III. Weak derivatives and Sobolev spaces

III.1 Definition and fundamental properties

We need to work with (in a sense) differentiable functions, but in an $\mathcal{L}^p(\Omega)$ setting. To this end, for $\Omega \subseteq \mathbb{R}^d$ open and $m \in \mathbb{N}$ define

$$\|f\|_{m,p} = \|f\|_{m,p,\Omega}$$

$$= \Big(\sum_{|\alpha| \le m} \|\partial_\alpha f\|_{p,\Omega}^p \Big)^{1/p}$$

for $1 \le p < \infty$ and

$$\|f\|_{m,\infty} = \max \{ \|\partial_\alpha f\|_{\infty,\Omega} \mid |\alpha| \le m \}$$

for functions $f \in C^m(\bar{\Omega})$

Here, $\alpha = (\alpha_1, \dots, \alpha_d)^T$ is a <u>multiindex</u>

$|\alpha| := \alpha_1 + \dots + \alpha_d$ the length of α

For $f \in C^m(\Omega)$, $\partial_\alpha f$ denotes the α-th partial derivative:

$$\partial_\alpha f(x) := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}(x)$$

Now, set $X = (C^m(\bar{\Omega}), \|\cdot\|_{m,p})$

Problem: X is not complete, no Banach-space for $1 \leq p < \infty$

Remedy: complete X to get $\tilde{X}$

$$\tilde{X} := \{ (f_j)_{j \in \mathbb{N}} \mid (f_j)_j \text{ is Cauchy in } X \}$$

with the <u>equivalence relation</u>

$$(f_j)_{j \in \mathbb{N}} \cong (g_j)_{j \in \mathbb{N}}$$

$$\Leftrightarrow \lim_{j \to \infty} \|f_j - g_j\|_{m,p} = 0$$

Define the norm on $\tilde{X}$:

$$\|(f_j)_{j\in\mathbb{N}}\|_{\tilde{X}} := \lim_{j\to\infty} \|f_j\|_{m,p}$$

<u>Note</u>: $(f_j)_{j\in\mathbb{N}}$ Cauchy in $X \Rightarrow \|(f_j)_{j\in\mathbb{N}}\|_{m,p}$ Cauchy in $\mathbb{R}$

$\Rightarrow \lim_{j\to\infty} \|f_j\|_{m,p}$ exists

<u>Further</u>: $\|\cdot\|_{\tilde{X}}$ compatible with the equivalence relation: let $(f_j)_{j\in\mathbb{N}}, (g_j)_{j\in\mathbb{N}} \in \tilde{X}$ with

$$\lim_{j\to\infty} \|f_j - g_j\|_{m,p} = 0$$

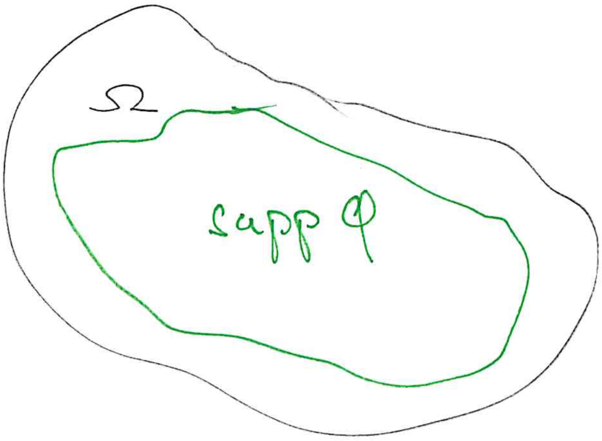
$$\Rightarrow \lim_{j\to\infty} \left| \|f_j\|_{m,p} - \|g_j\|_{m,p} \right| = 0$$

FA: $(\tilde{X}, \|\cdot\|_{\tilde{X}})$ is a Banach space

How to characterize $\tilde{X}$?

Let $f \in C^m(\Omega)$, α multiindex with $|\alpha| \leq m$ and $\varphi \in C_c^\infty(\Omega)$. Then

$$(3.1) \quad \int_\Omega \partial_\alpha f(x)\, \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x)\, \partial_\alpha \varphi(x)\, dx$$



This motivates the following notion. Let $f \in \mathcal{L}^p(\Omega)$, α multiindex,

Definition 3.1 (Weak derivative)

Assume the above notation. If there exists $f^{(\alpha)} \in \mathcal{L}^p(\Omega)$ s.t.

$$(3.2) \quad \int_\Omega f^{(\alpha)}(x)\, \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega f(x)\, D_\alpha \varphi(x)\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$, then $f^{(\alpha)}$ is called the weak α-th partial derivative. We denote it again by $\partial_\alpha f := f^{(\alpha)}$

Remarks 3.2

i) If $f^{(\alpha)} \in \mathcal{L}^p(\Omega)$ exists, then it is uniquely determined. Since, let $\tilde{f}^{(\alpha)} \in \mathcal{L}^p(\Omega)$ be another function fullfilling (3.2) it holds

$$\int_\Omega f^{(\alpha)}(x)\varphi(x)\,dx = \int_\Omega \tilde{f}^{(\alpha)}(x)\varphi(x)\,dx$$

for all $\varphi \in C_c^\infty(\Omega)$ $\Rightarrow$ $f^{(\alpha)} = \tilde{f}^{(\alpha)}$ a.e.

ii) If $f \in C^m(\Omega)$ s.t. $\|f\|_{m,p} < \infty$, then the classical derivatives $\partial_\alpha f$ equals the weak derivatives for $|\alpha| \leq m$

Definition 3.3 (Sobolev spaces)

Let $1 \le p \le \infty$ and $m \in \mathbb{N}$. Define

$$W^{m,p}(\Omega) = H^{m,p}(\Omega)$$

$$= \left\{ f \in \mathcal{L}^p(\Omega) \;\middle|\; \begin{array}{l} \text{all weak derivatives } \partial_\alpha f \in \mathcal{L}^p(\Omega) \\ \text{exist for } |\alpha| \le m \end{array} \right\}$$

For convenience: $W^{0,p}(\Omega) = H^{0,p}(\Omega) = \mathcal{L}^p(\Omega)$

Define $j : (\tilde{X}, \|\cdot\|_{\tilde{X}}) \longrightarrow (H^{m,p}(\Omega), \|\cdot\|_{m,p})$

$\boxed{1 \le p < \infty}$

$$j((f_j)_{j \in \mathbb{N}}) := \lim_{j \to \infty} f_j \quad \text{in } \mathcal{L}^p(\Omega)$$

and $\partial_\alpha f_j \xrightarrow[j\to\infty]{} f^{(\alpha)}$ for some $f^{(\alpha)} \in \mathcal{L}^p(\Omega)$

$f := \lim\limits_{j \to \infty} f_j$ fulfills (3.2) for $|\alpha| \le m$:

$$\int_\Omega \partial_\alpha f_j(x)\,\varphi(x)\,dx = (-1)^{|\alpha|} \int_\Omega f_j(x)\,\partial_\alpha \varphi(x)\,dx$$

$$\downarrow j \to \infty \qquad\qquad \downarrow j \to \infty$$

$$\int_\Omega f^{(\alpha)}(x)\,\varphi(x)\,dx \qquad (-1)^{|\alpha|} \int_\Omega f(x)\,\partial_\alpha \varphi(x)\,dx$$

with $f^{(\alpha)} = \lim_{j\to\infty} \partial_\alpha f_j$ in $\mathcal{L}^p(\Omega)$.

Then

$$\| (f_j)_{j\in\mathbb{N}} \|_{\tilde{X}} = \lim_{j\to\infty} \| f_j \|_{m,p} = \| J((f_j)_j) \|_{m,p}$$

$$= \| f \|_{m,p} = \Big(\sum_{|\alpha|\le m} \| f^{(\alpha)} \|_{m,p}^p \Big)^{1/p} .$$

Thus J is an isometry. Since $\tilde{X}$ is complete, then $J(\tilde{X}) \subseteq H^{m,p}(\Omega)$ complete

<u>Proposition 3.4</u> $1 \le p < \infty$, $m \in \mathbb{N}$

Let $f \in H^{m,p}(\Omega)$. Then there exist $f_j \in C^\infty(\Omega) \cap H^{m,p}(\Omega)$

s.t. $\| f - f_j \|_{m,p} \xrightarrow[j\to\infty]{} 0$

Corollary 3.5 $1 \leq p < \infty$, $m \in \mathbb{N}$

$(H^{m,p}(\Omega), \|\cdot\|_{m,p})$ is a Banach space

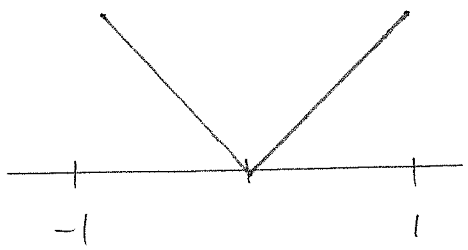
Proof:
Prop. 3.4 tells us that J is surjective. Thus $\hat{X}$ and $H^{m,p}(\Omega)$ are isometrically isomorphic.

Proposition 3.6 $m \in \mathbb{N}$

$W^{m,\infty}(\Omega)$ is a Banach-space

Example 3.7

$\Omega = (-1, 1) \subseteq \mathbb{R}$, $f(x) = |x|$



We show: $f \in H^{1,\infty}(\Omega)$ $(\Rightarrow f \in H^{1,p}(\Omega)$ for all $1 \le p \le \infty)$ (109)

note: $f \notin C^1(\Omega)$

Let $\varphi \in C_c^\infty(\Omega)$.

$$\int_{-1}^{1} f(x)\varphi'(x)dx = \int_{-1}^{0} f(x)\varphi'(x)dx + \int_{0}^{1} f(x)\varphi'(x)dx$$

$$= \int_{-1}^{0} -x\varphi'(x)dx + \int_{0}^{1} x\varphi'(x)dx$$

$$= \int_{-1}^{0} 1\varphi(x)dx - \int_{0}^{1} 1\varphi(x)dx + (-x\varphi(x))\Big|_{x=0}$$

$$- (x\varphi(x))\Big|_{x} = 0$$

$$= -\int_{-1}^{1} f'(x)\varphi(x)dx$$ with

$$f'(x) := \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

$f, f' \in L^\infty(\Omega)$ and fulfilling (3.2) ✓

$f \notin H^{2,p}(\Omega)$ for all $1 \le p \le \infty$

With Lemma 3.8 below, we show that

$f' \notin H^{1,p}(\Omega)$:

$$\int_{-1}^{1} f'(x)\,\varphi'(x)\,dx = \int_{-1}^{0} \underbrace{f'(x)}_{=-1}\varphi'(x)\,dx + \int_{0}^{1} \underbrace{f'(x)}_{=+1}\varphi'(x)\,dx$$

$$= -\int_{-1}^{0} 0\cdot\varphi(x)\,dx - \int_{0}^{1} 0\cdot\varphi(x)\,dx + \left(-1\,\varphi(x)\right)\Big|_{x=0}$$

$$- \left(1\,\varphi(x)\right)\Big|_{x=0}$$

$$= -2\varphi(0)$$

There exists no $f''_c \in \mathcal{L}^p(\Omega)$ with [illegible], but

$$\int_{-1}^{1} f''(x)\,\varphi(x)\,dx = 2\varphi(0) \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

<u>Lemma 3.8</u> $1 \le p \le \infty$, $k < m$, $k, m \in \mathbb{N}$

$f \in H^{m,p}(\Omega) \iff f \in H^{k,p}(\Omega)$ and

$\partial_\alpha f \in H^{m-k,p}(\Omega)$ for all

$|\alpha| \le m-k$

[illegible]

Moreover, if $f \in H^{m,p}(\Omega)$ and $|\alpha| + |\beta| \leq m$ then $\partial_{\alpha+\beta} f = \partial_\alpha (\partial_\beta f)$

Proof: simple exercise

Def. 3.9 $m \in \mathbb{N}$, $1 \leq p < \infty$

$$\overset{\circ}{H}{}^{m,p}(\Omega) := \left\{ f \in H^{m,p}(\Omega) \;\middle|\; \exists\, f_j \in C_c^\infty(\Omega) \quad \|f_j - f\|_{m,p} \xrightarrow[j\to\infty]{} 0 \right\}$$

$$= \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{m,p}} \subseteq H^{m,p}(\Omega)$$

III.2 Weak boundary values and trace operator.

Theorem 3.9 (trace operator)

Let $\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary. Then there exists an operator $\operatorname{tr}: H^{1,p}(\Omega) \to L^p(\partial\Omega)$,

$\|\operatorname{tr} f\|_{p,\partial\Omega} \leq C \|f\|_{1,p,\Omega}$ with

$\operatorname{tr} f = f|_{\partial\Omega}$ for $f \in H^{1,p}(\Omega) \cap C^0(\bar{\Omega})$

Proposition 3.10

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary. Then

$$\mathring{H}^{1,p}(\Omega) = \{ f \in H^{1,p}(\Omega) \mid \operatorname{tr} f = 0 \text{ in } L^p(\partial\Omega) \}$$

III.3 Further properties of Sobolev functions

Proposition 3.11 (Product and chain rule)

i) Let $\Omega \subseteq \mathbb{R}^d$ be open, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (setting $\frac{1}{\infty} = 0$). Let $f \in H^{m,p}(\Omega)$, $g \in H^{m,p'}(\Omega)$.

Then $f \cdot g \in H^{m,1}(\Omega)$ and $\partial_\alpha (f \cdot g)$ is given as in the classical case.

ii) Let $f \in H^{1,p}(\Omega)$ and $\phi \in C^{0,1}(\mathbb{R})$. Then $\phi \circ f \in H^{1,p}(\Omega)$ and

$$\nabla(\phi \circ f) = \phi'(f) \nabla f$$

(for ϕ' see Prop 3.14 below)

Proposition 3.12 (Poincaré)

$1 \leq p < \infty$

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded. Then there exists a constant $C_p > 0$ s.t.

$$\| f \|_{p,\Omega} \leq C_p \| \nabla f \|_{p,\Omega} \quad \text{for all } f \in \mathring{H}^{1,p}(\Omega)$$

Thus $\| \nabla f \|_{0,p,\Omega}$ is an equivalent norm to $\| f \|_{1,p,\Omega}$ on $\mathring{H}^{1,p}(\Omega)$

Proposition 3.13 (Gauss)

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary. Moreover, $f \in H^{1,p}(\Omega)$, $g \in H^{1,p'}(\Omega)$. Then

$$\int_\Omega f \partial_i g + \partial_i g f \, dx = \int_{\partial\Omega} f g \, \nu_i \, d H^{n-1} \qquad i = 1, \dots, d$$

$$\partial_i f := \frac{\partial f}{\partial x_i}$$

$\nu = (\nu_1, \dots, \nu_d)^T$ outer normal on $\partial\Omega$

Proposition 3.14

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary. Then for $m \in \mathbb{N}$

$$C^{m-1,1}(\bar{\Omega}) \cong H^{m,\infty}(\Omega)$$

Proposition 3.15 (Rellich)

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary, $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then for $f_j, f \in H^{m,p}(\Omega)$:

$$\text{if} \quad f_j \xrightarrow[j\to\infty]{} f \text{ in } H^{m,p}(\Omega)$$

$$\text{then} \quad f_j \xrightarrow[j\to\infty]{} f \text{ in } H^{m-1,p}(\Omega)$$

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III.4 Embeddings

For $H^{m,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ open we define the Sobolev number $s = s(m,p,d)$:

$$(3.3) \qquad s = m - d/p$$

and for $C^{k,\alpha}(\overline{\Omega})$, $0 \leq \alpha \leq 1$, $k \in \mathbb{N}$, $s = (k, \alpha)$

$$(3.4) \qquad s = k + \alpha$$

Recall: $\|f\|_k = \sum_{|\beta| \leq k} \|\partial_\beta f\|_{\infty, \overline{\Omega}}$

114b

Proposition 3.16

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary, $1 \leq p < \infty$.

Then $\{ u|_\Omega \mid u \in C_c^\infty(\mathbb{R}^d) \}$ dense in $H^{1,p}(\Omega)$.

and

$$\|f\|_{k,\alpha} = \|f\|_k + \sum_{|\beta|=k} [\partial_\beta f]_\alpha$$

$$[g]_\alpha := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|g(x)-g(y)|}{|x-y|^\alpha}$$

<u>Theorem 3.17</u>

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary,

$m_1 > m_2 \geq 0$, $1 \leq p_1, p_2 < \infty$.

i) If $s(m_1, p_1) \geq s(m_2, p_2)$

$$H^{m_1,p_1}(\Omega) \hookrightarrow H^{m_2,p_2}(\Omega),$$

i.e. $H^{m_1,p_1}(\Omega) \subseteq H^{m_2,p_2}$ and

$$\|f\|_{m_2,p_2} \leq C\,\|f\|_{m_1,p_1} \quad \text{for } f \in H^{m_1,p_1}(\Omega)$$

ii) If $s(m_1,p_1) > s(m_2,p_2)$ then

$$H^{m_1,p_1}(\Omega) \underset{\text{compact}}{\hookrightarrow} H^{m_2,p_2}(\Omega)$$

Example: i) $d = 2$, $m_1 = 1$, $m_2 = 0$, $p_1 = 2$

$$s(m_1, p_1) = m_1 - 1 = 0$$

$$s(m_2, p_2) = 0 - \frac{2}{p_2} < 0$$

$$\Rightarrow \quad H^{1,2}(\Omega) \underset{\text{compact}}{\hookrightarrow} L^{p_2}(\Omega) \quad \text{for all } 1 \le p_2 < \infty$$

Warning: false for $p_2 = \infty$

ii) $d = 3$, $m_1 = 1$, $m_2 = 0$, $p_1 = 2$

$$s(m_1, p_1) = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$s(m_2, p_2) = -\frac{3}{p_2}$$

$$\Rightarrow \quad H^{1,2}(\Omega) \hookrightarrow L^{p_2}(\Omega) \quad \text{for all } 1 \le p \le 6$$

$$H^{1,2}(\Omega) \underset{\text{compact}}{\hookrightarrow} L^{p_2}(\Omega) \quad \text{for all } 1 \le p < 6$$

Theorem 3.18

$\Omega \subseteq \mathbb{R}^d$ open with Lipschitz boundary

$m \geq 1, \quad 1 \leq p < \infty, \quad k \geq 0, \quad 0 \leq \alpha \leq 1$

i) $0 < \alpha < 1$. If $\quad m - \frac{d}{p} = s(m, p, d)$

$$= s(k, \alpha) = k + \alpha$$

then $\quad H^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\bar{\Omega})$

ii) If $\quad m - \frac{d}{p} = s(m, p, d) > s(k, \alpha) = k + \alpha$

then

$$H^{m,p}(\Omega) \underset{\text{compact}}{\hookrightarrow} C^{k,\alpha}(\bar{\Omega})$$

Examples:

i) $m = 1, \quad p > d \quad \Rightarrow \quad H^{1,p}(\Omega) \underset{\text{compact}}{\hookrightarrow} C^0(\bar{\Omega})$

ii) $m = 2, \quad p = 2$

$$H^{2,2}(\Omega) \underset{\text{compact}}{\hookrightarrow} C^0(\bar{\Omega}) \qquad \text{for } d \leq 3$$