Exercise 2a

Conditions: (1) $d(x,y) > 0\,,\; x \neq y$

(2) $d(x,y) = 0\,,\; x = y$

(3) $d(x,y) \leq d(x,z) + d(z,y)$

(4) $d(y,x) = d(x,y)$

i.) is a metric

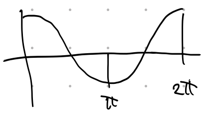
ii.) Not a metric, (1) is violated

iii.) Metric, and for $w_i = 1$ the Manhattan distance

iv.) Not a metric since (4) is violated

v.) Metric

Cos:



vi.) $y = \arccos(x)$

$y \in [0, \pi] \rightarrow (1)$ ✓

$x \in [-1, 1]$

Function is symmetric (4) ✓

$y, x > 0$

$$\frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum x_i^2}\sqrt{\sum y_i^2}} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum x_i^2 \sum y_i^2}} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum x_i^2 y_i^2}}\Bigg|_{x_i = y_i} = \frac{\sum_{i=1}^{n} x_i^2}{\sqrt{\sum x_i^4}}\Bigg|_{x_i = [2,2]} = \frac{4+4}{\sqrt{16+16}} = \frac{8}{\sqrt{32}} \neq 1$$

$\Rightarrow$ Not a metric, (2) is violated

vii.) Metric

Exercise 2b

Minkowski $:= d(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p} \quad p \in \mathbb{R}^+$

i.) $$d(ax, ay) = \left(\sum_{i=1}^{d} |ax_i - ay_i|^p\right)^{1/p} = \left(\sum_{i=1}^{d} |a(x_i - y_i)|^p\right)^{1/p} = \left(\sum |a|^p |x_i - y_i|^p\right)^{1/p}$$
$$= |a| \left(\sum |x_i - y_i|^p\right)^{1/p} = \underline{\underline{|a|\, d(x,y)}}$$

ii.) $$d(x+z, y+z) = \left(\sum |(x_i+z_i) - (y_i+z_i)|^p\right)^{1/p}$$
$$= \left(\sum |x_i + \cancel{z_i} - y_i - \cancel{z_i}|^p\right)^{1/p}$$
$$= \left(\sum |x_i - y_i|^p\right)^{1/p} = \underline{\underline{d(x,y)}}$$

Exercise 2c

vii.) $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases}$

$\Rightarrow$ vii.) is <u>not</u> homogen

$$d(ax,ay) = d\left(a\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}, a\begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}\right) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y \end{cases} = d(x,y) \neq |a| d(x,y)$$

Exercise 2d

$$\text{ii.)} := \sum x_i y_i (x_i - y_i)^2 = d(x,y)$$

$$\begin{aligned} d(x+z, y+z) &= \sum (x_i+z_i)(y_i+z_i)((x_i+z_i)-(y_i+z_i))^2 \\ &= \sum (x_i+z_i)(y_i+z_i)(x_i+\cancel{z_i}-y_i-\cancel{z_i})^2 \\ &= \sum (x_i+z_i)(y_i+z_i)(x_i-z_i)^2 \\ &= \sum (x_i y_i + z_i(x_i+y_i)+z_i^2)(x_i-y_i)^2 \\ &\neq d(x,y) \end{aligned}$$

$$\text{vi.)} := d(x,y) = \frac{2}{\pi} \operatorname{acos}\left(\frac{\sum x_i y_i}{\sqrt{\sum x_i^2}\sqrt{\sum y_i^2}}\right)$$

$$d(x+z, y+z) = \frac{2}{\pi} \operatorname{acos}\left(\underbrace{\frac{\sum (x_i+z_i)(y_i+z_i)}{\sqrt{\sum (x_i+z_i)^2}\sqrt{\sum (y_i+z_i)^2}}}_{(*)}\right)$$

$$(*) = \frac{\sum x_i y_i + z_i(x_i+y_i)+z_i^2}{\sqrt{\sum (x_i^2+2x_i z_i+z_i^2)(y_i^2+2y_i z_i+z_i)^2}}$$

$$\neq d(x,y)$$

<u>Translational invariance does not apply</u>