Ex 2

a)

$V'_\varphi = V_\varphi$

Prove $\exists$ scalar-valued $\psi(r)$: $\nabla \Delta V_\varphi = \psi e_r$

The Laplacian of a vector field in cylindrical coordinates is defined as $\Delta v = (\nabla \cdot \nabla) v = \nabla(\nabla \cdot v) - \nabla \times (\nabla \times v)$

$= \nabla \operatorname{div} v - \nabla \times \operatorname{rot} v$

We know from 1i and 1ii parts of it:

$\Delta v = \nabla \operatorname{div} v - \nabla \times \operatorname{rot} v$

$\overset{1i, 1ii}{=} 0 - \nabla \times \frac{1}{r} \partial_r (r v_\varphi(r)) e_z$

Define $\lambda(r) = \dots$

$= - \nabla \times \lambda(r) \cdot e_z$

$= - \nabla \lambda(r) \times e_z$

$= - \frac{d\lambda(r)}{dr} e_r \times e_z$

$= - \frac{d\lambda(r)}{dr} (e_r \times e_z)$

Note Exc. 1

$= - \frac{d\lambda(r)}{dr} e_\varphi$

$= - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r v_\varphi(r)) \right) e_\varphi$

$v = \frac{d}{dr} V_\varphi(r) e_\varphi = V'_\varphi(r) e_\varphi$

So

$\Delta v_\varphi = - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r v_\varphi(r)) \right)$

$\Rightarrow \Delta V'_\varphi = - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r V'_\varphi(r)) \right)$

$\Rightarrow$

Define $\hat{\lambda} := - \frac{d}{dr} \left(\frac{1}{r} \partial_r (r V'_\varphi(r)) \right)$

$\Rightarrow \nabla \Delta V'_\varphi = \nabla \hat{\lambda}$

We see $\hat{\lambda} = \hat{\lambda}(r)$. Therefore the gradient $\nabla \hat{\lambda}(r)$ is only in e_r-direction

$\Rightarrow$ There is a scalar valued functi. $\psi(r)$ //

Prove (*) equivalent to $-\frac{1}{Re} e_z \times \nabla \Delta V_\varphi(r) + \nabla \Pi = 0$

Proof:

$$-\frac{1}{Re} \Delta V + \nabla \Pi = 0 \qquad (*)$$

$$V = V_\varphi e_\varphi$$

$$\Delta V = \Delta V_\varphi'(r)\, e_\varphi$$

We have to show $\Delta V = \Delta V_\varphi'(r)\, e_\varphi \stackrel{!}{=} e_z \times \nabla \Delta V_\varphi(r)$

$$\Delta V = \Delta \frac{d V_\varphi(r)}{dr} e_\varphi$$

with $e_\varphi = e_z \times e_r$

$$\Delta V = \Delta \frac{d V_\varphi(r)}{dr} e_\varphi = \Delta \frac{d V_\varphi(r)}{dr} e_z \times e_r$$

$$= e_z \times \Delta \frac{d V_\varphi(r)}{dr} e_r$$

$$= e_z \times \nabla \Delta V_\varphi(r)$$

$$\Rightarrow \Delta V = e_z \times \nabla \Delta V_\varphi(r)$$

$$\stackrel{*}{\Rightarrow} -\frac{1}{Re} e_z \times \nabla \Delta V_\varphi(r) + \nabla \Pi = 0$$

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