Numerics of incompressible flows 1 (SS 25)
Homework 1

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Home work 1

Ex 1

$0 < R_1 < R_2$ $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid R_1 < \sqrt{x^2 + y^2} < R_2\}$

a.)
Couette flow: $(x, y, z) \mapsto (r, \varphi, z) := \left(\sqrt{x^2 + y^2}, \; \arctan\left(\frac{y}{x}\right), z\right)$

$$e_r(r, \varphi, z) := \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}, \quad e_\varphi(r, \varphi, z) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Prove:

1. $\nabla f = \partial_r f \, e_r + \frac{1}{r} \partial_\varphi f \, e_\varphi + \partial_z f \, e_z$

2. $\Delta f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_{\varphi\varphi} f + \partial_{zz} f$

Note:

For $v = v_x e_x + v_y e_y + v_z e_z = v_r e_r + v_\varphi e_\varphi + v_z e_z$

$$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

following holds: $\text{div } v = \frac{1}{r} \partial_r (r v_r) + \frac{1}{r} \partial_\varphi v_\varphi + \partial_z v_z$

Proof:

1. $\nabla f = \frac{\partial f}{\partial x} e_x + \frac{\partial f}{\partial y} e_y + \frac{\partial f}{\partial z} e_z$

$e_r = \cos(\varphi) e_x + \sin(\varphi) e_y$, $e_\varphi = -\sin(\varphi) e_x + \cos(\varphi) e_y$
$e_z = e_z$

By applying the chain rule to $f(x, y, z) = \hat{f}(r(x, y), \varphi(x, y), z)$
We get:

~~$\nabla f = \frac{\partial f}{\partial x} e_x + \frac{\partial f}{\partial y} e_y + \frac{\partial f}{\partial z} e_z$~~

~~$= \frac{\partial \hat f}{\partial r} \frac{\partial r}{\partial x} e_x + \frac{\partial \hat f}{\partial \varphi} \frac{\partial \varphi}{\partial y}$~~

- $\dfrac{\partial f}{\partial x} = \dfrac{\partial \hat{f}}{\partial r} \dfrac{\partial r}{\partial x} + \dfrac{\partial \hat{f}}{\partial \varphi} \dfrac{\partial \varphi}{\partial x}$

- $\dfrac{\partial f}{\partial y} = \dfrac{\partial \hat{f}}{\partial r} \dfrac{\partial r}{\partial y} + \dfrac{\partial \hat{f}}{\partial \varphi} \dfrac{\partial \varphi}{\partial y}$

- $\dfrac{\partial f}{\partial z} = \dfrac{\partial \hat{f}}{\partial z}$

Now we simplify the :

- $\dfrac{\partial r}{\partial x} = \partial_x \sqrt{x^2+y^2} = \dfrac{x}{\sqrt{x^2+y^2}}$

- $\dfrac{\partial r}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

- $\dfrac{\partial \varphi}{\partial x} = \partial_x \arctan\left(\dfrac{y}{x}\right) = -\dfrac{y}{x^2+y^2}$

- $\dfrac{\partial \varphi}{\partial y} = \partial_y \arctan\left(\dfrac{y}{x}\right) = \dfrac{x}{x^2+y^2}$

Rewrite:

$$\nabla f = \partial_r \hat{f}\, \dfrac{x}{\sqrt{x^2+y^2}} e_x + \partial_\varphi \hat{f}\left(-\dfrac{y}{x^2+y^2}\right)e_x + \partial_r \hat{f}\, \dfrac{y}{\sqrt{x^2+y^2}} e_y + \partial_\varphi \hat{f}\, \dfrac{x}{x^2+y^2} e_y + \partial_z \hat{f}\, e_z$$

$$= \partial_r \hat{f}\left(\dfrac{x}{\sqrt{x^2+y^2}} e_x + \dfrac{y}{\sqrt{x^2+y^2}} e_y\right) + \partial_\varphi \hat{f}\left(-\dfrac{y}{x^2+y^2} e_x + \dfrac{x}{x^2+y^2} e_y\right) + \partial_z \hat{f}\, e_z$$

If we assume $\cos\varphi = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi = \dfrac{y}{\sqrt{x^2+y^2}}$, $-\dfrac{y}{x^2+y^2} = -\dfrac{\sin\varphi}{r}$

$$\dfrac{x}{x^2+y^2} = \dfrac{\cos\varphi}{r}$$

then

$$\nabla f = \partial_r \hat{f}\begin{pmatrix}\cos\varphi\\\sin\varphi\\0\end{pmatrix} + \partial_\varphi \hat{f}\begin{pmatrix}-\sin\varphi\\\cos\varphi\\0\end{pmatrix} + \partial_z \hat{f}\, e_z$$

$$= \partial_r \hat{f}\, e_r + \partial_\varphi \hat{f}\, e_\varphi + \partial_z \hat{f}\, e_z.$$

So last thing is to prove the Assumptions.

- $\varphi = \arctan\left(\dfrac{y}{x}\right)$, $\cos\varphi = \cos\left(\arctan\left(\dfrac{y}{x}\right)\right) = \dfrac{x}{\sqrt{x^2+y^2}}$ ✓

- $\varphi = \arctan\left(\dfrac{y}{x}\right)$, $\sin\varphi = \sin\left(\arctan\left(\dfrac{y}{x}\right)\right) = \dfrac{y}{\sqrt{x^2+y^2}}$ ✓

- $r = \sqrt{x^2+y^2}$, $-\dfrac{\sin\varphi}{\sqrt{x^2+y^2}} = -\dfrac{y}{\sqrt{x^2+y^2}} \cdot \dfrac{1}{\sqrt{x^2+y^2}} = -\dfrac{y}{x^2+y^2}$ ✓

- $r = \sqrt{x^2+y^2}$, $\dfrac{\cos\varphi}{\sqrt{x^2+y^2}} = \dfrac{x}{x^2+y^2}$ ✓

Therefore

$$\nabla f = \partial_r \hat{f}\, e_r + \partial_\varphi \hat{f}\, e_\varphi + \partial_\varphi \hat{f}\, e_z$$ ✓

2.

$$\Delta f = \nabla \cdot \nabla f$$

We define
$$v := \nabla f = v_r \, e_r + v_\varphi \, e_\varphi + v_z \, e_z$$

where $v_r = \partial_r f$, $v_\varphi = \frac{1}{r} \partial_\varphi f$, $v_z = \partial_z f$

with the N.H it follows:

$$\operatorname{div}(\nabla f) = \frac{1}{r} \partial_r (r v_r) + \frac{1}{r} \partial_\varphi v_\varphi + \partial_z v_z$$

$$= \frac{1}{r} \partial_r (r \, \partial_r f) + \frac{1}{r} \partial_\varphi \frac{1}{r} \partial_\varphi f + \partial_z \partial_z f$$

$$= \frac{1}{r} \partial_r (r \, \partial_r f) + \frac{1}{r^2} \partial_{\varphi\varphi} f + \partial_{zz} f$$

b.)

Let $v : \Omega \to \mathbb{R}^3$ be smooth

to prove:
$$(v \cdot \nabla) v = \frac{1}{2} \nabla \|v\|^2 - v \times \operatorname{rot} v \qquad (\operatorname{rot} v = \nabla \times v)$$

Lets write both sides in index notation.

LHS: $\quad [(v \cdot \nabla) v]_i = \sum_j v_j \dfrac{\partial v_i}{\partial x_j} = \sum_j v_j \, \partial_j v_i$

RHS: ~~crossed out~~

• $\left[\frac{1}{2} \nabla \|v\|^2 \right]_i = \frac{1}{2} \partial_i \sum_j v_j^2 = \sum_j v_j \dfrac{\partial v_j}{\partial x_i} = \sum_j v_j \, \partial_i v_j$

• $[v \times \operatorname{rot} v]_i = \quad [v \times (\nabla \times v)]_i$

For better understanding:

$$(\nabla \times v)_i = \begin{cases} \partial_2 v_3 - \partial_3 v_2 & i=1 \\ \partial_3 v_1 - \partial_1 v_3 & i=2 \\ \partial_1 v_2 - \partial_2 v_1 & i=3 \end{cases}$$

$$\Rightarrow \quad [v \times (\nabla \times v)]_i = \begin{cases} v_2 (\partial_1 v_2 - \partial_2 v_1) - v_3 (\partial_3 v_1 - \partial_1 v_3) & i=1 \\ v_3 (\partial_2 v_3 - \partial_3 v_2) - v_1 (\partial_1 v_2 - \partial_2 v_1) & i=2 \\ v_1 (\partial_3 v_1 - \partial_1 v_3) - v_2 (\partial_2 v_3 - \partial_3 v_2) & i=3 \end{cases}$$

$\Rightarrow$ Let $i=1$: LHS: $v_1 \partial_1 v_1 + v_2 \partial_2 v_1 + v_3 \partial_3 v_1$

RHS: $v_1 \partial_1 v_1 + v_2 \partial_1 v_2 + v_3 \partial_1 v_3$
$\quad - (v_2 \partial_1 v_2 - v_2 \partial_2 v_1 - v_3 \partial_3 v_1 + v_3 \partial_1 v_3)$
$\quad = v_1 \partial_1 v_1 + v_2 \partial_2 v_1 + v_3 \partial_3 v_1$

$\Rightarrow$ LHS = RHS ~~crossed out~~ $(i = 2,3 \text{ analog})$