iii) **Prove** $\exists\, g = g(r): \quad v \times \mathrm{rot}(v) = \nabla g$

Proof

$$v \times \mathrm{rot}\, v = \begin{pmatrix} v_y \frac{1}{r}\partial_r (r v_\varphi(r)) - 0 \\ 0 - v_x \frac{1}{r}\partial_r (r v_\varphi(r)) \\ 0 \end{pmatrix}$$

$$= \left(v_\varphi(r) \frac{x}{r} \cdot \frac{1}{r} \partial_r (r v_\varphi(r)),\ v_\varphi(r) \frac{y}{r} \frac{1}{r} \partial_r (r v_\varphi(r)),\ 0 \right)^T$$

$$= \left(\frac{x}{r^2} v_\varphi(r)\, \partial_r (r v_\varphi(r)),\ \frac{y}{r^2} v_\varphi(r)\, \partial_r (r v_\varphi(r)),\ 0 \right)^T$$

$$= \frac{1}{r^2} v_\varphi(r)\, \partial_r (r v_\varphi(r)) \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$v = v_\varphi(r)\, e_\varphi(r, \varphi, z), \qquad \mathrm{rot}(v) = \frac{1}{r}\partial_r (r v_\varphi(r))\, e_z$$

For simplicity we define $\lambda(r) := \frac{1}{r}\partial_r (r v_\varphi(r))$

Therefore

$$v \times \mathrm{rot}\, v = v_\varphi(r)\, e_\varphi(r,\varphi,z) \times \lambda(r)\, e_z(r,\varphi,z)$$

$$= v_\varphi(r)\, \lambda(r) \cdot \left(e_\varphi(r,\varphi,z) \times e_z(r,\varphi,z) \right)$$

$$= v_\varphi(r)\, \lambda(r) \cdot \left(-e_r(r,\varphi,z) \right)$$

Note:
$e_\varphi = e_z \times e_r$
$\Rightarrow e_r = e_\varphi \times e_z$

We get for $g = g(r)$: $\nabla g(r) = \frac{dg}{dr} e_r$

So we get

$$v \times \mathrm{rot}\, v = \nabla g(r) \qquad \text{for } \frac{dg}{dr} = -v_\varphi(r)\, \lambda(r)$$

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