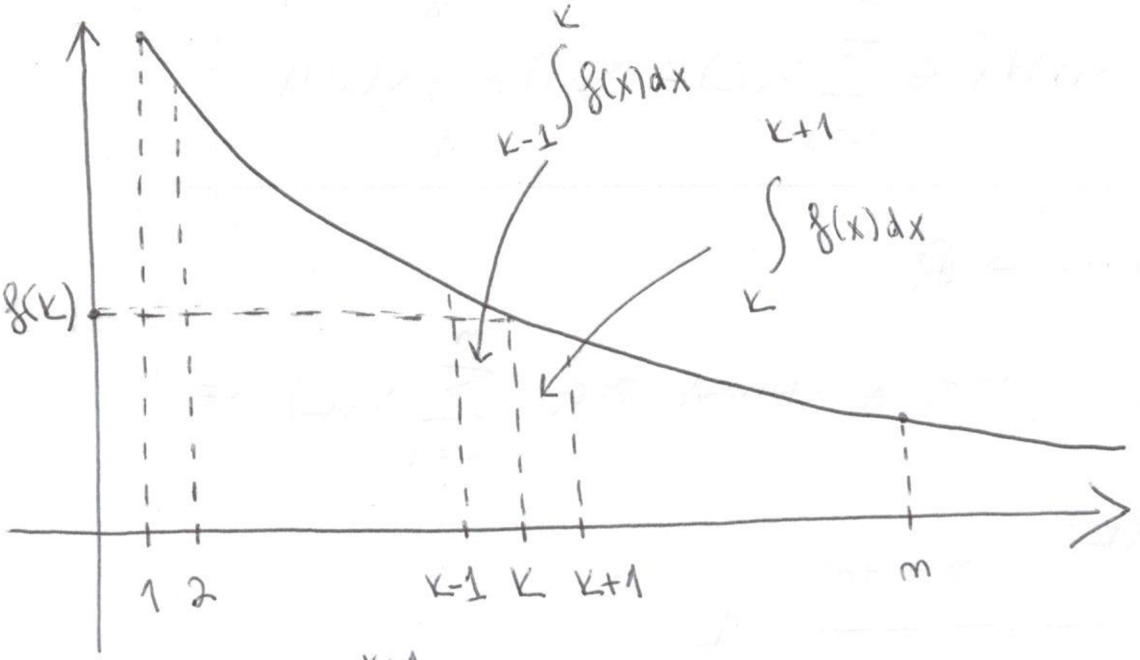
$f : [1, +\infty) \longrightarrow \mathbb{R}$ positiva, continua, monotona decrescente



$$\text{I} \quad f(k) \geq \int_{k}^{k+1} f(x)\,dx \qquad k = 1, \quad , m-1$$

$$\text{II} \quad f(k) \leq \int_{k-1}^{k} f(x)\,dx \qquad k = 2, \quad , m$$

$\boxed{\text{DAL DISEGNO}}$

$$\sum_{k=1}^{m} f(k) = \sum_{k=1}^{m-1} f(k) + f(m) \underset{\text{I}}{\geq} \sum_{k=1}^{m-1} \int_{k}^{k+1} f(x)\,dx + f(m)$$

$$\geq \int_{1}^{m} f(x)\,dx + f(m)$$

$$\sum_{k=1}^{m} f(k) = \sum_{k=2}^{m} f(k) + f(1) \underset{\text{II}}{\leq} \sum_{k=1}^{m} \int_{k-1}^{k} f(x)\,dx + f(1)$$

$$\leq \int_{1}^{m} f(x)\,dx + f(1)$$

$$\boxed{f(m) + \int_{1}^{m} f(x)\,dx \leq \sum_{k=1}^{m} f(k) \leq f(1) + \int_{1}^{m} f(x)\,dx}$$

FORMULA PER STIMARE SOMMATORIE

se $f : [1, \infty) \longrightarrow \mathbb{R}$ positiva, continua, monotona crescente

$$\boxed{f(1) + \int_1^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq f(n) + \int_1^n f(x)\,dx}$$

$g : [1, \infty) \longrightarrow \mathbb{R}$

g FORMULA ASINTOTICA CHIUSA PER $\sum_{k=1}^{n} f(k)$ SE

$$\frac{\sum_{k=1}^{n} f(k)}{g(n)} \xrightarrow{n \to +\infty} 1$$

SOMMATORIE NOTE (DA RICORDARE)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \frac{n^2(n+1)^2}{4}$$ (PRIMO TUTORAGGIO)

$a \neq 1 \quad \sum_{k=0}^{k} a^k = \frac{a^{n+1} - 1}{a - 1}$ ⚠ PARTE DA 0 NON DA 1

$$\sum_{k=1}^{n} 1 = n$$

(1) TROVARE UNA FORMULA ASINTOTICA CHIUSA

PER $H_n = \sum_{k=1}^{n} \frac{1}{k}$

$f : [1, \infty) \longrightarrow \mathbb{R} \qquad f(x) = \frac{1}{x}$ $\quad f$ continua (ovvio!)

$f(x) > 0 \quad \forall\, x \geq 1$ $\qquad f$ positiva

$f'(x) = -\frac{1}{x^2} < 0 \quad \forall\, x > 1 \Rightarrow$ f monotona decrescente

$$\Rightarrow \quad f(n) + \int_1^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(x)\,dx$$

$$\int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \left[\log x\right]_1^n = \log n - \log 1 = \log n$$

$$\Rightarrow \quad \frac{1}{n} + \log n \leq \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \log n$$

$g : [1, \infty) \longrightarrow \mathbb{R} \qquad g(x) = \log x$

$$\frac{\frac{1}{n} + \log n}{\log n} \leq \frac{\sum_{k=1}^{n} \frac{1}{k}}{\log n} \leq \frac{1 + \log n}{\log n}$$

$$\downarrow n \to +\infty \qquad\qquad \downarrow n \to +\infty$$

$$1 \qquad\qquad\qquad 1$$

SEGUE CHE $\dfrac{\sum_{k=1}^{n} f(k)}{g(n)} = \dfrac{\sum_{k=1}^{n} \frac{1}{k}}{\log n} \xrightarrow{n \to +\infty} 1$

QUINDI $\log n$ È UNA FORMULA ASINTOTICA CHIUSA

PER $H_n = \sum_{k=1}^{n} f(k)$

② TROVARE UNA STIMA PER $\sum_{k=1}^{n} k e^{-k^2}$

$f : [1, \infty) \longrightarrow \mathbb{R} \qquad f(x) = x e^{-x^2}$ continua (ovvio!)

$f(x) = \underset{\overset{\vee}{0}}{x}\,\underset{\overset{\vee}{0}}{e^{-x^2}} > 0 \quad \forall x \geq 1$ f positiva

$f'(x) = e^{-x^2} - 2x^2 e^{-x^2} = \underset{\overset{\wedge}{0}}{(1-2x^2)}\,\underset{\overset{\vee}{0}}{e^{-x^2}} < 0 \quad \forall x \geq 1$

$\Rightarrow$ f monotona decrescente

$\Rightarrow \quad f(n) + \int_1^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(x)\,dx$

$\int_1^n f(x)\,dx = \int_1^n x e^{-x^2} = \int_1^n \left(-\frac{1}{2}\right)(-2x e^{-x^2})\,dx =$

$= -\frac{1}{2} \int_1^n (-2x) e^{-x^2}\,dx = -\frac{1}{2} \left[e^{-x^2} \right]_1^n = -\frac{1}{2}\left(e^{-n^2} - e^{-1}\right)$

$= \frac{1}{2}\left(e^{-1} - e^{-n^2}\right)$

$$\Rightarrow \quad n e^{-n^2} + \frac{1}{2}(e^{-1} - e^{-n^2}) \leq \sum_{k=1}^{n} k e^{-k^2} \leq e^{-1} + \frac{1}{2}(e^{-1} - e^{-n^2})$$

$$\text{STIMA PER } \sum_{k=1}^{n} k e^{-k^2} \qquad \frac{3}{2}e^{-1} - \frac{1}{2}e^{-n^2}$$

③ TROVARE UNA STIMA PER $\sum_{k=1}^{n} \frac{1}{k^3}$

$f : [1, \infty) \longrightarrow \mathbb{R} \qquad f(x) = x^{-3} = \frac{1}{x^3}$ continua (ovvio!)

$f(x) = x^{-3} > 0 \quad \forall x \geq 1$ f positiva

$f'(x) = -3x^{-4} < 0 \quad \forall x \geq 1 \Rightarrow f$ monotona decrescente

$$\Rightarrow f(n) + \int_1^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(x)\,dx$$

$$\int_1^n f(x)\,dx = \int_1^n x^{-3}\,dx = \left[\frac{x^{-2}}{-2}\right]_1^n = -\frac{1}{2}n^{-2} + \frac{1}{2} = \frac{1}{2}(1 - n^{-2})$$

$$\Rightarrow n^{-3} + \frac{1}{2}(1 - n^{-2}) \leq \sum_{k=1}^{n} \frac{1}{k^3} \leq 1 + \frac{1}{2}(1 - n^{-2})$$

STIMA PER $\sum_{k=1}^{n} \frac{1}{k^3}$

4) TROVARE UNA STIMA PER $\sum_{k=1}^{n} k e^k$

$f : [1, \infty) \longrightarrow \mathbb{R} \quad f(x) = x e^x$ f continua

$f(x) = x e^x > 0 \quad \forall x \geq 1$ f positiva

$f'(x) = e^x + x e^x = (1+x) e^x > 0 \quad \forall x \geq 1$

$\Rightarrow$ f monotona crescente

$$\Rightarrow \quad f(1) + \int_1^n f(x)\,dx \leq \sum_{k=1}^{n} f(k) \leq f(n) + \int_1^n f(x)\,dx$$

$$\int_1^n f(x)\,dx = \int_1^n x e^x\,dx = \left[e^x(x-1)\right]_1^n = e^n(n-1)$$

$$\Rightarrow \quad e + e^n(n-1) \leq \sum_{k=1}^{n} k e^k \leq n e^n + e^n(n-1)$$

$$n e^n + e - e^n \leq \sum_{k=1}^{n} k e^k \leq 2n e^n - e^n$$

STIMA PER $\sum_{k=1}^{n} k e^k$

(5) TROVARE UNA FORMULA CHIUSA PER $\sum_{k=1}^{n} n\,2^{-k}$

$$\sum_{k=1}^{n}\left(n\,2^{-k}\right) = n\sum_{k=1}^{n} 2^{-k} = n\sum_{k=1}^{n}\left(\frac{1}{2}\right)^{k} =$$

$$= n\left(\sum_{k=0}^{n}\left(\frac{1}{2}\right)^{k} - \left(\frac{1}{2}\right)^{0}\right) = n\left(\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1} - 1\right) =$$

$$= n\left(\frac{\frac{1}{2^{n+1}} - 1}{\frac{1}{2} - 1} - 1\right) = n\left(\frac{\frac{1 - 2^{n+1}}{2^{n+1}}}{\frac{1-2}{2}} - 1\right)$$

$$= n\left(-\frac{1 - 2^{n+1}}{2^{n}} - 1\right) = n\left(-\frac{1}{2^{n}} + 2 - 1\right) = n\left(1 - \frac{1}{2^{n}}\right)$$

(6) TROVARE UNA FORMULA CHIUSA PER $\sum_{k=1}^{n}\sum_{j=1}^{m} k^{3} j$

$$\sum_{k=1}^{n}\sum_{j=1}^{m}\left(k^{3} j\right) = \left(\sum_{k=1}^{n} k^{3}\right)\left(\sum_{j=1}^{m} j\right) =$$ PRIMO TUTORAGGIO

$$= \left(\sum_{k=1}^{n} k\right)^{2}\left(\sum_{j=1}^{m} j\right) = \left(\frac{n(n+1)}{2}\right)^{2}\frac{m(m+1)}{2} =$$

$$= \frac{n^{2}(n+1)^{2}\,m(m+1)}{8}$$

④ TROVARE UNA FORMULA CHIUSA PER $\displaystyle\sum_{k=1}^{n}\sum_{j=1}^{m}(2k+3j)$

$$\sum_{k=1}^{n}\sum_{j=1}^{m}(2k+3j)=\sum_{k=1}^{n}\left(\sum_{j=1}^{m}(2k)+\sum_{j=1}^{m}(3j)\right)=$$

$$=\sum_{k=1}^{n}\left(2k\sum_{j=1}^{m}1+3\sum_{j=1}^{m}j\right)=\sum_{k=1}^{n}\left(2km+3\frac{m(m+1)}{2}\right)=$$

$$=\sum_{k=1}^{n}(2km)+\sum_{k=1}^{n}\left(3\frac{m(m+1)}{2}\right)=$$

$$=2m\sum_{k=1}^{n}k+3\frac{m(m+1)}{2}\sum_{k=1}^{n}1$$

$$=2m\frac{n(n+1)}{2}+3\frac{m(m+1)}{2}n$$

$$=m\,n(n+1)+\frac{3}{2}m\,n\,(m+1)$$

$$\prod_{k=1}^{n}x_k=x_1\cdot x_2\cdot\ \ \cdot x_k\cdot\ \ \cdot x_{n-1}\cdot x_n$$

$$n!=\prod_{k=1}^{n}k=1\cdot 2\cdot\ \ \cdot k\cdot\ \ (n-1)n$$

$a,b>0 \qquad \log(ab)=\log a+\log b$

$x_k>0 \qquad k=1,\ ,n$

$$\boxed{\log\left(\prod_{k=1}^{n}x_k\right)=\sum_{k=1}^{n}\log x_k}$$

FORMULA UTILE PER TROVARE FORMULE CHIUSE PER $\displaystyle\prod_{k=1}^{n}x_k$

(8) TROVARE UNA FORMULA CHIUSA PER $\prod_{k=1}^{n} e^{2k-1}$

$$\log\left(\prod_{k=1}^{n} e^{2k-1}\right) = \sum_{k=1}^{n} \log\left(e^{2k-1}\right) = \sum_{k=1}^{n} (2k-1) =$$

$$= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = 2\sum_{k=1}^{n} k - n = 2\,\frac{n(n+1)}{2} - n$$

$$= n(n+1) - n = n^2$$

$$\prod_{k=1}^{n} e^{2k-1} = e^{\log\left(\prod_{k=1}^{n} e^{2k-1}\right)} = e^{n^2}$$

(9) TROVARE UNA STIMA PER $\sum_{k=1}^{n} e^{1/k}$

$$\log\left(\prod_{k=1}^{n} e^{1/k}\right) = \sum_{k=1}^{n} \log e^{1/k} = \sum_{k=1}^{n} 1/k = H_n$$

$$\prod_{k=1}^{n} e^{1/k} = e^{\log\left(\prod_{k=1}^{n} e^{1/k}\right)} = e^{H_n}$$

DALL' ESERCIZIO (1)

$$\frac{1}{n} + \log n \leq H_n \leq 1 + \log n$$

QUINDI

$$e^{\frac{1}{n}+\log n} \leq e^{H_n} \leq e^{1+\log n}$$

$$e^{1/n}\, e^{\log n} \leq \prod_{k=1}^{n} e^{1/k} \leq e^{1}\, e^{\log n}$$

$$e^{1/n}\, n \leq \prod_{k=1}^{n} e^{1/k} \leq e\, n$$

STIMA PER $\prod_{k=1}^{n} e^{1/k}$

(10) TROVARE UNA FORMULA ASINTOTICA CHIUSA PER $\sum_{k=1}^{n} H_k$

$$\sum_{k=1}^{n} H_k = \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{k} \underset{\substack{\text{CAMBIO} \\ \text{DI INDICE}}}{=} \sum_{j=1}^{n} \sum_{k=j}^{n} \frac{1}{j} =$$

$$= \sum_{j=1}^{n} \left(\frac{1}{j} \sum_{k=j}^{n} 1 \right) = \sum_{j=1}^{n} \left(\frac{1}{j} (n-j+1) \right) =$$

$$= \sum_{j=1}^{n} \frac{1}{j}(n+1) - \sum_{j=1}^{n} \frac{1}{j} j =$$

$$= (n+1) \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} 1 = (n+1) H_n - n$$

DALL' ESERCIZIO (1)

$$\frac{1}{n} + \log n \leq H_n \leq 1 + \log n$$

OTTENGO QUINDI

$$(n+1)\left(\frac{1}{n} + \log n \right) - n \leq (n+1) H_n - n \leq (n+1)(1 + \log n) - n$$

$$1 + \frac{1}{n} + n\log n + \log n - n \leq \sum_{k=1}^{n} H_k \leq n + 1 + n\log n + \log n - n$$

$$g : [1, \infty) \longrightarrow \mathbb{R} \quad g(x) = x \log x$$

$$\frac{1+\frac{1}{n}+n\log n+\log n-n}{n\log n} \leq \frac{\sum_{k=1}^{n} H_k}{n\log n} \leq \frac{n+1+n\log n+\log n-n}{n\log n}$$

$$\frac{1+\frac{1}{n}+n\log n+\log n-n}{n\log n} \xrightarrow{n\to+\infty} 1 \qquad \frac{n+1+n\log n+\log n-n}{n\log n} \xrightarrow{n\to+\infty} 1$$

SEGUE CHE $\dfrac{\sum_{k=1}^{n} H_k}{g(n)} = \dfrac{\sum_{k=1}^{n} H_k}{n\log n} \xrightarrow{n\to+\infty} 1$

QUINDI $n\log n$ È UNA FORMULA ASINTOTICA CHIUSA

PER $\sum_{k=1}^{n} H_k = \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{1}{k}$

$$\sum_{k=1}^{n} \sum_{j=1}^{k} x_{kj} = \sum_{j=1}^{n} \sum_{k=j}^{n} x_{kj}$$

CAMBIO DI INDICE

$$\begin{array}{c|cccccc}
 & 1 & 2 & 3 & j & & n \\
\hline
1 & X_{11} & - & - & \cdot & & - \\
2 & X_{21} & X_{22} & - & \cdot & & - \\
3 & X_{31} & X_{32} & X_{33} & \cdot & & - \\
k & \cdot & \cdot & \cdot & \ddots & & \\
 & & & & X_{n-1,n-1} & & - \\
n & X_{n1} & X_{n2} & X_{n3} & \cdots & X & X_{nn}
\end{array}$$

STO SOMMANDO SUL TRIANGOLO IN FIGURA

(11) TROVARE UNA FORMULA CHIUSA PER $\sum_{k=1}^{n} k2^k$

$$\sum_{k=1}^{n}(k2^k) = \sum_{k=1}^{n}\left(\left(\sum_{j=1}^{k}1\right)2^k\right) = \sum_{k=1}^{n}\sum_{j=1}^{k}2^k =$$ CAMBIO DI INDICE

$$= \sum_{j=1}^{n}\sum_{k=j}^{n}2^k = \sum_{j=1}^{n}\left(\sum_{k=0}^{n}2^k - \sum_{k=0}^{j-1}2^k\right) =$$

$$= \sum_{j=1}^{n}\left(\frac{2^{n+1}-1}{2-1} - \frac{2^j-1}{2-1}\right) = \sum_{j=1}^{n}(2^{n+1}-2^j)$$

$$= \sum_{j=1}^{n}2^{n+1} - \sum_{j=1}^{n}2^j = 2^{n+1}\sum_{j=1}^{n}1 - \left(\sum_{j=0}^{n}2^j - 2^0\right)$$

$$= 2^{n+1}n - \frac{2^{n+1}-1}{2-1} + 1 = 2\,n + 2^{n+1} + 2$$