CS 237: *Probability in Computing* — Professors: Alina Ene and Tiago Januario
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Solutions to Midterm Practice Problems

Problem 1. A student must choose from the subjects, art, geology, or psychology, as possible electives. Among available options, she is always equally likely to choose art or psychology and twice as likely to choose geology.

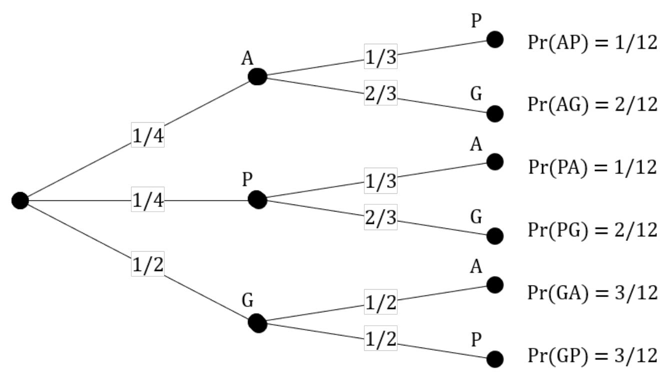
(a) If she has to choose one elective, what are the respective probabilities that she chooses art, geology, and psychology?

Solution. Let A denote art, P denote psychology, and G denote geology.

$$\Pr(A) = \Pr(P) = \frac{1}{4}, \quad \Pr(G) = \frac{1}{2}.$$

(b) If she has to choose two electives, what is the probability she chooses art and psychology?

Solution. The student's choices are illustrated using the tree diagram:



There are three pairs of electives the student can choose:

$$\Pr(\{A,P\}) = \Pr(AP) + \Pr(PA) = \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6};$$
$$\Pr(\{A,G\}) = \Pr(AG) + \Pr(GA) = \frac{1}{4}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12};$$
$$\Pr(\{P,G\}) = \Pr(PG) + \Pr(GP) = \frac{1}{4}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

Thus, the probability of the student choosing art and psychology is $\Pr(\{A,P\}) = 1/6$.

(c) For the two elective case, what is the probability she chose geology given that her other choice was art?

Solution. Let E_G and E_A be the events that she chose geology and that she chose art, respectively. By the definition of conditional probability,

$$\Pr(E_G \mid E_A) = \frac{E_G \cap E_A}{E_A} = \frac{\Pr(\{A, G\})}{\Pr(\{A, G\}) + \Pr(\{A, P\})} = \frac{5/12}{7/12} = \frac{5}{7}.$$

Problem 2. Among the students taking CS 237 this semester, 60% enjoy coffee, 70% enjoy tea, and 40% enjoy both. Prof. Tiago chooses a CS 237 student uniformly at random. What is the probability that the chosen student neither enjoys coffee nor tea?

Solution. Let C be the event that the chosen student enjoys coffee and T be the event that the chosen student enjoys tea. By Inclusion-Exclusion Principle,

$$1 - \Pr(C \cup T) = 1 - \Pr(C) - \Pr(T) + \Pr(C \cap T) = 1 - 0.6 - 0.7 + 0.4 = 0.1.$$

Problem 3. Let X be a real number chosen uniformly at random from $[2, 10]$. Find the probabilities of the following events:

(a) $X > 5$,

Solution. $\frac{10-5}{10-2} = \frac{5}{8}$.

(b) $5 < X < 7$,

Solution. $\frac{7-5}{10-2} = \frac{1}{4}$.

(c) $X^2 - 12X + 35 > 0$.

Solution. $X^2 - 12X + 35 > 0 \iff (X-5)(X-7) > 0 \iff X < 5 \vee X > 7$, so

$$\Pr(X < 5 \vee X > 7) = \Pr(X < 5) + \Pr(7 < X) = \frac{5-2}{10-2} + \frac{10-7}{10-2} = \frac{3}{4}.$$

Problem 4. Suppose a group of 8 couples (16 people) are invited to a party. Each person is assigned a seat around a circular table uniformly and independently at random.

(a) Count the number of distinct seating arrangements.

Solution. For each ordering of the 16 people, there are 16 identical seating arrangements which are given by rotating the circular table. This gives a total of $\frac{16!}{16} = 15!$ distinct arrangements.

(b) If Alice and Bob are both invited to the party, what is the probability that they are sitting next to each other?

Solution. Notice that Alice can be sitting to the left of Bob, or Bob to the left of Alice. Fix a pair of consecutive seats for Alice and Bob, after seating Alice and Bob there are 14! ways to seat the remaining 14 people. Since there are two ways to seat Alice and Bob, the probability that they are sitting next to each other is $2(\frac{14!}{15!}) = \frac{2}{15}$.
Alternatively, we can imagine Alice arrives first and sits in her seat. There are two seats out of the remaining fifteen that will result in Bob sitting next to her. This gives a probability of $\frac{2}{15}$.

(c) Suppose Jack and Jill arrive at the party and see that Alice and Bob are sitting next to each other. Assuming Alice and Bob are in their assigned seats, what is the probability that Jack and Jill will be sitting next to each other?

Solution. Since Alice and Bob are already seated, we can think of the circular table with Alice and Bob seated next to each other, as straight table with Alice at one end, Bob at the other, and 14 seats in between them. The question is now asking, given a straight table with 14 seats, what is the probability that Jack and Jill are seated next to each other?
There are 13 pairs of consecutive seats at the table given by seats $(i, i+1)$ for each i from 1 to 13, and for each pair of consecutive seats we can have (Jack, Jill) or (Jill, Jack). After seating Jack and Jill there are 12! ways to seat the remaining people. Since there are 14! ways to seat everyone, the probability in question is given by

$$\frac{2 \cdot 13 \cdot 12!}{14!} = \frac{2}{14} = \frac{1}{7}.$$

Problem 5. Mr. Smith has two children of different ages. (Assume boys and girls are equally likely.) Consider the following events:

$$\begin{aligned} A &= \text{“both children are boys,”} \\ B &= \text{“at least one child is a boy,”} \\ C &= \text{“older child is a boy.”} \end{aligned}$$

(a) Calculate probabilities for $\Pr(A \mid B)$ and $\Pr(A \mid C)$.

Solution.

$$\begin{aligned} &\Pr(A) = 1/4; \quad \Pr(B) = 3/4; \quad \Pr(C) = 1/2 \\ &\Pr(A \cap B) = \Pr(A \cap C) = \Pr(A) = 1/4 \\ &\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1}{3}; \quad \Pr(A \mid C) = \frac{\Pr(A \cap C)}{\Pr(C)} = \frac{1}{2} \end{aligned}$$

(b) Let A and B be two disjoint events with $\Pr(A) > 0$ and $\Pr(B) > 0$. Are A and B independent?

Solution. $\Pr(A \cap B) = 0$ but $\Pr(A) \cdot \Pr(B) \neq 0$, so no.

Problem 6 (Children in a Line)**.** Ten children (five boys and five girls) are standing in a line. All possible ways in which they might line up are equally likely. All children have different names. For each item below, compute the probability of the specified event.

(a) Children appear in the line in alphabetical order by name.

Solution. The probability is $\frac{1}{10!}$.
There are 10! ways of ordering 10 people, and only one of them is the alphabetical order by name.

(b) All girls precede all boys.

Solution. The probability is $\frac{5! \cdot 5!}{10!}$. There are 5! ways to order the girls, and for each of them, there are 5! ways to order the boys.

(c) Between any two girls there are no boys (that is, the girls stand together in an uninterrupted block).

Solution. The probability is $\frac{6! \cdot 5!}{10!}$.
Let's call boys b_1, b_2, b_3, b_4, b_5. Since girls cannot be separated, let's represent them as a single object, G. There are 6! ways to order $b_1, b_2, b_3, b_4, b_5, G$, and for each of them, there are 5! ways to order the girls.

(d) Children alternate by gender in the line.

Solution. The probability is $\frac{2 \cdot 5! \cdot 5!}{10!}$.
The gender of the first child has 2 possibilities. For each of them, there are 5! to order the girls, and for each of those possibilities, there are 5! ways to order the boys.

(e) Neither the boys nor the girls stand together in an uninterrupted block.

Solution. The probability is $1 - \left(\frac{6! \cdot 5!}{10!} + \frac{6! \cdot 5!}{10!} - \frac{2 \cdot 5! \cdot 5!}{10!}\right) = 1 - \frac{5! \cdot 5!}{9!}$.
Let B be the event that boys stand together in an uninterrupted block, and G be the event that girls stand together in an uninterrupted block. Then the event of interest is $\overline{B} \cap \overline{G} = \overline{B \cup G}$ (the equality holds by De Morgan's law). By applying first the Complement Rule and then Inclusion-Exclusion Principle, we get

$$\Pr(\overline{B \cup G}) = 1 - \Pr(B \cup G) = 1 - (\Pr(B) + \Pr(G) - \Pr(B \cap G)).$$

From (c), we have that $\Pr(B) = \Pr(G) = \frac{6! \cdot 5!}{10!}$. Similarly to (b), $\Pr(B \cap G) = 2 \cdot \frac{5! \cdot 5!}{10!}$.

Problem 7 (Gold and silver coins)**.** You have two fair coins: silver and gold.

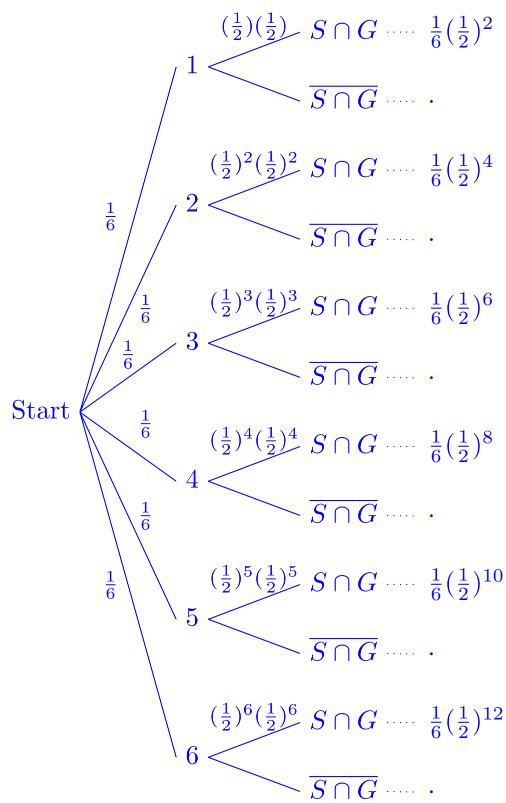
(a) You toss each coin 6 times. Let S be the event that all tosses of the silver coin are heads. Let G be the event that all tosses of the gold coin are heads. Are S and G independent?

Solution. To prove that S and G are independent, we will show that the definition of independence holds: $\Pr(S \cap G) = Pr(S) \cdot \Pr(G)$.
The sample space Ω consists of 2^{12} outcomes since we have 12 coin tosses total and 2 possibilities for each coin toss. The probability that $S \cap G$ occurs, that is, that all outcomes are heads is $\frac{1}{2^{12}}$.
There are 2^6 ways for gold coin tosses to be all heads: 1 possibility for the gold coin tosses and 2^6 possibilities for the silver coin tosses. So, $\Pr(S) = \frac{2^6}{2^{12}} = \frac{1}{2^6}$. Similarly, $\Pr(G) = \frac{1}{2^6}$.
We see that $\Pr(S \cap G) = \frac{1}{2^{12}} = \frac{1}{2^6} \cdot \frac{1}{2^6} = \Pr(S) \cdot \Pr(G)$, so S and G are independent.

(b) You roll a standard die. Let random variable D be the number (from 1 to 6) that you rolled. You toss each coin D times. Let S be the event that all tosses of the silver coin are heads. Let G be the event that all tosses of the gold coin are heads.

(i) Draw a partial tree diagram for this random experiment. Include all outcomes for the die. For the coins, instead of including all outcomes, only mark whether S and G occurred. Include all branches where S and G occurred and omit the rest.

Solution.



(ii) Are S and G independent?

Solution. To prove that S and G are not independent, we will show that the definition of independence does not hold for S and G, that is, that $\Pr(S \cap G) \neq \Pr(S) \cdot \Pr(G)$. From the tree diagram,

$$\Pr(S \cap G) = \frac{1}{6} \cdot \left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \frac{1}{2^{10}} + \frac{1}{2^{12}} \right) = \frac{1}{6} \cdot \frac{4^5 + 4^4 + 4^3 + 4^2 + 4 + 1}{64^2}.$$

Since nothing in the problem changes if we switch gold and silver, $\Pr(S) = \Pr(G)$. The tree diagram for $\Pr(S)$ would contain the first two levels of the diagram above. From the modified tree diagram (or, equivalently, by the Law of Total Probability), we get:

$$\Pr(S) = \sum_{k=1}^{6} \Pr(S \cap [D = k]) = \frac{1}{6} \cdot \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} \right) = \frac{1}{6} \cdot \frac{63}{64}.$$

Let $N = 4^5 + 4^4 + 4^3 + 4^2 + 4 + 1$. To compare $\Pr(S \cap G)$ and $\Pr(S) \cdot \Pr(G)$, we can evaluate N using the formula for the sum of geometric series or simply notice that:

$$\frac{1}{6} \cdot \frac{4^5 + 4^4 + 4^3 + 4^2 + 4 + 1}{64^2} \neq \left(\frac{1}{6}\right)^2 \cdot \frac{63^2}{64^2}$$

because

$$4^5 + 4^4 + 4^3 + 4^2 + 4 + 1 \neq \frac{63 \cdot 63}{6}.$$

The latter inequality holds because the left-hand side is an integer and the right-hand-side is equal to $\frac{21 \cdot 63}{2}$, which is not integral.

Since $\Pr(S \cap G) \neq \Pr(S) \cdot \Pr(G)$, events S and G are not independent.

(c) We perform the same random experiment as in part (b) and define D as before. Compute the probability that $D = 1$ given that we got exactly one heads from $2D$ coin tosses (D for the silver coin and D for the gold coin). **(Show your work and explain what laws of probability you are using.)**

Solution. Let E be the event that we get one heads from the $2D$ coin tosses. By the product rule,

$$\Pr([D = 1] \cap E) = \Pr(D = 1) \cdot \Pr(E \mid D = 1) = \frac{1}{6} \cdot \frac{|\{\mathtt{HT}, \mathtt{TH}\}|}{|\{\mathtt{HT}, \mathtt{TH}, \mathtt{HH}, \mathtt{TT}\}|} = \frac{1}{6} \cdot \frac{1}{2}.$$

Notice that once we condition on $D = 1$, that is, on rolling 1, we know that we tossed each coin once. We can consider the general case when we roll k, where $k = 1, 2, 3, 4, 5,$ or 6. Then we have $2k$ coin tosses. There are $2k$ possibilities for where one heads can occur in a sequence of $2k$ coin tosses, and there are 2^{2k} possible outcomes of $2k$ coin tosses. So, $\Pr(E \mid D = k) = \frac{2k}{2^{2k}}$.

By the law of total probability,

$$\Pr(E) = \sum_{k=1}^{6} \Pr([D = k] \cap E) = \sum_{k=1}^{6} \Pr(D = k) \cdot \Pr(E \mid D = k) = \frac{1}{6} \sum_{k=1}^{6} \frac{2k}{2^{2k}}.$$

By the definition of conditional probability,

$$\Pr(D = 1 \mid E) = \frac{\Pr([D = 1] \cap E)}{\Pr(E)} = \frac{\frac{1}{6} \cdot \frac{1}{2}}{\frac{1}{6} \sum_{k=1}^{6} \frac{2k}{2^{2k}}} = \frac{1}{\sum_{k=1}^{6} \frac{4k}{2^{2k}}} = \frac{1}{4\left(\frac{1}{2^2} + \frac{2}{2^4} + \frac{3}{2^6} + \frac{4}{2^8} + \frac{5}{2^{10}} + \frac{6}{2^{12}}\right)}.$$

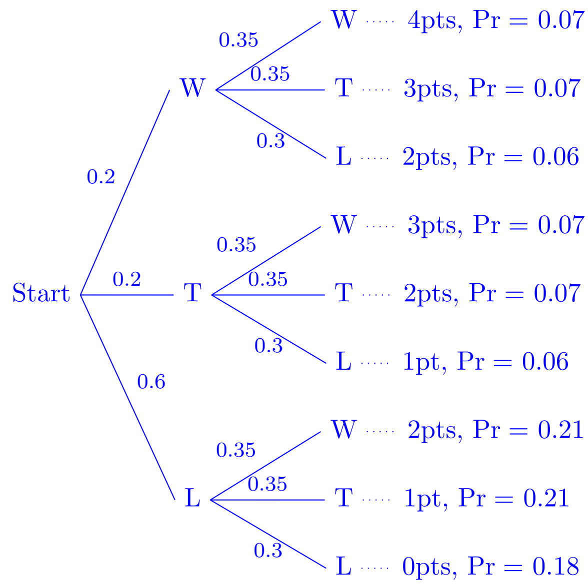
Another solution is to first use Bayes' rule to get

$$\Pr(D = 1 \mid E) = \frac{\Pr(D = 1) \cdot \Pr(E \mid D = 1)}{\Pr(E)},$$

and then to apply the Law of Total Probability to compute $\Pr(E)$. You could also compute $\Pr(E)$ by drawing a tree diagram.

Problem 8. The BU soccer team has two games scheduled this weekend. The team has probability 0.4 of not losing the first game, and it has probability 0.7 of not losing the second game, independently of the first. If the team does not lose a particular game, it is equally likely to win or tie. The team receives 2 points for a win, 1 point for a draw, and 0 points for a loss. Let X be the number of points that the team wins this weekend. Find the PDF and CDF of X.

Solution. We can solve the problem using a tree diagram.



Using the tree diagram above, we get the PDF and CDF of X:

$$f_X(x) = \begin{cases} 0.18 & x = 0 \\ 0.27 & x = 1 \\ 0.34 & x = 2 \\ 0.14 & x = 3 \\ 0.07 & x = 4 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ 0.18 & 0 \leq x < 1 \\ 0.45 & 1 \leq x < 2 \\ 0.79 & 2 \leq x < 3 \\ 0.93 & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

Problem 9. For each of the following piece-wise functions, determine whether or not it is a valid PDF for a continuous random variable. If it isn't, please explain why.

(a)

$$f_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{3}x & 0 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

Solution. No, because the total area under the function is clearly not equal to 1.

(b)

$$f_X(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 6 \\ \frac{1}{5} & 6 \leq x \leq 8 \end{cases}$$

Solution. No, because the function does not have a defined value for all values of x (i.e. no "otherwise" section).

(c)

$$f_X(x) = \begin{cases} -\frac{2}{3}x + \frac{4}{3} & x < 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution. No, because the total area under the function is clearly not equal to 1, and when $x < 0, f_X(x) > 1$.

(d)

$$f_X(x) = \begin{cases} \frac{1}{6}x & 0 \leq x \leq 3 \\ 250 & 3 \leq x \leq 3.001 \\ 0 & \text{otherwise} \end{cases}$$

Solution. Yes. The function is clearly non-negative, and the area under the entire function is 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)dx &= \int_0^3 \frac{1}{6}xdx + \int_3^{3.001} 250dx \\ &= \left[\frac{1}{12}x^2\right]_0^3 + [250x]_3^{3.001} \\ &= \left(\frac{1}{12}(3)^2 - \frac{1}{12}(0)^2\right) + (250(3.001) - 250(3)) \\ &= 0.75 + 0.25 = 1 \end{aligned}$$

(e)

$$f_X(x) = \begin{cases} 0.3 & x = 3 \\ 0.7 & x = 5 \\ 0 & \text{otherwise} \end{cases}$$

Solution. No, this would be a valid discrete PDF, but as a continuous PDF, the area under the curve is 0.

Problem 10. A continuous random variable M has the following PDF:

$$f_M(x) = \begin{cases} x^2 - 2x + 1 & \text{if } 1 \leq x \leq 2 \\ \frac{1}{3} & \text{if } 5 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

Find:

(a) the CDF of M

Solution. $$\int_1^y x^2 - 2x + 1 \text{ d}x = \left[\frac{x^3}{3} - x^2 + x\right]_1^y = \frac{y^3}{3} - y^2 + y - \frac{1}{3}$$

$$\int_5^y \frac{1}{3} \text{ d}x = \left[\frac{x}{3}\right]_5^y = \frac{y-5}{3}$$

$$\mathrm{CDF}_M(x) = \begin{cases} 0 & x < 1 \\ \frac{x^3}{3} - x^2 + x - \frac{1}{3} & 1 \le x \le 2 \\ \frac{1}{3} & 2 < x < 5 \\ \frac{x-4}{3} & 5 \le x \le 7 \\ 1.0 & x > 7 \end{cases}$$

Note that when $5 \le x \le 7$, we have $\frac{1}{3} + \frac{x-5}{3} = \frac{x-4}{3}$.

(b) $\Pr\left(\frac{1}{2} \le M \le 6\right)$.

Solution. $\mathrm{CDF}_M(6) - \mathrm{CDF}_M\left(\frac{1}{2}\right) = \frac{2}{3} - 0 = \frac{2}{3}$

Problem 11. A deck of cards consists of 6 red and 5 black cards. A second deck of cards consists of 9 red cards. A deck is selected uniformly at random and a card is drawn uniformly at random from the deck and found to be red. What is the probability that the card was drawn from the first deck?

Solution. Let D_1 be the event that we chose deck 1 and R be the event that we chose red card. We use Bayes' theorem and then apply the law of total probability:

$$\begin{aligned} \Pr(D_1|R) &= \frac{\Pr(R|D_1) \cdot \Pr(D_1)}{\Pr(R)} \\ &= \frac{\Pr(R|D_1) \cdot \Pr(D_1)}{\Pr(R|D_1) \cdot \Pr(D_1) + \Pr(R|\overline{D_1}) \cdot \Pr(\overline{D_1})} \\ &= \frac{(6/11) \cdot (1/2)}{(6/11) \cdot (1/2) + (1) \cdot (1/2)} \\ &= \frac{6}{17} \end{aligned}$$

Problem 12. Defaulting on a loan means failing to pay back the loan. In ProbabilityVille, 5% of the residents will default on their loan. The Central Bank of ProbabilityVille has developed a test to predict whether a person will default. The test is not perfect though:

- If the person will default, there is a 6% chance that the test will predict that the person will not default.
- If the person will not default, there is a 2% chance that the test will predict that the person will default.

A person is chosen uniformly at random from the residents of ProbabilityVille. Given that the test predicts that the person will default, what is the probability that the person will truly default?

Solution. Let D be the event that the chosen person defaults, and T be the event that the test predicts that the person defaults. By Bayes' theorem:

$$\begin{aligned}
\Pr(D|T) &= \frac{\Pr(T|D)\cdot\Pr(D)}{\Pr(T)} \\
&= \frac{\Pr(T|D)\cdot\Pr(D)}{\Pr(T|D)\cdot\Pr(D)+\Pr(T|\overline{D})\cdot\Pr(\overline{D})} \\
&= \frac{0.94\cdot 0.05}{0.94\cdot 0.05+0.02\cdot 0.95} \\
&= \frac{47}{66}
\end{aligned}$$

Problem 13. We have a deck of cards (52 in total). Let's play some card drawing games.

(a) What is the probability of drawing 5 hearts with replacement?

Solution. There is a $\frac{13}{52}$ chance of drawing a heart once, and all draws with replacement are independent.

$$\Pr(5\text{ hearts}) = \left(\frac{13}{52}\right)^5$$

(b) What is the probability of drawing 5 spades without replacement?

Solution. There are $\binom{13}{5}$ ways to draw 5 spades without replacement, and there are $\binom{52}{5}$ total ways to draw 5 cards without replacement.

$$\Pr(5\text{ spades}) = \frac{\binom{13}{5}}{\binom{52}{5}}$$

(c) Given that we already have 2 Kings in hand, what is the probability of drawing another three cards with the same rank from the remaining 50 cards without replacement?

Solution. There are $\binom{50}{3}$ total ways to draw the next 3 cards without replacement. With 2 Kings already in hand, the next 3 cards we pick have to be from one of the other 12 ranks (since you can't have 5 Kings), and there are $\binom{12}{1}$ ways to pick that rank. Each of the other 12 ranks still has all 4 of its cards in the deck, so there are $\binom{4}{3}$ ways to pick 3 cards from that rank.

$$\Pr(\text{next 3 from same rank}) = \frac{\binom{12}{1}\binom{4}{3}}{\binom{50}{3}}$$