Numerics of incompressible flows 1 (SS 25)
Homework 2

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Note: Hand in your assignments in pairs of two or three, either on paper at the start of the exercise class or online via StudOn. Ensure that your names are written on your assignments.

Exercises are optional, but if you finish the course with at least 50% of the attainable points, you will be awarded 1 grade better in the final exam (provided you pass with at least a 4.0). Similarly, at least 75% of the attainable points will earn you 2 grades better.

Exercise 1: Couette Flow II ((2+3+3)+2)

Notation & setting of this exercise sheet are the same as in Ex. 1 from the previous exercise sheet. We now make an additional symmetry assumption for the smooth vector field v:

$$v(x,y,z) = v_\varphi(r) e_\varphi(r,\varphi,z).$$

Also, we define a smooth scalar-valued function $p : \Omega \to \mathbb{R}$ and also assume $p(x,y,z) = p(r)$ for symmetry reasons.

a) Prove that

i) $\operatorname{div} v = 0$,

ii) $\operatorname{rot} v = \frac{1}{r}\partial_r(r v_\varphi(r)) e_z$, and

iii) there exists $g = g(r)$ such that $v \times \operatorname{rot} v = \nabla g$.

Note: You can use without proof that $e_\varphi = e_z \times e_r$ and that the following relation holds for two three-dimensional vector fields A, B:

$$\operatorname{rot}(A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + (\operatorname{div} B)A - (\operatorname{div} A)B$$

b) Conclude that if the solution has the assumed symmetry the Navier-Stokes equations with zero right-hand side may be written as

$$-\frac{1}{\mathrm{Re}}\Delta v + \nabla \pi = 0 \qquad \text{with } \pi = p + \frac{1}{2}\|v\|^2 - g. \tag{$*$}$$

Exercise 2: Couette Flow III (4+3+3)

We want to find a solution of the equation $(*)$.

a) Let $V_\varphi' = v_\varphi$. Prove that there is a scalar-valued $\psi(r)$ such that $\nabla \Delta V_\varphi = \psi e_r$ and conclude that $(*)$ is equivalent to

$$-\frac{1}{\mathrm{Re}} e_z \times \nabla \Delta V_\varphi(r) + \nabla \pi = 0.$$

b) Argue, why a) and the symmetry of p implies $\pi \equiv \text{const}$ and $\psi = 0$.

c) Using b), give an explicit formula for v if we prescribe a constant angular velocity on both cylinder surfaces: $v_\varphi(R_i) := R_i \omega_i, i = 1, 2$.