Complete and Compact Description of Fault Tolerant Quantum Gate Operations for Topological Majorana Qubit Systems

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Among the list of major threats to quantum computation, quantum decoherence poses one of the largest because it generates losses to the environment within a computational system which can not be recovered via error correction methods. Basing the computational states on topological phases offer a promising solution to this problem by removing the possibility of local decoherence entirely. Here, the qubit is based on non-local Majorana fermions which are topological by nature, and the gate operations are generated by swapping or braiding the positions of said Majorana. The algorithmic calculation for such gate operations are well known; however, the opposite process of gates to braids is underdeveloped with many different possible qubit definitions and gates. In the article, the braid calculation is recapitulated, taking notice of all patterns generated for multiqubit systems, and a complete list of gates for a general n-qubit system is listed with transformations between possible definitions as well. **Context**, **Need**, **Task**, **Object**, **Findings**, Conclusions, Perspectives *Summary of subjects, conclusions and results***146; <500 words**

1. INTRODUCTION

The standard superconducting (SC) qubits that form the basis of computation for the state of the art systems at Google, IBM, Baidu, and Alibaba have generated great strides for the field of quantum computation (QC). They satisfy the criteria for a “good enough” qubit because one possesses a sufficient degree of control over the individual qubit state, and they scale better than other options **CITE: SCALABILITY/ CONTROL OF SC QB**. However, there may exist an absolute ceiling for the scalability and tolerance of SC qubit based systems due to the problem of decoherence. Of course, this problem is shared among all quantum systems indiscriminately, but it particularly threatens the future of QC since this field deals with the manipulation of information. One would hope to minimize these effects in order to maintain the integrity of the information provided to the computer. If the computer flipped a bit or deleted information without the user knowing, what good is this system?

To ameliorate the situation, one may bolster the system by secluding the qubits from the environment as much as possible. Whatever decoherence that remains may be dealt with using a number of error correcting methods. However, these require the assumption that, after all environmental effects are considered, the true state of the system remains a linear combination of qubit Hamiltonian energy eigenstates **CITE: ERROR CORRECTION METHODS**. This would mean that the true state of the system is recoverable via a unitary transformation. In other words, we hope that the state remains pure when all is said and done. In actuality, environmental fluctuations force the qubit into a mixed state which is not at all a linear combination of the energy eigenstates of the qubit Hamiltonian. Furthermore, the system is not represented by a state vector, but a mixed state density matrix instead **CITE: ERROR CORRECTION METHODS**.

For these reasons, an alternate solution to the decoherence problem has been proposed which makes use of topological states of matter [1]. As their name suggests, these are condensed matter systems equipped with a degenerate ground state manifold based on some associated topology that is separated from the remaining spectrum by an energy gap. One can form a computational basis from this ground state manifold and construct a quantum computer based on a topologically invariant parameter with states which can not couple via local perturbations. Such a computer would be fault tolerant or decoherence proof [2].

There exist copious examples of well studied systems

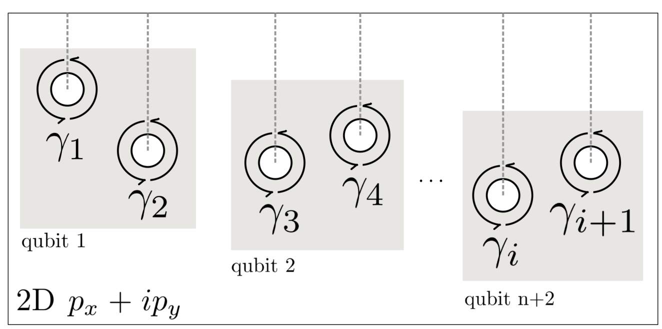

FIG. 1. Majorana Fermion Setup. The system is located in a 2D $p_x + ip_y$- wave SC which is induced on the surface of a TI by an s-wave SC adjacent to the system. The qubits are defined by collecting two MF into one fermionic operator, depicted here as grey boxes. Vortices are made by allowing a magnetic field to penetrate the system with some strength in between the two critical values. Braids are made by switching positions of the vortices where one must cross the arbitrary branch cuts, depicted as dotted lines.

with varying flavors and temperaments that exhibit these topological qualities. A promising candidate and the subject of this work makes use of non local Majorana fermion (MF) pairs within 2D $p_x + ip_y$ wave SC formed by adhering a 3D type-II, s-wave SC to a topological insulator (TI) [3]. These exotic quasi particles emerge wherever there is a transition from one topological state to another **CITE: MF EMERGING ON EDGES OF SYSTEM KITEAV?**. For this particular system, these topological transition points correspond to wherever the SC order parameter becomes zero or equivalently wherever there is a gap closure in the spectrum. On the bulk boundary of the system, the topology changes from the nontrivial 2D SC to trivial vacuum, and the associated Hamiltonians of these two phases cannot be smoothly connected without closing the gap. Therefore, on the boundary of the system, there will be a chiral MF **CITE: CHIRAL MF ON BOUNDARY**. Additionally, this system is a type-II SC permits local gap closures for points where the magnetic field is in between the first and second critical field values. These are localized normal phase points which are accompanied by Abrikosov vortices. Again, the Hamiltonians cannot be smoothly connected without gap closure, and a MF is hosted here. In both cases, the gap closure is a mechanism which allows for the trivial (vacuum) and topological SC Hamiltonians to be smoothly connected **CITE: HOMOTOPY OF HAMS**. The equal parts particle-hole excitations at these boundary points are self-annihilating and are a MFs **CITE: MF EQUAL PARTS ANNHILIATING AND CREATING**.

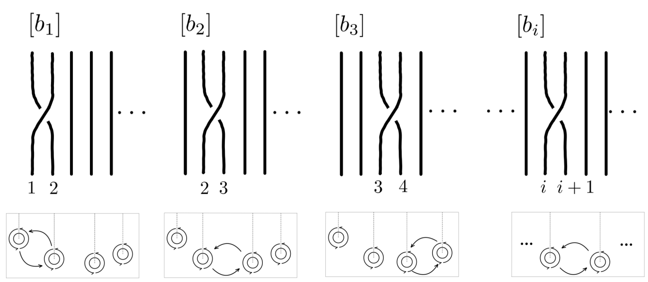

FIG. 2. Braid Group. B_k may be divided into equivalence classes, $[b_i]$, where the equivalence relation is defined as the braids which may be deformed into one another via some smooth operation. This system only has access to such classes of braids. Each braid is defined as the movement of one vortex around another.

These modes exist purely two dimensionally and, due to this fact, MF exchange statistics may be qualitatively different from 3D statistics. In three dimensions, k indistinguishable particles exchange symmetrically (Bosons) or antisymmetrically (Fermions) following the Permutation group, P_k; however, 2D particles may exchange with any phase change in between 0 and π, obeying anyonic statistics instead. In most cases, if these modes correspond to a degenerate ground state, particle exchanges will still simply impart a phase factor. However, in some special cases these exchanges rotate a manifold spanned by the degenerate ground states. In this situation these exchanges do not commute and are said to be Non-Abelian. The 2D MFs exchange in such a way which follows the Braid group, B_k, which have unitary operator representations, $U_i^{(k)}$ [4][2].

The methods which outline direct calculation of unitary operators from a braid are well documented in several other publications [2][5][6][7][4][8]. However, these sources only provide limited examples and make no attempt to compile a complete list of unitary gates. To expand upon this subject, one of the gaps that needs to be filled is the delineation of the full extent of unitary operations possible. Furthermore, because of the arbitrary nature of qubit definition in this setting, these sources appear to construct different gates. To connect the various definitions of qubits and construct the most complete picture, the transformations between definitions must be specified as well. Additionally, the opposite algorithmic process, quantum gates to MF braids, is currently underdeveloped. This opposite directional operation is required for the construction of an accessible quantum computer. Quantum computation and quantum information science is a diverse field with mathematicians, computer scientists, and physicists, and, for this reason, a large portion of research in this field is unconcerned with the physical realization of a qubit and its associated gate operations. Instead, this research is performed with no particular system in mind.

This work serves to supplement that which is already written on the subject of MF braids by expanding the possible gate operations and then compacting them into succinct mathematical form. Here, all possible linear representations of braids are calculated for the two and four MF cases. Following these direct calculations and the insights gained from them, all possible gates are formed for arbitrary numbers of qubits and general forms are stated.

2. DESCRIPTION OF SYSTEM

The typical setup for a primitive SC topological computer is pictorially depicted in Figure 1 where a 2D SC region has been induced in the surface of the topological insulator. A magnetic field penetrates the system, closing the gap, and inducing Abrikosov vortices which host the MFs. To sufficiently separate the ground state from excited states, one may also grow pin sites [3]. As one vortex encircles another, the path taken generates a total Berry phase of π to the state of the system. This winding number is represented by branch cuts depicted as dotted lines in Figure 1, starting at each vortex and ending somewhere on the borders of the region [9]. These cuts are made arbitrarily and will not affect the total calculation as long as everything is consistent.

One may only interact with these vortices through some macroscopic means, the only actions one may take in regards to the Majorana operators, γ_i, are,

1. *Relabelling:* $\gamma_i \rightarrow \gamma_j$
2. *Crossing branch cuts:* $\gamma_i \rightarrow -\gamma_j$

which may be accomplished by a physical exchange of MFs or an exchange and crossing branch cuts.

BRAID CALCULATION

The first step in this calculation is to define the MF operators. The 2D SC system is special in that the gap closure at Dirac points occurs precisely where the Fermi level sits **CITE: GAP CLOSURE AT FERMI LEVEL**. For this reason, the Bogolon ground state excitations in the SC system are equal parts creation of electron and hole,

$$\gamma_i = a^\dagger + a$$
$$\gamma_{i+1} = i(a^\dagger - a),$$

where each pair of γ's are associated with a, a highly non-local fermionic operator [8].One method of determining the occupation number of the fermionic operator is as follows. When one is ready to take a measurement of this system, bringing the vortices together removes the degeneracy in the ground state. The resulting two energy eigenstate are situated above and below the Fermi surface. The occupation number associated with this operator represents the number of quasi-electrons populating the final lower energy state once a fusion is made between the two MFs [2]. The computational states $|0\rangle$ and $|1\rangle$ are defined to be the occupation of this level and are the qubit state upon measurement.

Now, one establishes the algebra associated with the creation and annihilation operators,

$$\{a_1, a_1^\dagger\} = 1 \tag{1}$$
$$\{a_1, a_1\} = \{a_1^\dagger, a_1^\dagger\} = 0. \tag{2}$$

These relationships are of course motivated by Pauli exclusion, i.e. $(a_1^\dagger)^2 |0\rangle = 0$ and $(a_1 a_1^\dagger + a_1^\dagger a_1)|0\rangle = |0\rangle$. The MF operators then inherit their own commutation relationship from eq. 1 and eq. 2 [8],

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}, \tag{3}$$

yielding two rules for Majorana operators:

1. *Two identical γ operators in a row will annihilate to 1.*
2. *Exchanging any two adjacent operators produces a negative sign.*

The representation that maps the braid group to linear operators, $\rho : B_k \rightarrow \mathcal{LO}$, is an exponentiation of the MF operators [8][2][9],

$$\rho([b_i]) = U_i = e^{\frac{\pi}{4}\gamma_i\gamma_{i+1}} = \frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_{i+1}). \tag{4}$$

One may verify that this operator indeed has the correct action on the MF by performing a similarity transform on some arbitrary γ_k,

$$U_i^\dagger \gamma_k U_i = \frac{1}{2}(1 + \gamma_{i+1}\gamma_i)\gamma_k(1 + \gamma_i\gamma_{i+1}).$$

If $k \neq i$ or $i+1$, meaning the transformation is on a MF uninvolved with the braid, then one would expect γ_k to be invariant under this transformation,

$$U_i^\dagger \gamma_k U_i = \gamma_k.$$

Commutation rules from eq. 3 are used to move and cancel the operators throughout. If the transformation is being performed on operators that are involved in the braid,

$$U_i^\dagger \gamma_i U_i = \gamma_{i+1},$$

and,

$$U_i^\dagger \gamma_{i+1} U_i = -\gamma_i.$$

These three transformations demonstrate that U_i does in fact represent a braid between adjacent MF i and $i+1$. Since $\gamma_i \rightarrow \gamma_{i+1}$ and $\gamma_{i+1} \rightarrow -\gamma_i$, U_i represents a counterclockwise braid where γ_{i+1} crosses a branch cut as depicted in Figure 2.

In all that follows, qubits are formed from MF pairs which are adjacent to one another. This is done arbitrarily, and there is no standard method of defining fermionic operators. One may make any choice here and the resulting gates may look different upon first glance. However, as shown in the final section, the choice of definition are connected via a similarity transformation on the $\mathcal{LO}$ representation of braids, and, once one set is calculated, all others are recoverable. For this reason, we have chosen to collect them adjacently, as in Figure 1, to maintain some form of notational simplicity.

Defining two adjacent MF in terms of qubit operators,

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} a_1 \\ a_1^\dagger \end{pmatrix}$$

and, by taking the inverse,

$$\begin{pmatrix} a_1 \\ a_1^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

the qubit operators may be written in terms of MF operators. These form the computational basis, $\{|0\rangle, |1\rangle\} = \{|0\rangle, a_1^\dagger |0\rangle\}$. To calculate the matrix representation of $U_1^{(2)}$, simply rewrite it in terms of the qubit operators, and operate on all members of the basis to determine the braid's effect on the ground state **CITE: BRAID CALC**. In this document, the number within the superscript parenthesis labels the number of MFs within the computational system, and the subscript labels the braid in accordance with Figure 2. The $|0\rangle$ state transforms as follows,

$$\begin{aligned} U_1^{(2)} |0\rangle &= \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)|0\rangle \\ &= \frac{1}{\sqrt{2}}(1+i\{a_1^\dagger + a_1\}\{a_1^\dagger - a_1\})|0\rangle \\ &= \frac{1}{\sqrt{2}}(1+i)|0\rangle . \end{aligned}$$

The same method is used to find the effect on $|1\rangle$,

$$\begin{aligned} U_1^{(2)} |1\rangle &= \frac{1}{\sqrt{2}}(1+i\{a_1^\dagger a_1^\dagger - a_1^\dagger a_1 + a_1 a_1^\dagger - a_1 a_1\})|1\rangle \\ &= \frac{1}{\sqrt{2}}(1+i\{-a_1^\dagger a_1 + a_1 a_1^\dagger\})a_1^\dagger|0\rangle \\ &= \frac{1}{\sqrt{2}}(a_1^\dagger + i\{-a_1^\dagger a_1 a_1^\dagger + a_1 a_1^\dagger a_1^\dagger\})|0\rangle \\ &= \frac{1}{\sqrt{2}}(a_1^\dagger - i a_1^\dagger\{1 - a_1^\dagger a_1\})|0\rangle \\ &= \frac{1}{\sqrt{2}}(a_1^\dagger - i a_1^\dagger)|0\rangle \\ &= \frac{1}{\sqrt{2}}(1-i)|1\rangle . \end{aligned}$$

In both cases the commutation rules, eqs. 1 and 2 have been used [8]. Dividing out a global phase factor of $e^{i\frac{\pi}{4}}$, the counterclockwise braid of adjacent MF $U_1^{(2)}$ in matrix form is,

$$U_1^{(2)} = \mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

This and the opposite braid, $U_1^{(2)\dagger}$ are the only two braids accessible to a two MF system.

Using analogous methods from above, one may specify more gate operations by collecting more MF into the computational region of the 2D SC. For a four MF system, form a new basis,

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} = \{|0\rangle, a_2^\dagger|0\rangle, a_1^\dagger|0\rangle, a_1^\dagger a_2^\dagger|0\rangle\},$$

and MF operators [8],

$$\begin{aligned} \gamma_1 &= a_1^\dagger + a_1 \\ \gamma_2 &= i(a_1^\dagger - a_1) \\ \gamma_3 &= a_2^\dagger + a_2 \\ \gamma_4 &= i(a_2^\dagger - a_2). \end{aligned}$$

To reiterate, these definitions collect adjacent MF under a single qubit operator. The possible equivalence classes of braids in the B_4 group are depicted in Figure 3 with qubit definitions included as light and dark grey shading.

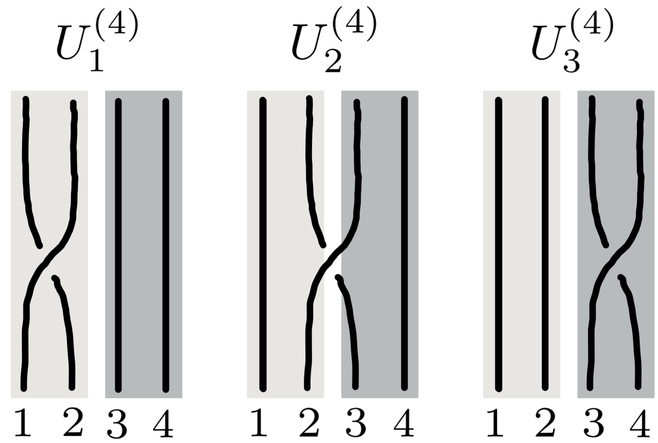

FIG. 3. Four Majorana Braids. The set of all braids available to a four MF system are encapsulated in this picture. Any possible combination of braids will be comprised of and deformed from some combination of the above braids. Qubits 1 and 2 are highlighted light and dark grey respectively.

Transforming the operators in an identical way to the two MF case produces linear operators,

$$U_1^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \mathbf{S} \otimes \mathbf{I},$$

$$U_3^{(4)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} = \mathbf{I} \otimes \mathbf{S},$$

and,

$$U_2^{(4)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \mathbf{I} + i\mathbf{X} \otimes \mathbf{X}).$$

One important observation is that $U_1^{(4)}$ and $U_3^{(4)}$, the odd labelled braids, involve MF defined under the same qubit operator while $U_2^{(4)}$, the only even labelled braid, shares MF from different qubits. This observation provides the qualitative difference between the equivalence classes of a given braid group. The specific scalar elements within the matrix are completely determined by how one defines the qubit operators from MF; however, the locations of each scalar value, i.e. diagonalized or coupling, are determined by whether or not MFs are shared between the qubits.

This fact is constant regardless of how one chooses the initial qubit definitions.

The peculiar locations of non zero entries within $U_2^{(4)}$ are explained via parity, which, in this context, refers to the even or odd number of electrons in the SC bulk [7][4]. When two vortices are fused in the measurement process, the energy level may only be filled by a single, unpaired electron. The number of electrons should remain unchanged throughout the braid, so the only states that may couple via braiding are those with equivalent parity. For this reason, a state with even parity, $|11\rangle$ cannot become a state of odd parity, $|10\rangle$. The $U_2^{(4)}$ transformation above demonstrates this property as even states become coupled with even states and odd states become coupled odd states, e.g.

$$|00\rangle \longrightarrow \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle).$$

Typically, one works within either the even or odd computational subspaces, so that a useful single qubit may be defined [7][2]. This is done by redefining the computational states as $|\tilde{0}\rangle \equiv |00\rangle$ and $|\tilde{1}\rangle \equiv |11\rangle$, and ignoring the odd subspace. The transformations from above become,

$$\tilde{U}_1^{(4)} = \tilde{U}_3^{(4)} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \mathbf{S},$$

and,

$$\tilde{U}_2^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\mathbf{I} + i\mathbf{X}),$$

where the tilde denotes the even space reduced matrices. Once this convention is employed, a single qubit may be defined using four MF rather than two.

These gates are decomposed above and shown to be intimately related to the Phase gate and Bloch sphere x-axis rotation operator,

$$R_x(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbf{I} - i\sin\left(\frac{\theta}{2}\right)\mathbf{X},$$

evaluated for a specific rotational value, $\theta = -\frac{\pi}{2}$. When defining a qubit from four MF, this braid generates quarter rotations about the Bloch sphere's x-axis, and, in combination with the phase gate braids, it is only possible for the qubit state to visit the six poles of the Bloch sphere as shown in Figure 4. One may therefore create any of the quarter rotation Pauli $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ gates. As an example, since the Hadamard gate places the state at the positive x-axis, it should be automatically assumed that some combination of the two possible braids will place the qubit state there as well. This corresponds to the third row qubit operation in Figure 4. This single qubit case highlights that $\mathbf{S}$ and $\mathbf{R}_x(-\frac{\pi}{2})$ emerge naturally as native gates. This property extends to higher dimensional computational spaces.

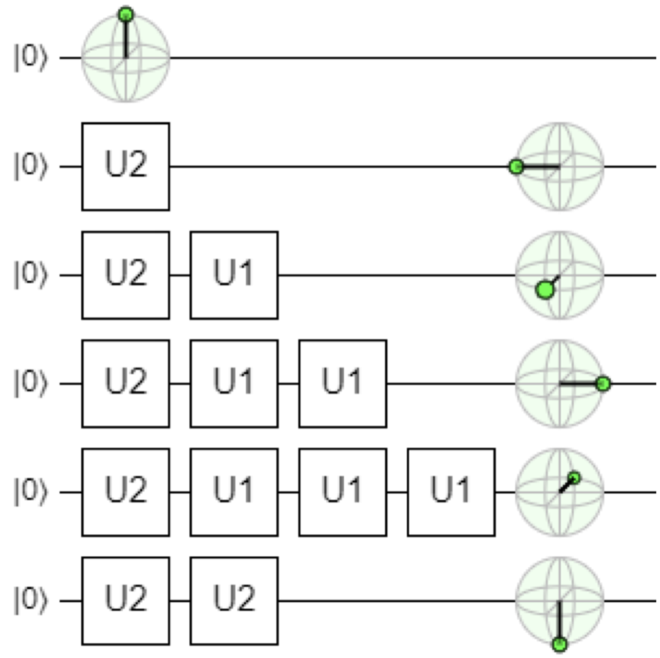

FIG. 4. Quarter Rotation Pauli Spin Braids. A system of four Majorana fermions defining a single qubit has two braids one may create. These two braids only allow the state to visit the six poles of the Bloch sphere meaning the only gates available are quarter rotation Pauli $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ gates. The third row in this image depicts a Hadamard gate and the final row depicts an $\mathbf{X}$ gate. This image was rendered using *https://algassert.com/quirk*.

The two and four MF systems reveal a pattern regarding the calculation of even and odd braids. When braiding two MF from the same qubit, the diagonal matrix imparts a phase factor on the $|1\rangle$ state defined within the qubit operator which contains the MF being swapped. For example, when braiding γ_1 and γ_2 defined under a_1, the computational states $|1...\rangle$ receive the phase factor i and states $|0...\rangle$ are unaffected. The ellipses in the above kets are meant to demonstrate that it is irrelevant how the other states are populated when using the U_1 braid. Similarly, the coupling braids mix adjacent qubits in the ket. For example, the U_2 braid couples even states $|00...\rangle$ and $|11...\rangle$, or odd states $|01...\rangle$ and $|10...\rangle$ equivalently, with a phase factor of i placed off diagonal. These observations almost allow one to easily construct the matrix form for any number of MF or qubits.

Increasing the number of MF once more and implementing these observations, the compact forms of a six

MF system are,

$$\tilde{U}_1^{(6)} = \mathbf{S} \otimes \mathbf{I}$$
$$\tilde{U}_2^{(6)} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \mathbf{I} + i\mathbf{X} \otimes \mathbf{X})$$
$$\tilde{U}_3^{(6)} = \mathbf{I} \otimes \mathbf{S}$$
$$\tilde{U}_4^{(6)} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \mathbf{I} + i\mathbf{I} \otimes \mathbf{X})$$
$$\tilde{U}_5^{(6)} = \mathbf{S} \oplus i\mathbf{S}^\dagger,$$

where now patterns become noticeable for the compacted form of each gate as well. Aside from the final two "ancillary" qubit braids, the odd braids contain a phase gate in the position of the tensor product that will only apply the gate to the qubit for which the braiding occured. The even braids are similar in that double tensor $\mathbf{X}$ product occurs in the tensor product in such a way that the $\mathbf{R}_x(\frac{\pi}{2})$ is applied only to the qubits which shared the MFs.

In other words, for a general n qubit system, the non-ancillary odd gates will fall into the following pattern for a general system,

$$\tilde{U}_1^{(n+2)} = \mathbf{S} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \dots$$
$$\tilde{U}_3^{(n+2)} = \mathbf{I} \otimes \mathbf{S} \otimes \mathbf{I} \otimes \dots$$
$$\tilde{U}_5^{(n+2)} = \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{S} \otimes \dots$$
$$\vdots$$

and,

$$\tilde{U}_2^{(n+2)} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \mathbf{I} \otimes \dots + i\mathbf{X} \otimes \mathbf{X} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \dots)$$
$$\tilde{U}_4^{(n+2)} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \mathbf{I} \otimes \dots + i\mathbf{I} \otimes \mathbf{X} \otimes \mathbf{X} \otimes \mathbf{I} \otimes \dots)$$
$$\tilde{U}_6^{(n+2)} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \mathbf{I} \otimes \dots + i\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{X} \otimes \mathbf{X} \otimes \dots)$$
$$\vdots$$

The total number of gates within each tensor product is exactly equal to n, the number of non-ancillary qubits.

The ancillary braids break out of these established patterns due to the fact that the even space reduction process partially excludes the states of the final qubit. For this reason, if one were to opt out of reducing the computational state out of personal preference, the above pattern would hold for all braids for a given system. However, here the final even and odd braids break the pattern. The last even braid in a system, which share MFs from the last data qubit and the ancillary, has the effect of applying the $\mathbf{X}$ gate to the final data qubit only,

$$\tilde{U}_{finaleven}^{(n+2)} = \frac{1}{\sqrt{2}}(\mathbf{I} \otimes \dots \otimes \mathbf{I} + i\mathbf{I} \otimes \dots \otimes \mathbf{X}).$$

The final odd braid, which swaps MF within the ancillary qubit, results in patterned tensor sum of the phase gate with its adjoint,

$$\tilde{U}_3^{(4)} = \mathbf{S}$$
$$\tilde{U}_5^{(6)} = \mathbf{S} \oplus i\mathbf{S}^\dagger$$
$$\tilde{U}_7^{(8)} = \mathbf{S} \oplus i\mathbf{S}^\dagger \oplus i\mathbf{S}^\dagger \oplus \mathbf{S}$$
$$\tilde{U}_9^{(10)} = \mathbf{S} \oplus i\mathbf{S}^\dagger \oplus i\mathbf{S}^\dagger \oplus \mathbf{S} \oplus i\mathbf{S}^\dagger \oplus \mathbf{S} \oplus \mathbf{S} \oplus i\mathbf{S}^\dagger.$$

GENERALIZING GATES

To completely generalized these observations, for any $n \times n$ quantum gate acting on a system of $\frac{n}{2}$ qubits, one needs $n+2$ MFs. For this number of MFs, there are $2(n+1)$ equivalence classes of braids accessible including their undo action. This collection of braids may be further divided into the $n+2$ odd diagonal gates and n even coupling gates. These braids only allow the multi qubit state to visit the 6 poles of their individual Bloch spheres; no matter how they are entangled. Therefore, unitary gates which place the single qubit state anywhere in between these poles have no representation as a braid, and they are not possible with this quantum computer. Finally, one may condense the patterns for the ith braid observed into compact equations. For non-ancillary braids ($i < n$), odd and even braids take the forms,

$$\tilde{U}_i^{(n+2)} = \mathbf{I}^{\otimes \frac{1}{2}(i-1)} \otimes \mathbf{S} \otimes \mathbf{I}^{\otimes \frac{1}{2}(n-i-1)}$$
$$\tilde{U}_i^{(n+2)} = \frac{1}{\sqrt{2}}(\mathbf{I}^{\otimes \frac{n}{2}} + i\mathbf{I}^{\otimes(\frac{i}{2}-1)} \otimes \mathbf{X}^{\otimes 2} \otimes \mathbf{I}^{\otimes \frac{1}{2}(n-i-2)}),$$

respectively, and the ancillary ($i \geq n$) odd and even braids take the form,

$$\tilde{U}_i^{(n+2)} = \tilde{U}_{i-2}^{(n)} \oplus i\tilde{U}_{i-2}^{(n)\dagger}$$
$$\tilde{U}_i^{(n+2)} = \frac{1}{\sqrt{2}}(\mathbf{I}^{\otimes \frac{n}{2}} + i\mathbf{I}^{\otimes(\frac{i}{2}-1)} \otimes \mathbf{X}).$$

DEFINITION TRANSFORMATION

As mentioned previously, the qubits here are defined by pairing MF adjacently, but the claim is that the particular choice is arbitrary.

Let the collection of gates for a four MF system with our definition be D_0, where,

$$D_0 = \{\mathbf{S} \otimes \mathbf{I}, \mathbf{I} \otimes \mathbf{S}, \mathbf{R}_x(-\frac{\pi}{2})\}.$$

Here, we are not using the even subspace reduced matrices. An alternate definition, D_1 would be to collection

γ_1, γ_3 and γ_2, γ_4 together. One simply performs a similarity transformation on D_0 with the gate which moves γ_2 to the γ_3 spot, $D_1 = U_2 D_0 U_2^\dagger$,

$$U_2^{(4)} U_1 U_2^{(4)\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\mathbf{I} \otimes \mathbf{I} - i\mathbf{X} \otimes \mathbf{Y}),$$

$$U_2^{(4)} U_2 U_2^{(4)\dagger} = U_2^{(4)},$$

$$U_2^{(4)} U_1 U_2^{(4)\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\mathbf{I} \otimes \mathbf{I} - i\mathbf{Y} \otimes \mathbf{X}).$$

This change in definition is depicted in Figure **FIG**. This supports the general notion that qubit coupling will only occur when sharing MF because, under this definition, all braids will result in MF swapping between qubits.

An alternate definition, D_2, could be where the middle and outer two MF are collected as in Figure **FIG**. To acquire this definition from D_0, one would make the $D_0 \to D_1$ transformation and then swap the new γ_3 and γ_4. In math, let $U_{i'}$ be the new gates under the D_1 definition,

$$\begin{aligned} U_{3'} D_1 U_{3'}^\dagger &= U_{3'} \{U_2 D_0 U_2^\dagger\} U_{3'}^\dagger \\ &= U_2 U_3 \{U_2^\dagger U_2\} D_0 \{U_2^\dagger U_2\} U_3^\dagger U_2^\dagger \\ &= U_2 U_3 D_0 U_3^\dagger U_2^\dagger, \end{aligned}$$

where we have used unitarity of the operators in the last line. This transformation acts on the D_0 gates like so,

$$\begin{aligned} U_1 &\to \frac{1}{\sqrt{2}} (\mathbf{I} \otimes \mathbf{I} - i\mathbf{X} \otimes \mathbf{Y}) \\ U_2 &\to \mathbf{I} \otimes \mathbf{S} \\ U_3 &\to \frac{1}{\sqrt{2}} (\mathbf{I} \otimes \mathbf{I} - i\mathbf{Y} \otimes \mathbf{X}), \end{aligned}$$

where again the U_2 gate is the only one which does not share MF. It is even possible to switch the definitions of MF within each qubit and to swap qubit one and two using the same process. With a full description of braids and transformations, this system is fully characterized.

CONCLUSION

Currently, quantum computers have the ability to perform actions that classical computers cannot, and, for this reason, the field is growing rapidly with many groups pushing what is possible. However, the state of the art systems currently used for computation face the same scalability hurdle caused by decoherence. Thus, the search for fault tolerance in multi qubit systems is a necessary next step for the field of quantum information and quantum computation. The results in this document will allow scientists to possess some intuition when implementing an example of a fault tolerant quantum computing system. Of course, the above results are "less universal" than those implemented in a conventional system; however, as research into topological states of matter continues, a better system may be implemented with similar calculations and expanded qubit gates. These systems will hopefully provide the qubit of the 22_{nd} century.

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