CS 237: Probability in Computing

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POISSON DISTRIBUTION

- The last distribution that we will introduce is the Poisson distribution
- The Poisson distribution is a **discrete** distribution that is widely used in applications
- The Poisson distribution is related to the Binomial distribution (more on this shortly)

POISSON DISTRIBUTION

- The Poisson distribution is used in scenarios where we are counting the number of events that occur in a particular interval of time, and there are a large number of events, each with a small probability of occurring
  - The number of jobs that arrive at a server in an hour
  - The number of emails I receive in a day
  - The number of earthquakes in San Francisco in a century
  - The number of large meteorites that hit Earth in a year
  - The number of photons captured by a detector in 10 minutes
  - …

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POISSON DISTRIBUTION

- Range: $\mathbb{N} = \{0, 1, 2, \ldots\}$

The number of events that occur

POISSON DISTRIBUTION

- Range: $\mathbb{N} = \{0,1,2,\ldots\}$
- Parameter: $\lambda > 0$
- Notation: $X \sim \mathrm{Poisson}(\lambda)$

The number of events that occur

$\mathrm{Ex}(X) = \mathrm{Var}(X) = \lambda$

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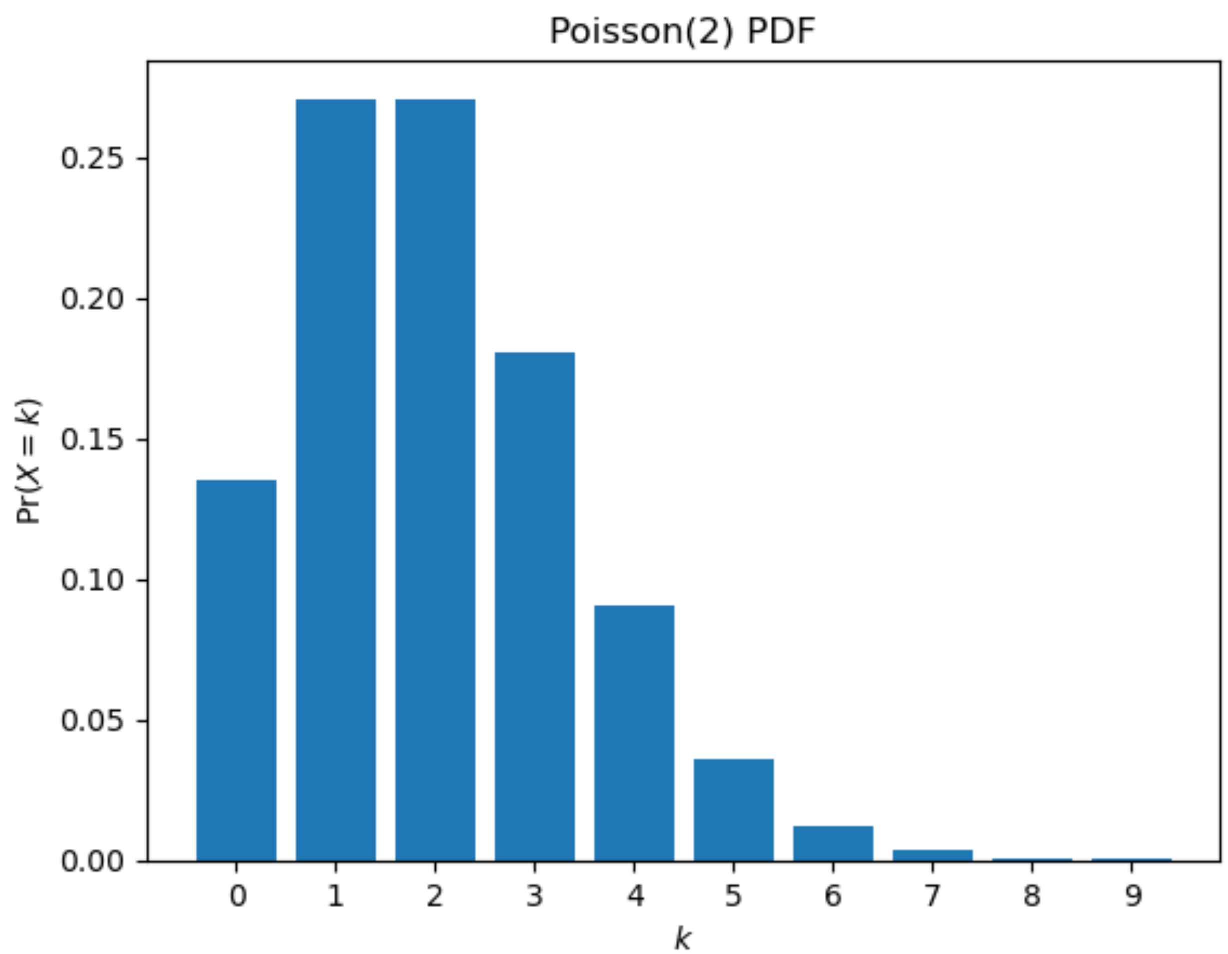
PDF

$$\Pr(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad \text{for all } k \in \mathbb{N}$$

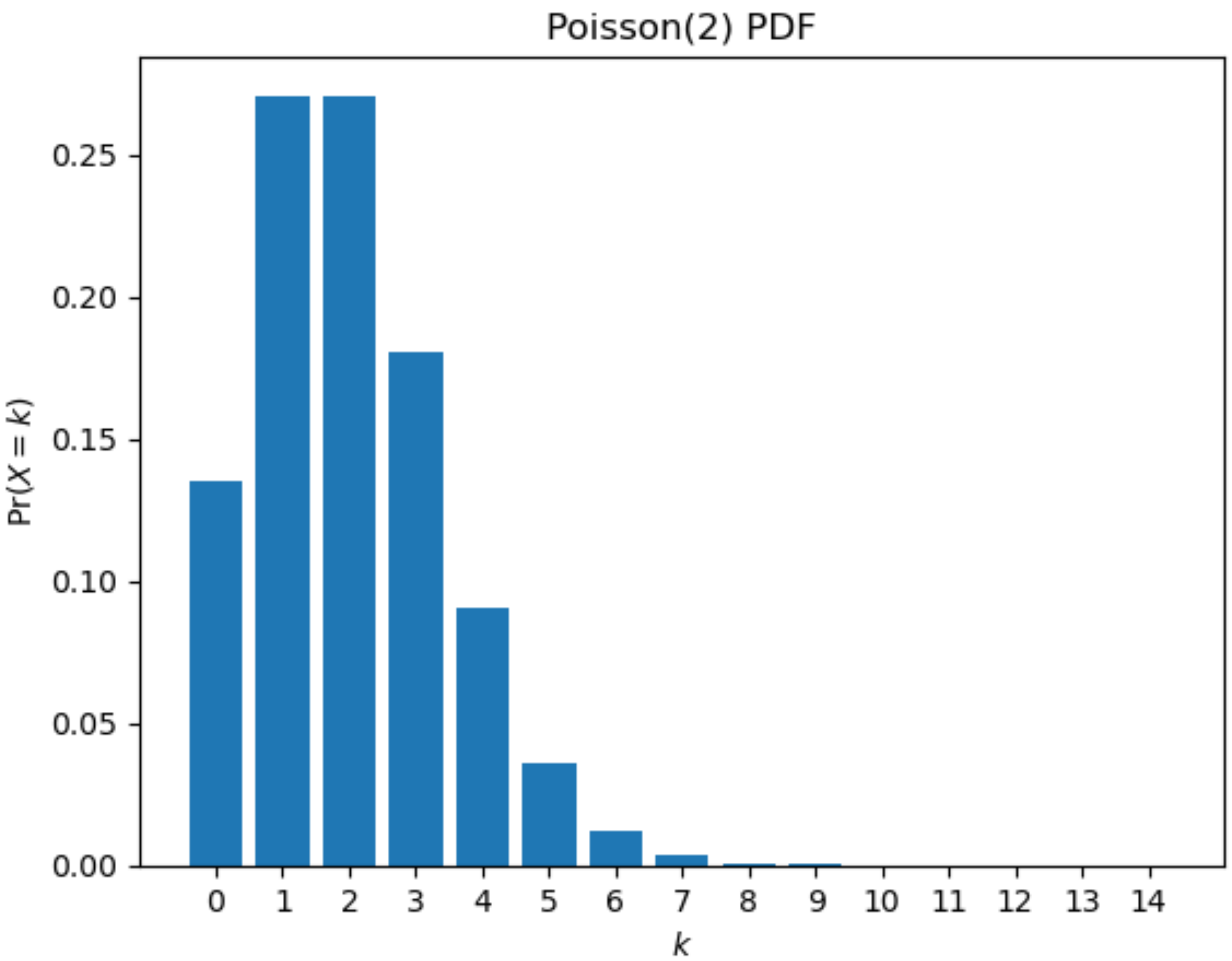
Convention: $0! = 1$

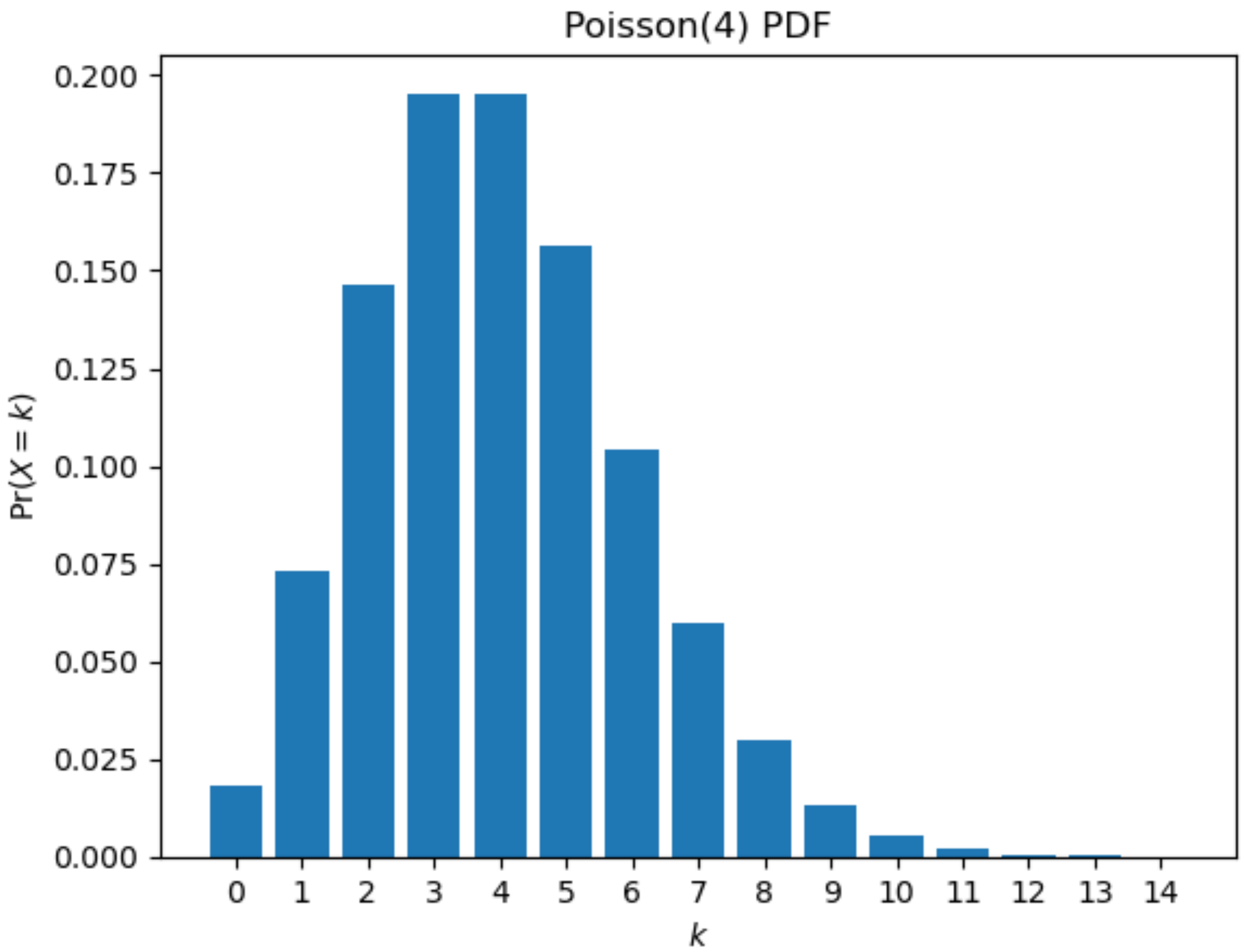


POISSON DISTRIBUTION

Poisson(λ) **PDF**

$$\boxed{\Pr(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \text{ for all } k \in \mathbb{N}}$$





POISSON DISTRIBUTION

$\text{Poisson}(\lambda)$ **PDF:** $\Pr(X = k) = e^{-\lambda} \cdot \dfrac{\lambda^k}{k!}$ for all $k \in \mathbb{N}$

Expectation and Variance:

$$\text{Ex}(X) = \lambda \qquad \text{Ex}(X^2) = \lambda(1 + \lambda) \qquad \text{Var}(X) = \lambda$$

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Can calculate them directly from the PDF

POISSON DISTRIBUTION

- The Poisson distribution is used in scenarios where we are counting the number of events that occur in a particular interval of time, and there are a large number of events, each with a small probability of occurring
- We can interpret λ as the rate of occurrence of these rare events
  - Number of jobs arriving at a server: $\lambda = 20$ (jobs per hour)
  - Number of emails: $\lambda = 15$ (emails per day)
  - Number of earthquakes: $\lambda = 2$ (earthquakes per century)
  - …

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Poisson Paradigm (Law of Rare Events):

Let $A_1, A_2, \ldots, A_n$ be independent events, and let $p_i = \Pr(A_i)$.

Let X be the number of these events that occur.

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Then X is **approximately** $\text{Poisson}(\lambda)$ with $\lambda = p_1 + p_2 + \ldots + p_n$.

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Example: X = number of earthquakes in San Francisco in a year

Each event A_i corresponds to an earthquake occurring at a specific time during the year and location in San Francisco

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Example: X = no. of heads in n coin tosses $\sim \mathrm{Binomial}(n, p)$

$A_i = i$-th coin toss is heads $\quad \Pr(A_i) = p$

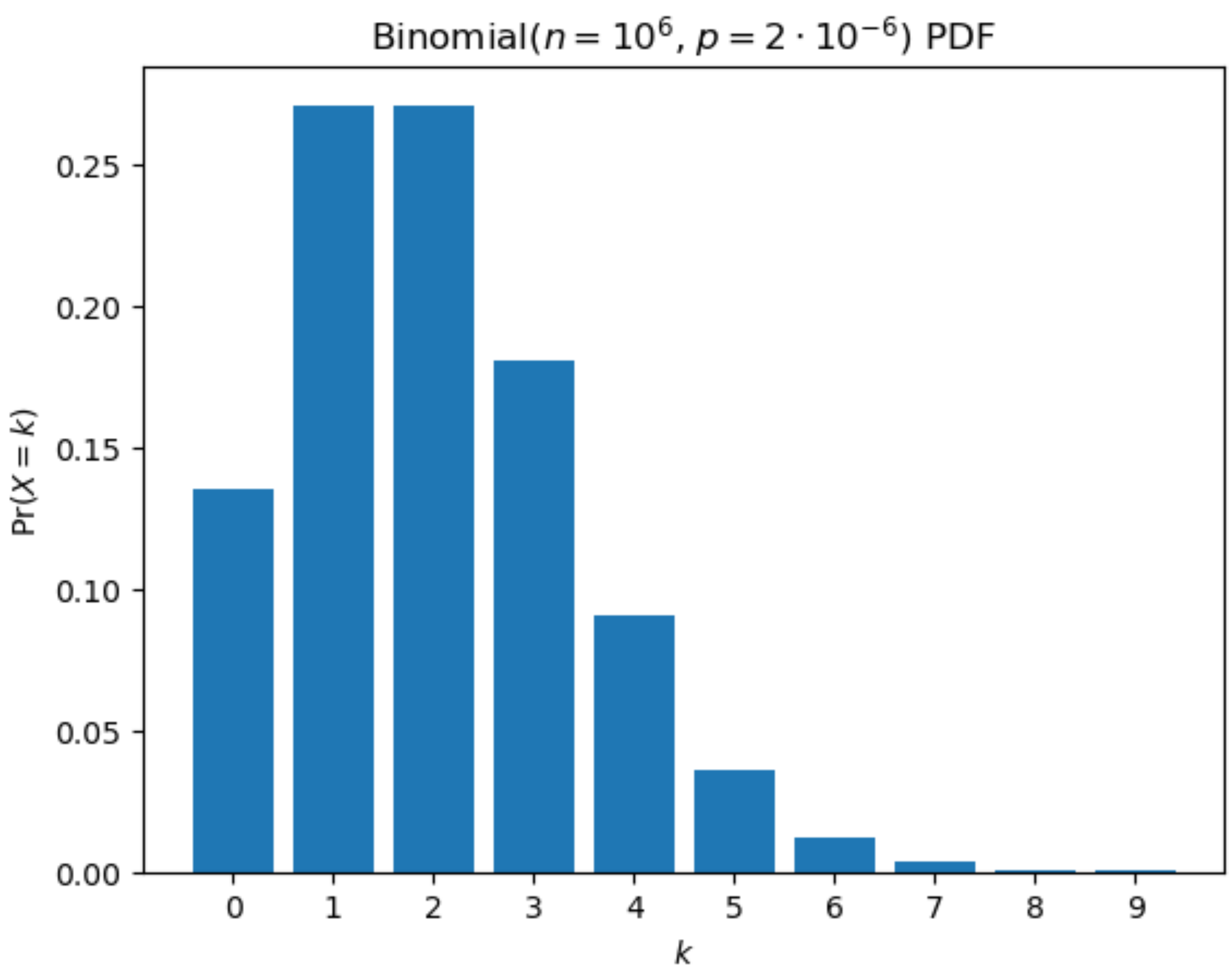
If n is large and p is small: X is approximately $\mathrm{Poisson}(\lambda = np)$

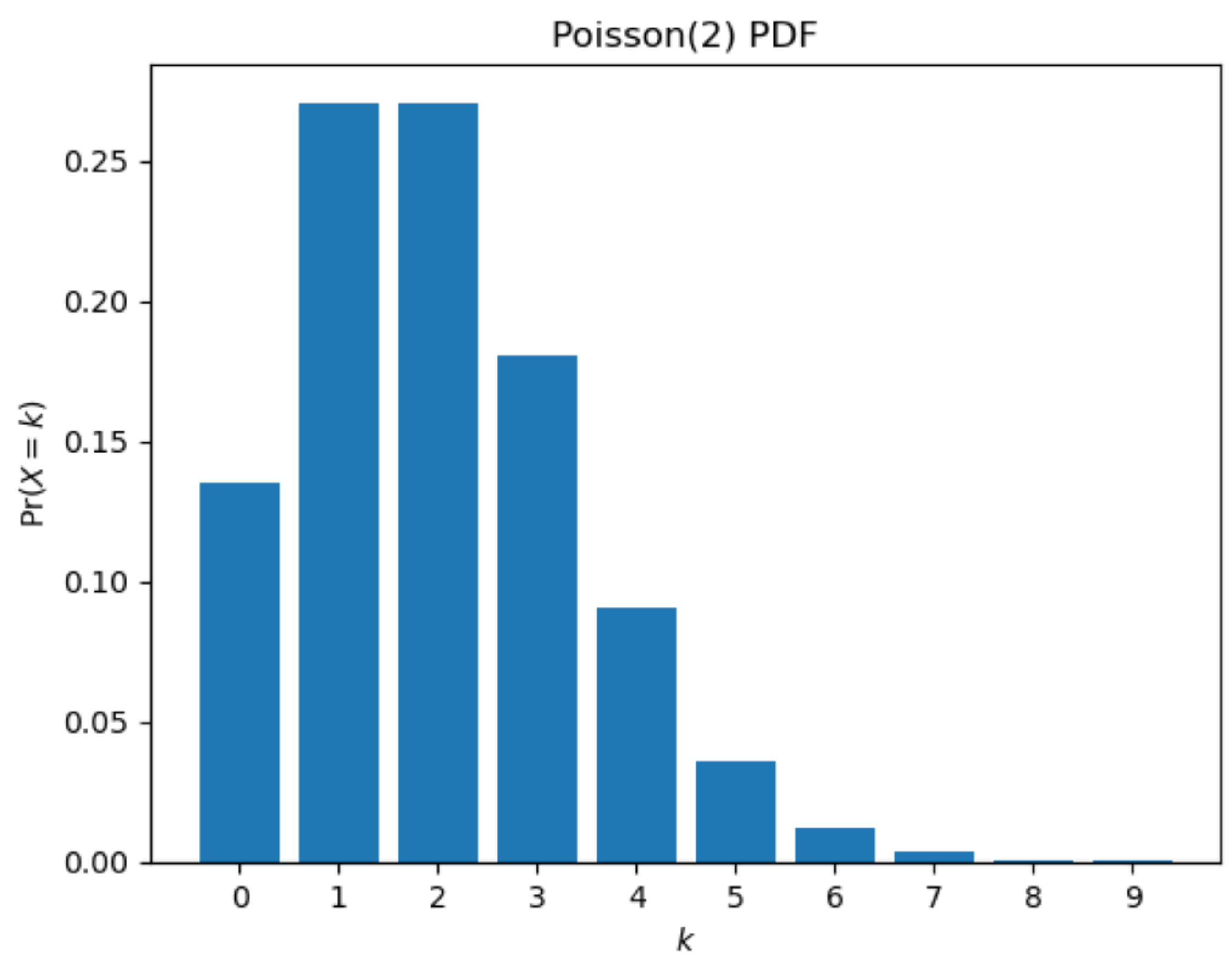
Very useful approximation in practice

X = no. of successes in n independent trials $\sim \text{Binomial}(n, p)$

If n is large and p is small: X is approximately $\text{Poisson}(\lambda = np)$

Example: $n = 10^6, p = 2 \cdot 10^{-6}, \lambda = np = 2$





X = no. of successes in n independent trials $\sim \text{Binomial}(n, p)$

If n is large and p is small: X is approximately $\text{Poisson}(\lambda = np)$

Example: Each day, $n =$ 1 million people independently decide whether to visit a certain website, and each person visits with probability $p = 2 \cdot 10^{-6}$.

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Suppose we are interested in finding, e.g., $\Pr(X \geq 3)$

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Suppose we are interested in finding, e.g., $\Pr(X \geq 3)$

$$\Pr(X \geq 3) = 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2)$$

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$$\Pr(X \geq 3) = 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2)$$
$$= 1 - \sum_{k=0}^{2} \binom{n}{k} p^k (1-p)^{n-k}$$

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Exact calculation is difficult due to numerical errors

We can use the Poisson approx. to find very good estimates

X = no. of successes in n independent trials $\sim \text{Binomial}(n, p)$

If n is large and p is small: X is approximately $\text{Poisson}(np)$

Example: Each day, a million people independently decide whether to visit a certain website, and each person visits with probability $2 \cdot 10^{-6}$

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$\boxed{\text{Poisson}(\lambda)\ \textbf{PDF:}\ \Pr(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}\ \text{ for all } k \in \mathbb{N}}$

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$$\begin{aligned}\Pr(X \geq 3) &= 1 - \Pr(X = 0) - \Pr(X = 1) - \Pr(X = 2) \\ &\approx 1 - e^{-2} - e^{-2} \cdot 2 - e^{-2} \cdot \frac{2^2}{2!} \\ &= 1 - 5e^{-2} \approx 0.3233\end{aligned}$$

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POISSON PARADIGM

The Poisson approximation can be used even in settings where the events are not necessarily independent, as long as they are "not very dependent"

Poisson Paradigm:

Let $A_1, A_2, \ldots, A_n$ be "weakly dependent" events.

Let X be the number of these events that occur.

Suppose that the number of events n is large.

Suppose that the events are rare, i.e., each $p_i = \Pr(A_i)$ is small.

Then X is **approximately** $\text{Poisson}(\lambda)$ with $\lambda = p_1 + p_2 + \ldots + p_n$.

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Example (Birthday problem): There are m people in the room. Each person's birthday is equally likely to be any day of the year, independently of the others.

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Example (Birthday problem): There are m people in the room. Each person's birthday is equally likely to be any day of the year, independently of the others.

What is the Pr that at least two people have the same birthday?

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What is the Pr that at least two people have the same birthday?

Direct calculation is possible but difficult

Poisson approximation gives us very good estimates even for fairly small m (e.g., $m = 23$)

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Example ($m = 3$): A_1 : (Alice, Bob) A_2 : (Alice, Charlie) A_3 : (Bob, Charlie)

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The events are not independent due to overlaps between pairs

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Events for non-overlapping pairs are independent

(Alice, Bob), (Charlie, Dave), (Ella, Frank), …

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$n = \binom{m}{2}$ pairs of people

Example: For $m = 23$, we have $n = \binom{23}{2} = 253$ pairs

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A_i: event that the i-th pair of people have the same birthday

$n = \binom{m}{2}$ pairs of people $\quad p = \Pr(A_i) = ?$

Assume for simplicity: number of birthdays = 365 (no leap years)

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$n = \binom{m}{2}$ pairs of people $\quad p = \Pr(A_i) = \frac{1}{365}$

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$X \approx \text{Poisson}(\lambda = np) \quad \Pr(X \geq 1) = 1 - \Pr(X = 0) \approx 1 - e^{-\lambda}$

For $m = 23$: $\Pr(X \geq 1) \approx 1 - e^{-\lambda} \approx 0.5$

Very good approx.

POISSON PARADIGM

Example (Near-birthday problem): There are m people in the room. Each person's birthday is equally likely to be any day of the year, independently of the others.

What is the Pr that at least two people have birthdays **within one day of each other** (i.e., same day or one day apart)?

Direct calculation is even more difficult

Poisson approximation gives us very good estimates even for fairly small m (e.g., $m = 14$)

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Example (Near-birthday problem): There are m people in the room. Each person's birthday is equally likely to be any day of the year, independently of the others.

What is the Pr that at least two people have birthdays **within one day of each other** (i.e., same day or one day apart)?

$X =$ number of pairs of people with birthdays within one day

A_i: event that i-th pair of people have birthdays within one day

$n = \binom{m}{2}$ pairs of people $\qquad p = \Pr(A_i)$

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$n = \binom{m}{2}$ pairs of people $\qquad p = \Pr(A_i) = \frac{3}{365}$

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If Alice's birthday is Jan 1st:

Bob's birthday needs to be Dec 31st, Jan 1st, or Jan 2nd

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For $m = 14$: $\Pr(X \geq 1) \approx 1 - e^{-\lambda} \approx 0.5266$

Very good approx.