MTH2003

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

January 2019

Differential Equations

Module Leader: Prof. Vadim N. Biktashev

Duration: 2 HOURS.

Answer Section A (50%) and any TWO of the three questions in Section B (25% for each).

Marks shown in questions are merely a guideline. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED NOTE** examination.

USEFUL FORMULAS AND GRAPHS

These can be used without proofs, unless a proof is explicitly asked for.

Wronskian of two functions $y_1(x)$, $y_2(x)$ is

$$W[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Variation of Parameters formula: if $L = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + P(x)\dfrac{\mathrm{d}}{\mathrm{d}x} + Q(x)$ and $L[y_1] = 0$, $L[y_2] = 0$, and $W(x) \equiv W[y_1, y_2](x) \neq 0$, then the solution of

$$L[y] = R(x)$$

is given by

$$y(x) = -y_1(x)\int \frac{y_2(x)R(x)}{W(x)}\,\mathrm{d}x + y_2(x)\int \frac{y_1(x)R(x)}{W(x)}\,\mathrm{d}x.$$

Laplacian in polar coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2}$$

An important boundary-value problem

$$y'' + \lambda y = 0, \qquad y(0) = y(2\pi), \quad y'(0) = y'(2\pi)$$

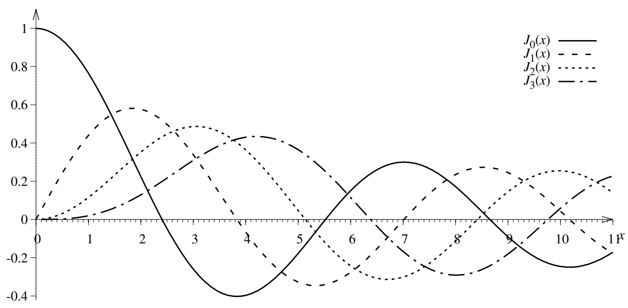
has solutions $\lambda = n^2$, $n \in \mathbb{Z}_{\geq 0}$, $\phi_0(x) = 1$, $\phi_n^{(1)}(x) = \cos(nx)$, $\phi_n^{(2)}(x) = \sin(nx)$.

Bessel equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

Solutions that are bounded as $x \to 0$ are known as Bessel functions of the first kind,

$$y(x) = J_n(x).$$



SECTION A

1. (a) Find the general solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 3x^2\, e^{-2x}.$$

Find also the solution of this equation satisfying $y(0) = y(1) = 0$. (10)

(b) Verify that $y_1(x) = e^x$ is a solution of the differential equation

$$x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = 0, \quad x \neq 0,$$

and find its general solution. Write down a solution $y_2(x)$ that is linearly independent of $y_1(x)$. Confirm it by using the Wronskian. (10)

(c) Use a trial function of the form $y = \sum_{n=0}^{\infty} a_n x^n$ to find the solution of the differential equation

$$\frac{d^2y}{dx^2} - y = 0.$$

Find the recurrence relation for the coefficients a_n. Deduce from it the explicit formulas for the coefficients in the series for solutions $y_1(x)$ and $y_2(x)$, where $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 0$, $y_2'(0) = 1$. Hence find $y_1(x)$ and $y_2(x)$ in elementary functions. *Marks will be awarded only for solutions using the prescribed method!* (10)

(d) Show that *all* the eigenvalues λ_n and eigenfunctions $\phi_n(x)$ for the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \qquad y'(-1) = 0, \qquad y'(1) = 0$$

are given by $\lambda_n = (n\pi/2)^2$, where $n \in \{0, 1, 2, \dots\}$, and $\phi_n = \cos(n\pi x/2)$ for even n and $\phi_n = \sin(n\pi x/2)$ for odd n. (10)

(e) Using the results of part (d), write down the general solution of the partial differential equation with boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u_x(-1,t) = 0,\ u_x(1,t) = 0,$$

in the form of an infinite series of terms of the form $X(x)T(t)$. Find also the solution that satisfies the initial conditions $u(x,0) = 2\cos^2(\pi x)$. For this solution, find $u(1,1)$ to 7 significant figures (you should not need a calculator for this). (10)

[50]

SECTION B

2. (a) Find the typical solution (depending on an arbitrary constant) and the special solutions (any other solutions that are not captured by the typical solution) of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos(x)\,|y-1|^{1/2}\,.$$

Find a solution of this equation satisfying the initial condition $y(0) = 2$ and show that it is unique in some interval containing the point $x = 0$. Find also a solution of this equation satisfying the initial condition $y(0) = 1$ and state whether this solution is unique in some interval containing the point $x = 0$. (15)

(b) Peano's theorem states that an initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \qquad y(x_0) = y_0$$

is guaranteed to have a solution in some interval of x including point x_0, if function $f(x,y)$ is defined and continuous in a rectangle in the (x,y) plane, including point (x_0,y_0) in its interior.

State a further assumption that is required by Picard's theorem guaranteeing also the uniqueness of the solution to this problem.

(5)

(c) Determine the maximal interval in which the solution to the initial-value problem with $y(0) = 2$ found in part (a) of this question is unique. (5)

[**25**]

3. (a) Explain why the method of Frobenius is applicable to the equation

(Eq3a) $$x^2y'' + xy' - x^2y = 0, \qquad x > 0,$$

and use this method to show that a solution of this equation, bounded as $x \to 0$, can be written in the form

(#) $$y_1 = x^m \sum_{n=0}^{\infty} a_{2n} x^{2n},$$

with an appropriately chosen m which you need to identify, and that the coefficients a_n satisfy the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+2)^2}.$$

(15)

(b) Show that the coefficients satisfying the above recurrence relation are given by the explicit formula

$$a_{2n} = \frac{a_0}{(2^n n!)^2}.$$

(5)

(c) State Abel's identity for a ODE

$$y'' + p(x)y' + q(x)y = 0$$

(you are not required to prove it). State what form this identity takes for the equation (Eq3a). Use this to show that any solution of (Eq3a) that is linearly independent of the solution (#) found in part (a), cannot be an analytical function bounded as $x \to 0$.

(5)

[**25**]

4. (a) Motion of the membrane of a circular drum is described by a wave equation

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \qquad x^2 + y^2 \leq 1, \qquad t \geq 0,$$

with boundary conditions $u(x, y, t) = 0$ whenever $x^2 + y^2 = 1$. Consider polar coordinates (ρ, θ) in the (x, y) plane, $x = \rho \cos \theta$, $y = \rho \sin \theta$. Use the method of separation of variables, a.k.a. generalized Fourier method, to describe the general solution of this equation, in the form of a double-infinite series of "modes", that is, solutions in the form $u(\rho, \theta, t) = R(\rho)Q(\theta)T(t)$. Assume without proof that time dependence can be only of the form $T(t) = A \cos(\omega t) + B \sin(\omega t)$ for arbitrary A, B but appropriately chosen $\omega > 0$. (12)

(b) This circular drum will sound a number of incommensurable frequencies. What are the two lowest frequencies in its spectrum? Give the answers to 2 significant figures. (6)

(c) For each of the lowest frequency modes, identified in part (b), consider the particular solutions $u(\rho, \theta, t)$ in which only that mode is present. Describe the shape of their nodal lines, that is, loci of the points (ρ, θ) on the drum such that $u(\rho, \theta, t) = 0$ for all $t > 0$. (7)

[25]