Homework 1

Ex 1

$0 < R_1 < R_2$ $\qquad \Omega = \{(x,y,z) \in \mathbb{R}^3 \mid R_1 < \sqrt{x^2+y^2} < R_2\}$

a.) Couette flow: $(x,y,z) \mapsto (r,\varphi,z) := \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right), z\right)$

$$e_r(r,\varphi,z) := \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}, \quad e_\varphi(r,\varphi,z) = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Prove:

1. $\nabla f = \partial_r f e_r + \frac{1}{r} \partial_\varphi f e_\varphi + \partial_z f e_z$
2. $\Delta f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_{\varphi\varphi} f + \partial_{zz} f$

Note:

for $v = v_x e_x + v_y e_y + v_z e_z = v_r e_r + v_\varphi e_\varphi + v_z e_z$

$e_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

following holds: $\operatorname{div} v = \frac{1}{r} \partial_r (r v_r) + \frac{1}{r} \partial_\varphi v_\varphi + \partial_z v_z$

Proof:

1. $$\nabla f = \frac{\partial f}{\partial x} e_x + \frac{\partial f}{\partial y} e_y + \frac{\partial f}{\partial z} e_z$$

$e_r = \cos(\varphi) e_x + \sin(\varphi) e_y$, $\quad e_\varphi = -\sin(\varphi) e_x + \cos(\varphi) e_y$
$e_z = e_z$

By applying the chain rule to $f(x,y,z) = \hat{f}(r(x,y), \varphi(x,y), z)$

We get:

- $$\frac{\partial f}{\partial x} = \frac{\partial \hat{f}}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \hat{f}}{\partial \varphi} \frac{\partial \varphi}{\partial x}$$
- $$\frac{\partial f}{\partial y} = \frac{\partial \hat{f}}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial \hat{f}}{\partial \varphi} \frac{\partial \varphi}{\partial y}$$
- $$\frac{\partial f}{\partial z} = \frac{\partial \hat{f}}{\partial z}$$

Now we simplify ~~this~~:

- $\frac{\partial r}{\partial x} = \partial_x \sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$

- $\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$

- $\frac{\partial \varphi}{\partial x} = \partial_x \arctan\left(\frac{y}{x}\right) = -\frac{y}{x^2+y^2}$

- $\frac{\partial \varphi}{\partial y} = \partial_y \arctan\left(\frac{y}{x}\right) = \frac{x}{x^2+y^2}$

Rewrite:

$$\nabla f = \partial_r \hat{f} \frac{x}{\sqrt{x^2+y^2}} e_x + \partial_\varphi \hat{f} \left(-\frac{y}{x^2+y^2}\right) e_x + \partial_r \hat{f} \frac{y}{\sqrt{x^2+y^2}} e_y + \partial_\varphi \hat{f} \frac{x}{x^2+y^2} e_y + \partial_z \hat{f} e_z$$

$$= \partial_r \hat{f} \left(\frac{x}{\sqrt{x^2+y^2}} e_x + \frac{y}{\sqrt{x^2+y^2}} e_y\right) + \partial_\varphi \hat{f} \left(-\frac{y}{x^2+y^2} e_x + \frac{x}{x^2+y^2} e_y\right) + \partial_z \hat{f} e_z$$

If we assume $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$, $-\frac{y}{x^2+y^2} = -\frac{\sin\varphi}{r}$

$$\frac{x}{x^2+y^2} = \frac{\cos\varphi}{r}$$

then

$$\nabla f = \partial_r \hat{f} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} + \partial_\varphi \hat{f} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} + \partial_\varphi \hat{f} e_z$$

$$= \partial_r \hat{f} \, e_r + \partial_\varphi \hat{f} \, e_\varphi + \partial_\varphi \hat{f} e_z .$$

So last thing is to prove the Assumptions.

- $\varphi = \arctan\left(\frac{y}{x}\right)$, $\cos\varphi = \cos\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{x}{\sqrt{x^2+y^2}}$ ✓
- $\varphi = \arctan\left(\frac{y}{x}\right)$, $\sin\varphi = \sin\left(\arctan\left(\frac{y}{x}\right)\right) = \frac{y}{\sqrt{x^2+y^2}}$ ✓
- $r = \sqrt{x^2+y^2}$, $-\frac{\sin\varphi}{\sqrt{x^2+y^2}} = -\frac{y}{\sqrt{x^2+y^2}} \cdot \frac{1}{\sqrt{x^2+y^2}} = -\frac{y}{x^2+y^2}$ ✓
- $r = \sqrt{x^2+y^2}$, $\frac{\cos\varphi}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$ ✓

Therefore

$$\nabla f = \partial_r \hat{f} e_r + \partial_\varphi \hat{f} e_\varphi + \partial_\varphi \hat{f} e_z$$

✓

2.

$\Delta f = \nabla \cdot \nabla f$

we define $v := \nabla f = v_r e_r + v_\varphi e_\varphi + v_z e_z$

where $v_r = \partial_r \hat{f}$, $v_\varphi = \frac{1}{r} \partial_\varphi \hat{f}$, $v_z = \partial_z \hat{f}$

with the Note it follows:

$$\operatorname{div}(\nabla f) = \frac{1}{r} \partial_r (r v_r) + \frac{1}{r} \partial_\varphi v_\varphi + \partial_z v_z$$
$$= \frac{1}{r} \partial_r (r \partial_r \hat{f}) + \frac{1}{r} \partial_\varphi \frac{1}{r} \partial_\varphi \hat{f} + \partial_z \partial_z \hat{f}$$
$$= \frac{1}{r} \partial_r (r \partial_r \hat{f}) + \frac{1}{r^2} \partial_\varphi \hat{f} + \partial_{zz} \hat{f}$$

6.)

Let $v: \Omega \to \mathbb{R}^3$ be smooth

to prove:

$$(v \cdot \nabla) v = \frac{1}{2} \nabla \|v\|^2 - v \times \operatorname{rot} v \qquad (\operatorname{rot} v = \nabla \times v)$$

Lets write both sides in index notation.

LHS $[(v \cdot \nabla) v]_i = \sum_j v_j \frac{\partial v_i}{\partial x_j} = \sum_j v_j \partial_j v_i$

RHS

- $[\frac{1}{2} \nabla \|v\|^2]_i = \frac{1}{2} \partial_i \sum_j v_j^2 = \sum_j v_j \frac{\partial v_j}{\partial x_i} = \sum_j v_j \partial_i v_j$

- $[v \times \operatorname{rot} v]_i = [v \times (\nabla \times v)]_i$

For better understanding:

$$(\nabla \times v)_i = \begin{cases} \partial_2 v_3 - \partial_3 v_2 & i=1 \\ \partial_3 v_1 - \partial_1 v_3 & i=2 \\ \partial_1 v_2 - \partial_2 v_1 & i=3 \end{cases}$$

$$\Rightarrow [v \times (\nabla \times v)]_i = \begin{cases} v_2 (\partial_1 v_2 - \partial_2 v_1) - v_3 (\partial_3 v_1 - \partial_1 v_3) & i=1 \\ v_3 (\partial_2 v_3 - \partial_3 v_2) - v_1 (\partial_1 v_2 - \partial_2 v_1) & i=2 \\ v_1 (\partial_3 v_1 - \partial_1 v_3) - v_2 (\partial_2 v_3 - \partial_3 v_2) & i=3 \end{cases}$$

$\Rightarrow$ Let $i=1$: LHS: $v_1 \partial_1 v_1 + v_2 \partial_2 v_1 + v_3 \partial_3 v_1$

RHS: $v_1 \partial_1 v_1 + v_2 \partial_1 v_2 + v_3 \partial_1 v_3$
$- (v_2 \partial_1 v_2 - v_2 \partial_2 v_1 - v_3 \partial_3 v_1 + v_3 \partial_1 v_3)$
$= v_1 \partial_1 v_1 + v_2 \partial_2 v_1 + v_3 \partial_3 v_1$

$\Rightarrow$ LHS = RHS ($i = 2,3$ analog)