Section Review Exercises

1. What is a mathematical system?
2. What is an axiom?
3. What is a definition?
4. What is an undefined term?
5. What is a theorem?
6. What is a proof?
7. What is a lemma?
8. What is a direct proof?
9. What is the formal definition of "even integer"?
10. What is the formal definition of "odd integer"?
11. What is a subproof?
12. How do you disprove a universally quantified statement?

Exercises

1. Give an example (different from those of Example 2.1.1) of an axiom in Euclidean geometry.
2. Give an example (different from those of Example 2.1.2) of an axiom in the system of real numbers.
3. Give an example (different from those of Example 2.1.1) of a definition in Euclidean geometry.
4. Give an example (different from those of Example 2.1.2) of a definition in the system of real numbers.
5. Give an example (different from those of Example 2.1.3) of a theorem in Euclidean geometry.
6. Give an example (different from those of Example 2.1.5) of a theorem in the system of real numbers.
7. Prove that for all integers m and n, if m and n are even, then $m + n$ is even.
8. Prove that for all integers m and n, if m and n are odd, then $m + n$ is even.
9. Prove that for all integers m and n, if m and n are even, then mn is even.
10. Prove that for all integers m and n, if m and n are odd, then mn is odd.
11. Prove that for all integers m and n, if m is odd and n is even, then mn is even.
12. Prove that for all integers m and n, if m and $m + n$ are even, then n is even.
13. Prove that for all rational numbers x and y, $x + y$ is rational.
14. Prove that for all rational numbers x and y, xy is rational.
15. Prove that for every rational number x, if $x \neq 0$, then $1/x$ is rational.
16. If a and b are real numbers, we define $\max\{a, b\}$ to be the *maximum* of a and b or the common value if they are equal. Prove that for all real numbers d, d_1, d_2, x,

$$\text{if } d = \max\{d_1, d_2\} \text{ and } x \geq d, \text{ then } x \geq d_1 \text{ and } x \geq d_2.$$

17. Justify each step of the following direct proof, which shows that if x is a real number, then $x \cdot 0 = 0$. Assume that the following are previous theorems: If a, b, and c are real numbers, then $b + 0 = b$ and $a(b + c) = ab + ac$. If $a + b = a + c$, then $b = c$.

    Proof $x \cdot 0 + 0 = x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$; therefore, $x \cdot 0 = 0$.
18. If X and Y are nonempty sets and $X \times Y = Y \times X$, what can we conclude about X and Y? Prove your answer.
19. Prove that $X \cap Y \subseteq X$ for all sets X and Y.
20. Prove that $X \subseteq X \cup Y$ for all sets X and Y.
21. Prove that if $X \subseteq Y$, then $X \cup Z \subseteq Y \cup Z$ for all sets X, Y, and Z.
22. Prove that if $X \subseteq Y$, then $X \cap Z \subseteq Y \cap Z$ for all sets X, Y, and Z.
23. Prove that if $X \subseteq Y$, then $Z - Y \subseteq Z - X$ for all sets X, Y, and Z.
24. Prove that if $X \subseteq Y$, then $Y - (Y - X) = X$ for all sets X and Y.
25. Prove that if $X \cap Y = X \cap Z$ and $X \cup Y = X \cup Z$, then $Y = Z$ for all sets X, Y, and Z.
26. Prove that $\mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y)$ for all sets X and Y.
27. Prove that $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$ for all sets X and Y.
28. Prove that if $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$, then $X \subseteq Y$ for all sets X and Y.
29. Disprove that $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$ for all sets X and Y.
30. Give a direct proof along the lines of the second proof in Example 2.1.13 of the statement

$$X \cap (Y - Z) = (X \cap Y) - (X \cap Z) \quad \text{for all sets } X, Y, \text{ and } Z.$$

(In Example 2.1.11, we gave a direct proof of this statement using the definition of set equality.)

In each of Exercises 31–43, if the statement is true, prove it; otherwise, give a counterexample. The sets X, Y, and Z are subsets of a *universal set U. Assume that the universe for Cartesian products is $U \times U$.*

31. For all sets X and Y, either X is a subset of Y or Y is a subset of X.
32. $X \cup (Y - Z) = (X \cup Y) - (X \cup Z)$ for all sets X, Y, and Z.
33. $\overline{Y - X} = X \cup \overline{Y}$ for all sets X and Y.
34. $Y - Z = (X \cup Y) - (X \cup Z)$ for all sets X, Y, and Z.
35. $X - (Y \cup Z) = (X - Y) \cup Z$ for all sets X, Y, and Z.
36. $\overline{X - Y} = \overline{Y - X}$ for all sets X and Y.
37. $\overline{X \cap Y} \subseteq X$ for all sets X and Y.
38. $(X \cap Y) \cup (Y - X) = Y$ for all sets X and Y.
39. $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ for all sets X, Y, and Z.
40. $\overline{X \times Y} = \overline{X} \times \overline{Y}$ for all sets X and Y.
41. $X \times (Y - Z) = (X \times Y) - (X \times Z)$ for all sets X, Y, and Z.
42. $X - (Y \times Z) = (X - Y) \times (X - Z)$ for all sets X, Y, and Z.
43. $X \cap (Y \times Z) = (X \cap Y) \times (X \cap Z)$ for all sets X, Y, and Z.
44. Prove the associative laws for sets [Theorem 1.1.21, part (a)].
45. Prove the commutative laws for sets [Theorem 1.1.21, part (b)].
46. Prove the distributive laws for sets [Theorem 1.1.21, part (c)].
47. Prove the identity laws for sets [Theorem 1.1.21, part (d)].
48. Prove the complement laws for sets [Theorem 1.1.21, part (e)].
49. Prove the idempotent laws for sets [Theorem 1.1.21, part (f)].
50. Prove the bound laws for sets [Theorem 1.1.21, part (g)].
51. Prove the absorption laws for sets [Theorem 1.1.21, part (h)].
52. Prove the involution law for sets [Theorem 1.1.21, part (i)].
53. Prove the 0/1 laws for sets [Theorem 1.1.21, part (j)].
54. Prove De Morgan's laws for sets [Theorem 1.1.21, part (k)].

In Exercises 55–63, $\triangle$ denotes the symmetric difference operator defined as $A \triangle B = (A \cup B) - (A \cap B)$, where A and B are sets.

55. Prove that $A \triangle B = (A - B) \cup (B - A)$ for all sets A and B.
56. Prove that $(A \triangle B) \triangle A = B$ for all sets A and B.
★57. Prove or disprove: If A, B, and C are sets satisfying $A \triangle C = B \triangle C$, then $A = B$.
58. Prove or disprove: $A \triangle (B \cup C) = (A \triangle B) \cup (A \triangle C)$ for all sets A, B, and C.
59. Prove or disprove: $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$ for all sets A, B, and C.
60. Prove or disprove: $A \cup (B \triangle C) = (A \cup B) \triangle (A \cup C)$ for all sets A, B, and C.
61. Prove or disprove: $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$ for all sets A, B, and C.
62. Is $\triangle$ commutative? If so, prove it; otherwise, give a counterexample.
★63. Is $\triangle$ associative? If so, prove it; otherwise, give a counterexample.

Theorem 1.1.21 *Let U be a universal set and let A, B, and C be subsets of U. The following properties hold.*

(a) *Associative laws:*

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) *Commutative laws:*

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) *Distributive laws:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) *Identity laws:*

$$A \cup \varnothing = A, \quad A \cap U = A$$

(e) *Complement laws:*

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \varnothing$$

(f) *Idempotent laws:*

$$A \cup A = A, \quad A \cap A = A$$

(g) *Bound laws:*

$$A \cup U = U, \quad A \cap \varnothing = \varnothing$$

(h) *Absorption laws:*

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) *Involution law:*

$$\overline{\overline{A}} = A$$

(j) *0/1 laws:*

$$\overline{\varnothing} = U, \quad \overline{U} = \varnothing$$

(k) *De Morgan's laws for sets:*

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Proof The proofs are left as exercises (Exercises 44–54, Section 2.1) to be done after more discussion of logic and proof techniques.