Numerical methods for incompressible flows I

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Contents

0 Prologue

Flows are ubiquitous, some examples

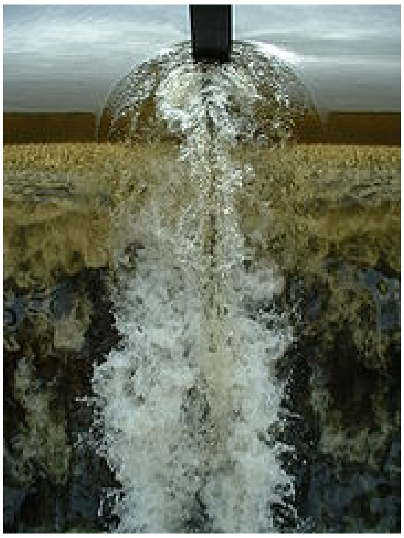


Figure 1: Flow encountered in all days life.

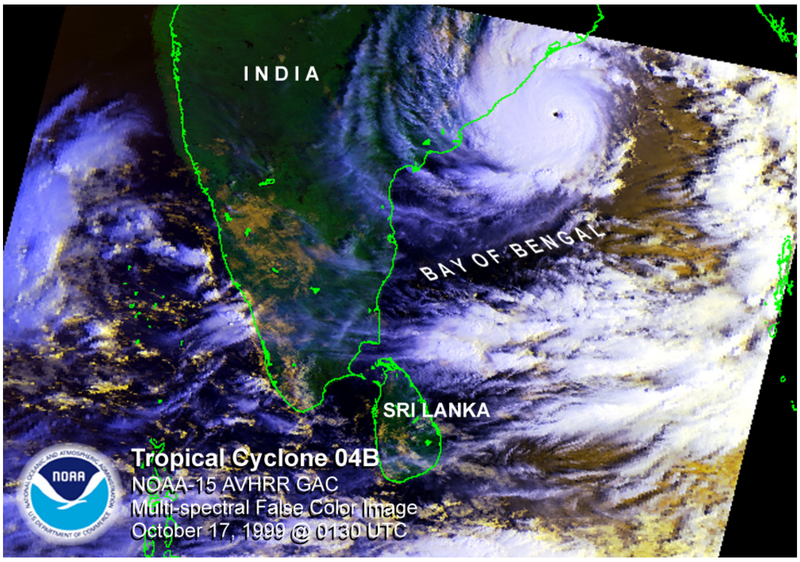


Figure 2: Cyclone over Asia.

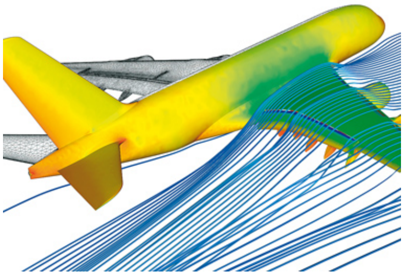


Figure 3: Numerical simulation of flow around Airbus A380. Source: EADS/DLR.

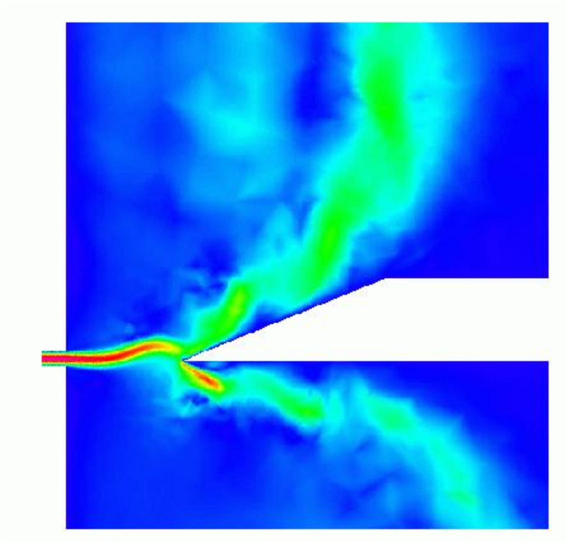


Figure 4: Edge tone or how does the tone in a recorder is generated: the air jet from left hits the labium, becomes instable and starts oscillating at a certain frequency.

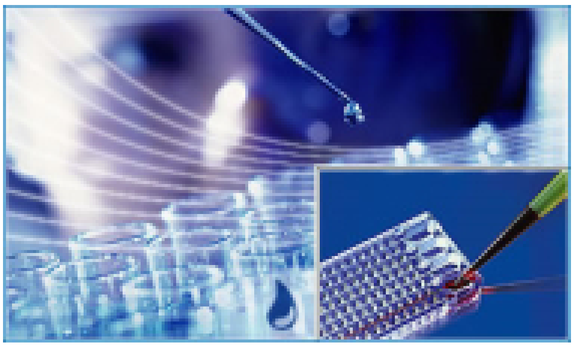


Figure 5: Micro flow, *lab-on-a-chip*. Source: Boehringer.

1 Introduction

1.1 Brief derivation of the Navier–Stokes equations

Continuum hypothesis:

Fluid is composed of a continuum of “fluid particles” making up the domain $\Omega(t) \subset \mathbb{R}^d$ at any

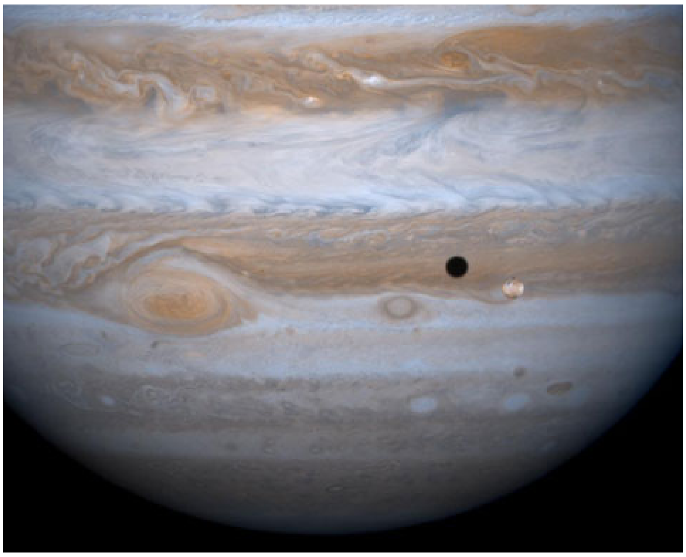


Figure 6: Bands of clouds on Jupiter. Source: NASA.

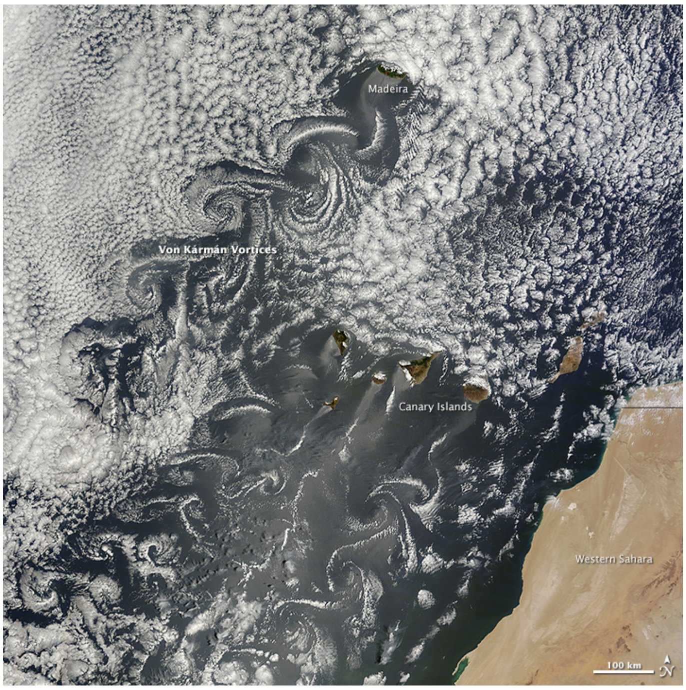


Figure 7: Von Karman vortex shedding in the wake of islands. Source: eltiempohoy.es

time instant $t > 0$.

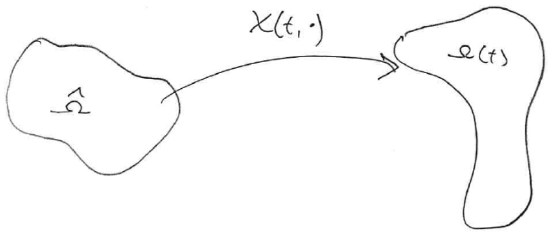
The behavior of the fluid is described by a diffeomorphism $\chi(t, \cdot),\ t > 0$

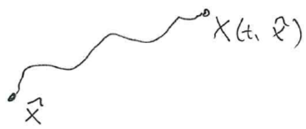
$$\chi : [0, T] \times \hat{\Omega} \to \Omega(t)$$

$$\hat{\Omega} = \Omega(0) \text{ a "nice" reference domain}$$

i.e. open, bounded and with smooth boundary.

The trajectory of a "fluid particle" initially at position $\hat{x}$ is described by $X(t,\hat{x}),\ t \in [0,T],\ \hat{x} \in \hat{\Omega}$ with $X(0,\hat{x}) = \hat{x}$ and $X(t,\hat{x}) = \chi(t,\hat{x})$.





Remark: In this setting $\Omega(t)$ will have the same topology as $\hat{\Omega}$. Thus, for instance "breakup" of a drop is excluded in this setting.

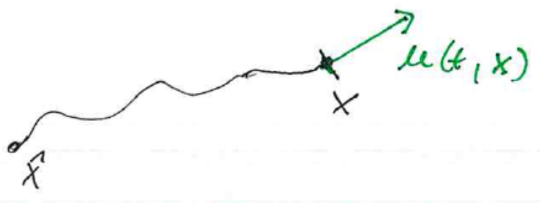
Physical quantities:

- $\rho = \rho(t,x)$ fluid's "density"
- $u = (u_1, ..., u_d)^T = u(t,x)$ "velocity"
- $p = p(t,x)$ "pressure"

with $x \in \Omega(t)$.
Velocity u is given as the temporal derivative of a particle's trajectory:

$$u(t,x) := \frac{d}{dt}\chi(t,\hat{x}) \tag{1.1}$$

with $x = \chi(t,\hat{x})$.

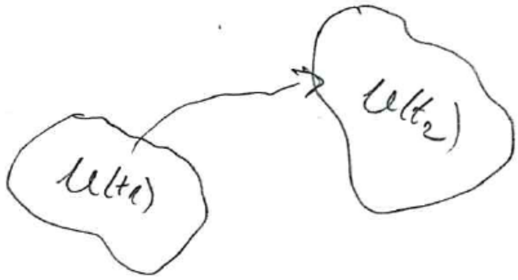


Derivation of a "complete" set of equations: Consider an arbitrary, smoothly bounded **test volume** $\hat{U} \subset \hat{\Omega}$, $U(t) := \chi(t,\hat{U}) \subset \Omega(t)$.

<u>**1. conservation property:** mass</u>

The mass $M(t)$ contained in $U(t)$ at time instant t:

$$M(t) = \int\limits_{U(t)} \rho(t,x)\ dx$$



Mass is conserved over time:

$$0 = \frac{d}{dt} M(t) = \frac{d}{dt} \int\limits_{U(t)} \rho(t,x)\ dx.$$

We need:

Theorem 1.1 (Reynolds' Transport Theorem)**.**
Let $\Phi : Q_T \to I\!R$ be smooth and $U(t) = \chi(t, \hat{U}),\ t \in [0,T]$ smooth . Then:

$$\frac{d}{dt} \int\limits_{U(t)} \Phi(t,x)\ dx = \int\limits_{U(t)} \frac{\partial \Phi}{\partial t}(t,x) + \nabla \cdot \big(\Phi(t,x) u(t,x)\big)\ dx$$

$Q_T := \{(t,x) \mid t \in [0,T],\ x \in \Omega(t)\}$

Note:

$$\nabla \cdot \big(\Phi u\big) := \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \big(\Phi u_i\big) = \sum_{i=1}^{d} \partial_i \big(\Phi u_i\big)$$

In order to prove the above theorem we need a couple of lemmas.

Lemma 1.2.

$Gl(d) := \{A \in I\!R^{d\times d} \mid A$ regular$\}$

Define:

$$\varphi \;:\; Gl(d) \to I\!R$$

$$A \mapsto det(A)$$

It holds:

φ is differentiable and

$$D\varphi(A)(U) = det(A)\; tr(UA^{-1})$$

for $U \in I\!R^{d\times d}$, $tr\,(B) = \sum\limits_{i=1}^{d} B_{ii}$, $B \in I\!R^{d\times d}$.

Note:

$$D\varphi(A) \;:\; I\!R^{d\times d} \to I\!R$$

$$U \mapsto D\varphi(A)(U) = \sum_{i,j=1}^{d} \frac{\partial \varphi}{\partial A_{i,j}}(A) U_{i,j}$$

Proof:

We will show: φ is continuously differentiable in direction of the $i-j-$th coefficient $\Rightarrow \varphi$ continuously differentiable in Gl$(d) \subset I\!R^{d\times d}$

(we could also argue: det is a polynomial of degree d w.r.t. the coefficients of A)

We start by computing

$$\varphi(A + \varepsilon\tau^{ij}) - \varphi(A) =?$$

where τ^{ij} denotes the matrix consisting of coefficients 0 except for the $i-$th row and $j-$th column, where $\tau^{i,j} = 1$.

$$\begin{aligned}
\varphi(A + \varepsilon\tau^{ij}) - \varphi(A) &= \det(A + \varepsilon\tau^{ij}) - \det(A) \\
&= \det((I + \varepsilon\tau^{ij}A^{-1})A) - \det(A) \\
&= \det(I + \varepsilon\tau^{ij}A^{-1})\det(A) - \det(A) \\
&= \big[\det(I + \varepsilon\tau^{ij}A^{-1}) - 1\big]\det(A)
\end{aligned}$$

begin{**side computation**} what is $\tau^{ij}A^{-1}$?

Let $M \in I\!R^{d\times d}$ with

$$M = \begin{bmatrix} -m_1^T- \\ \vdots \\ -m_d^T- \end{bmatrix}, \qquad m_i \in I\!R^d, \quad i = 1, ..., d$$

$k, l \in \{1, \ldots, d\}$:

$$(\tau^{ij}M)_{kl} = \sum_{r=1}^{d} (\tau^{ij})_{kr} M_{rl} = \sum_{r=1}^{d} \delta_{ik}\delta_{jr} M_{rl}$$

$$= \delta_{ik} M_{jl} = \begin{cases} M_{jl} & \text{if } k = i \\ 0 & \text{else} \end{cases}$$

i.e.

$$\tau^{ij}M = \begin{pmatrix} 0 \\ -m_j^T- \\ 0 \end{pmatrix} \leftarrow i-\text{th row}$$

end{**side computation**}

Since the determinant is a **multilinear** mapping w.r.t. the rows,

$$\det(I + \varepsilon\tau^{ij}A^{-1}) = \det(I) + \varepsilon\det([\star\star])$$

with

$$[\star\star] = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ A^{-1}_{j1} & & \ldots & & A^{-1}_{jd} \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \leftarrow i-\text{th row}$$

$\Rightarrow \det([\star\star]) = A^{-1}_{ji} \overset{\text{check}}{=} \mathrm{tr}(\tau^{ij}A^{-1})$

Furthermore:

$$\varphi(A + \varepsilon\tau^{ij}) - \varphi(A) = \det(A + \varepsilon\tau^{ij}) - \det(A) = \varepsilon \cdot \mathrm{tr}(\tau^{ij}A^{-1})\det(A)$$

$\Rightarrow \varphi$ continuously, partially differentiable with

$$\partial_{A_{ij}}\varphi(A) = \mathrm{tr}(\tau^{ij}A^{-1})\det(A)$$

$\Rightarrow \varphi$ continuously differentiable and

$$D\varphi(A)(U) = \sum_{i,j=1}^{d} A^{-1}_{ji}\det(A)U_{ij}$$

$$= \mathrm{tr}(UA^{-1})\det(A)$$

$\square$

Lemma 1.3.

It holds:

$$\partial_t \chi^{-1}(t,x) = -D\chi^{-1}(t,x)\partial_t \chi(t,\hat{x})$$

with $\hat{x} = \chi^{-1}(t,x)$.

Proof:

For $\hat{x} \in \hat{\Omega}$ we have $\chi^{-1}(t,\chi(t,\hat{x})) = \hat{x} \quad \forall t \in [0,T]$

$\Rightarrow$

$$\begin{aligned} 0 &= \frac{d}{dt}\chi^{-1}(t,\chi(t,\hat{x})) \\ &= \frac{\partial}{\partial t}\chi^{-1}(t,x) + D\chi^{-1}(t,x)\frac{\partial}{\partial t}\chi(t,\hat{x}) \end{aligned}$$

with $x = \chi(t,\hat{x})$.

$\Rightarrow$

$$\partial_t \chi^{-1}(t,x) = -\underbrace{D\chi^{-1}(t,x)}_{\in \mathbb{R}^{d\times d}}\underbrace{\partial_t \chi(t,\hat{x})}_{\in \mathbb{R}^d}$$

$\square$

Corollary 1.4 (ad Lemma 1.2)**.**

$$\frac{d}{dt}\det(D\chi(t,\hat{x})) = \det(D\chi(t,\hat{x}))\cdot \mathit{div}\,(u(t,x))$$

with $x = \chi(t,\hat{x})$

Proof:

Define:

- $\eta(t) := D\chi(t,\hat{x})$
- $\varphi(A) := \det(A)$

$\Rightarrow \det(D\chi(t,\hat{x})) = \varphi(\eta(t))$

$$\begin{aligned} \frac{d}{dt}\det(D\chi(t,\hat{x})) &= D\varphi(D\chi)\frac{\partial}{\partial t}D\chi(t,\hat{x}) \\ &= \det(D\chi)\mathrm{tr}\Big(D\dot{\chi}(t,\hat{x})D\chi^{-1}(t,x)\Big) \end{aligned}$$

with $x = \chi(t,\hat{x})$ and $D\dot{\chi}(t,\hat{x}) = \frac{\partial}{\partial t}D\chi$

On the other hand: $u(t,x) = \partial_t\chi(t,\hat{x})$ at $x = \chi(t,\hat{x})$, i.e.

$$u(t,x) = \partial_t \chi(t,\chi^{-1}(t,x))$$

$\Rightarrow$

$$D_x u(t,x) = D_{\hat{x}}\partial_t\chi(t,\hat{x})\underbrace{\frac{\partial \hat{x}}{\partial x}(t,x)}_{D\chi^{-1}(t,x)}$$

Therefore:

$$D\dot{\chi}D\chi^{-1} = \nabla u := Du$$

and moreover:

$$\mathrm{tr}(D\dot{\chi}D\chi^{-1}) = \mathrm{tr}(\nabla u) = \mathrm{div}\ (u)$$

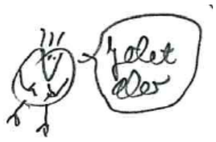
□

Since the above formula is crucial, we write it as a separate conclusion:

Corollary 1.5.

$$D\dot{\chi}(t,\hat{x})D\chi^{-1}(t,x) = \nabla u(t,x)$$

with $x = \chi(t,\hat{x})$.



Proof (Reynolds' Transport Theorem):

$$\int\limits_{U(t)} \Phi(t,x)\ dx \overset{\text{transformation rule}}{=} \int\limits_{\hat{U}} \Phi(t,\chi(t,\hat{x}))|\det(D\chi(t,\hat{x}))|\ d\hat{x}$$

$\Rightarrow$

$$\begin{aligned}
\frac{d}{dt}\int\limits_{U(t)} \Phi(t,x)\ dx &= \int\limits_{\hat{U}} \frac{d}{dt}\Big(\Phi(t,\chi(t,\hat{x}))|\det(D\chi(t,\hat{x}))|\Big)\ d\hat{x} \\
&= \int\limits_{\hat{U}} \frac{d}{dt}\Phi(t,\chi(t,\hat{x}))|\det(D\chi(t,\hat{x}))| + \Phi(t,\chi(t,\hat{x}))\cdot\frac{d}{dt}|\det(D\chi(t,\hat{x}))|\ d\hat{x} \\
&\overset{\substack{\text{may assume} \\ \det(D\chi(t,\hat{x}))>0}}{=} \int\limits_{\hat{U}} \Big[(I) + \Phi(t,\chi(t,\hat{x}))\underbrace{\mathrm{div}(u(t,\chi(t,\hat{x})))}_{\text{by Cor. 1.4}}\Big]\cdot\det(D\chi(t,\hat{x}))\ d\hat{x} \\
&= (*)
\end{aligned}$$

with

$$(I) = \frac{\partial}{\partial t}\Phi(t,x) + D\Phi(t,x)\underbrace{\frac{\partial}{\partial t}\chi(t,\hat{x})}_{u(t,x)}$$

(in coordinates: $D\Phi u = \sum\limits_{i=1}^{d} u_i \frac{\partial}{\partial x_i}\Phi =: u\cdot\nabla\Phi$)

$$\Rightarrow (*) = \int\limits_{\hat{U}} \Big(\partial_t\Phi(t,x) + u(t,x)\cdot\nabla\Phi(t,x) + \Phi(t,x)\mathrm{div}(u(t,x))\Big)\cdot\det(D\chi(t,\hat{x}))\ d\hat{x}$$

with $x = \chi(t, \hat{x})$ for short. Now:

$$
\begin{aligned}
u \cdot \nabla\Phi + \Phi \operatorname{div}(u) &= \nabla \cdot (\Phi u) \\
&= \sum_{i=1}^{d} \partial_i (\Phi u_i)
\end{aligned}
$$

Therefore:

$$
(*) = \int_{\hat{U}} \Big(\partial_t \Phi + \nabla \cdot (\Phi u)\Big) \det(D\chi)\ d\hat{x} \overset{\text{Trafo}}{=} \int_{U(t)} \partial_t \Phi + \nabla \cdot (\Phi u)\ dx
$$

$\square$

Back to the conservation of mass, p. 4

$$
M(t) := \int_{U(t)} \rho(t, x)\ dx
$$

$$
0 = \frac{d}{dt} M(t) = \int_{U(t)} \partial_t \rho + \nabla \cdot (\rho u)\ dx
$$

Proposition 1.6 (Fundamental Lemma of the calculus of variations)**.**
$\Omega \subset I\!R^d$ open, $\Phi : \Omega \to I\!R$ continuous. If

$$
\int_U \Phi(x)\ dx = 0
$$

for all open, smoothly bounded $U \subset \Omega$, then

$$
\Phi(x) = 0 \qquad \forall x \in \Omega
$$

Proof: calculus.

Now, let $t \in [0, T]$ be arbitrary and $U \subset I\!R^d$ open and smooth. Define $\hat{U} := \chi^{-1}(t, U) \Rightarrow U(t) = U$.

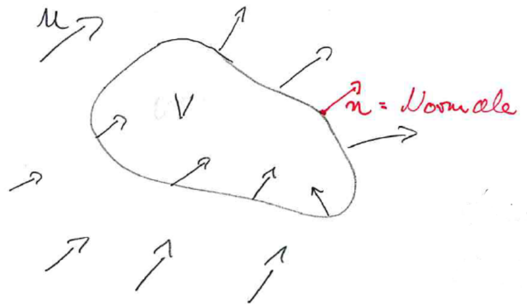
Since U is arbitrary, we may conclude

Conservation of mass in differential form:

$$
\partial_t \rho(t, x) + \nabla \cdot (\rho(t, x) u(t, x)) = 0 \tag{1.2}
$$

$\forall (t, x) \in Q_T$.

Interpretation: Let V be time independent. Then



$$\frac{d}{dt}\int\limits_V \rho\ dx = \int\limits_V \partial_t\rho\ dx = -\int\limits_V \nabla\cdot(\rho u)\ dx$$
$$\stackrel{\text{Gauß}}{=} -\int\limits_{\partial V} \rho u\cdot n\ do_x$$

In words: the mass contained in V can only change by the amount of mass flowing in/out of V through the boundary ∂V.

<u>**2. conservation property:** momentum</u>

Recall: for a "point mass" M we have

$$\overrightarrow{\text{momentum}} = M\cdot\ \overrightarrow{\text{velocity}}$$

Applied to our case: momentum I in $\hat{U}\subset\hat{\Omega}$ is given by:

$$I(t) = \int\limits_{U(t)} \rho u\ dx \qquad \in I\!R^d$$

Newton's 2nd law now states:

change of momentum with time = sum of all (inner + outer) forces

$$\frac{d}{dt}I(t) = \underbrace{\int\limits_{U(t)} \rho f\ dx}_{\text{outer forces}} + \underbrace{\int\limits_{\partial U(t)} F\ do_x}_{\text{inner forces}} \tag{1.3}$$

Cauchy's Theorem: F is of the form

$$F(n) = Sn$$

with $S(t,x)\in I\!R^{d\times d}$ symmetric and n = the outward pointing normal to $\partial U(t)$.

Applying Reynolds' Transport Theorem and Prop. 1.6 to each component of Eq. (1.3) yields (with $\Phi := \rho u_i,\ i = 1,...,d$):

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(\rho u_i u) = \rho f_i + (\nabla\cdot S)_i$$

Closer look:

$$\int\limits_{\partial U(t)} F_i \; do_x = \int\limits_{\partial U(t)} (S(t,x)n)_i \; do_x$$
$$\stackrel{\text{Gauss}}{=} \int\limits_{U(t)} \sum_{j=1}^{d} \partial_j (S(t,x)_{ij}) \; dx$$

$$\nabla \cdot S = \begin{pmatrix} \partial_1 S_{11} + \ldots + \partial_d S_{1d} \\ \\ \partial_1 S_{d1} + \ldots + \partial_d S_{dd} \end{pmatrix} \quad \text{and} \quad u_i u = u_i \begin{pmatrix} u_1 \\ \\ u_d \end{pmatrix}$$

In vector/matrix notation:

$$\begin{bmatrix} \nabla \cdot (\rho u_1 u) \\ \vdots \\ \nabla \cdot (\rho u_d u) \end{bmatrix} = \nabla \cdot (\rho u \otimes u)$$

where $\otimes$ denotes the dyadic product:

$$(a \otimes b)_{i,j} = a_i b_j \quad \text{for } a, b \in I\!R^d$$

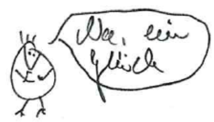
Summarizing we get

$$\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) = \rho f + \nabla \cdot S \tag{1.4}$$

($S =$ the stress tensor).

From now on, we will make our life easier by only considering **incompressible** fluids:

$$\nabla \cdot u = 0 \tag{1.5}$$



Then from Eq. (1.2) one can deduce

$$\partial_t \rho + \nabla \cdot (\rho u) = \partial_t \rho + u \cdot \nabla \rho + \underbrace{\rho \nabla \cdot u}_{=0} = 0$$

If $\rho(t = 0, \cdot) \equiv$ constant and if we have "compatible" boundary data, it follows that $\rho(t,x) \equiv$ const. $\forall t, x$. Thus, we assume hereafter

$$\rho \equiv \text{const.} \tag{1.6}$$

System Eqs. (1.2), (1.4), (1.5) is not yet closed, a "constitutive" law for $S = S(t, x, u, \nabla u, \ldots)$ is missing.

With the help of the observer invariance principle one can show: $S = S(t, x, \nabla u, ...)$. S can be split into a trace free part and the rest:

$$S = T - pI$$

with $\mathrm{tr}(T) = 0$ and $p = -\frac{1}{d}\mathrm{tr}(S)$. The variable p is called "pressure". We now consider the simplest form for T:

$$T(t, x, \nabla u) = \mu D(u) := \mu(\nabla u + \nabla u^T) \tag{1.7}$$

where $\mu > 0$ is a constant. Fluids with the property Eq. (1.7) are called **Newtonian** fluids. **Note:** $T = \mu(\nabla u + \nabla u^T)$ is symmetric and trace free.

Critical view on the assumptions so far:

- Eq. (1.5) $\nabla \cdot u = 0$ never fulfilled exactly. However, a very good approximation for many liquids but also for **slow** flow of gases (without change of volume). Here, "slow" means slow compared to the speed of sound.
- Eq. (1.7) Newtonian fluid: very good approximation for e.g. water and –more generally– many liquids consisting of "small" molecules, in contrast to polymers like ketchup, tooth paste, ...

Let's summarize our findings so far. The flow of an **incompressible**, **Newtonian** fluid is described by a velocity vector field u and a scalar pressure field p fulfilling

$$\begin{cases} \rho(\partial_t u + \nabla \cdot (u \otimes u)) = \rho f + \nabla \cdot S = \rho f + \mu \nabla \cdot D(u) - \nabla p^1 \\ \nabla \cdot u = 0 \end{cases} \tag{1.8}$$

Observing $\nabla \cdot u = 0$ one may equivalently write

$$\begin{aligned} \nabla \cdot (u \otimes u)_i &= \partial_j(u_i u_j) = \partial_j u_i u_j + u_i \partial_j u_j \\ &= (u \cdot \nabla u)_i + (u \underbrace{\mathrm{div}(u)}_{=0})_i \\ &= (u \cdot \nabla u)_i \end{aligned}$$

and also

$$\begin{aligned} \nabla \cdot D(u)_i &= \partial_j(\partial_j u_i + \partial_i u_j) = \partial_{jj} u_i + \partial_i \partial_j u_j \\ &= (\Delta u)_i + (\nabla \underbrace{\mathrm{div}(u)}_{=0})_i \\ &= (\Delta u)_i \end{aligned}$$

[1]Note: $\nabla \cdot (p\ Id) = \nabla p$

Here we have used Einstein's summation convention: when an index appears twice in a single term, it implies summation of that term over all the values of the index.
Thus:

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla p = \rho f \\ \nabla \cdot u = 0 \end{cases} \tag{1.9}$$

For further reading:
M. Gurtin: *An Introduction to Continuum Mechanics.*

Physical interpretation of the various terms:

$$\begin{aligned} \rho(\partial_t u + u \cdot \nabla u) &= \text{inertial forces} \\ -\mu \Delta u &= \text{viscous dissipation} \\ \nabla p &= \text{pressure gradient} \end{aligned}$$

System Eq. (1.8) (or equivalently Eq. (1.9)) consists of $(d+1)$ equations (d equations for the momentum, 1 divergence constraint $\nabla \cdot u = 0$) for $(d+1)$ unknowns ($(u_1, ..., u_d)$ and p).
System has to be closed by initial and boundary conditions (see below).

1.2 Non dimensionalization

Eq. (1.9) is written in terms of physical quantities with physical units:

$$\begin{aligned} [u] &= \frac{m}{s} \\ [p] &= \frac{N}{m^2} = Pa \\ [\rho] &= \frac{kg}{m^3} \\ [\mu] &= \frac{N}{m^2} s = Pa \cdot s \end{aligned}$$

It is much more convenient to have the equations in non dimensional form, because for instance

- usually much less parameters,
- the relative strength/influence of each term can be identified.

To non dimensionalize[2], choose "characteristic" quantities: L – length , T – time, U – velocity (often appropriate $U = \frac{L}{T} \Leftrightarrow T = \frac{L}{U}$)
Then define:

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{T}, \quad \bar{u} = \frac{u}{U}$$

[2]For a general approach see Buckingham's Π Theorem.

"-"–quantities are now without physical units. Accordingly, the derivatives transform like

$$\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial \bar{t}}, \qquad \frac{\partial}{\partial x_j} = \frac{1}{L}\frac{\partial}{\partial \bar{x}_j}, \qquad \nabla = \frac{1}{L}\bar{\nabla}, \qquad \mathrm{div} = \frac{1}{L}\bar{\mathrm{div}}, \qquad \Delta = \frac{1}{L^2}\bar{\Delta}$$

so that, for example, $\bar{\Delta} = \sum_{i=1}^{d} \frac{\partial^2}{\partial \bar{x}_i^2}$
Upon inserting into the second equation of Eq. (1.9)

$$0 = \nabla \cdot u = \frac{1}{L}U\bar{\nabla} \cdot \bar{u} \quad \Leftrightarrow \quad \bar{\nabla} \cdot \bar{u} = 0$$

Inserting into the first equation of Eq. (1.9): (assume for the moment $f = 0$)

$$\rho U(\frac{1}{T}\partial_{\bar{t}}\bar{u} + \frac{U}{L}\bar{u} \cdot \bar{\nabla}\bar{u}) - \frac{\mu}{L^2}U\bar{\Delta}\bar{u} + \frac{1}{L}\bar{\nabla}p = 0$$

Set $\frac{U}{L} = \frac{1}{T}$, divide by $\frac{\rho U^2}{L}$:

$$\partial_{\bar{t}}\bar{u} + \bar{u} \cdot \bar{\nabla}\bar{u} - \frac{\mu}{\rho}\frac{1}{LU}\bar{\Delta}\bar{u} - \frac{1}{\rho U^2}\bar{\nabla}p = 0$$

Define $\nu = \frac{\mu}{\rho}$ the kinematic viscosity, Re:= $\frac{LU}{\nu}$ the Reynolds number. Following what we have done, Re has to be dimensionless. Let's check:

$$[\mathrm{Re}] = \frac{\frac{m^2}{s}}{\frac{m^2}{s}} = 1$$

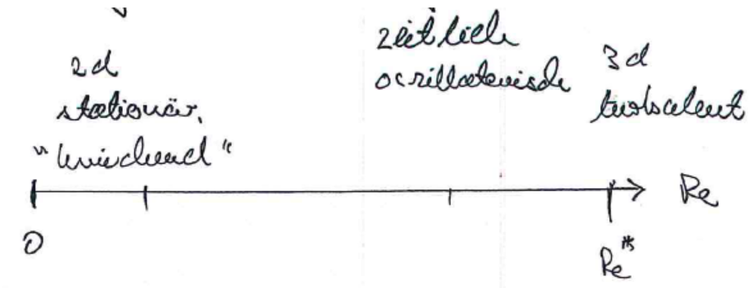
(note $N = \frac{kg \cdot m}{s^2}$)
Last, the pressure term: set $P := \rho U^2$ and $\bar{p} := \frac{1}{P} = \frac{1}{\rho U^2}p$. Then:

$$\begin{cases} \partial_{\bar{t}}\bar{u} + \bar{u} \cdot \bar{\nabla}\bar{u} - \frac{1}{\mathrm{Re}}\bar{\Delta}\bar{u} + \bar{\nabla}\bar{p} = 0 \\ \bar{\nabla} \cdot \bar{u} = 0 \end{cases} \tag{1.10}$$

Hereafter, since most of the time we are going to work with the non–dim quantities, "-" is dropped for better readability.
Non dimensionalization of right hand side f: appropriate scaling very much dependent on physical situation.

Important observation:

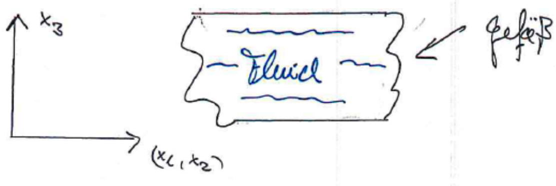
Theorem 1.7 (Similarity Theorem)**.**
Given 2 (real) flows. If the 2 physical settings lead to non dimensional systems with the same Reynolds no. Re and admit the same data (domain, right–hand side, initial/boundary conditions etc.), then the flows are **equivalent***, i.e. after a corresponding transformation of t, x, u, p they are the same.*



$Re^* \approx 10^3 - 10^4$

1.3 Reynolds no. for some flows

Flow	Re
floating microorganisms	$10^{-5} - 10^{-2}$
dropping of honey	$\ll 1$
blood flow in the aorta	ca. 10^3
stirring coffee in a mug	ca. $10^3 - 10^4$
car at 50 km/h	ca. 10^6
creek	10^6
cyclone	ca. 10^{11}



1.4 Some examples

a) Hydrostatic $\overset{\text{def.}}{\Leftrightarrow} u \equiv 0$

right hand side f = gravity $= -\rho g \vec{e}_3, \quad g = 9.81 \frac{m}{s^2}$.

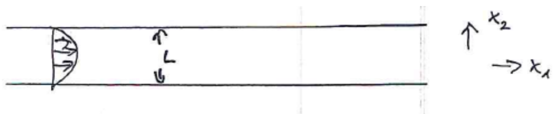
Since $u \equiv 0$ non–dimenzionalization of Eq. $(1.10)_1$ does not really make sense here. So let's stay with the equations in physical units:

$$\nabla p = -\rho g \vec{e}_3$$

$$p = C - \rho g x_3$$

independent of the shape of the vessel - **hydrostatic pressure**.

b) Hagen-Poiseuille flow



long 2d pipe with diameter L.

Assumption:

$$f = 0, \quad u = \begin{pmatrix} u_1(x_2) \\ 0 \end{pmatrix}$$

Then (1.5) is fulfilled automatically. Inserting into $(1.10)_1$:

$$\partial_t u = 0, \quad u \cdot \nabla u = u_1 \partial_{x_1} \begin{pmatrix} u_1(x_2) \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow -\frac{1}{\text{Re}} \begin{pmatrix} \Delta u_1(x_2) \\ 0 \end{pmatrix} + \nabla p = 0$$

$$\Rightarrow \partial_{x_2} p = 0 \Rightarrow p = p(x_1)$$

So that:

$$-\frac{1}{\text{Re}} u_1''(x_2) + \partial_{x_1} p(x_1) = 0$$

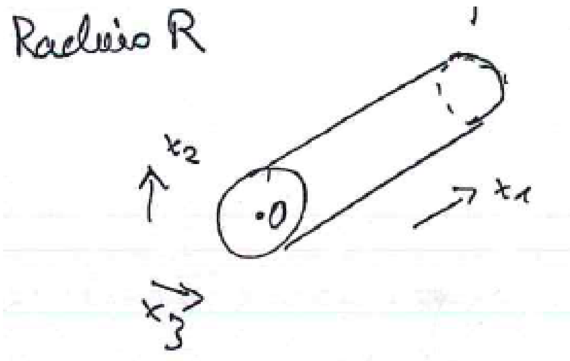
for all $(x_1, x_2) \in$ pipe. Now, since p does not depend on x_2, $\partial_{x_1} p(x_1) \equiv \lambda \equiv$ const.:

$$u_1''(x_2) = \text{Re}\ \lambda$$

With boundary condition ("no–slip"): $u_1(0) = u_1(L) = 0$

$$\text{(1.11)} \qquad \Rightarrow \quad \left.\begin{aligned} u_1(x_2) &= -\tfrac{\mathrm{Re}\,\lambda}{2} x_2(1-x_2) \\ p(x_1) &= C + \lambda x_1 \end{aligned}\right\}$$

3d–pipe, circular cross section:



Try: $u = (u_1(x_2,x_3),0,0)^T$ same computation as above yields:

$$\Delta_{(x_2,x_3)} u_1 = \lambda\ \mathrm{Re}$$

Let us try a rotationally symmetric form (will be justified a posteriori):

$$u_1(x_2,x_3) = u_1(r), \quad r = \sqrt{x_2^2 + x_3^2}, \quad \Delta u_1 = \frac{1}{r}\partial_r(r\partial_r u_1)$$

Then:

$$\frac{1}{\mathrm{Re}}\Delta u_1 = \lambda \ \Leftrightarrow \ \frac{1}{r}\partial_r(r\partial_r u_1) = \lambda\ \mathrm{Re}$$

$$\Rightarrow \partial_r(r\partial_r u_1) = \lambda\ \mathrm{Re}\ r \quad \Rightarrow \quad r\partial_r u_1 = \frac{\lambda\ \mathrm{Re}}{2} r^2 + C_1$$

$$\Rightarrow \partial_r u_1 = \frac{\lambda\ \mathrm{Re}}{2} r + \underbrace{\frac{C_1}{r}}_{\text{drop}} \quad \Rightarrow \quad u_1 = \frac{\lambda\ \mathrm{Re}}{4} r^2 + C_2$$

No–slip condition at $r = 1$: $u_1(1) = 0$

$$\text{(1.12)} \qquad \left.\begin{aligned} u_1 &= -\tfrac{\lambda\ \mathrm{Re}}{4}(1-r^2) \\ p &= C + \lambda x_1 \end{aligned}\right\}$$

In dimensional form:

$$\begin{aligned} v := u_{1,\ \mathrm{phys}} &= -\frac{U\lambda}{4}\mathrm{Re}\ \frac{R^2 - \tilde r^2}{R^2} \\ &= -\frac{\lambda}{4}\frac{U^2 R}{\nu}\frac{R^2 - \tilde r^2}{R^2} = \frac{\nabla_{\mathrm{phys}} q (\tilde r^2 - R^2)}{4\mu} \end{aligned}$$

Where we have used

$$\tilde r = Rr, \quad L = R, \quad q := p_{\mathrm{phys}} = Pp = U^2\rho p \quad \Rightarrow \quad \lambda\rho\frac{U^2}{R} = \nabla_{\mathrm{phys}} q \quad \Rightarrow \quad \lambda = \frac{R}{\rho U^2}\nabla_{\mathrm{phys}} q$$

If l is the length of the pipe, then

$$v = \frac{\delta p}{4\mu l}(R^2 - \tilde{r}^2) \tag{1.13}$$

with $\delta p =$ "pressure drop" $=$ pressure(inflow) - pressure(outflow)
We also see that for a given pressure gradient λ, the velocity behaves like $\frac{1}{\mu}R^2$.

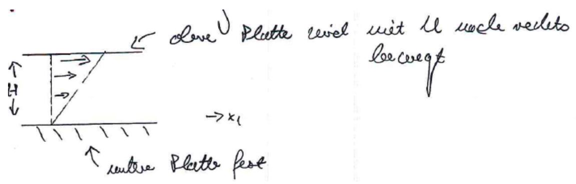
Q: how big is the throughput? **A: exercise.**

Remarks:

i) Flow given by Eq. (1.12) is a solution of the Navier–Stokes equations for all $\mathrm{Re} > 0$. However, for large $\mathrm{Re} = \mathrm{O}(10^4)$ this solution becomes instable. The flow, which can be observed in reality, is then turbulent.

ii) The solution of the above exercise relates the total flow (throughput) $Q = \delta p \cdot f(\mu)$, to the viscosity μ. This provides a means to experimentally access the viscosity.

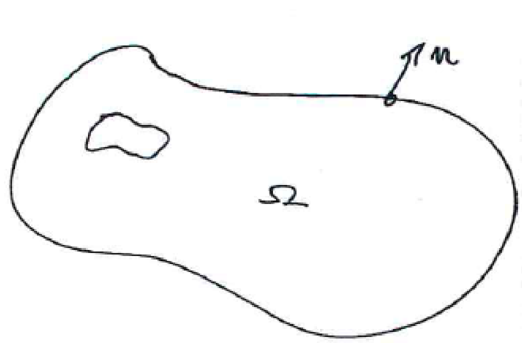
c) Shear flow



Try: $u = (u_1(x_2), 0, 0)$, $\nabla p = 0$, $u_1(1) = 1, u_1(0) = 0$
Inserting into (1.10) yields

$$-u_1''(x_2) = 0 \quad \Rightarrow \quad u_1(x_2) = x_2$$

1.5 Mathematically “sensible” problem formulation



$\Omega \subset I\!R^d$ domain, i.e. Ω open, connected (bounded, smooth). $(0,T)$, $T > 0$, time interval (possibly $T = \infty$), $Q_T := (0,T) \times \Omega$, $\Gamma_T := (0,T) \times \partial\Omega$

i) Instationary Navier–Stokes equations: find (u,p) fulfilling

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p &= f \quad \text{in } Q_T \\ \nabla \cdot u &= 0 \quad \text{in } Q_T \\ \text{Initial condition:} \qquad u(0,\cdot) &= u_0 \quad \text{in } \Omega \\ \text{Dirichlet boundary condition:} \qquad u &= u_D \quad \text{on } \Gamma_T \end{aligned} \tag{1.14}$$

Note: no initial, no boundary condition for p!

ii) Instationary Stokes equations: find (u,p) fulfilling

$$\begin{aligned} \partial_t u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p &= f \quad \text{in } Q_T \\ \nabla \cdot u &= 0 \quad \text{in } Q_T \\ u(0,\cdot) &= u_0 \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \Gamma_T \end{aligned} \tag{1.15}$$

iii) Stationary Navier–Stokes equations: find (u,p) fulfilling

$$\begin{aligned} u \cdot \nabla u - \frac{1}{\mathrm{Re}} \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \partial\Omega \end{aligned} \tag{1.16}$$

iv) Stationary Stokes equations: find (u,p) fulfilling

$$\begin{aligned} -\frac{1}{\mathrm{Re}} \Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \partial\Omega \end{aligned} \tag{1.17}$$

Remarks:

i) Necessary condition for u_D:

$$\int\limits_{\partial\Omega} u_D \cdot n = 0 \qquad\qquad n = \text{outward pointing normal to } \partial\Omega$$

"What flows in, must flow out". Reasoning:

$$\int\limits_{\partial\Omega} u_D \cdot n = \int\limits_{\partial\Omega} u \cdot n = \int\limits_{\Omega} \nabla \cdot u = \int\limits_{\Omega} 0 = 0$$

ii) pressure p only determined up to an additive constant.

2 Existence, uniqueness and regularity of solutions

2.1 The stationary Stokes equations

$$(1.17)\begin{cases} -\Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u &= u_D \quad \text{on } \partial\Omega \end{cases}$$

For simplicity, assume for the moment $u_D \equiv 0$

Goal: Derive a *weak* formulation of the problem. To this end, multiply $(1.17)_1$ by $\varphi \in C_0^\infty(\Omega)^d$, div $\varphi \equiv 0$ and integrate over Ω:

$$\int_\Omega -\Delta u \cdot \varphi + \int_\Omega \nabla p \cdot \varphi = \int_\Omega f \cdot \varphi$$

Integration by parts: (repeated indices are summed up over)

$$\begin{aligned} \int_\Omega f_i \varphi_i &= \int_\Omega -\partial_{jj} u_i \varphi_i \; dx + \int_\Omega \partial_i p \varphi_i \\ &= \int_\Omega \partial_j u_i \partial_j \varphi_i \; dx - \int_\Omega p \partial_i \varphi_i \\ &= \int_\Omega \nabla u : \nabla \varphi \; dx - \int_\Omega p \; \mathrm{div}(\varphi) \; dx = \int_\Omega \nabla u : \nabla \varphi \; dx. \end{aligned}$$

Suggests the following choice of trial and test functions:

$$\begin{aligned} X &:= \mathring{H}^{1,2}(\Omega)^d \\ V &:= \{v \in X \mid \mathrm{div} \; v = 0\} \end{aligned}$$

with the norm

$$\|v\|_X := \|\nabla v\|_{L^2(\Omega)} =: \|\nabla v\|$$

If Ω is open, bounded and has Lipschitz boundary then

$$V = \overline{\{v \in C_0^\infty(\Omega)^d \mid \mathrm{div} \;\; v = 0\}}^{\|\cdot\|_X}$$

Thanks to Poincaré's inequality the norm $\|\cdot\|_X$ is equivalent to $\|\cdot\|_{H^{1,2}}$ on X. Moreover, V is a **closed** subspace of X.

Inner product on X:

$$(v, w)_X := \int_\Omega \nabla v : \nabla w \; dx.$$

Let $V^\perp$ be the orthogonal complement of V in X w.r.t. $(\cdot, \cdot)_X$:

$$\begin{aligned} V^\perp &:= \{v \in X \mid (v, w)_X = 0 \; \forall w \in V\} \\ &= \{v \in X \mid \int_\Omega \nabla v : \nabla w \; dx = 0 \; \forall w \in X \text{ with } \mathrm{div}(w) \equiv 0\} \end{aligned}$$

General result from functional analysis: $X = V \perp V^\perp$

Define the **bilinear form** $a : X \times X \to I\!R$

$$a(v,w) := (v,w)_X = \int_\Omega \nabla v : \nabla w \; dx$$

and the **linear form** $l : X \to I\!R$

$$\langle l, v\rangle := \int_\Omega f v \; dx.$$

Definition 2.1 (Stokes problem, weak formulation, 1st version).

$f \in L^2(\Omega)$ (or, more generally $l \in V'$[3])

Find $u \in V$ fulfilling

$$a(u,v) = \langle l, v\rangle \quad \forall v \in V \tag{2.1}$$

Lemma 2.2.

Problem (2.1) admits a unique solution.

Proof:

$a(\cdot,\cdot)$ is **continuous** (i.e. $|a(v,w)| \le C\|v\|_X \cdot \|w\|_X$, here: $C = 1$) and **coercive**, that is ($a(v,v) \ge \alpha\|v\|_X^2$, here: $\alpha = 1$) Then use Lax-Milgram. □

Problem: Where is the pressure p?

Stokes problem, weak formulation 2nd version:

$$Y := L_0^2(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q(x)\; dx = 0\}.$$

Note: for $v \in X$ and $q \in Y \cap C^\infty(\Omega)$ we have

$$\int_\Omega \nabla q(x) \cdot v(x)\; dx = -\int_\Omega q(x) \cdot \operatorname{div}(v(x))\; dx.$$

Define the bilinear form $b : Y \times X \to I\!R$

$$b(q,v) := -\int_\Omega q(x)\; \operatorname{div}(v(x))\; dx.$$

Definition 2.3 (Stokes problem, 2nd version: mixed formulation).

Let $l \in X'$ be given. Find $(u,p) \in X \times Y$ with

$$\begin{cases} a(u,v) + b(p,v) = \langle l, v\rangle & \forall v \in X \\ b(q,u) = 0 & \forall q \in Y \end{cases} \tag{2.2}$$

[3] $V' := \{l : V \to I\!R \mid l \text{ linear, continuous}\}$

Clear: (u,p) solution of (2.2) $\Rightarrow u$ solution of Eq. (2.1) (since $b(q,u) = 0\ \forall q \in Y \Rightarrow$ (by definition) $u \in V$)

Other direction: next section.

Why $L_0^2(\Omega)$? p only determined up to an additive constant by Eq. (1.17) Condition $\int_\Omega p\ dx = 0$ fixes this degree of freedom (see below).

2.2 Theory of saddle point problems

Further reading:

- W. Hackbusch - *Elliptic Differential Equations - Theory and Numerical Treatment* (also available in German)
- D. Braess - *Finite Elements* (also in German)
- Quarteroni/Valli - *Numerical approximation of partial differential equations*
- Brezzi/Fortin - *Mixed and hybrid finite element methods*

General setting: X, Y Hilbert spaces

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}$$
$$b(\cdot,\cdot) : Y \times X \to \mathbb{R}$$

$a(\cdot,\cdot)$, $b(\cdot,\cdot)$ continuous, i.e. there exist $C_1, C_2 \geq 0$ with

$$|a(v,w)| \leq C_1 \|v\|_X \|w\|_X \quad \forall v, w \in X$$
$$|b(q,v)| \leq C_2 \|q\|_Y \|v\|_X \quad \forall v \in X,\ q \in Y$$

Let $f \in X'$ and $g \in Y'$ be given. We consider the following **saddle point problem**: find $(u,p) \in X \times Y$ fulfilling

$$\left\{\begin{array}{ll} a(u,v) + b(p,v) = \langle f, v\rangle & \forall v \in X \\ b(q,u) = \langle g, q\rangle & \forall q \in Y \end{array}\right. \tag{2.3}$$

Theorem 2.4.
Let $a(\cdot,\cdot)$ be symmetric and coercive. Then: $(u,p) \in X \times Y$ is solution of (2.3) $\Leftrightarrow$

$$J(u,q) \leq J(u,p) \leq J(v,p) \qquad \forall v \in X, q \in Y, \tag{2.4}$$

that is to say, (u,p) is a **saddle point** *of J, where*

$$J(v,q) := \frac{1}{2} a(v,v) + b(q,v) - \langle f,v\rangle - \langle g,q\rangle$$

$$(2.4) \Leftrightarrow J(u,p) = \min_{v\in X} J(v,p) = \max_{q\in Y} \min_{v \in X} J(v,q)$$

Proof: elementary but lengthy.

Operator notation: by the Lax–Milgram theorem there are linear, bounded operators A, B

$$A : X \to X'$$
$$B : Y \to X'$$

defined by

$$\langle Av, \varphi\rangle_{X'\times X} = a(v,\varphi) \qquad \forall v, \varphi \in X$$
$$\langle Bq, \varphi\rangle_{X'\times X} = b(q,\varphi) \qquad \forall q \in Y, \varphi \in X$$

As for any bounded operator, for B there exists an **adjoint** $B^* : X \to Y'$, that is

$$\langle q, B^*v\rangle_{Y\times Y'} = b(q,v) = \langle Bq, v\rangle_{X'\times X} \qquad \forall v \in X, q \in Y.$$

Using this notation, Eq. (2.3) can equivalently be written: Find $(u,p) \in X \times Y$ fulfilling

$$\left\{\begin{aligned} Au + Bp &= f \quad &&(\text{in } X') \\ B^*u &= g &&(\text{in } Y') \end{aligned}\right. \tag{2.5}$$

If A is invertible, i.e. $A^{-1} \in L(X', X)$, then from Eq. $(2.5)_1$

$$u = A^{-1}(f - Bp)$$

and from Eq. (2.6_2):

$$g = B^*u = B^*A^{-1}(f - Bp) = B^*A^{-1}f - \underbrace{B^*A^{-1}B}_{=:S}\,p$$

(This is 1 step of Gaußian elimination of the 2x2 operator system (2.5))
We find:

$$\left\{\begin{aligned} Sp &= B^*A^{-1}f - g \\ u &= A^{-1}(f - Bp) \end{aligned}\right. \tag{2.6}$$

$S := B^*A^{-1}B$ is called **Schur complement operator**

$$S = B^*A^{-1}B : Y \to Y'.$$

Clear:

$$(u,p) \text{ solution of Eq. (2.5)} \Rightarrow (u,p) \text{ solution of Eq. (2.6)}$$

Likewise:

$$(u,p) \text{ solution of Eq. (2.6)} \Rightarrow (u,p) \text{ solution of Eq. (2.5)}$$

Eq. (2.5) (or equivalently Eq. (2.6)) uniquely solvable for all $f \in X', g \in Y' \Leftrightarrow S^{-1} \in L(Y', Y)$

Warning: B not bijective in general

Example: in $\mathbb{R}^d$

$$A := \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathrm{Gl}(3)$$

$$A^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^4, \quad B^* = (1,1)$$

$$S = (1,1) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1,1) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 5 \in \ \mathrm{Gl}(1)$$

Let us introduce $V_0 := V = \ker B^* = \{v \in X \mid b(q,v) = 0 \ \forall q \in Y\}$.
$V_0 \subset X$ closed subspace.

$$\Rightarrow X = V_0 \perp V_\perp \text{ with } V_\perp = (V_0)^\perp = \{v \in X \mid (v,\varphi) = 0 \ \forall \varphi \in V_0\}.$$

The decomposition $X = V_0 \perp V_\perp$ induces a corresponding decomposition in the dual space.

[4] Trick: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\det = ad - bc$

Lemma 2.5.

It holds

1) $X' = V_0' \oplus V_\perp'$ *where*

$$V_0' := \{v' \in X' \mid \langle v', v\rangle = 0 \ \forall v \in V_\perp\}$$
$$V_\perp' := \{v' \in X' \mid \langle v', v\rangle = 0 \ \forall v \in V_0\}$$

2) *the Riesz isomorphism* $j : X \to X'$ *(defined by* $\langle j(v), \varphi\rangle = (v, \varphi)_X$*) maps*

$$j : V_0 \to V_0'$$
$$j : V_\perp \to V_\perp'$$

onto.

3) $V_0' \perp V_\perp'$ *with respect to the inner product*

$$(v', \varphi')_{X'} := (j^{-1}(v'), j^{-1}(\varphi'))_X \quad \forall v', \varphi' \in X'$$

4) $\|v'\|^2_{X'} = \|v_0'\|^2_{X'} + \|v_\perp'\|^2_{X'}$ *where* $v' = \underbrace{v_0'}_{\in V_0'} + \underbrace{v_\perp'}_{\in V_\perp'}$

Proof:

1) Let $v' \in X'$. For any $v \in X$ we have the decomposition $v = \underbrace{v_0}_{\in V_0} + \underbrace{v_\perp}_{\in V_\perp}$.

Define:

$$\langle v_0', v\rangle := \langle v', v_0\rangle$$
$$\langle v_\perp', v\rangle := \langle v', v_\perp\rangle$$

Since $\|v\|^2_X = \|v_0\|^2_X + \|v_\perp\|^2_X$ it follows

$$v_0', \ v_\perp' \in X'.$$

Moreover: $\langle v_0', v\rangle = 0 \ \forall v \in V_\perp \Rightarrow v_0' \in V_0'$. Likewise: $v_\perp' \in V_\perp'$.

Furthermore:

$$\langle v_0' + v_\perp', v\rangle = \langle v', v_0\rangle + \langle v', v_\perp\rangle = \langle v', v\rangle$$

saying that $v_0' + v_\perp' = v'$. Remains to show: the sum of spaces is a direct sum.

To this end, let $v' \in V_0' \cap V_\perp'$

$$\Rightarrow \langle v', v\rangle = \underbrace{\langle v', v_0\rangle}_{=0} + \underbrace{\langle v', v_\perp\rangle}_{=0} = 0 \quad \text{for all } v \in V$$

$\Rightarrow v' = 0$ in X'.

Rest: left to the reader. $\square$

This decomposition also induces a decomposition of operators A, B. Let $v = \underbrace{v_0}_{\in V_0} + \underbrace{v_\perp}_{\in V_\perp} \in X$.

$$Av =: v' = \underbrace{v'_0}_{\in V'_0} + \underbrace{v'_\perp}_{\in V'_\perp}$$

Now define operators $A_{00}, A_{0\perp}, A_{\perp 0}, A_{\perp\perp}$ by:

$$\begin{aligned} A_{00} &: V_0 \to V'_0 \\ A_{00} v_0 &:= \tilde{v}'_0 \qquad Av_0 = \tilde{v}'_0 + \tilde{v}'_\perp \end{aligned}$$

$$\begin{aligned} A_{0\perp} &: V_\perp \to V'_0 \\ A_{0\perp} v_\perp &:= \hat{v}'_0 \qquad Av_\perp = \hat{v}'_0 + \hat{v}'_\perp \end{aligned}$$

$$\begin{aligned} A_{\perp 0} &: V_0 \to V'_\perp \\ A_{\perp 0} v_0 &:= \tilde{v}'_\perp \qquad Av_0 = \tilde{v}'_0 + \tilde{v}'_\perp \end{aligned}$$

$$\begin{aligned} A_{\perp\perp} &: V_\perp \to V'_\perp \\ A_{\perp\perp} v_\perp &:= \hat{v}'_\perp \qquad Av_\perp = \hat{v}'_0 + \hat{v}'_\perp \end{aligned}$$

Moreover, for $X \ni v = v_0 + v_\perp$ and

$$X' \ni v' = v'_0 + v'_\perp = Av$$

we have:

$$v' = v'_0 + v'_\perp = \underbrace{(A_{00} v_0 + A_{0\perp} v_\perp)}_{\in V'_0} + \underbrace{(A_{\perp 0} v_0 + A_{\perp\perp} v_\perp)}_{\in V'_\perp}$$

or in compact notation:

$$\begin{array}{c} V'_0 \ni \\ V'_\perp \ni \end{array} \begin{bmatrix} v'_0 \\ v'_\perp \end{bmatrix} = \begin{bmatrix} A_{00} & A_{0\perp} \\ A_{\perp 0} & A_{\perp\perp} \end{bmatrix} \begin{bmatrix} v_0 \\ v_\perp \end{bmatrix}$$

Since $B^* : V_0 \perp V_\perp \to Y'$ satisfies $B^*_{|V_0} = 0$, operator B^* can be decomposed as

$$(B^*_0, B^*_\perp) = (0, B^*_\perp).$$

Now, operator B: let $q \in Y$, then we have $Bq \in V'_\perp$, because

$$\langle Bq, \varphi \rangle_{X' \times X} = \langle q, B^* \varphi \rangle = \langle q, 0 \rangle = 0$$

for $\varphi \in \ker(B^*) = V_0$, that is $\mathrm{im}(B) \subset V'_\perp$ by definition of $V'_\perp$ (see p. 23).

This means, B can be written as

$$B = \begin{pmatrix} 0 \\ B \end{pmatrix} \begin{array}{l} \in V'_0 \\ \in V'_\perp \end{array}.$$

Using these decompositions Eq. (2.5) admits a very intuitive splitting:

$$(2.7) \qquad \begin{cases} A_{00}u_0 + A_{0\perp}u_\perp = f_0 & \in V_0' \\ A_{\perp 0}u_0 + A_{\perp\perp}u_\perp + Bp = f_\perp & \in V_\perp' \\ B^* u_\perp = g & \in Y' \end{cases}$$

or in an operator*vector notation

$$(2.8) \qquad \begin{bmatrix} A_{00} & A_{0\perp} & 0 \\ A_{\perp 0} & A_{\perp\perp} & B \\ 0 & B^* & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_\perp \\ p \end{bmatrix} = \begin{bmatrix} f_0 \\ f_\perp \\ g \end{bmatrix}$$

Now it is easy to characterize the solvability of the saddle point problem.

Theorem 2.6.

Problem (2.5) is uniquely solvable for all $f \in X', g \in Y'$, iff

A_{00} is invertible : A_{00}^{-1} exists, $A_{00}^{-1} \in L(V_0', V_0)$ and B is invertible in the sense: $B^{-1} \in L(V_\perp', Y)$

Proof:

" $\Leftarrow$ " $(B^*)^{-1} = (B^{-1})^* \in L(Y', V_\perp)$.

First solve for $u_\perp$:

$$B^* u_\perp = g$$

Then for u_0:

$$A_{00}u_0 = f_0 - A_{0\perp}u_\perp$$

Finally get p by:

$$Bp = f_\perp - A_{\perp 0}u_0 - A_{\perp\perp}u_\perp$$

" $\Rightarrow$ " Choose $f \in X'$ with $f_\perp = 0$ and $f_0 \in V_0'$ arbitrary, $g = 0$.

By assumption, there exists $(u_0, u_\perp, p)$ solution of the system. From Eq. (2.7):

$$B^* u_\perp = 0 \Rightarrow u_\perp \in V_0 \Rightarrow u_\perp = 0$$

Then the first "row" in Eq. (2.7) reads:

$$A_{00}u_0 = f_0.$$

Since $f_0 \in V_0'$ is arbitrary, we conclude A_{00} is invertible and then (by the principle of uniform boundedness)

$$A_{00}^{-1} \in L(V_0', V_0).$$

Next: $f = 0, g \in Y'$ arbitrary.

Same argument as above: $(B^*)^{-1} \in L(Y', V_\perp)$

Again, $B^{-1} \in L(V_\perp', Y)$

$\square$

Characterizing invertibility:

Theorem 2.7.
Let U, W be Hilbert spaces

$$t : U \times W \to \mathbb{R} \quad \textit{bilinear, continuous}$$

Define $T : U \to W'$ by

$$\langle Tx, y\rangle_{W' \times W} = t(x, y) \quad \forall x \in U, y \in W$$

The following statements are equivalent:

i) T^{-1} *exists and* $T^{-1} \in L(W', U)$

ii) there exists $\alpha > 0$, s.t. for all $x \in U$:

$$\sup_{0 \neq y \in W} \frac{|t(x, y)|}{\|y\|_W} \geq \alpha \|x\|_U$$

and there exists also $\alpha' > 0$, s.t. for all $y \in W$:

$$\sup_{0 \neq x \in U} \frac{|t(x, y)|}{\|x\|_U} \geq \alpha' \|y\|_W$$

iii) There exists $\alpha > 0$ fulfilling

$$\sup_{0 \neq y \in W} \frac{|t(x, y)|}{\|y\|_W} \geq \alpha \|x\|_U$$

and

$$\sup_{0 \neq x \in U} \frac{|t(x, y)|}{\|x\|_U} > 0$$

for all $y \neq 0$.

Proof:

i)$\Rightarrow$ ii) Let $x \in U$. We may assume $x \neq 0$

$$\sup_y \frac{|t(x,y)|}{\|y\|_W} = \sup_y \frac{|\langle Tx, y\rangle|}{\|y\|_W} = \sup_y \frac{|\langle TT^{-1}x', y\rangle|}{\|y\|_W} = \sup_y \frac{|\langle x', y\rangle|}{\|y\|_W}$$

$$= \|x'\|_{W'} \overset{\substack{x=T^{-1}x' \\ x' \in W'}}{=} \|Tx\|_{W'} = \frac{\|x\|_U}{\frac{\|x\|_U}{\|Tx\|_{W'}}}$$

$$\geq \|x\|_U \inf_{0 \neq \hat{x} \in U} \Big(\frac{\|\hat{x}\|_U}{\|T\hat{x}\|_{W'}} \Big)^{-1}$$

$$\overset{\substack{\hat{x}=T^{-1}\hat{x}' \\ \hat{x}' \in W'}}{=} \|x\|_U \inf_{0 \neq \hat{x}' \in W'} \Big(\frac{\|T^{-1}\hat{x}'\|_U}{\|\hat{x}'\|_{W'}} \Big)^{-1}$$

$$= \|x\|_U \Big(\sup_{\hat{x}'} \frac{\|T^{-1}\hat{x}'\|_U}{\|\hat{x}'\|_{W'}} \Big)^{-1}$$

$$= \|x\|_U \frac{1}{\|T^{-1}\|_{L(W',U)}}$$

This means iia) is fulfilled with $\alpha = \frac{1}{\|T^{-1}\|}$.

Same arguments applied to T^* show iib) with

$$\alpha' = \frac{1}{\|(T^*)^{-1}\|} = \frac{1}{\|T^{-1}\|} = \alpha$$

Note: from functional analysis we know that T invertible $\Leftrightarrow$ T^* invertible.

ii) $\Rightarrow$ iii) ✓

iii)$\Rightarrow$ i) First show: $\mathrm{im}(T) \subset W'$ is a **closed** subspace.

To this end, let $y'_k \in \mathrm{im}(T)$ a sequence with $y'_k \overset{k\to\infty}{\longrightarrow} y'$ in W'. This means that there exist $x_k \in U$ with $y'_k = Tx_k$. Since it is convergent, $(y'_k)_{k\in\mathbb{N}}$ is also a Cauchy sequence. Now:

$$\alpha\|x_k - x_m\|_U \overset{iiia)}{\leq} \sup_{0\neq z \in W} \frac{|t(x_k - x_m, z)|}{\|z\|_W} = \sup_z \frac{\langle T(x_k - x_m), z\rangle}{\|z\|_W}$$
$$= \|T(x_k - x_m)\|_{W'} = \|y'_k - y'_m\|_{W'} \overset{k,m\to\infty}{\longrightarrow} 0$$

We have thus proven that $(x_k)_{k\in\mathbb{N}}$ is also a Cauchy sequence. Therefore, there exists $x \in U$ with

$$x_k \overset{k\to\infty}{\longrightarrow} x \ \text{ in } U$$

Thanks to the continuity of T:

$$Tx \overset{k\to\infty}{\longleftarrow} Tx_k = y'_k \overset{k\to\infty}{\longrightarrow} y'$$

Thus: $Tx = y'$, i.e. $y' \in \text{im}(T)$.

This shows that $\text{im}(T)$ is closed.

Next: let us assume that T is not surjective, i.e. $\text{im}(T) \neq W'$. Then there is $z_0' \in \text{im}(T)^\perp$, $z_0' \neq 0$.

Let $z_0 := j_W^{-1} z_0' \in W$ with $j_W : W \to W'$ the Riesz isomorphism.

Use iiib) with $y = z_0$:

$$0 < \sup_{0 \neq x \in U} \frac{|t(x, z_0)|}{\|x\|_U} = \sup_x \frac{|\langle Tx, z_0 \rangle|}{\|x\|_U} \overset{\text{Def. } (\cdot,\cdot)_{W'}}{=} \sup_x \frac{|(Tx, j_W z_0)_{W'}|}{\|x\|_U}$$

$$= \sup_x \frac{|(Tx, z_0')_{W'}|}{\|x\|_U} = 0 \quad \lightning$$

We therefore conclude that T is surjective.

T injective: follows from iiia)

We have thus shown that T is bijective $\Rightarrow T^{-1}$ exists and $\Rightarrow T^{-1} \in L(W', U)$

$\square$

Corollary 2.8.

The saddle point problem Eq. (2.3) *(or equivalently Eq.* (2.5)*) admits a unique solution* (u, p) *for all* $f \in X', q \in Y'$*, iff there exist* $\alpha, \beta > 0$ *such that*

i) $\sup_{0 \neq x_0 \in V_0} \frac{|a(v_0, x_0)|}{\|x_0\|_X} \geq \alpha \|v_0\|_X \quad \forall v_0 \in V_0$

$\sup_{0 \neq v_0 \in V_0} \frac{|a(v_0, x_0)|}{\|v_0\|_X} > 0 \quad \forall 0 \neq x_0 \in V_0$

ii) The Ladyzenskaja–Babuska–Brezzi condition (LBB) holds:

$$\sup_{0 \neq v \in X} \frac{|b(q, v)|}{\|v\|_X} \geq \beta \|q\|_Y \quad \forall q \in Y$$

Proof:

Use Theorem 2.6 to show

$$i) \overset{\text{Thm. 2.7, } U=W=V_0}{\Leftrightarrow} A_{00}^{-1} \in L(V_0', V_0), \qquad B^{-1} \in L(V_\perp', Y) \overset{\text{Thm. 2.7, } U=Y,\ W=V_\perp}{\Rightarrow} ii)$$

Remains to show:

$$ii) \Rightarrow B^{-1} \in L(V_\perp', Y)$$

As in the proof of the above theorem (iii) $\Rightarrow$ i)): $\text{im}(B) \subset V_\perp'$ closed.

We have:

$$V_0' = j_X(V_0) = j_X(\ker(B^*)) \overset{\text{FA}}{=} \text{im}(B)^\perp$$

Since $\text{im}(B)$ closed, we infer:

$$\text{im}(B) = (V_0')^\perp = V_\perp'$$

Injectivity clear by ii).

Therefore: B bijective, continuous $\Rightarrow B^{-1} \in L(V_\perp', Y)$

$\square$

Remark:

i) Second condition in Corollary 2.8 i) superfluous, if $a(\cdot,\cdot)$ symmetric.

ii) Sufficient condition for i) in Corollary 2.8: $a(\cdot,\cdot)$ symmetric and coercive on V_0. In particular, if $a(\cdot,\cdot)$ symmetric and coercive on X.

2.3 Application to the stationary Stokes problem

Recall:

- $X = \mathring{H}^{1,2}(\Omega)^d$, $Y = L_0^2(\Omega)$
- $a(u,v) := \int\limits_\Omega \nabla u(x) : \nabla v(x)\; dx$ symmetric, coercive on X
- $b(q,v) := -\int\limits_\Omega q(x) \cdot \operatorname{div}(v(x))\; dx$
- $\langle Bq, v\rangle = b(q,v) = \langle q, B^* v\rangle$

To show unique solvability of the stationary Stokes problem, by Corollary 2.8 the only thing that remains to be shown is the LBB condition.
Note: now one can see why we need to define the pressure space Y as $Y = L_0^2(\Omega)$: LBB implies that B is injective. Taking a constant function, for instance $q \equiv 1$, would give, however, $Bq = \nabla q = \nabla 1 = 0$.
Rewriting the LBB condition:
For $q \in L_0^2(\Omega)$ define $\nabla q \in H^{-1}$ by

$$\langle \nabla q, v\rangle_{X' \times X} := -\int\limits_\Omega q(x) \cdot \operatorname{div}(v(x))\; dx$$

where $H^{-1} := \left(\mathring{H}^{1,2}(\Omega)^d\right)' = X'$ and

$$\begin{aligned} \|\nabla q\|_{H^{-1}} &:= \sup_{0 \neq v \in X} \frac{|\langle \nabla q, v\rangle|}{\|\nabla v\|} \\ &\leq \sup_v \|q\| \cdot \frac{\left(\int\limits_\Omega |\operatorname{div}(v(x))|^2\; dx\right)^{\frac{1}{2}}}{\|\nabla v\|} \\ &\leq C\|q\| \end{aligned}$$

LBB:

$$\sup_{0 \neq v \in X} \frac{|\int\limits_\Omega q \cdot \operatorname{div}(v)\; dx|}{\|\nabla v\|} \geq \beta \|q\|$$

$$\Leftrightarrow \qquad \|\nabla q\|_{H^{-1}} \geq \beta\|q\|$$

Lemma 2.9.

Let $\beta > 0$. The following statements are equivalent:

i) $\|\nabla q\|_{H^{-1}} \geq \beta \|q\| \quad \forall q \in L^2_0(\Omega)$

ii) For $q \in L^2_0(\Omega)$ *there exists* $v \in \mathring{H}^{1,2}(\Omega)^d$ *with* $q = div(v)$ *and* $\|\nabla v\| \leq \frac{1}{\beta}\|q\|$.

Proof:

"**ii)⇒i)**" Let $q \neq 0$ be given. Choose v as in ii), $v \neq 0$. Define $\tilde{v} := \frac{v}{\|\nabla v\|}$.

Then:

$$\begin{aligned}\|\nabla q\|_{H^{-1}} = \sup_{0 \neq w \in X} \frac{|b(q,w)|}{\|\nabla w\|} &\geq |b(q, \tilde{v})| \\ &= |\int_\Omega q \cdot \operatorname{div}(\frac{v}{\|\nabla v\|})\ dx| = |\int_\Omega \frac{|q(x)|^2}{\|\nabla v\|}\ dx| \\ &\geq \beta \|q\|.\end{aligned}$$

"**i)⇒ii)**" i) is just the LBB condition for $b(\cdot,\cdot)$. By Thms. 2.6, 2.7 we infer

$$(B^*)^{-1} \in L(Y', V_\perp) \qquad \text{with} \qquad \|(B^*)^{-1}\| \leq \frac{1}{\beta}.$$

Set $v := (B^*)^{-1} q$ (where we identify $(L^2_0)'$ with L^2_0).
$\Rightarrow \operatorname{div}(v) = B^*(B^*)^{-1} q = q$ and

$$\|\nabla v\| = \|v\|_X \leq \|(B^*)^{-1}\| \cdot \|q\| \leq \frac{1}{\beta}\|q\|$$

□

Theorem 2.10 (Nečas)**.**

Let Ω have Lipschitz boundary, i.e. $\partial\Omega \in C^{0,1}$. Then LBB is fulfilled with some $\beta = \beta(\Omega) > 0$. Then clearly, the Stokes problem

$$\begin{aligned} a(u,v) + b(p,v) &= \langle f, v\rangle \quad \forall v \in X \\ b(q,u) &= \langle g, q\rangle \quad \forall q \in Y \end{aligned}$$

admits a unique solution $(u,p) \in X \times Y$ for $f \in X', g \in Y'$ and the following estimate holds:

$$\|\nabla u\| + \|p\| \leq C(\Omega)\Big[\|f\|_{H^{-1}} + \|g\|\Big]$$

Proof: Nečas, rather technical.

□

Let's come back to the relation between the 1st and 2nd weak formulation for Stokes.

Corollary 2.11.

Let $u \in V_0$ be the unique solution of Eq. (2.1).

Then there exists a uniquely determined pressure $p \in Y$, such that $(u, p) \in X \times Y$ solves (2.2).

Proof:

Let $l = f \in X'$ and $u \in V = V_0$ be solution of

$$a(u, v) = \langle l, v \rangle \quad \forall v \in V_0.$$

Define $\langle f_\perp, v \rangle := \langle l, v \rangle - a(u, v) \;\; \forall v \in X$

It follows: $f_\perp \in V_\perp'$, since $\langle f_\perp, v \rangle = 0 \;\; \forall v \in V_0$ (see Lemma 2.5)

Thanks to the LBB condition we have $B^{-1} \in L(V_\perp', Y)$. Thus there exists a unique p, $Y \ni p = B^{-1} f_\perp$, i.e. $Bp = f_\perp$, with

$$\begin{aligned} \langle Bp, v \rangle &= \langle f_\perp, v \rangle \quad && \forall v \in X \\ \Leftrightarrow b(p, v) &= \langle l, v \rangle - a(u, v) \quad && \forall v \in X \\ \Leftrightarrow a(u, v) + b(p, v) &= \langle l, v \rangle \quad && \forall v \in X \end{aligned}$$

□

Regularity: If f, g "more regular", we also expect the solution (u, p) to be "more regular". Indeed:

Theorem 2.12.

$k \in \mathbb{N}, k \geq 0, H^0 := L^2(\Omega)$.

Let $\partial\Omega \in C^{k+1,1}$. Then there is a $C > 0$, such that for $f \in H^{k,2}(\Omega)^d$, $g \in H^{k+1,2}(\Omega)$:

$$u \in X \cap H^{k+2,2}(\Omega)^d, p \in Y \cap H^{k+1,2}(\Omega)$$

and

$$\|u\|_{H^{k+2,2}} + \|p\|_{H^{k+1,2}} \leq C \Big[\|f\|_{H^{k,2}} + \|g\|_{H^{k+1,2}} \Big]$$

Note that the regularity w.r.t. f and g is optimal. To see this, take $u \in H^{k+2,2}(\Omega)^d$, $p \in H^{k+1,2}(\Omega)$ and insert them into the Stokes equation. Then

$$f := -\Delta u + \nabla p \in H^{k,2}(\Omega)^d$$

and

$$g := \nabla \cdot u \in H^{k+1,2}(\Omega).$$

General boundary condition u_D:

Consider $u_D \in H^{1,2}(\Omega)^d$ with $\operatorname{div}(u_D) = 0$.

The Stokes problem with boundary condition u_D: Find $u \in H^{1,2}(\Omega)^d, p \in L^2_0(\Omega)$ fulfilling

$$\left.\begin{aligned} -\Delta u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned}\right\} \text{in } \Omega$$

$$u = u_D \text{ on } \partial\Omega$$

can be formulated in a weak form as: Find (u, p) such that

$$u = u_D + \tilde{u}, \quad \tilde{u} \in (\mathring{H}^{1,2}(\Omega))^d$$

and

$$\begin{aligned} a(u_D + \tilde{u}, v) + b(p, v) &= \langle l, v\rangle \qquad \forall v \in X \\ b(q, u_D + \tilde{u}) &= 0 \qquad\qquad \forall q \in L^2_0(\Omega) \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} a(\tilde{u}, v) + b(p, v) &= \langle l, v\rangle - a(u_D, v) \qquad \forall v \in X \\ b(q, \tilde{u}) &= 0 \qquad\qquad \forall q \in L^2_0(\Omega) \end{aligned}$$

2.4 The stationary Navier–Stokes equations

Reading:

- R. Temam: *Navier-Stokes Equations*, North-Holland 1984
- P. Galdi: *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, Springer 2011

For the rest of this chapter: $\Omega \subseteq I\!R^d$ open, bounded with Lipschitz boundary and $d \leq 3$.

Recall the stationary Navier-Stokes equations (1.16) : find (u, p) fulfilling

$$\begin{aligned} u \cdot \nabla u - \frac{1}{\mathrm{Re}}\Delta u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Recall

Lemma 2.13 (Generalized Hölder inequality). *Let $p_i \geq 1, i = 1 \dots n$ with $1/p_1 + \dots + 1/p_n = 1$ and $f_i \in L^{p_i}(\Omega), i = 1, \dots n$. Then $f := f_1 \cdot f_2 \cdots f_n \in L^1(\Omega)$ and*

$$\int_\Omega |f_1 \cdot f_2 \cdots f_n| \, dx \leq \|f_1\|_{L^{p_1}(\Omega)} \cdot \|f_2\|_{L^{p_2}(\Omega)} \cdots \|f_n\|_{L^{p_n}(\Omega)}.$$

Lemma 2.14. *For $u, v, w \in X$ the function $u \cdot \nabla v\, w \in L^1(\Omega)$ and*

$$n(u; v, w) := \int_\Omega u \cdot \nabla v\, w\, dx \leq C\|u\|_X \|v\|_X \|w\|_X.$$

Furthermore, if $u \in V$ the trilinear form $n(\cdot; \cdot, \cdot)$ is antisymmetric with respect to the last two arguments, i.e.

$$n(u; v, w) = -n(u; w, v)$$

and thus in particular

$$n(u; v, v) = 0.$$

Proof. By Lemma 2.13

$$\int_\Omega u \cdot \nabla v\, w dx \leq \|u\|_{L^4(\Omega)} \|v\|_X \|w\|_{L^4(\Omega)}.$$

The first result then follows by the embedding

$$H^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$$

for $1 - d/2 \geq -d/q$, in particular for $q \leq 6$.

The second part of the lemma is proved by integration by parts:

$$\int_\Omega u \cdot \nabla v\, w dx = \int_\Omega u_i \partial_i v_j w_j dx = -\int_\Omega \underbrace{\partial_i u_i v_j w_j}_{=0} + u_i v_j \partial_i w_j dx = -\int_\Omega u \cdot \nabla w\, v\, dx.$$

□

Like in the Stokes case we have two weak formulations of the stationary Navier-Stokes problem. The first one is formulated in V. Thus, the pressure is not present.

Definition 2.15 (Navier-Stokes problem, weak formulation, 1st version)**.**
Let $l \in X'$. Find $u \in V$ fulfilling

$$\frac{1}{Re} a(u, v) + n(u; u, v) = \langle l, v \rangle \quad \forall v \in V \tag{2.9}$$

The second weak formulation is again a mixed formulation.

Definition 2.16 (Navier-Stokes problem, 2nd version: mixed formulation)**.**
Let $l \in X'$ be given. Find $(u, p) \in X \times Y$ with

$$\begin{cases} \frac{1}{Re} a(u, v) + n(u; u, v) + b(p, v) = \langle l, v \rangle & \forall v \in X \\ b(q, u) = 0 & \forall q \in Y \end{cases} \tag{2.10}$$

Proposition 2.17. *The solutions of Def. 2.15 and Def. 2.16 are equivalent in the following sense: If (u, p) is a solution of (2.10), then u is a solution of (2.9). Conversely, if u is a solution of (2.9), then there exists $p \in Y$ such that (u, p) is a solution of (2.10).*

Proof. First part clear (like for Stokes).

Now, let $u \in V$ be a solution of (2.9). Then by Lemma 2.14 $n(u; u, \cdot) : X \to X'$, $v \mapsto n(u, u, v)$ is linear and continuous. By Corollary 2.11 there exists a pressure $p \in Y$ such that

$$\frac{1}{Re} a(u, v) + b(p, v) = \langle l, v \rangle - n(u; u, v),$$
$$\text{div } u = 0$$

for all $v \in X, q \in Y$. $\square$

Since both definitions are equivalent, we are going to show the existene of a solution $u \in V$ according to Def. 2.15 and recover the pressure later by the above lemma. This simplifies the analysis a bit.

The proof of existence of u will be constructive by a Galerkin procedure.

To this end, observe that X is a separable Hilbert space, so is $V \subseteq X$. Therefore we can find linearly independent functions $\phi_i \in V$, $i = 1, 2, \ldots$ such that with $V_m := \text{span}\{\phi_i \mid i = 1, \ldots m\}$

$$\overline{\bigcup_{m \in I\!N} V_m} = V.$$

We define the finite dimensional Galerkin problem by:

Definition 2.18 (Galerkin problem)**.**
Let $m \in I\!N$. Find $u_m \in V_m$ such that

$$\frac{1}{Re} a(u_m, v_m) + n(u_m; u_m, v_m) = \langle l, v_m \rangle \quad \forall v_m \in V_m \tag{2.11}$$

We will proceed by the following steps:

1. Show that the Galerkin problem has a solution u_m for all $m \in I\!N$.
2. Prove that u_m stays in a bounded set in V (energy estimate).
3. Extract a convergent subsequence $(u_{m_k})_{k \in I\!N}$.
4. Pass to the limit in the equation.

1st step. To start with, let us rewrite the Galerkin problem (2.11) as a nonlinear system of equations in $I\!R^m$. Since $u_m \in V_m$, there are coefficients $\alpha_m = (\alpha_{1,m}, \ldots \alpha_{m,m})^T \in I\!R^m$ with

$$u_m = \sum_{i=1}^{m} \alpha_{i,m} \phi_i.$$

Then Eq. (2.11) is equivalent to the following system of equations in $I\!R^m$:

$$G(\alpha_m) = 0,$$

where $G : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$G(\alpha_m)_i := \frac{1}{Re} a(u_m, \phi_i) + n(u_m; u_m, \phi_i) - \langle l, \phi_i \rangle \quad \forall i = 1, \dots m. \tag{2.12}$$

Does this equation have a solution?

Theorem 2.19 (Brouwer's[5]Fixed Point Theorem)**.** *Let $R > 0$ and $F : \overline{B}_R \to \overline{B}_R$ be continuous, where $\overline{B}_R \subseteq \mathbb{R}^m$. Then F has a fixed point $\alpha_0 \in \overline{B}_R$, i.e.*

$$F(\alpha_0) = \alpha_0.$$

Corollary 2.20. *Let $G : \mathbb{R}^m \to \mathbb{R}^m$ be continuous. Moreover, assume there exists a constant $k > 0$ such that*

$$G(\alpha) \cdot \alpha > 0$$

for $|\alpha| = k$. Then there exists α_0 such that

$$G(\alpha_0) = 0.$$

Proof. Assume that G does not have a zero in $\overline{B}_k \subseteq \mathbb{R}^m$. Then the mapping F,

$$F(\alpha) := -k \frac{G(\alpha)}{|G(\alpha)|}$$

is well defined, continuous and maps $\overline{B}_k$ into itself. We can now apply Brouwer's Fixed Point Theorem to infer the existence of α_0 with $F(\alpha_0) = \alpha_0$. Taking the norm of both sides shows

$$|\alpha_0| = k \frac{|G(\alpha_0)|}{|G(\alpha_0)|} = k$$

that α_0 has norm k. Now taking the inner product in the fixed point equation with α_0, one sees

$$k^2 = |\alpha_0|^2 = F(\alpha_0) \cdot \alpha_0 = -k \frac{G(\alpha_0)}{|G(\alpha_0)|} \cdot \alpha_0 < 0,$$

since $G(\alpha_0) \cdot \alpha_0 > 0$. This is a contradiction. □

In order to apply Corollary 2.20 to the Galerkin problem (2.11), we have to show for G defined by Eq. (2.12):

- $G : \mathbb{R}^m \to \mathbb{R}^m$ is continuous.
- $G(\alpha) \cdot \alpha > 0$ for $|\alpha| = k$ and some $k > 0$.

[5] Luitzen Egbertus Jan Brouwer (1881 – 1966), dutch mathematician.

The first assumption, continuity of G, is very simple to show, since G is a quadratic function in α. So, let's check the second condition:

$$\begin{aligned}
G(\alpha)\cdot\alpha &= \sum_{i=1}^{m} G(\alpha)_i\alpha_i = \sum_{i=1}^{m}\alpha_i\Big(\frac{1}{Re}a(u_m,\phi_i)+n(u_m;u_m,\phi_i)-\langle l,\phi_i\rangle\Big)\\
&= \frac{1}{Re}a(u_m,u_m)+n(u_m;u_m,u_m)-\langle l,u_m\rangle\\
&= \frac{1}{Re}a(u_m,u_m)-\langle l,u_m\rangle \geq \frac{1}{Re}a(u_m,u_m)-\|l\|_{X'}\|u_m\|_X\\
&= \frac{1}{Re}\|u_m\|_X^2-\|l\|_{X'}\|u_m\|_X \geq \frac{1}{Re}\|u_m\|_X^2-\frac{Re}{2}\|l\|_{X'}^2-\frac{1}{2Re}\|u_m\|_X^2\\
&= \frac{1}{2Re}\|u_m\|_X^2-\frac{Re}{2}\|l\|_{X'}^2.
\end{aligned}$$

Thus, $G(\alpha)\cdot\alpha > 0$ if $\|u_m\|_X > Re\|l\|_{X'}$. Since V_m is finite dimensional, all norms on V_m are equivalent. Thus, there exists a $k>0$, such that the above estimates holds if $|\alpha| = k$.

Note that the crucial step proving the estimate was the antisymmetry of the trilinear form

$$n(u_m;u_m,u_m) = 0.$$

2nd step. Energy estimate.

We have established the existence of a solution $u_m \in V_m$ of Eq. (2.11). Now test this equation by u_m. Using again the antisymmetry of the trilinear form we get

$$\frac{1}{Re}a(u_m,u_m) = \langle l,u_m\rangle \leq \|l\|_{X'}\|u_m\|_X;$$

Dividing by $\|u_m\|_X$:

$$\frac{1}{Re}\|u_m\|_X \leq \|l\|_{X'}.$$

3rd step. Convergent subsequence.

Since V is a Hilbert space and thus reflexive and since u_m is uniformly bounded in V, there is a subsequence $(m_k)_{k\in\mathbb{N}}$ such that

$$u_{m_k} \overset{k\to\infty}{\rightharpoonup} u \qquad \text{weakly in } V \quad \text{for some } u\in V.$$

For ease of notation, the subsequence will be denoted again by u_m.

Moreover, because the embedding $H^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$ is compact, the convergence is strong in $L^4(\Omega)$:

$$u_m \overset{m\to\infty}{\rightarrow} u \qquad \text{strongly in } L^4(\Omega).$$

4th step. Passage to the limit.

Fix $m_0 \in \mathbb{N}$. Then

$$\frac{1}{Re}a(u_m,v) \overset{m\to\infty}{\longrightarrow} \frac{1}{Re}a(u,v),$$

$$n(u_m;u_m,v) \overset{m\to\infty}{\longrightarrow} n(u;u,v)$$

for all $v \in V_{m_0}$. Since m_0 is arbirtrary, by density it follows that the above convergences also hold for all $v \in V$ and therefore

$$\frac{1}{Re} a(u, v) + n(u; u, v) = \langle l, v \rangle \quad \forall v \in V.$$

All together, we have thus proved:

Theorem 2.21. *Let $l \in X'$. Then there exists a solution $u \in V$ of Eq.* (2.9)*. Furthermore, there is a pressure $p \in Y$ such that (u, p) is a solution of* (2.10)*.*

Moreover, u fulfills the energy estimate

$$\|u\|_X \leq Re \|l\|_{X'} \tag{2.13}$$

Note that the energy estimate for u follows from the one for u_m and the weakly semi-lower continuity of the norm.

(Non–) uniqueness.

Uniqueness can only be proved (and expected) under very restrictive smallness assumptions on $l \in X'$.

Let $u, \tilde{u} \in V$ be two solutions. Subtracting Eq. (2.9) for u and $\tilde{u}$, respectively, yields:

$$\frac{1}{Re} a(u - \tilde{u}, v) + n(u; u, v) - n(\tilde{u}; \tilde{u}, v) = 0 \qquad \forall v \in V. \tag{2.14}$$

Set $w = u - \tilde{u}$. Thanks to the trilinearity of $n(\cdot; \cdot, \cdot)$ one computes

$$n(u; u, v) - n(\tilde{u}; \tilde{u}, v) = n(u; u - \tilde{u}, v) + n(u; \tilde{u}, v) - n(\tilde{u}; \tilde{u}, v) = n(u; w, v) - n(w; \tilde{u}, v).$$

Upon taking $v = w$ one gets

$$n(u; u, w) - n(\tilde{u}; \tilde{u}, w) = n(u; w, w) - n(w; \tilde{u}, w) = -n(w; \tilde{u}, w).$$

Using this in Eq. (2.14) we infer

$$\frac{1}{Re} \|w\|_X^2 = n(w; \tilde{u}, w).$$

Using the bound on the trilinear from from Lemma 2.14 and the energy estimate for $\tilde{u}$ one arrives at

$$\frac{1}{Re} \|w\|_X^2 \leq C \|w\|_X^2 \|\tilde{u}\|_X \leq C\, Re \|w\|_X^2 \|l\|_{X'}$$

and then

$$\left(\frac{1}{Re} - C\, Re \|l\|_{X'} \right) \|w\|_X^2 \leq 0.$$

If the expression in the paranthesis $\left(\frac{1}{Re} - C\, Re \|l\|_{X'} \right) > 0$, then we can conclude $\|w\|_X \leq 0$, which is only possible if $w = 0$, which means $u = \tilde{u}$.

Theorem 2.22. *There is a constant $C > 0$ such that, if $\|l\|_{X'} \leq \frac{1}{C\, Re^2}$, then the solution of Problem 2.9 is unique.*

For more general data, uniqueness cannot be expected. There are physical examples of non-uniqueness in the stationary case. Even stronger: For certain settings one can rigorously show that there exists more than one solution (without an explicit formula), see for instance Temam.

2.5 The instationary (Navier–) Stokes equations

The linear case (instationary Stokes equations) is not much harder than the heat equation. However, with some peculiarities regarding temporal regularity for $t \searrow 0$.
For the nonlinear problem (instationary Navier-Stokes equations) existence of a weak solution can be shown for $d = 2, 3$ for instance.
Uniqueness and regularity for $d = 2$ not too difficult. For $d = 3$, the question of uniqueness and regularity is still open despite intense research over the last decades, see also

http://www.claymath.org/millennium-problems/navier%E2%80%93stokes-equation

3 Finite element approximation of the Stokes problem

3.1 Approximation of general saddle point problems

Recall: X, Y Hilbert spaces, $f \in X', g \in Y'$.
We are looking for $(u, p) \in X \times Y$ with

$$\left\{ \begin{aligned} a(u,v) + b(p,v) &= \langle f, v\rangle \quad && \forall v \in X \\ b(q,u) &= \langle g, q\rangle && \forall q \in Y \end{aligned} \right. \tag{3.1}$$

"mixed formulation".

Why not working directly in $V = V_0$? One could think of choosing $V_h \subset V$ and formulate the discrete problem according to the first weak form of the Stokes system: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \quad \forall v_h \in V_h$$

(in case of $g \equiv 0$).
However,

$$V_h \subset V = \{v \in \mathring{H}^{1,2}(\Omega)^d \mid \operatorname{div}(v) = 0\}$$

would require for $v_h \in V_h$:

$$\operatorname{div}(v_h) \equiv 0$$

Could work. However, construction of such a subspace rather difficult.
Instead, we are working with the mixed formulation: Choose $X_h \subset X$, $Y_h \subset Y$ (finite dimensional)
We are looking for $(u_h, p_h) \in X_h \times Y_h$ fulfilling

$$\left\{ \begin{aligned} a(u_h,v_h) + b(p_h,v_h) &= \langle f, v_h\rangle \quad && \forall v_h \in X_h \\ b(q_h,u_h) &= \langle g, q_h\rangle && \forall q_h \in Y_h \end{aligned} \right. \tag{3.2}$$

Hereafter: $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ continuous and $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ fulfill the solvability conditions from Corollary 2.8 i), ii).

Analogously as in the continuous case, define operators

$$A_h : X_h \to X_h'$$
$$\langle A_h v_h, w_h \rangle_{X_h' \times X_h} := a(v_h, w_h) \qquad \forall v_h, w_h \in X_h$$

$$B_h : Y_h \to X_h'$$
$$\langle B_h q_h, v_h \rangle_{X_h' \times X_h} := b(q_h, v_h) \qquad \forall q_h \in Y_h,\ v_h \in X_h$$

Furthermore, B_h^* by

$$B_h^* : X_h \to Y_h'$$
$$\langle B_h^* v_h, q_h \rangle_{Y_h' \times Y_h} := b(q_h, v_h) \qquad \forall q_h \in Y_h,\ v_h \in X_h$$

We identify Y_h' with a subspace of Y' via:

Let $g_h \in Y_h'$, **extend** g_h on Y' by

$$(g_h)_{|Y_h^\perp} = 0$$

This means for $B_h^* v_h \in Y'$:

$$\langle B_h^* v_h, q \rangle := \langle B_h^* v_h, P_{Y_h} q \rangle = b(P_{Y_h} q, v_h) \qquad \forall v_h \in X_h,\ q \in Y$$

with $P_{Y_h} : Y \to Y_h$ the **orthogonal projection**.

Lemma 3.1.

Problem (3.2) admits a unique solution for all $f \in X_h'$, $g \in Y_h'$, iff there are constants $\alpha_h, \beta_h > 0$ such that

i) $\sup\limits_{0 \neq x_0 \in V_{0,h}} \frac{|a(v_0,x_0)|}{\|x_0\|_X} \geq \alpha_h \|v_0\|_X \qquad \forall v_0 \in V_{0,h}$

ii) $\sup\limits_{0 \neq v_h \in X_h} \frac{|b(q_h,v_h)|}{\|v_h\|_X} \geq \beta_h \|q_h\|_Y \qquad \forall q_h \in Y_h$

Here, $V_{0,h} := \mathit{ker}(B_h^*) = \{v_h \in X_h \mid b(q_h, v_h) = 0\ \forall q_h \in Y_h\}$

Proof:

Apply Corollary 2.8 to X_h, Y_h. Note: 2nd condition in i) dispensable, since $\dim V_{0,h} < \infty$

$\square$

Remark 3.2.

i) $a(\cdot,\cdot)$ *coercive* $\Rightarrow$ *condition i) above.*

ii) *the LBB condition for* X_h, Y_h **does not** *follow from the one for* X, Y*!*

iii) *condition ii) with* $\beta_h > 0 \Leftrightarrow B_h$ *injective*

$$\Rightarrow \dim Y_h \leq \dim X_h.$$

However, unfortunately not the other way round.

iv) *In general:*

$$V_{0,h} \not\subset V_0$$