, payoffs are infinite series: $(2, 2\delta, 2\delta^2, \ldots)$, and the final payoff is:

$$\begin{aligned} 2 + 2\delta + 2\delta^2 + \ldots &= 2(1 + \delta + \delta^2 + \ldots) \\ &= 2 \cdot \frac{1}{1-\delta} \end{aligned}$$

ROBERT AUMANN

Consider, now, the following strategy, called the *Grim Trigger* strategy.

Start by cooperating. If the other player defects at some round t, switch to defecting forever.

If both players play the Grim Trigger, they each get a payoff of:

$$2 \cdot \left({}^{1}/{}_{1-\delta}\right)$$

Note, now, that as long as $\delta > 0.5$, no agent has an incentive to deviate.

In other words, the Grim Trigger is a Nash equilibrium!

Infinite games admit other equilibria!