CS 237: Probability in Computing

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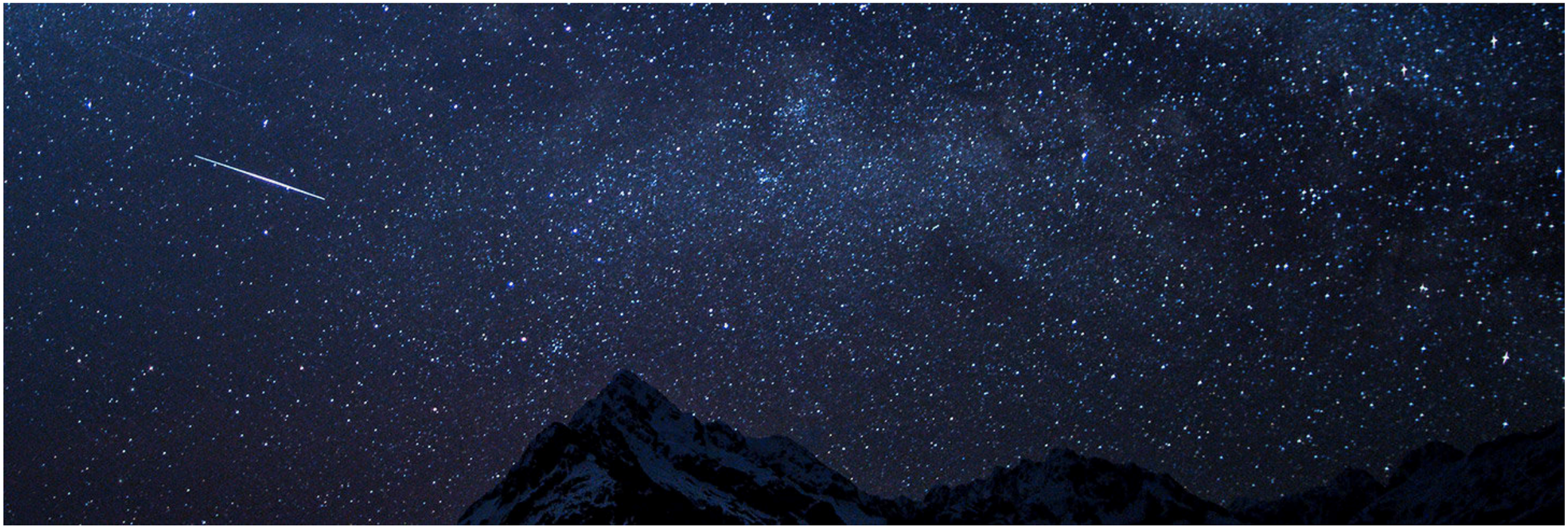
Boston University



A shooting star

EXPONENTIAL DISTRIBUTION

- Models the amount of time until some event of interest takes place
- Examples:
  - Time until we observe a shooting star
  - Time until a taxi or bus arrives
  - Time waiting in line at the post office
  - Time until a new job arrives at a server
  - Time until a hard drive breaks down
  - …

EXPONENTIAL DISTRIBUTION

- Range: $[0,\infty)$ The amount of time until some event happens

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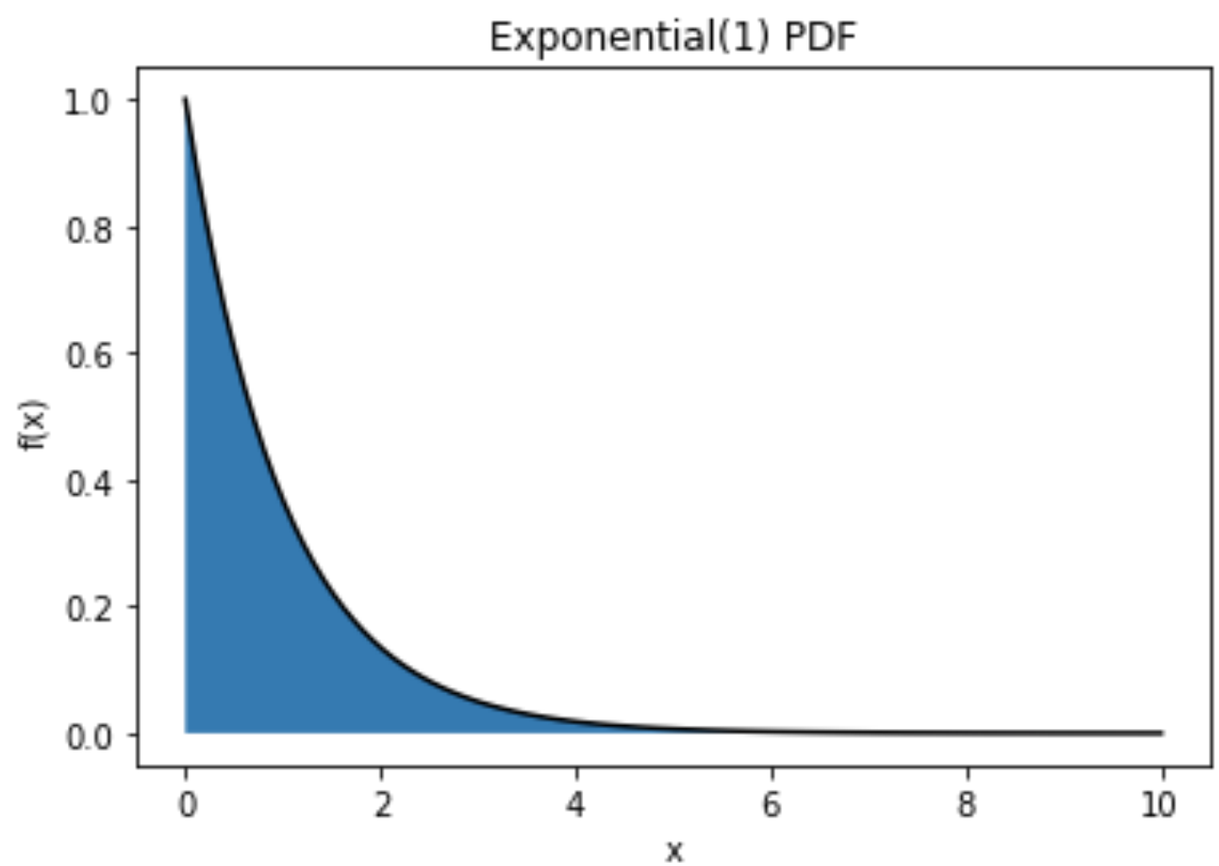
- PDF and CDF:

PDF

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



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Recall: $F(x) = \int_{-\infty}^{x} f(t)dt$

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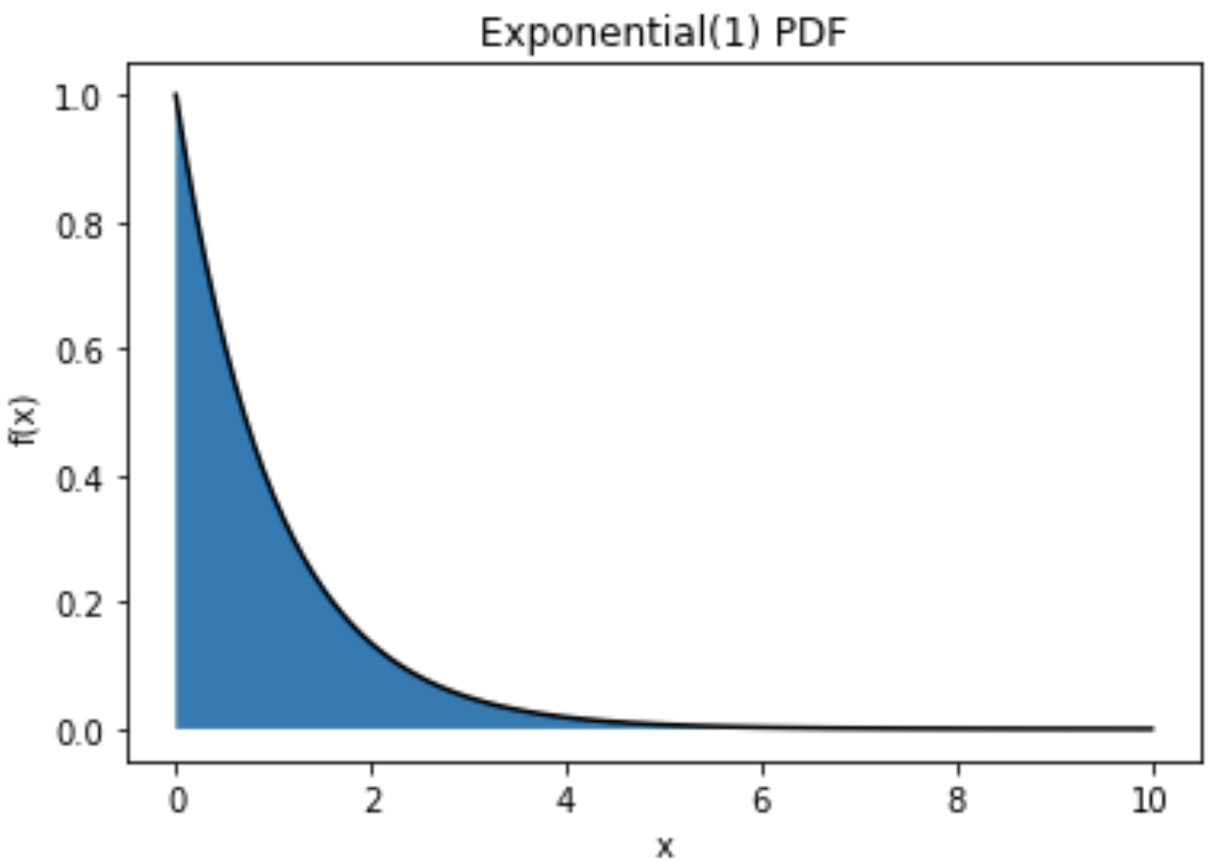

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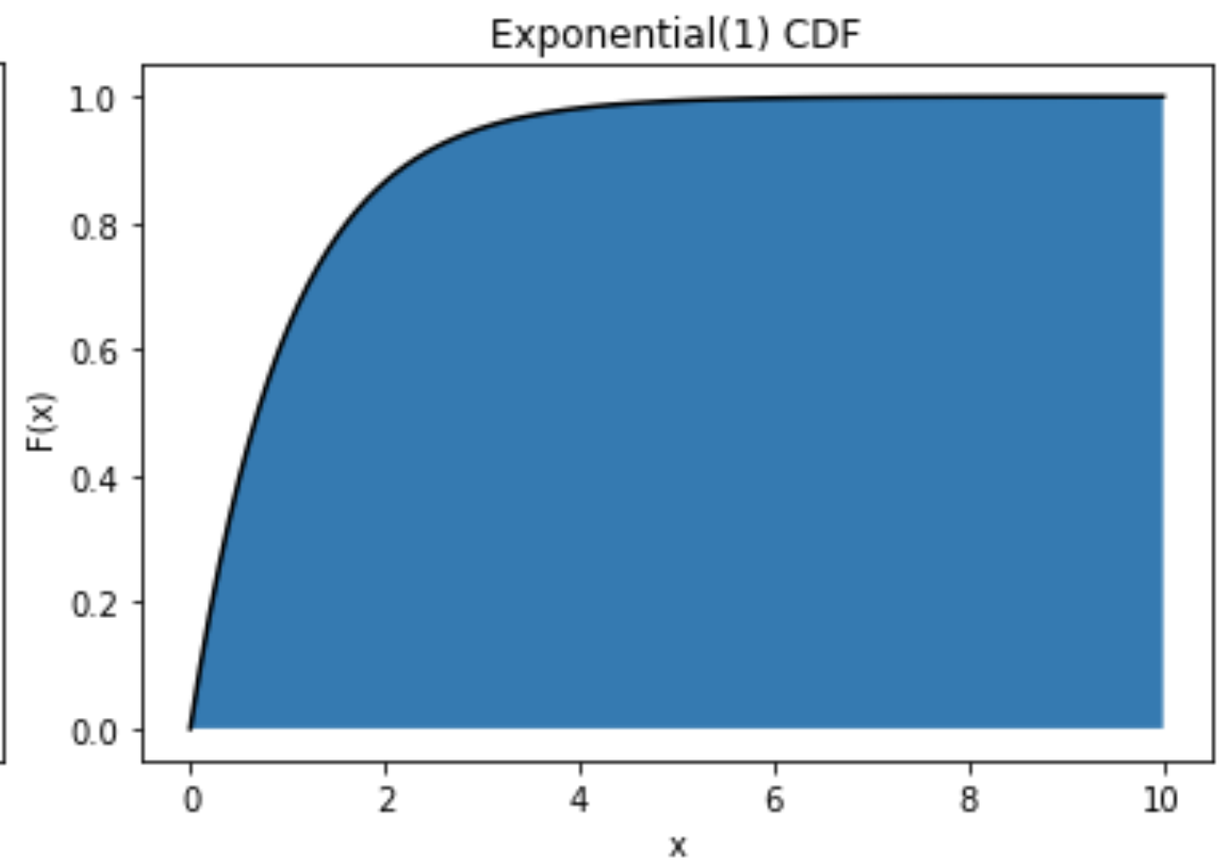

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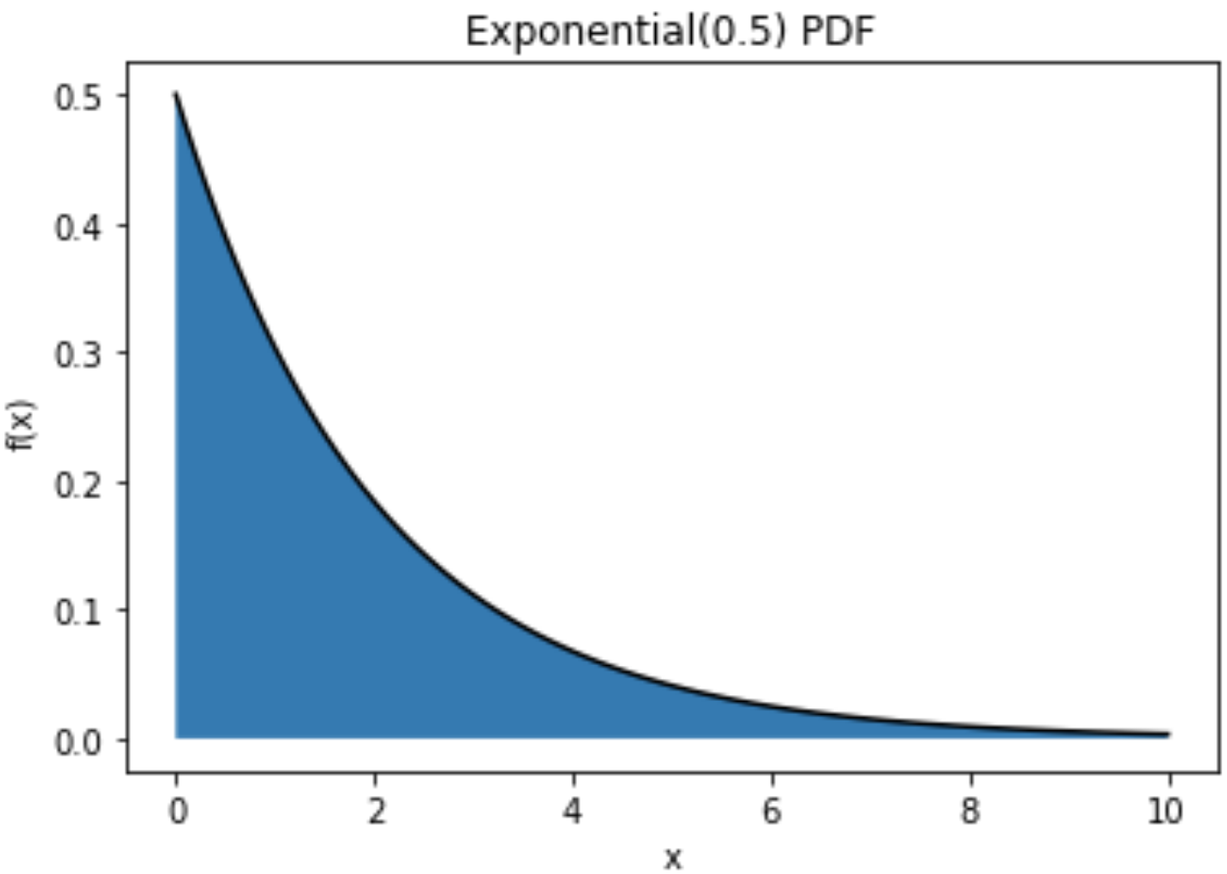

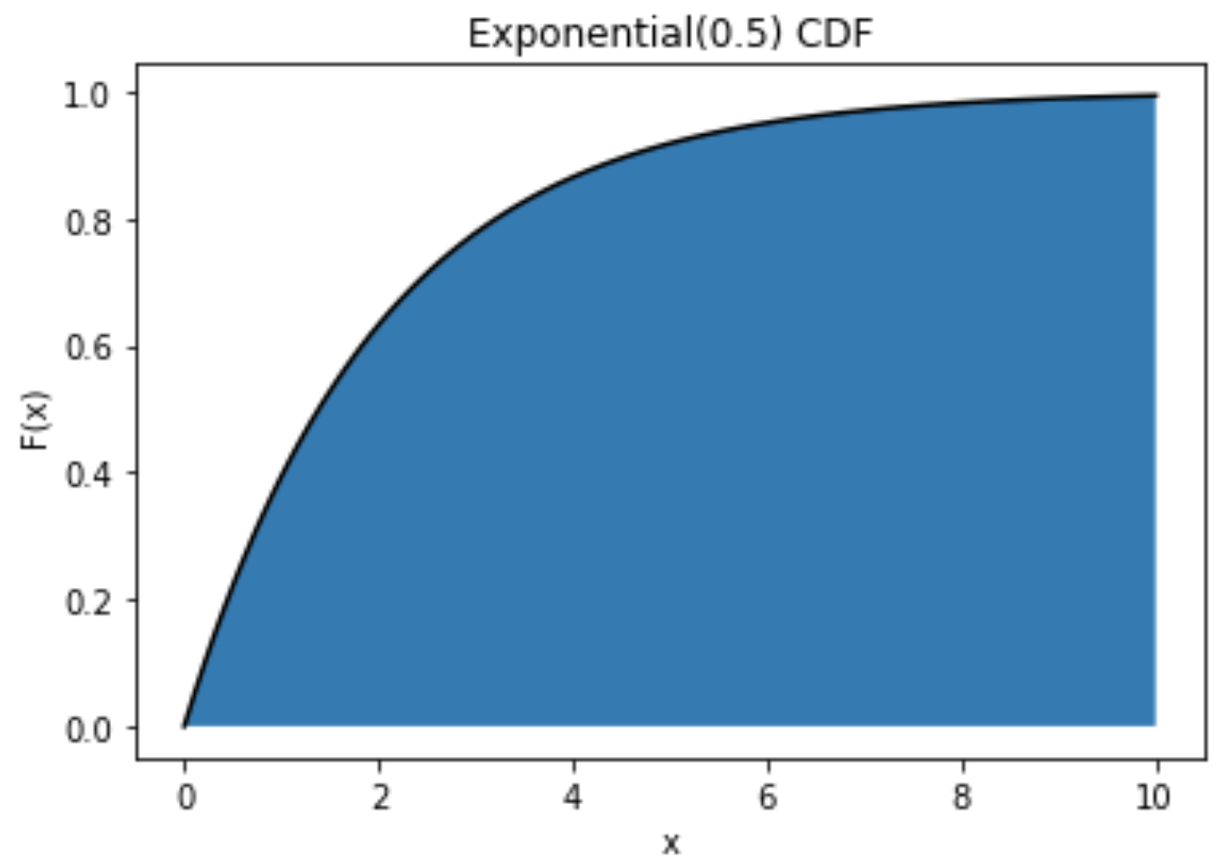

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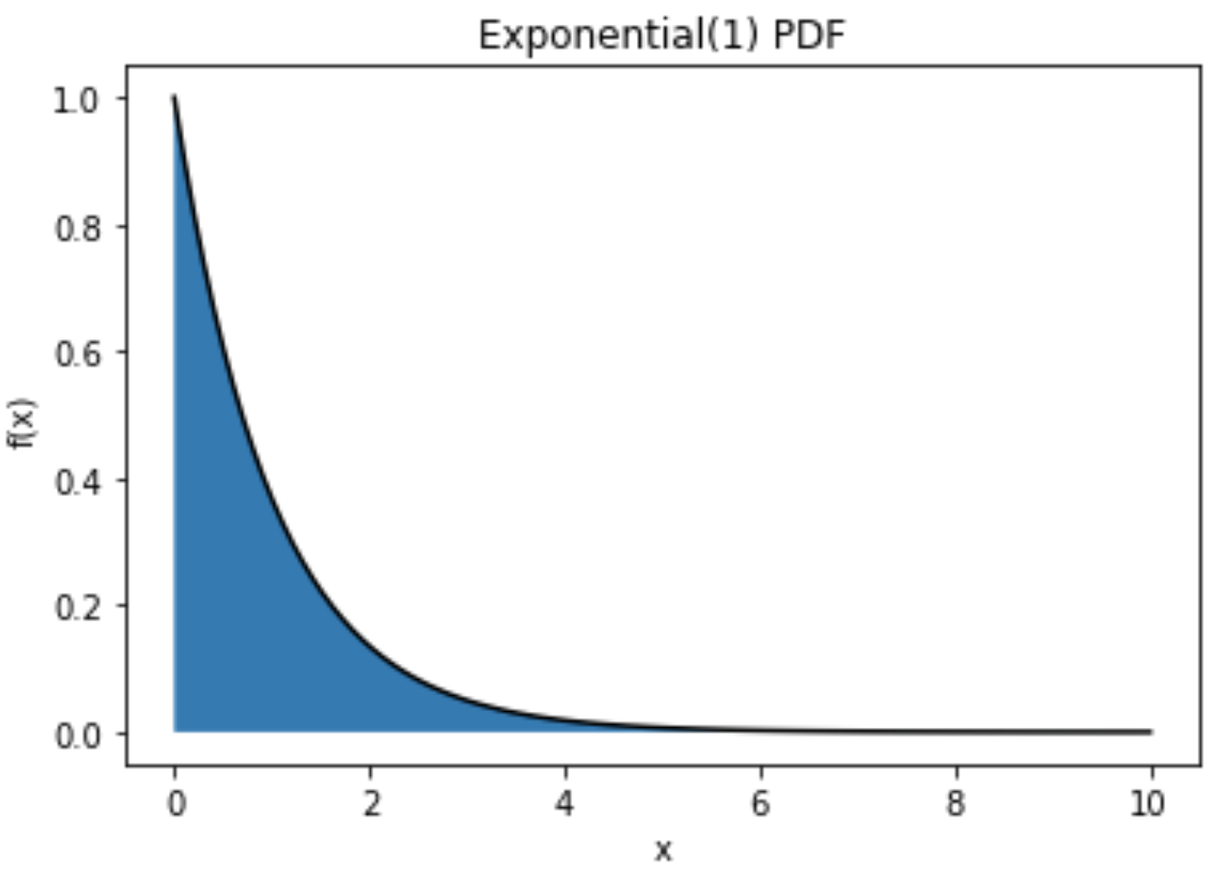

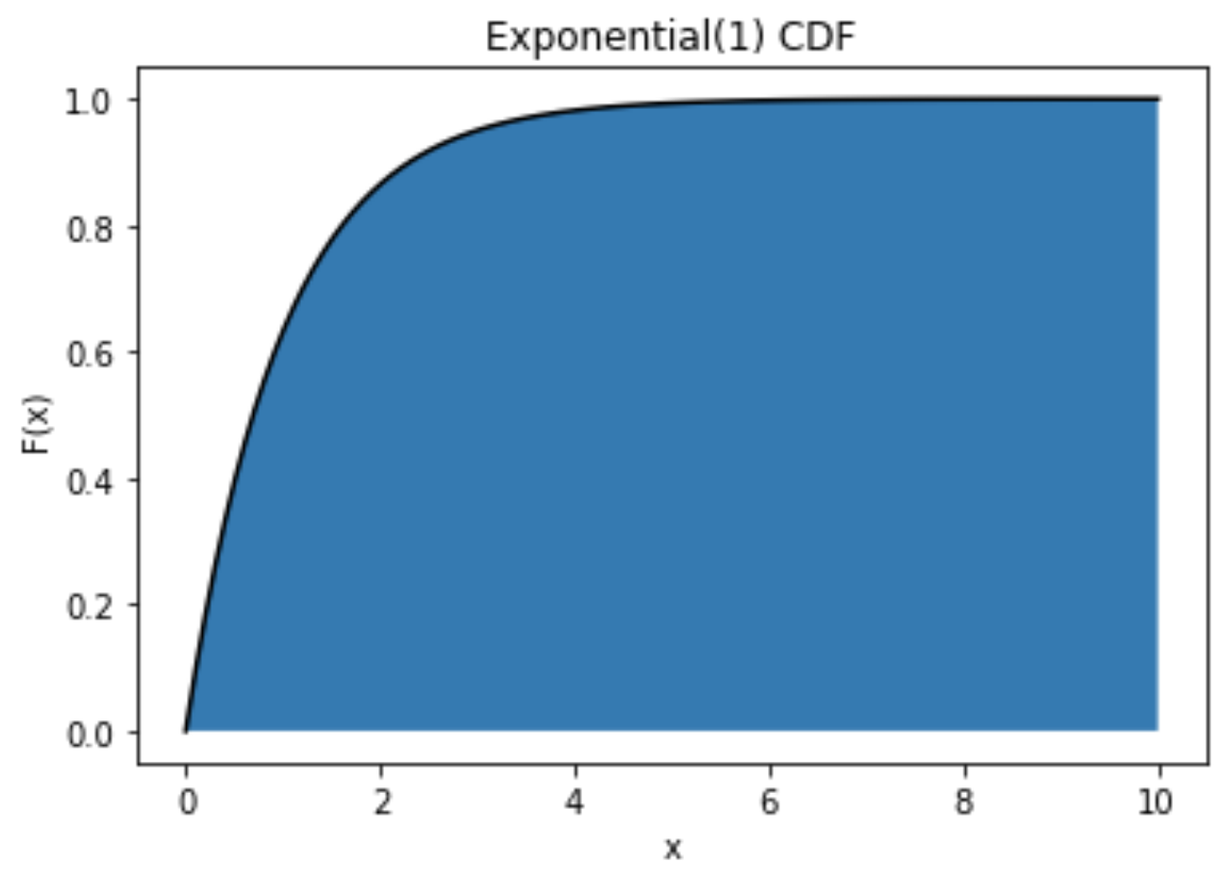

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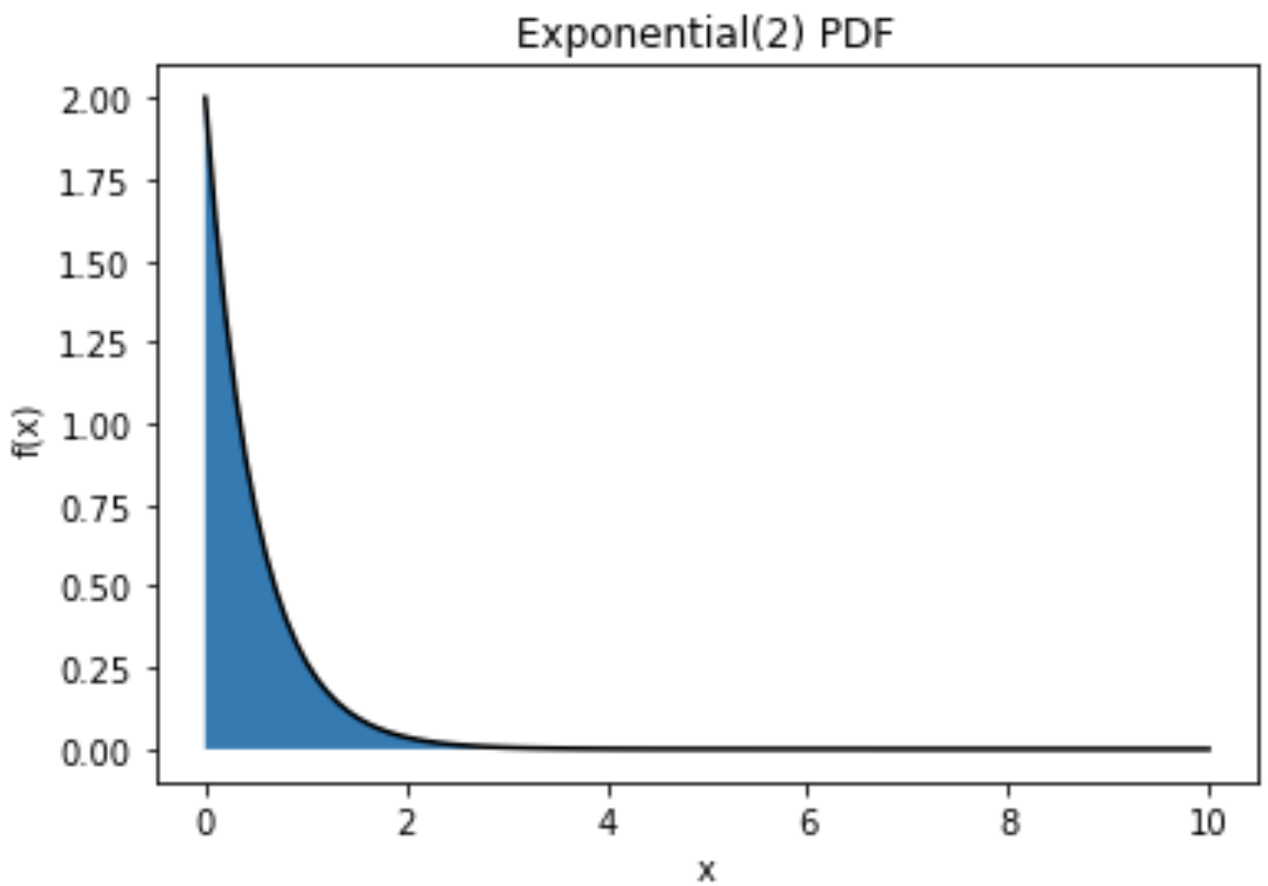

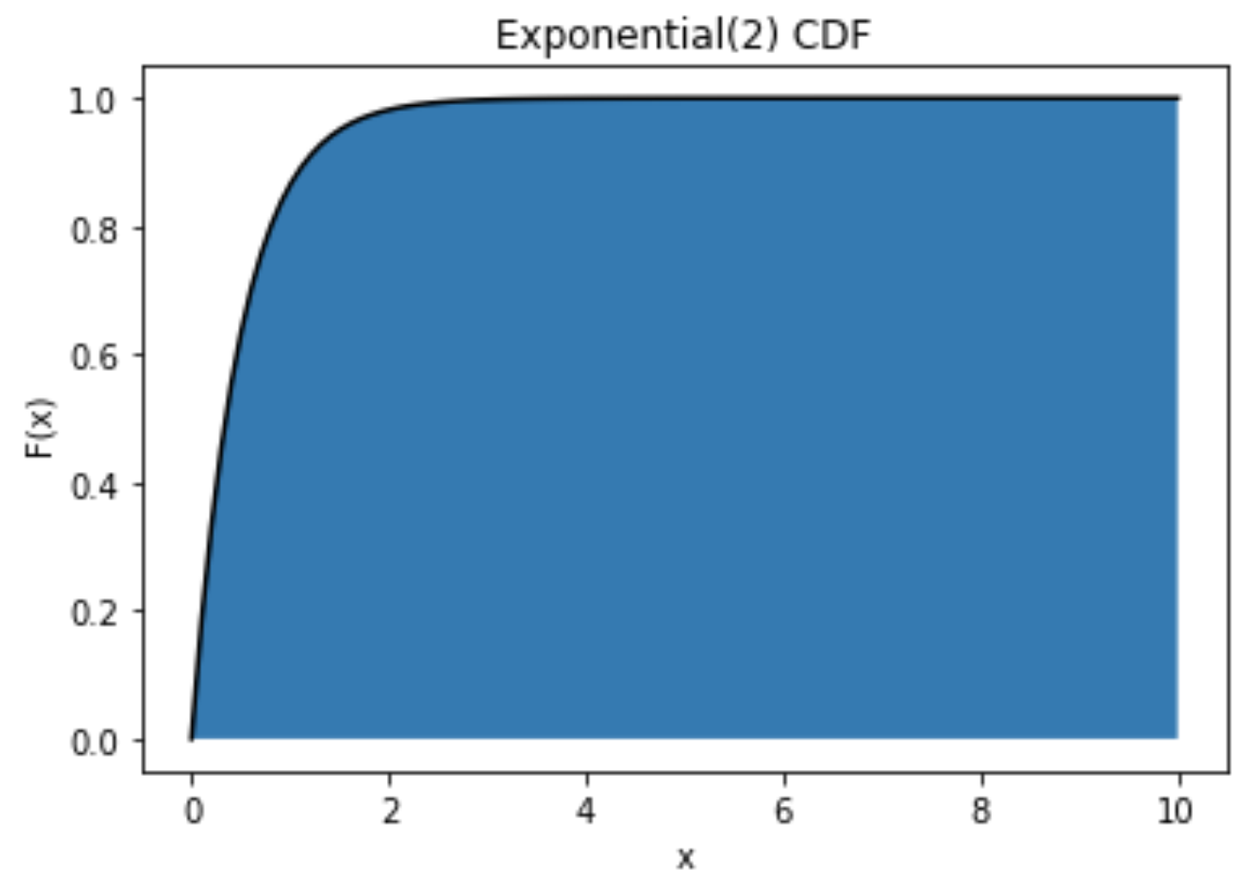

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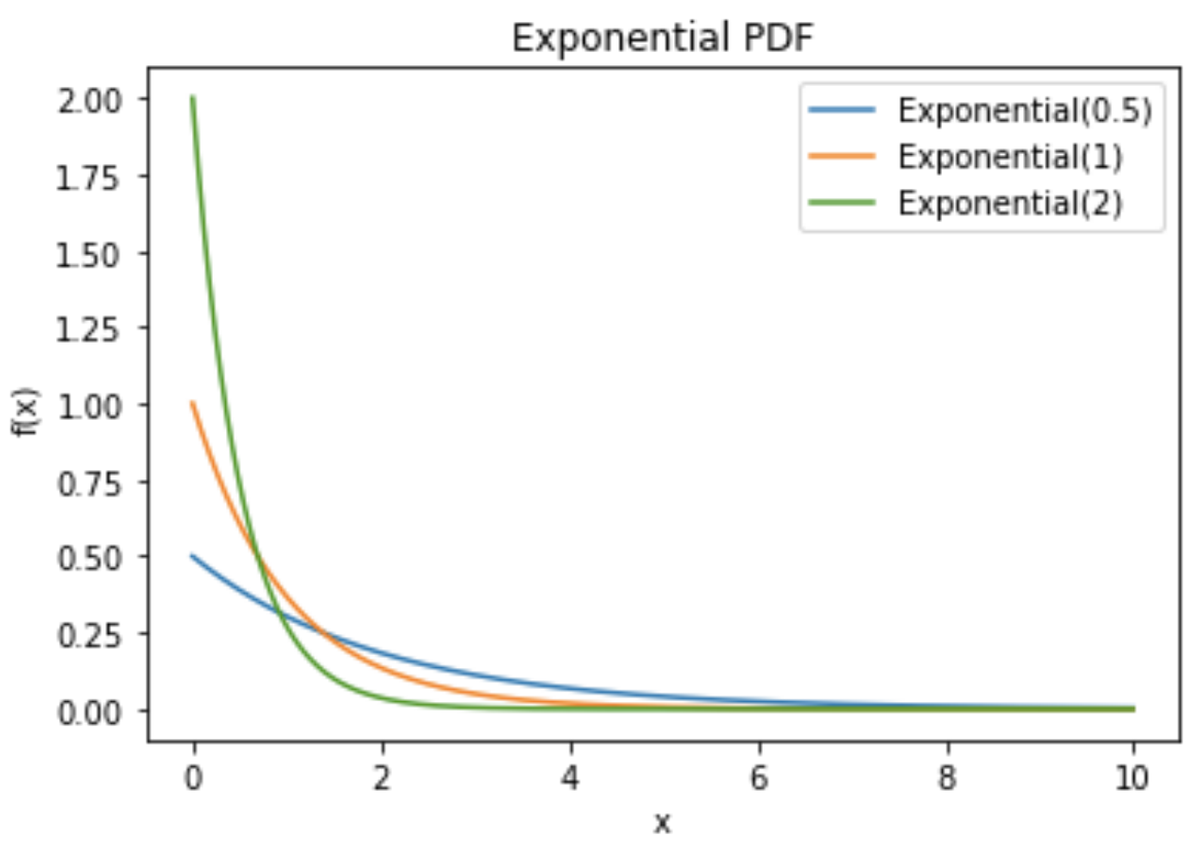

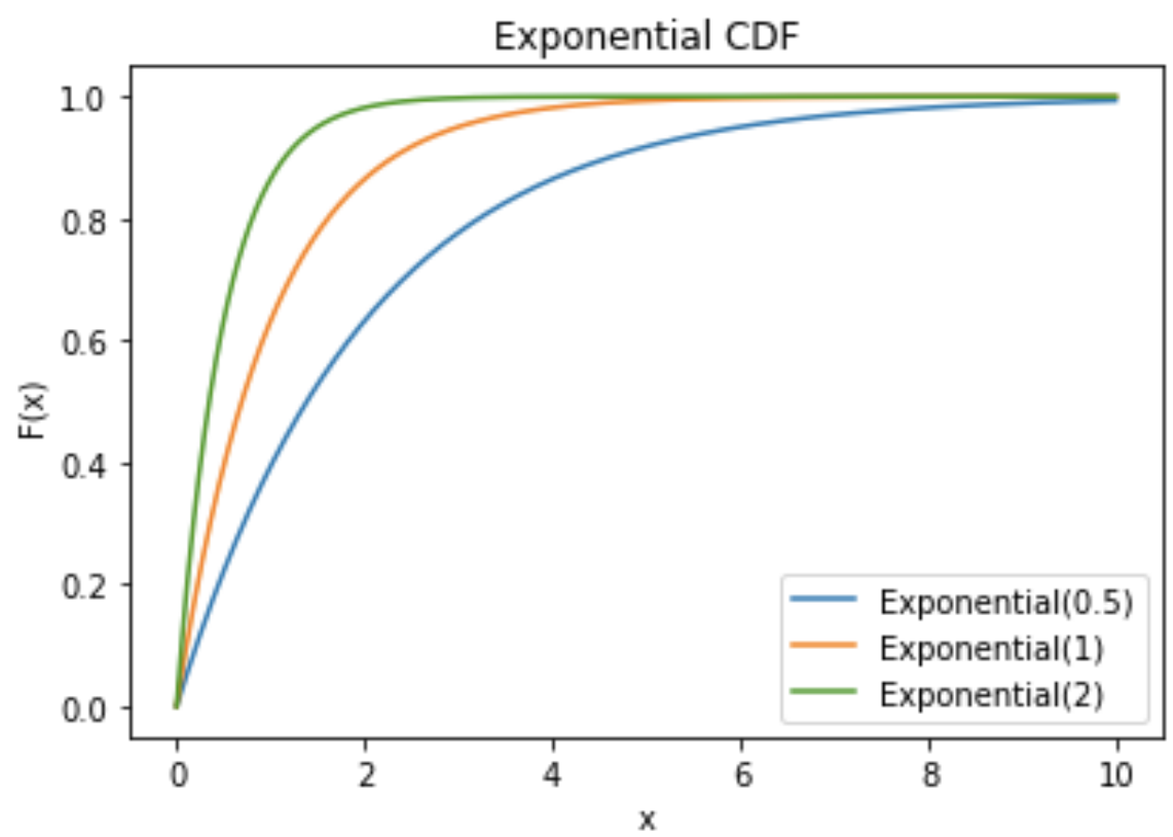

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- PDF: $f(x) = \lambda \cdot e^{-\lambda x}$ for all $x \geq 0$
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- Expectation and Variance:

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- Calculating Ex and Var requires integration by parts (not expected of you in this class)

EXPONENTIAL DISTRIBUTION: EXAMPLE

- The time is currently 11am
- We are interested in the time from now until we see a shooting star
- Suppose we model the quantity of interest as an Exponential random variable with expectation 10 days
- What is the probability that we will see a shooting star for the first time sometime between noon and 11pm?

EXPONENTIAL DISTRIBUTION: EXAMPLE

- Let $X =$ time (in days) from now until we see a shooting star

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- By our modeling assumption:

$$X \sim \text{Exponential}(\lambda) \qquad \text{Ex}(X) = 10 \text{ days}$$

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- By our modeling assumption:

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Using this information, we can find the value of the parameter:

$$\text{Ex}(X) = \frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10}$$

EXPONENTIAL DISTRIBUTION: EXAMPLE

- Let $X =$ time (in days) from now until we see a shooting star
- By our modeling assumption: $X \sim \text{Exponential}\left(\frac{1}{10}\right)$
- The event of interest is that X is at least 1 hour away but not more than 12 hours away from now

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$$1 \text{ hour} = \frac{1}{24} \text{ days} \qquad 12 \text{ hours} = \frac{1}{2} \text{ days}$$

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CDF: $F(x) = 1 - e^{-\lambda x}$

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1 hour = $\frac{1}{24}$ days 12 hours = $\frac{1}{2}$ days

$$\Pr\left(\frac{1}{24} \leq X \leq \frac{1}{2}\right) =$$

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$$\begin{aligned}
\Pr(1/24 \leq X \leq 1/2) &= \Pr(X \leq 1/2) - \Pr(X \leq 1/24) \\
&= F(1/2) - F(1/24) \\
&= \left(1 - e^{-\frac{1}{10}\cdot\frac{1}{2}}\right) - \left(1 - e^{-\frac{1}{10}\cdot\frac{1}{24}}\right) \\
&= e^{-\frac{1}{10}\cdot\frac{1}{24}} - e^{-\frac{1}{10}\cdot\frac{1}{2}}
\end{aligned}$$

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EXPONENTIAL DISTRIBUTION: EXAMPLE

CDF: $F(x) = 1 - e^{-\lambda x}$

- Alternatively, we can measure X in hours instead of days:
- By our modeling assumption:

$$X \sim \text{Exponential}(\lambda) \qquad \text{Ex}(X) = 10 \text{ days} = 240 \text{ hours}$$

$$\text{Ex}(X) = \frac{1}{\lambda} = 240 \Rightarrow \lambda = \frac{1}{240}$$

- The event of interest is that X is at least 1 hour away but not more than 12 hours away from now

$$\Pr(1 \leq X \leq 12) = F(12) - F(1)$$

$$= \left(1 - e^{-\frac{1}{240}\cdot 12}\right) - \left(1 - e^{-\frac{1}{240}\cdot 1}\right) = e^{-\frac{1}{240}} - e^{-\frac{1}{20}}$$

ASK THE AUDIENCE

- The post office has a single counter
- The time (in minutes) the clerk takes per customer has an Exponential(0.1) distribution
- When you arrive, the clerk is helping another customer and there is no one else in front of you
- What is the expected time you will wait in line?

(A) < 10 mins (B) 10 mins (C) > 10 mins

ASK THE AUDIENCE

- Let X = service time (in mins) for the customer in front of you
- Let t = time (in mins) the customer has been served already when you arrived at the post office (note: **not** a random variable)
- We are given that $X \geq t$
- Your waiting time is $W = X - t$

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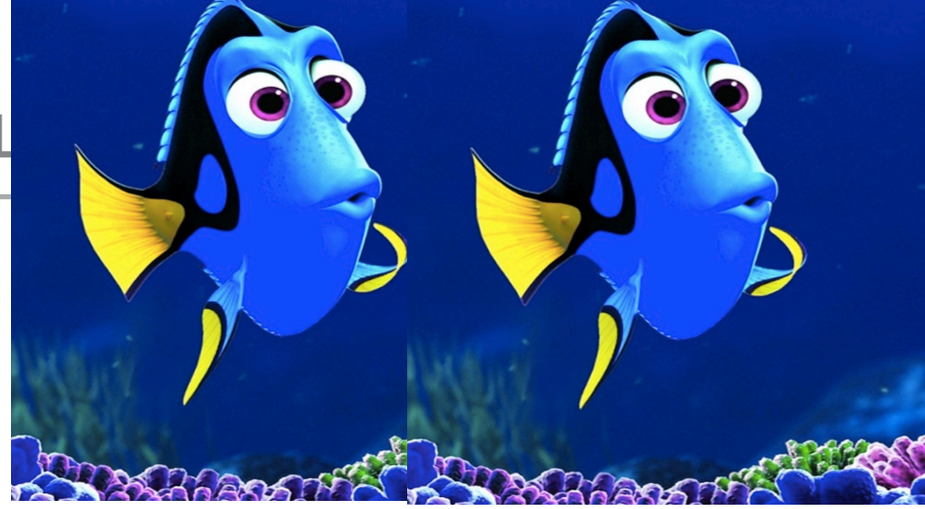
- We know $X \sim \text{Exponential}(0.1)$. What is the distribution of W?
  - Exponential distribution is memoryless (similarly to the Geometric)
  - $W \sim \text{Exponential}(0.1)$ and thus $\text{Ex}(W) = \dfrac{1}{0.1} = 10$ minutes



AMNESIA ONCE AGAIN

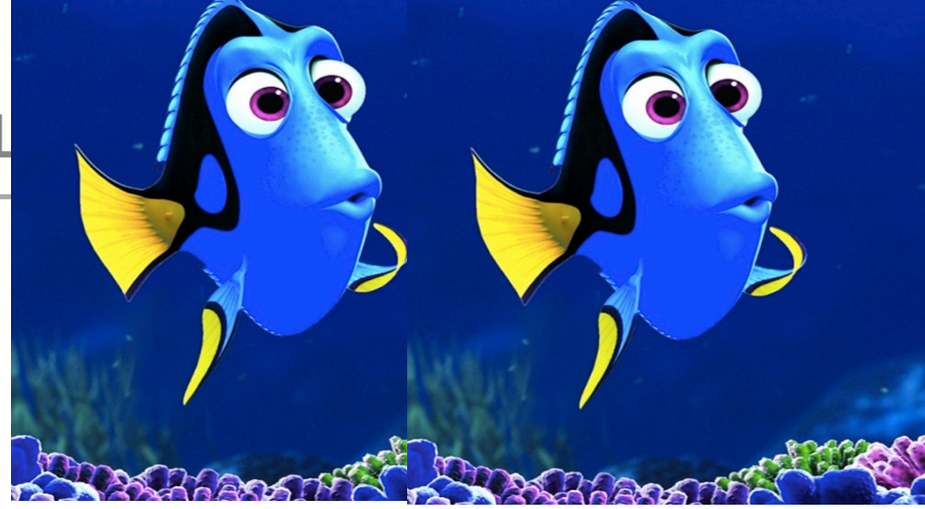
Thm (memoryless property): Let $X \sim \text{Exponential}(\lambda)$

For all $x \geq 0$ and $t \geq 0$: $\Pr(X \geq x + t \mid X \geq t) = \Pr(X \geq x)$

wait an additional x minutes

wait x minutes

Very important property for applications



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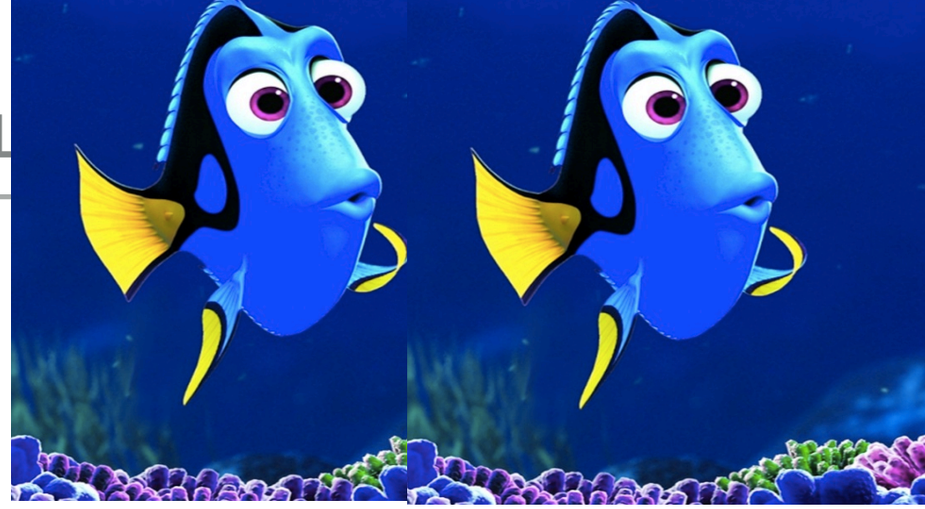

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Recall: $\Pr(X \geq x) = 1 - F(x) = e^{-\lambda x}$

$$\Pr(X \geq x+t \mid X \geq t) = \frac{\Pr([X \geq x+t] \cap [X \geq t])}{\Pr(X \geq t)} = \frac{\Pr(X \geq x+t)}{\Pr(X \geq t)}$$

$$= \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x} = \Pr(X \geq x)$$

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EXPONENTIAL AND GEOMETRIC

	Exponential(λ)	Geometric(p)
Range	$x \in [0,\infty)$	$n \in \{1,2,3,\cdots\} = \mathbb{N}\setminus\{0\}$
CDF	$1 - e^{-\lambda x}$	$1 - (1-p)^n$
$\mathrm{Ex}(X)$	$\frac{1}{\lambda}$	$\frac{1}{p}$
$\mathrm{Var}(X)$	$\frac{1}{\lambda^2}$	$\frac{1-p}{p^2}$
Memoryless?	Yes	Yes

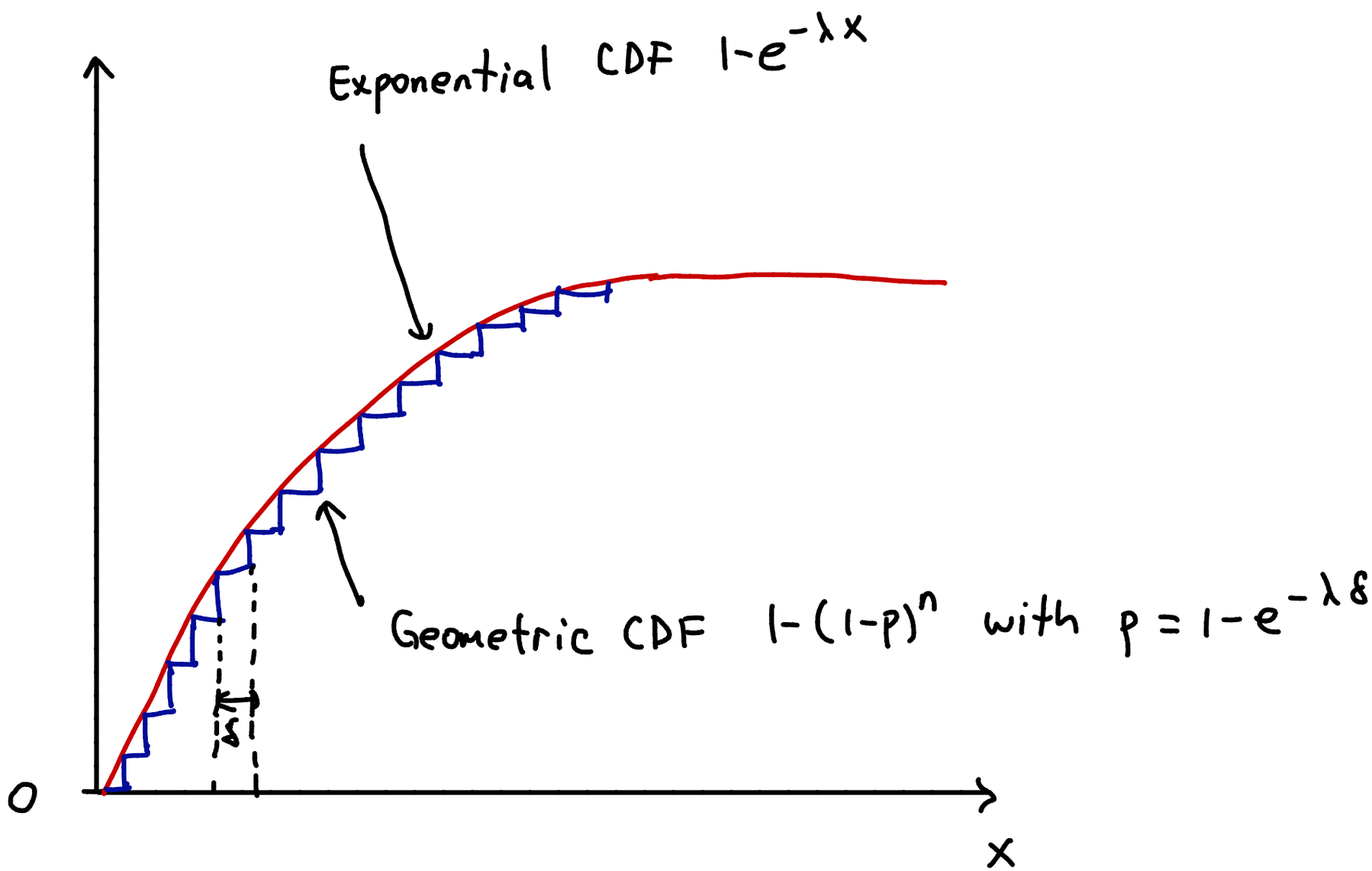
EXPONENTIAL AND GEOMETRIC

- The Exponential distribution can be viewed as a continuous approximation of the Geometric distribution
- We will illustrate this relationship next, by plotting the Exponential and Geometric CDFs

EXPONENTIAL AND GEOMETRIC

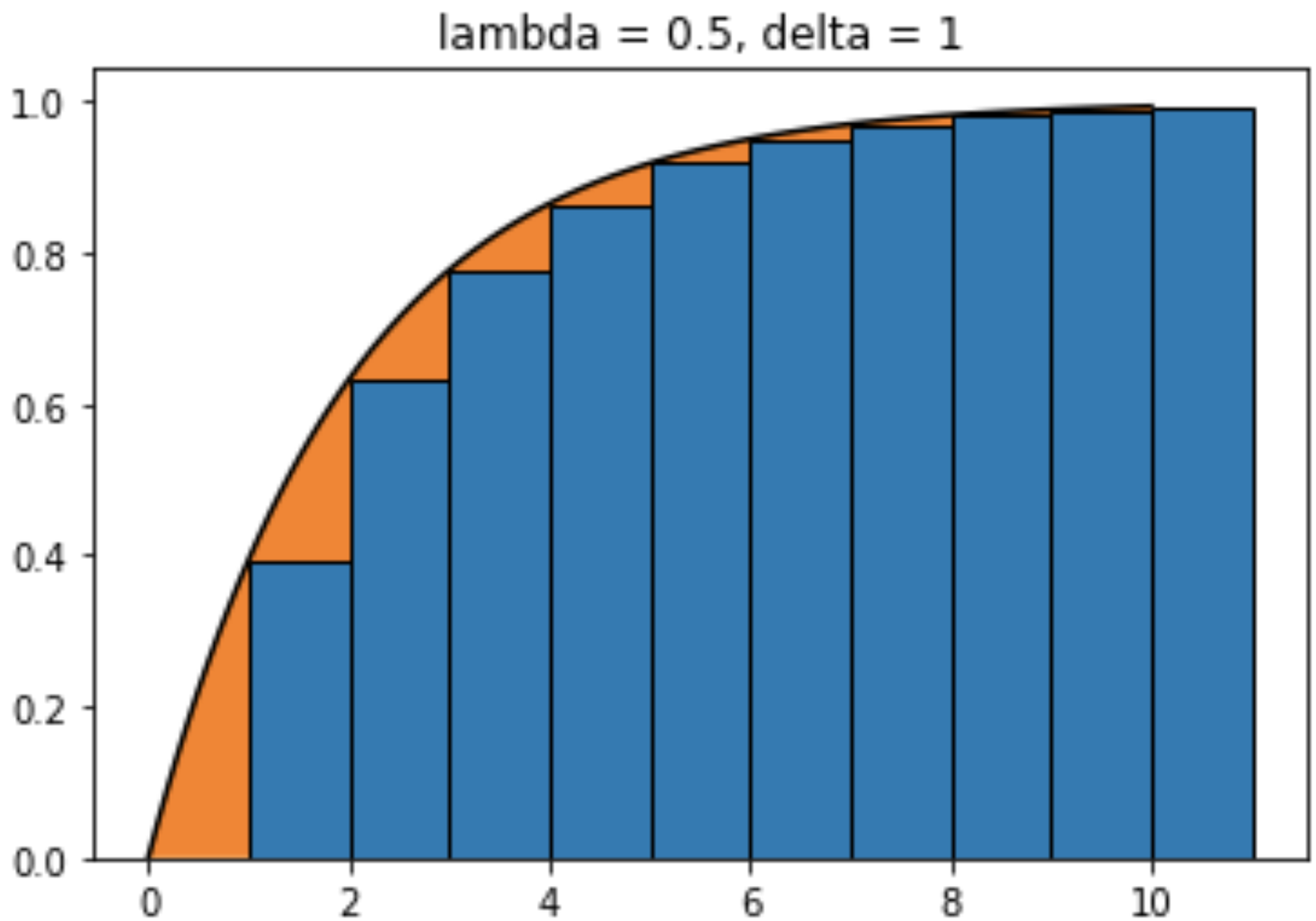
Think of the x-axis as time (in seconds, say)

Each Geometric trial happens every δ seconds



EXPONENTIAL AND GEOMETRIC

As the time delta between geometric trials becomes smaller, geometric and exponential CDFs get closer and closer



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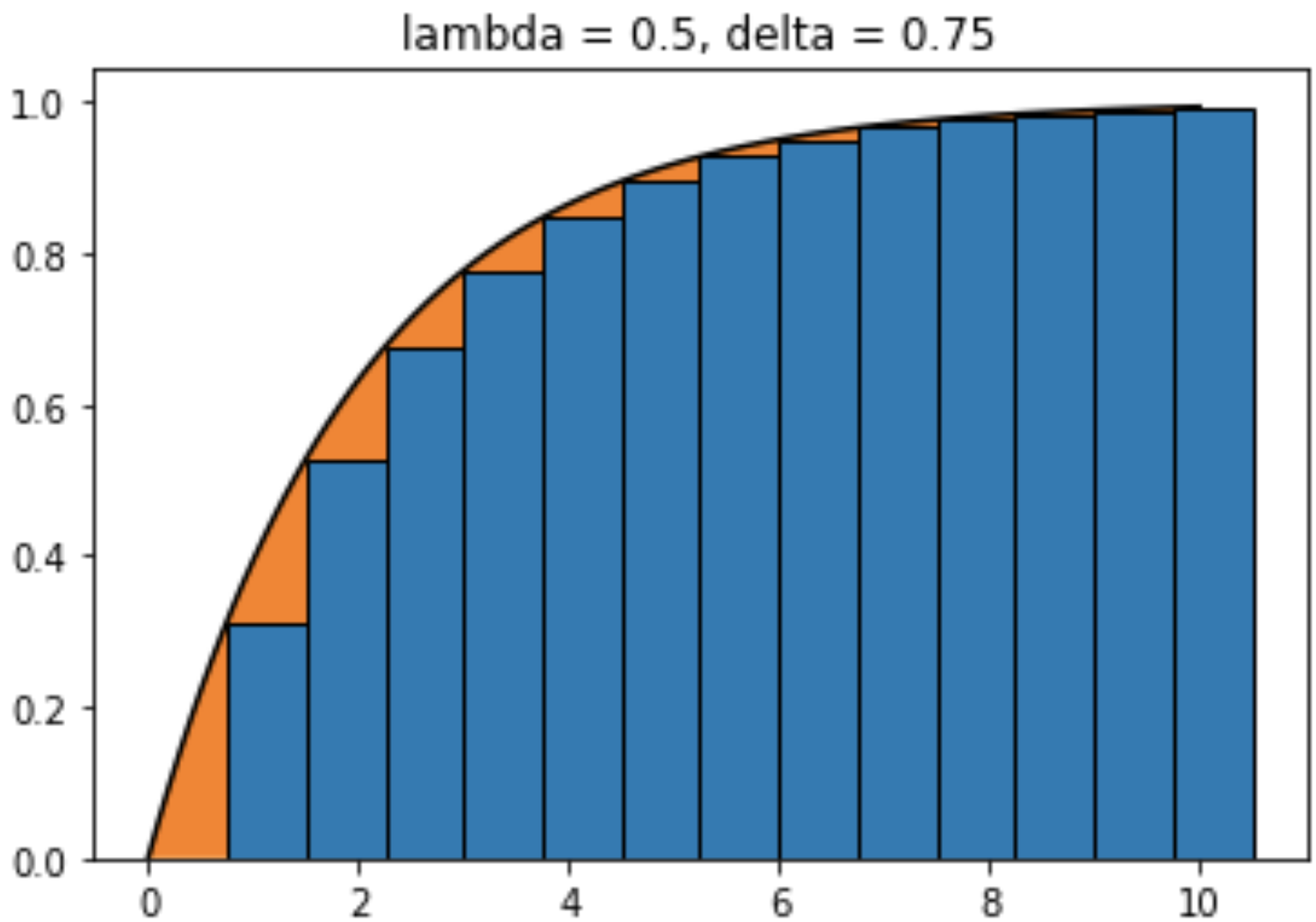

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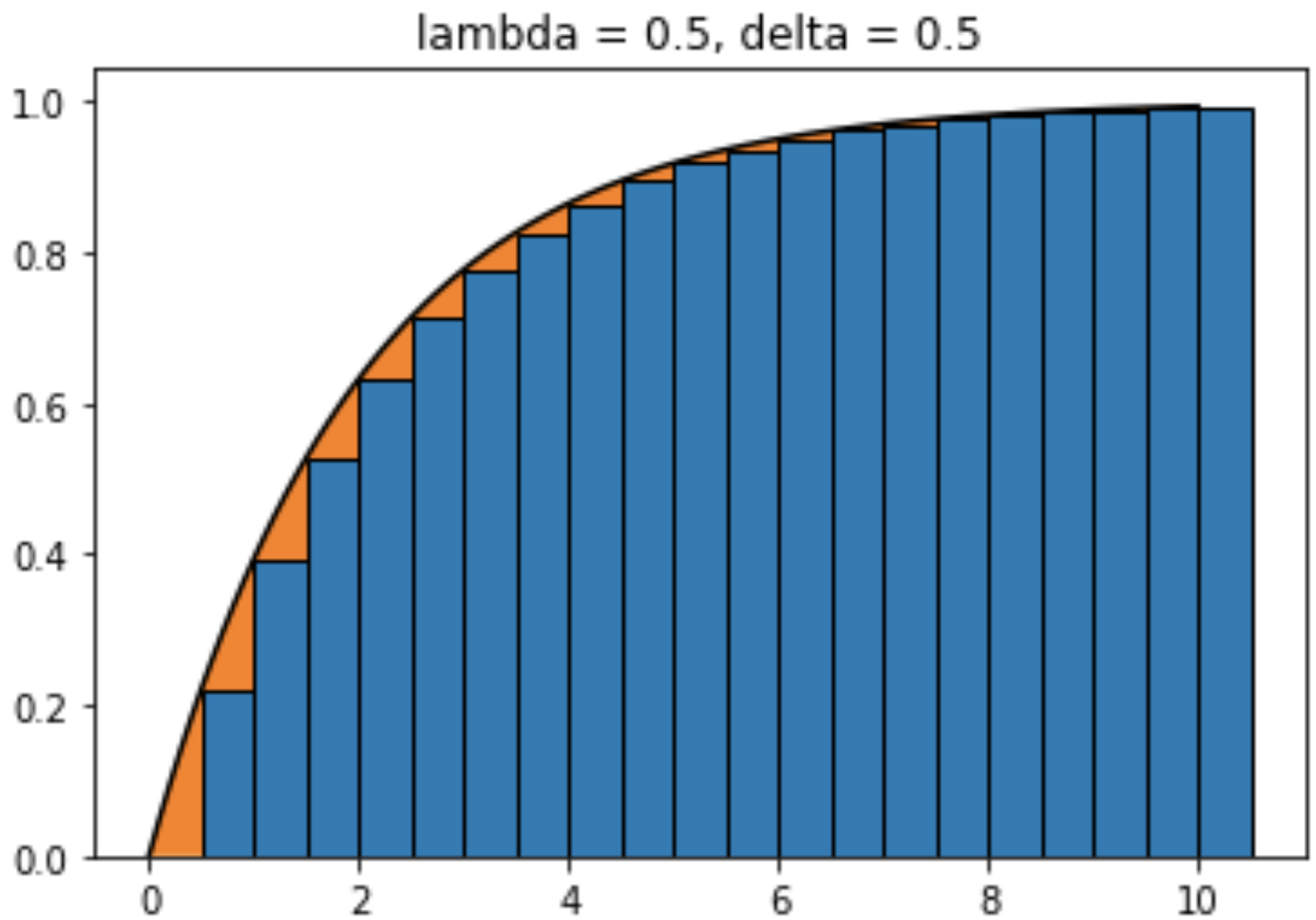

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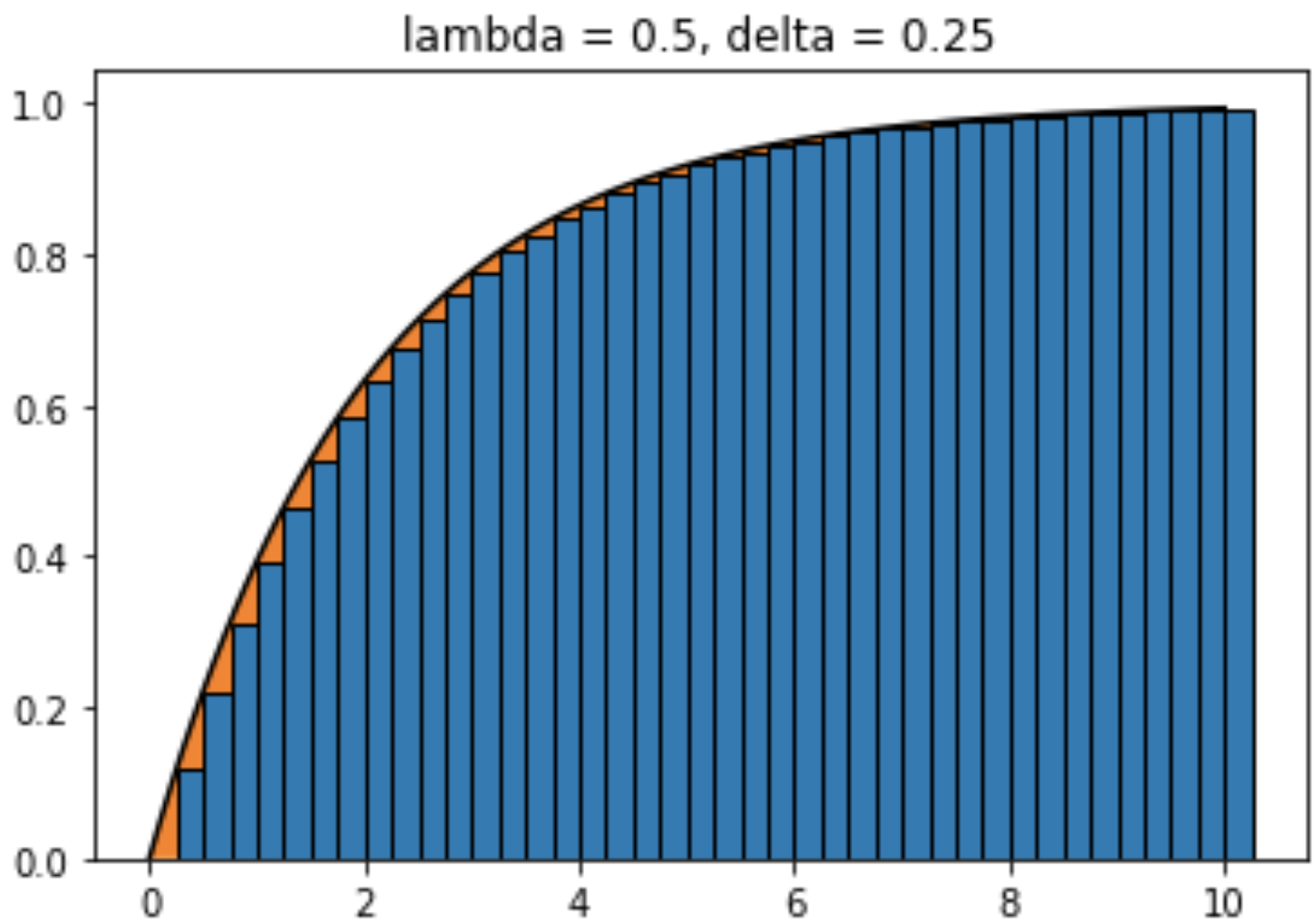

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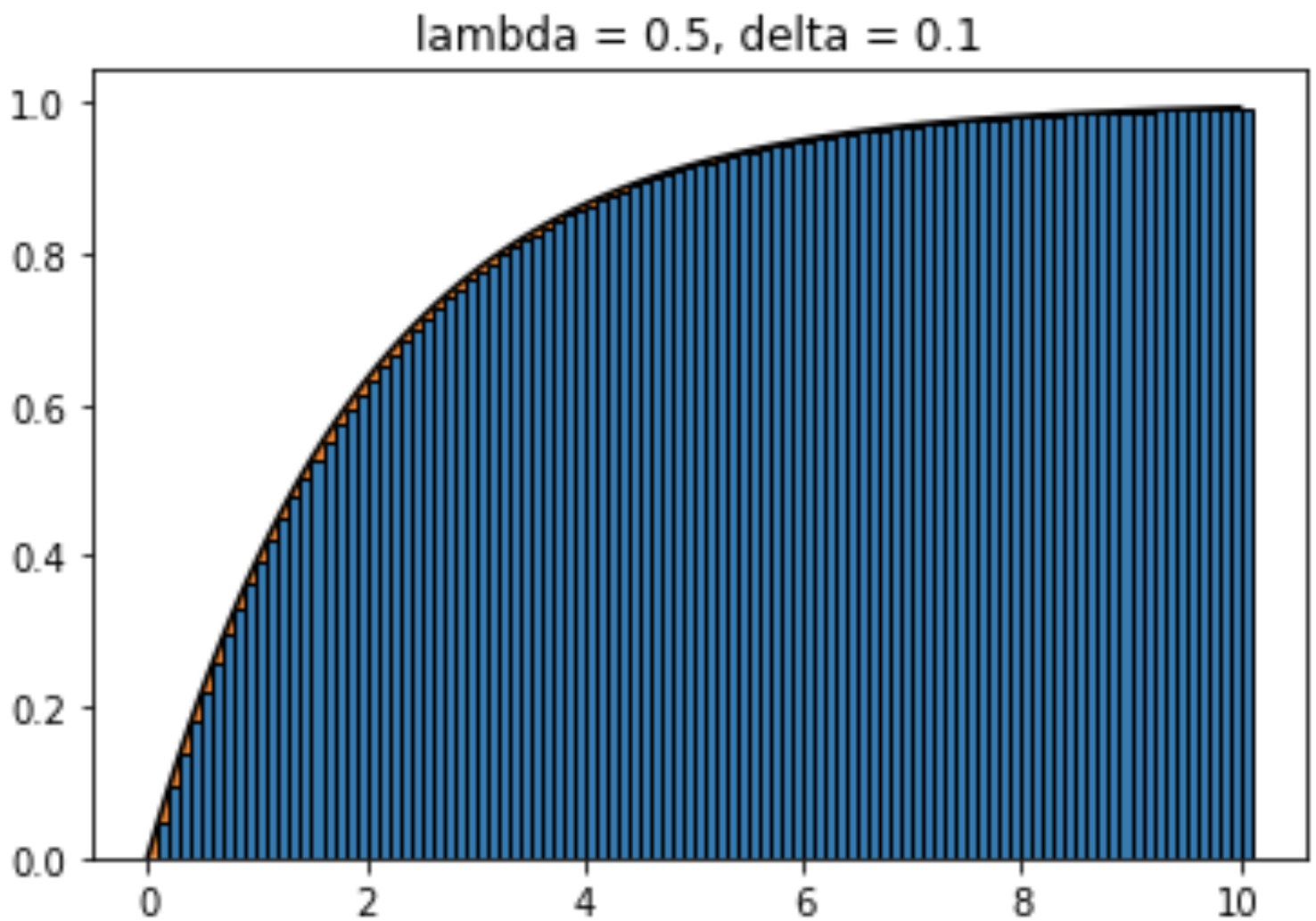
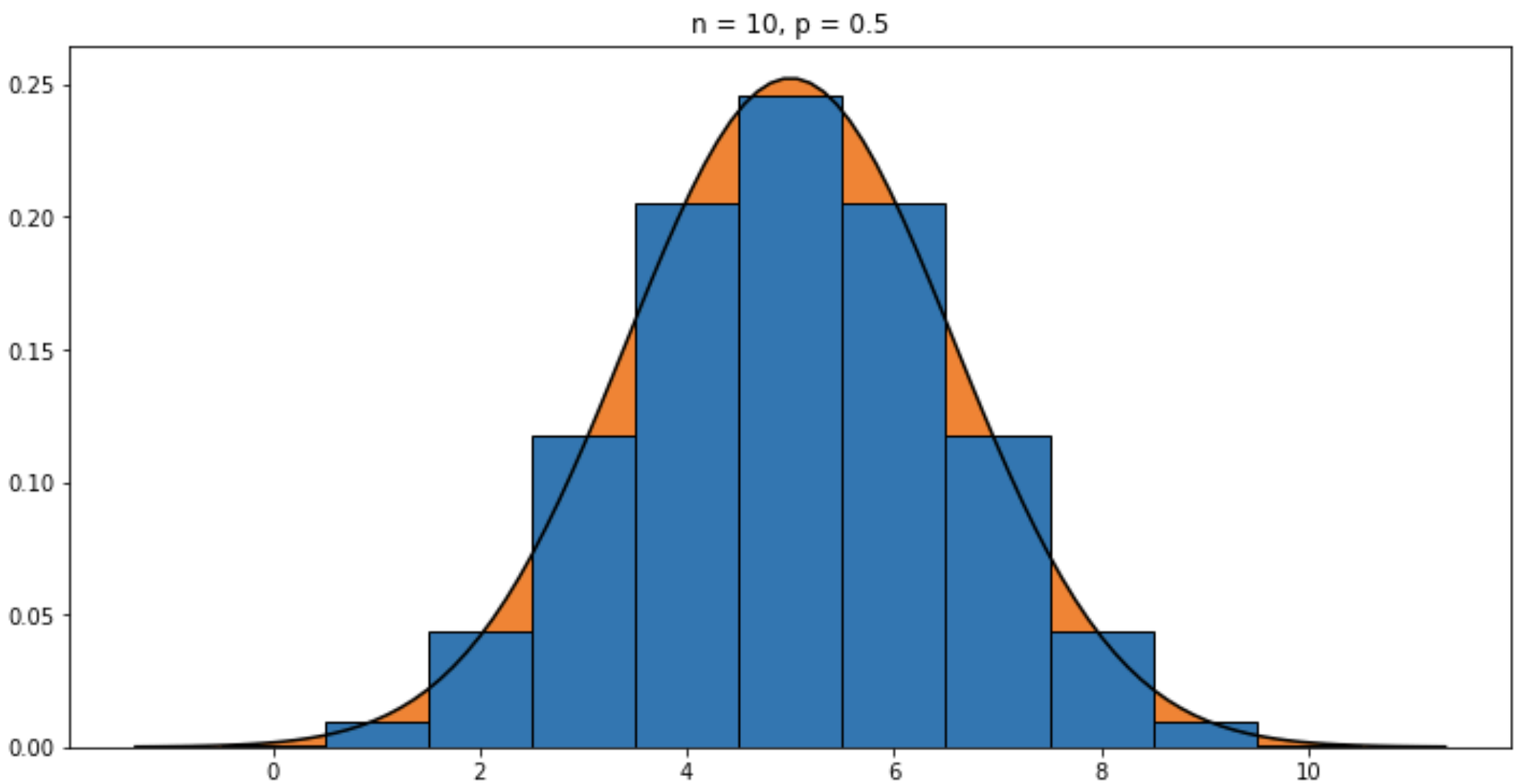


BINOMIAL AND NORMAL

Plot of Binomial and Normal distributions with the same Ex and Var: Binomial(n, p) and Normal(np, np(1-p))



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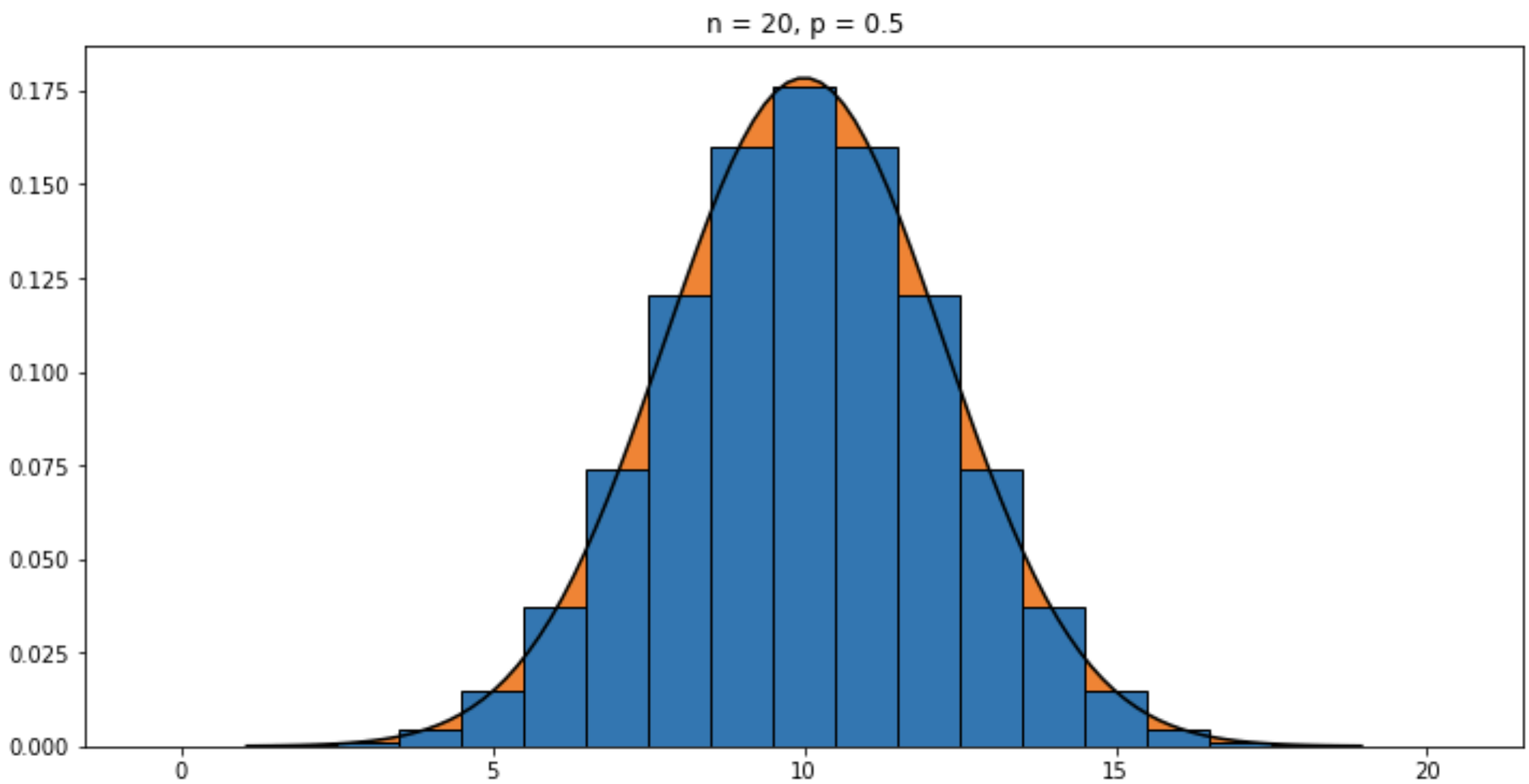

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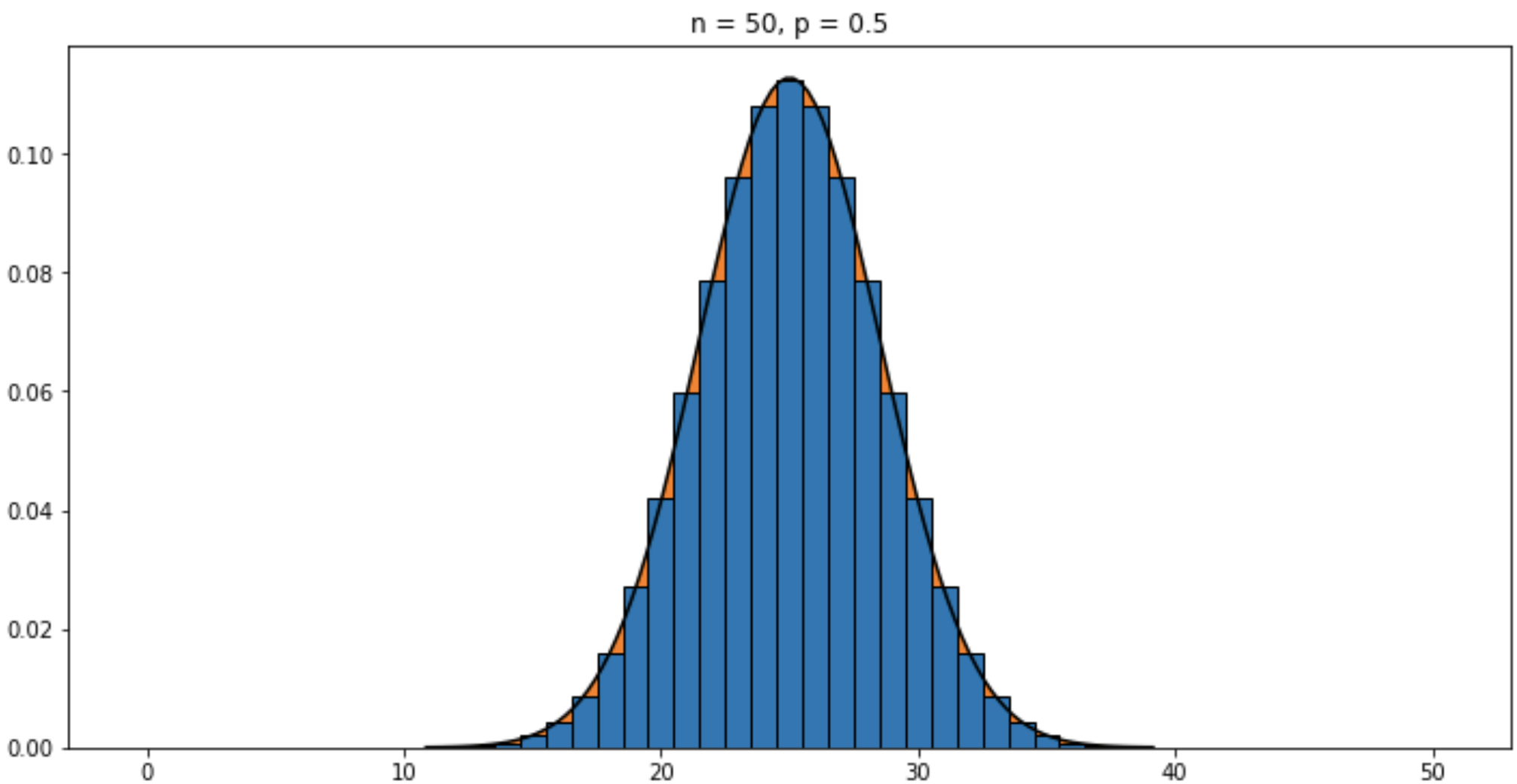

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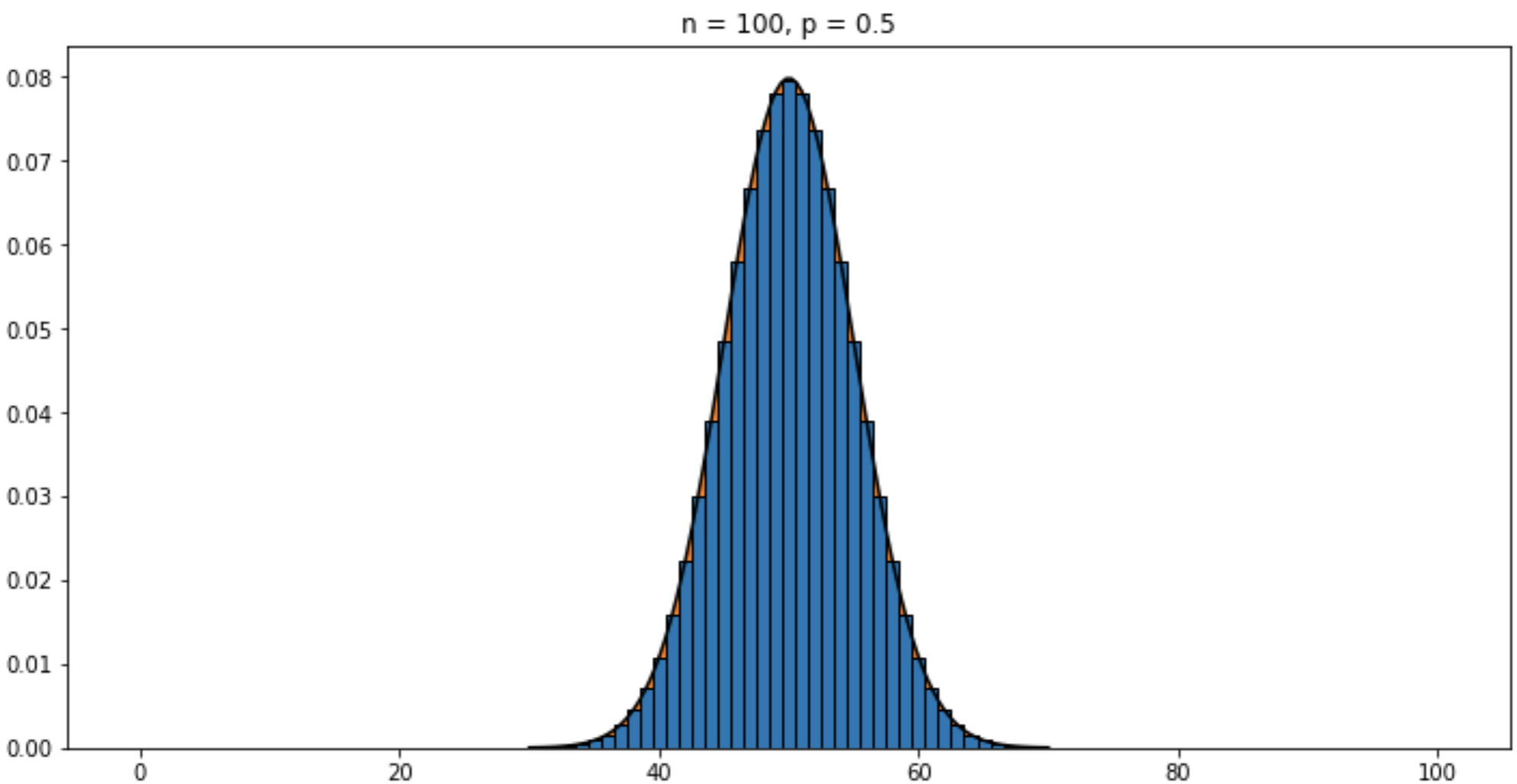

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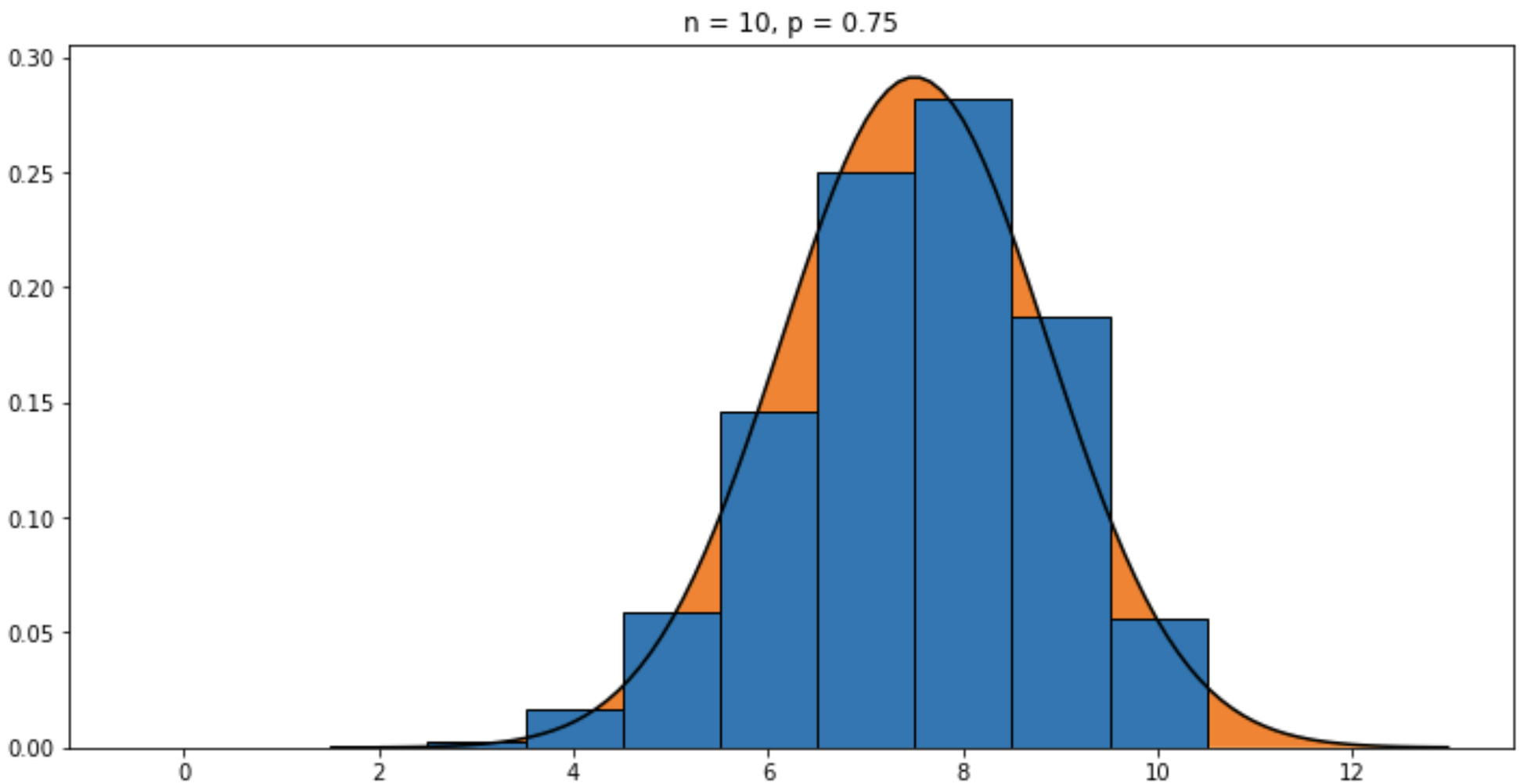

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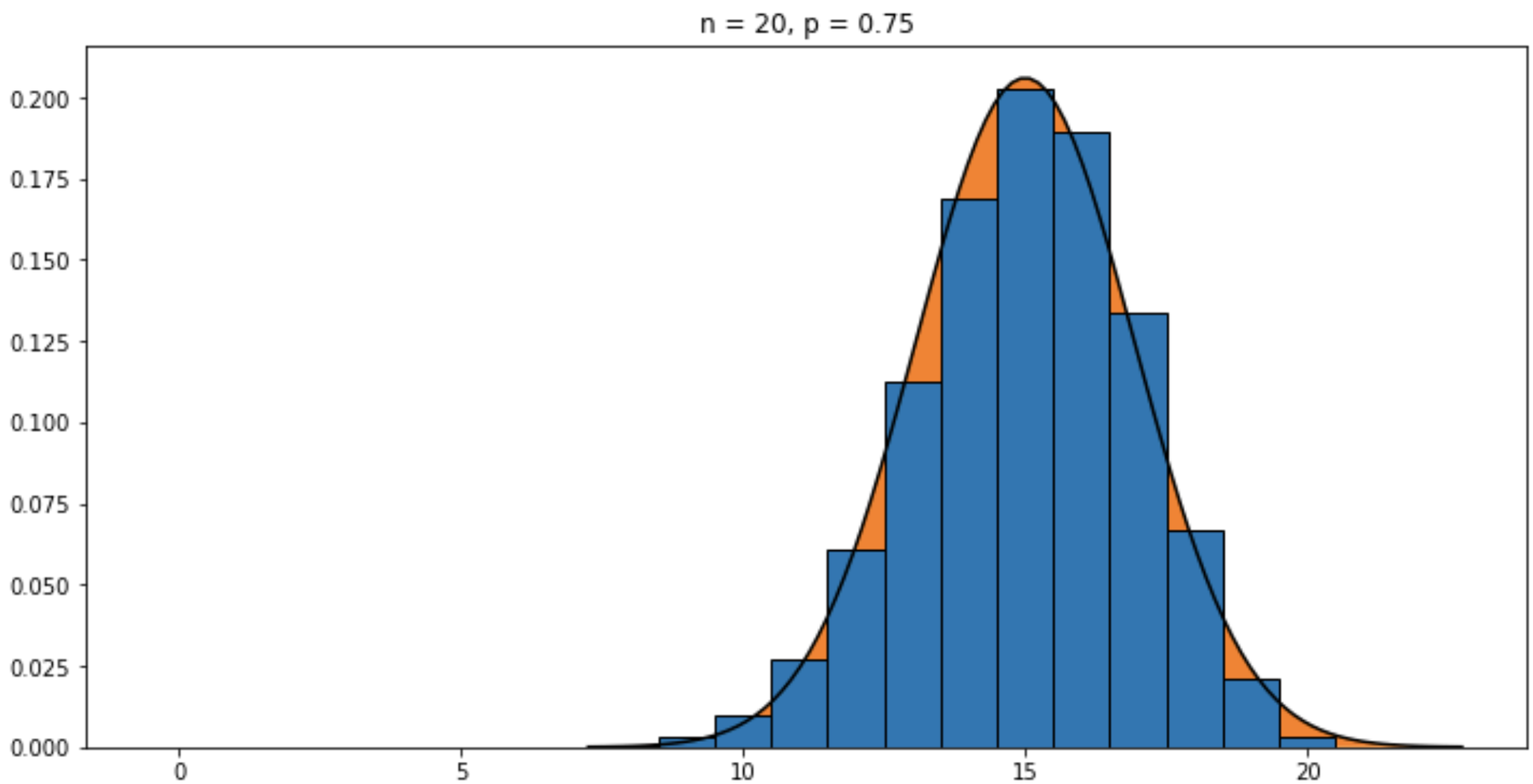

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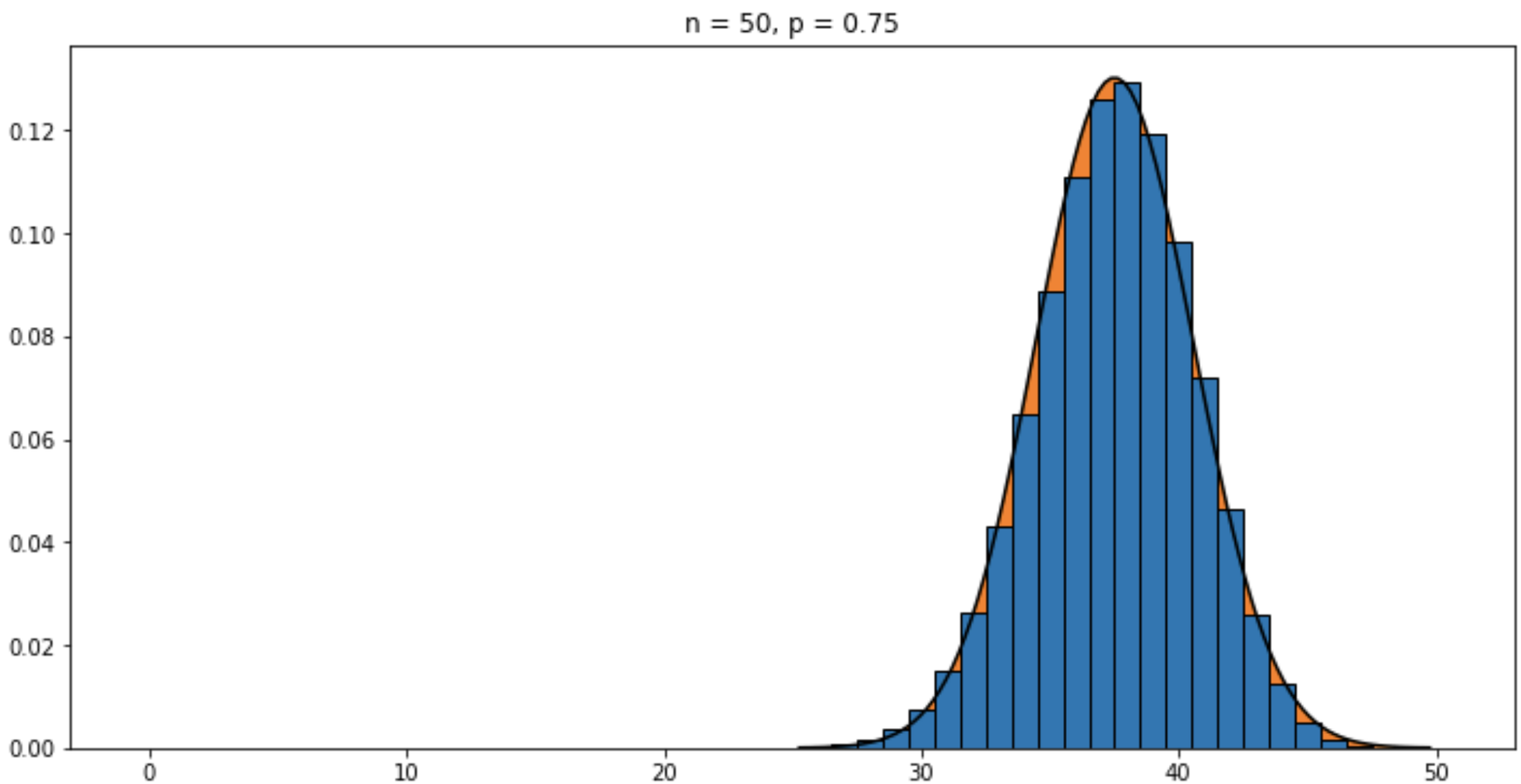
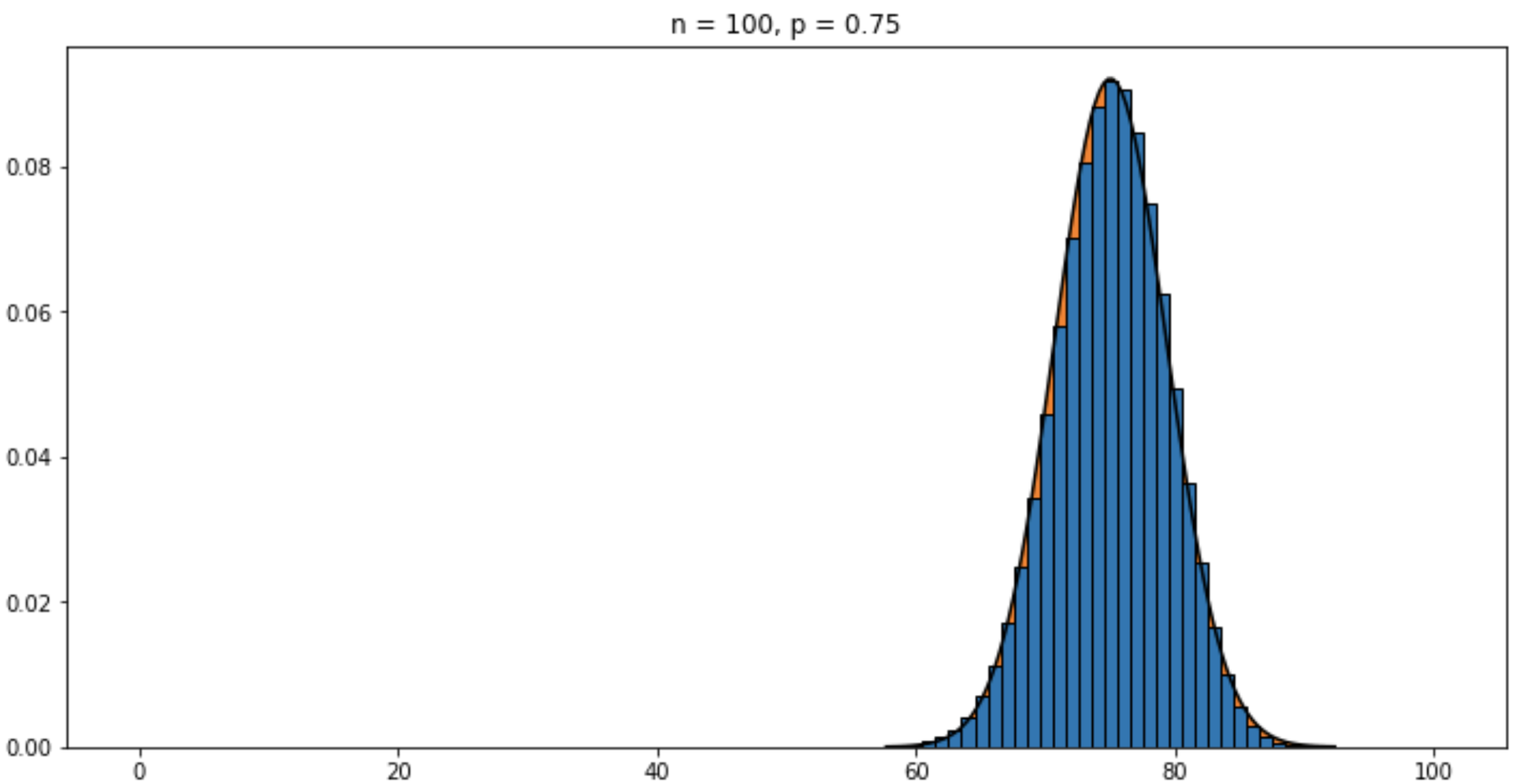


BINOMIAL AND NORMAL

Plot of Binomial and Normal distributions with the same Ex and Var: Binomial(n, p) and Normal(np, np(1-p))