Numerics of incompressible flows 1 (SS 25)
Homework 2

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Homework 2

Exercise 1

$v(x,y,z) = v_\varphi(r) \cdot e_\varphi(r,\varphi,z)$

$p : \Omega \to \mathbb{R} \qquad p(x,y,z) = p(r)$

a)

i) Prove $\operatorname{div} v = 0$

Proof: With the Note from Ex 1 we have

$$\operatorname{div} v = \frac{1}{r}\partial_r(r v_r) + \frac{1}{r}\partial_\varphi v_\varphi + \partial_z v_z$$

Since $v = v_\varphi e_\varphi$ we can say ~~v_r, v_z~~ $v_r = v_z = 0$.

$$\Rightarrow \operatorname{div} v = \frac{1}{r}\partial_\varphi v_\varphi$$

Since v_φ ~~does~~ is not dependent to φ, the derivate in respect to φ is 0 $(\partial_\varphi v_\varphi = 0)$

$$\Rightarrow \operatorname{div} v = 0$$

ii) according to i) $v_z = 0$

$$\text{rot}\, v = \begin{pmatrix} \partial_y v_z & - \partial_z v_y \\ \partial_z v_x & - \partial_x v_z \\ \partial_x v_y & - \partial_y v_x \end{pmatrix} = \begin{pmatrix} -\partial_z v_y \\ \partial_z v_x \\ \partial_x v_y - \partial_y v_x \end{pmatrix}$$

we further know

$$v = v_\varphi(r) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = -\sin\varphi \, v_\varphi(r) e_x + \cos\varphi \cdot v_\varphi(r) e_y + 0 e_z$$

$$= -\frac{y}{r} v_\varphi(r) e_x + \frac{x}{r} v_\varphi(r) e_y = v_x e_x + v_y e_y + 0 e_z$$

$$\Rightarrow \quad v_x = -\frac{y}{r} v_\varphi(r) \qquad v_y = \frac{x}{r} v_\varphi(r)$$

→ these components are independent of z, thus:

$$\text{rot}(v) = \begin{pmatrix} 0 \\ 0 \\ \partial_x v_y - \partial_y v_x \end{pmatrix} = (\partial_x v_y - \partial_y v_x) e_z .$$

$$\to \partial_x v_y = \left(\frac{d}{dx} \frac{x}{r(x,y)}\right) v_\varphi(r) + \frac{x}{r} \partial_r v_\varphi(r) \frac{d}{dx} r(x,y)$$

$$= \frac{1 \cdot r - x \frac{d}{dx} r(x,y)}{r^2} v_\varphi(r) + \frac{x}{r} \partial_r v_\varphi(r) \frac{x}{r}$$

$$= \frac{r - \frac{x^2}{r}}{r^2} v_\varphi(r) + \frac{x^2}{r^2} \partial_r v_\varphi(r)$$

$$\to \partial_y v_x = -\frac{r - \frac{y^2}{r}}{r^2} v_\varphi(r) - \frac{y^2}{r^2} \partial_r v_\varphi(r)$$

$$\Rightarrow \partial_x v_y - \partial_y v_x = \left(\frac{r - \frac{x^2}{r}}{r^2} + \frac{r - \frac{y^2}{r}}{r^2}\right) v_\varphi(r) + \frac{x^2 + y^2}{r^2} \partial_r v_\varphi(r)$$

$$= \frac{2r - \frac{(x^2+y^2)}{r}}{r^2} v_\varphi(r) + \partial_r v_\varphi(r) = \frac{1}{r} v_\varphi(r) + \partial_r v_\varphi(r)$$

where $x^2+y^2 = r^2$.

→ with $\frac{1}{r} \partial_r (r v_\varphi(r)) e_z = (\frac{1}{r} v_\varphi(r) + \partial_r v_\varphi(r)) e_z$ we conclude

$$\text{rot}(v) = (\partial_x v_y - \partial_y v_x) e_z = (\frac{1}{r} v_\varphi(r) + \partial_r v_\varphi(r)) e_z$$

$\blacksquare$

∂_r

iii)

<u>Prove</u> $\exists\, g = g(r) : \quad v \times rot(v) = \nabla g$

<u>Proof</u>

$$v \times rot\, v = \begin{pmatrix} v_y \frac{1}{r}\partial_r (r v_\varphi(r)) - 0 \\ 0 - v_x \frac{1}{r}\partial_r (r v_\varphi(r)) \\ 0 \end{pmatrix}$$

$$= \left(v_\varphi(r) \frac{x}{r} \cdot \frac{1}{r} \partial_r (r v_\varphi(r)),\ v_\varphi(r) \frac{y}{r} \frac{1}{r} \partial_r (r v_\varphi(r)),\ 0 \right)^T$$

$$= \left(\frac{x}{r^2} v_\varphi(r)\, \partial_r (r v_\varphi(r)),\ \frac{y}{r^2} v_\varphi(r)\, \partial_r (r v_\varphi(r)),\ 0 \right)^T$$

$$= \frac{1}{r^2} v_\varphi(r)\, \partial_r (r v_\varphi(r)) \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$v = v_\varphi(r)\, e_\varphi(r, \varphi, z) \quad , \quad rot(v) = \frac{1}{r} \partial_r (r v_\varphi(r))\, e_z$$

For simplicity we define $\lambda(r) := \frac{1}{r} \partial_r (r v_\varphi(r))$

Therefore

$$v \times rot\, v = v_\varphi(r)\, e_\varphi(r, \varphi, z) \times \lambda(r)\, e_z(r, \varphi, z)$$

$$= v_\varphi(r)\, \lambda(r) \cdot \left(e_\varphi(r, \varphi, z) \times e_z(r, \varphi, z) \right)$$

$$= v_\varphi(r)\, \lambda(r) \cdot \left(- e_r(r, \varphi, z) \right)$$

Note:
$e_\varphi = e_z \times e_r$
$\Rightarrow e_r = e_\varphi \times e_z$

We get for $g = g(r)$: $\nabla g(r) = \frac{dg}{dr} e_r$

So we get

$$v \times rot\, v = \nabla g(r) \quad \text{for} \quad \frac{dg}{dr} = - v_\varphi(r)\, \lambda(r)$$

//

b.) Prove: If solution has assumed symmetry, then:

$$-\frac{1}{Re}\Delta v + \nabla \Pi = 0 \qquad \text{with } \Pi = p + \frac{1}{2}\|v\|^2 - g$$

Proof:

We use Homework 1b:

$$(v \cdot \nabla) v = \frac{1}{2}\nabla\|v\|^2 - v \times \operatorname{rot} v$$

and put in the results from Ex 1iii:

$$(v \cdot \nabla) v = \frac{1}{2}\nabla\|v\|^2 - \nabla g \qquad \text{with } \frac{dg}{dr} = -V_\varphi(r)\lambda(r)$$

The Navier stokes equation with 0 on the rightside,

can be written as

$$-\frac{1}{Re}\Delta v + (v \cdot \nabla) v + \nabla p = 0 \text{ where } p \text{ is pressure.}$$

Therefore we get

$$-\frac{1}{Re}\Delta v + \left(\frac{1}{2}\nabla\|v\|^2 - \nabla g\right) + \nabla p = 0$$

This can be rewritten

as:

$$-\frac{1}{Re}\Delta v + \nabla\left(\frac{1}{2}\|v\|^2 - g + p\right) = 0$$

~~and~~ which is

$$-\frac{1}{Re}\Delta v + \nabla \Pi = 0 \qquad \text{where } \Pi = p + \frac{1}{2}\|v\|^2 - g$$

$$\text{and } \frac{dg}{dr} = -V_\varphi(r)\lambda(r)$$

Ex 2

a)
$V'_\varphi = V_\varphi$

Prove $\exists$ ~~...~~ scalar-valued $\psi(r)$: $\nabla \Delta V_\varphi = \psi e_r$

The Laplacian of a vector field in cylindrical coordinates is definded as $\Delta v = (\nabla \cdot \nabla) v = \nabla(\nabla \cdot v) - \nabla \times (\nabla \times v)$

$$= \nabla \operatorname{div} v - \nabla \times \operatorname{rot} v$$

We know from 1i and 1ii parts of it:

$$\Delta v = \nabla \operatorname{div} v - \nabla \times \operatorname{rot} v$$

$$\xrightarrow{1i,\ 1ii} = 0 - \nabla \times \frac{1}{r} \partial_r (r v_\varphi(r)) e_z$$

Define $\lambda(r) = \dots$

$$= - \nabla \times \lambda(r) \cdot e_z$$

$$= - \nabla \lambda(r) \times e_z$$

$$= - \frac{d\lambda(r)}{dr} e_r \times e_z$$

$$= - \frac{d\lambda(r)}{dr} (e_r \times e_z)$$

$$\xrightarrow{\text{Note Exc. 1}} = - \frac{d\lambda(r)}{dr} e_\varphi$$

$$= - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r v_\varphi(r)\right) e_\varphi$$

~~$\frac{d}{dr} V_\varphi(r) = V_\varphi(r)$~~ $\quad v = \frac{d}{dr} V_\varphi(r) e_\varphi = V_\varphi(r) e_\varphi$

So
$$\Delta v_\varphi = - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r v_\varphi(r)) \right)$$

$$\Rightarrow \Delta V'_\varphi = - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r V'_\varphi(r)) \right)$$

$\Rightarrow$ ~~Define~~

Define $\hat{\lambda} := - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r V'_\varphi(r)) \right)$

$$\Rightarrow \nabla \Delta V'_\varphi = \nabla \hat{\lambda}$$

We see $\hat{\lambda} = \hat{\lambda}(r)$. Therefore the gradient $\nabla \hat{\lambda}(r)$ is only in e_r-direction

$\Rightarrow$ There is a scalar valued functi. $\psi(r)$ //

Prove (*) equivalent to $-\frac{1}{Re} e_z \times \nabla \Delta V_\varphi(r) + \nabla \Pi = 0$

Proof:

$$-\frac{1}{Re} \Delta V + \nabla \Pi = 0 \qquad (*)$$

$$V = V_\varphi e_\varphi$$

$$\Delta V = \Delta V_\varphi'(r)\, e_\varphi$$

We have to show $\Delta V = \Delta V_\varphi'(r)\, e_\varphi \overset{!}{=} e_z \times \nabla \Delta V_\varphi(r)$

$$\Delta V = \Delta \frac{d V_\varphi(r)}{dr} e_\varphi$$

with $e_\varphi = e_z \times e_r$

$$\Delta V = \Delta \frac{d V_\varphi(r)}{dr} e_\varphi = \Delta \frac{d V_\varphi(r)}{dr} e_z \times e_r$$

$$= e_z \times \Delta \frac{d V_\varphi(r)}{dr} e_r$$

$$= e_z \times \nabla \Delta V_\varphi(r)$$

$$\Rightarrow \Delta V = e_z \times \nabla \Delta V_\varphi(r)$$

$$\overset{*}{\Rightarrow} -\frac{1}{Re} e_z \times \nabla \Delta V_\varphi(r) + \nabla \Pi = 0$$

5.) $$-\frac{1}{Re} e_z \times \nabla \Delta V_\varphi(r) + \nabla \Pi = 0$$

$$\overset{2a}{\Rightarrow} \quad -\frac{1}{Re} e_z \times \psi(r) e_r + \nabla \Pi = 0$$

$e_\varphi = e_z \times e_r$

$$\Rightarrow \quad -\frac{1}{Re} \psi(r)\, e_\varphi + \nabla \Pi = 0$$

$$\Pi = p + \frac{1}{2} \|v\|^2 - g$$

Since $p = p(r)$ and $g = g(r)$

$$\Rightarrow \quad \Pi = \Pi(r)$$

$$\Rightarrow \quad \nabla \Pi = \frac{d\Pi}{dr} e_r$$

$$\Rightarrow \quad -\frac{1}{Re} \psi(r)\, e_\varphi + \frac{d\Pi}{dr} e_r = 0$$

Since e_φ and e_r are linear independent

$$\Rightarrow \quad -\frac{1}{Re} \psi(r) = 0 \quad \wedge \quad \frac{d\Pi}{dr} = 0$$

$$\Rightarrow \quad \psi = 0 \quad \wedge \quad \Pi \equiv const$$