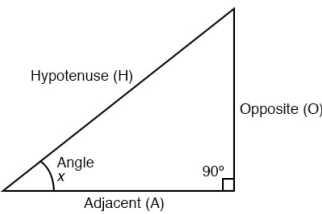


Student Copy
Grade 9 Day 6

Trigonometric Functions

Trigonometric functions describe the relationship between the angles of a triangle and the lengths of its sides. They are mainly used with right-angled triangles and later with the unit circle.



- In a right-angled triangle shown,
- Hypotenuse: the longest side (opposite the right angle)
 - Opposite side: the side opposite angle x
 - Adjacent side: the side next to angle x
 -

The three main trigonometric functions are sine, cosine and tangent. These are defined as follows:

$$\sin x = \frac{\text{opposite side of angle } x}{\text{hypotenuse}}$$

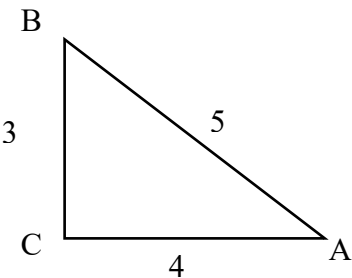
$$\cos x = \frac{\text{adjacent side to angle } x}{\text{hypotenuse}}$$

$$\tan x = \frac{\text{opposite side of angle } x}{\text{adjacent side to angle } x}$$

Example 1. In $\triangle ABC$, $\angle C$ is the right angle.

$$\sin A = 4/5 \qquad \cos A = 3/5 \qquad \tan A = 3/4$$

$$\sin B = 3/5 \qquad \cos B = 4/5 \qquad \tan B = 4/3$$



Using the Unit Circle

The unit circle has radius 1 and is centered at the origin.

For an angle α ,

- $\cos \alpha = x\text{-coordinate}$
- $\sin \alpha = y\text{-coordinate}$
- $\tan \alpha = x/y$

This helps define trigonometric functions for all angles, not just those in triangles.

Given a circle of radius r. The sine, cosine, tangent are defined as follows:

$$\sin \alpha = \frac{y}{r}; \cos \alpha = \frac{x}{r}; \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{y}{x}.$$

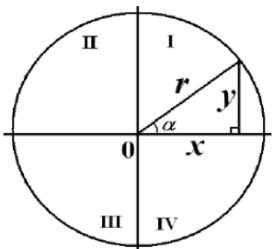
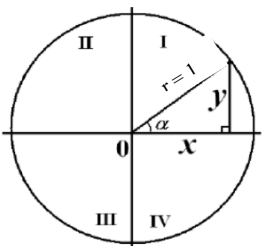
The other three trigonometric functions, cosecant, secant, and cotangent are:

$$\csc \alpha = \frac{1}{\sin \alpha}; \sec \alpha = \frac{1}{\cos \alpha}; \cot \alpha = \frac{1}{\tan \alpha} = \frac{\cos \alpha}{\sin \alpha}$$

Since r is always positive, it follows from this definition that the cosine function is negative if the angle α is in quadrants II or III and the sine is negative in quadrants III and IV. Note that: sine and cosine, tangent and cotangent, secant and cosecant, are cofunctions of each other. In other words, $\sin \alpha = \cos (90^\circ - \alpha)$; $\sec \alpha = \csc (90^\circ - \alpha)$; $\tan \alpha = \cot (90^\circ - \alpha)$.

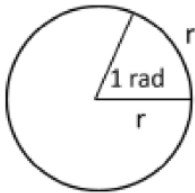
Example, $\cos 5^\circ = \sin 85^\circ$.

Note that: $\sin \alpha \csc \alpha = 1$; $\cos \alpha \sec \alpha = 1$; $\tan \alpha \cot \alpha = 1$.



Radians and degrees:

A degree, usually denoted by symbol $^{\circ}$, is a measurement of a plane angle, defined so that a full rotation is 360 degrees. An alternative measurement is the radian: One radian is the angle subtended at the center of a circle by an arc whose length is equal to the radius of the circle. We can convert between radians and degrees via



$$1 \text{ radian} = \frac{180^{\circ}}{\pi} \approx 57.296^{\circ}; \quad 1^{\circ} = \frac{\pi}{180} \text{ rad}.$$

For instance, we can convert the measures of some special angles from degrees to radians: $30^{\circ} = 30 \cdot$

$$\frac{\pi}{180} = \frac{\pi}{6}.$$

Special and Quadrantal Angles

Special angles are angles whose trigonometric values are well-known and exact. They come from the 30° – 60° – 90° and 45° – 45° – 90° triangles and from the unit circle. The main special angles are: 0° , 30° , 45° , 60° , 90° (in radians: 0 , $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$)

Quadrantal are angles in **standard position** whose **terminal side lies on one of the coordinate axes** (x- or y-axis). Some quadrantal angles: 0° ; 90° ; 180° ; 270° ; 360° .

Angle	$0 = 0^{\circ}$	$\frac{\pi}{6} = 30^{\circ}$	$\frac{\pi}{4} = 45^{\circ}$	$\frac{\pi}{3} = 60^{\circ}$	$\frac{\pi}{2} = 90^{\circ}$	$\pi = 180^{\circ}$
Sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
Cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
Tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞	0
Csc	∞	2	$\sqrt{2}$	$2\sqrt{3}/3$	1	∞
Sec	1	$2\sqrt{3}/3$	$\sqrt{2}$	2	∞	-1
Cot	∞	$\sqrt{3}$	1	$\sqrt{3}/3$	0	∞

Any trigonometric function whose argument (angle) is $n \cdot 90^{\circ} \pm \alpha$ can be written simply in term of α . Here are some special cases in the table below:

α and $-\alpha$	α and $\pi - \alpha$	α and $\pi/2 - \alpha$	α and $\alpha \pm \pi$
$\sin (-\alpha) = -\sin \alpha$	$\sin (\pi - \alpha) = \sin \alpha$	$\sin (\pi/2 - \alpha) = \cos \alpha$	$\sin (\alpha \pm \pi) = -\sin \alpha$
$\cos (-\alpha) = \cos \alpha$	$\cos (\pi - \alpha) = -\cos \alpha$	$\cos (\pi/2 - \alpha) = \sin \alpha$	$\cos (\alpha \pm \pi) = -\cos \alpha$
$\tan (-\alpha) = -\tan \alpha$	$\tan (\pi - \alpha) = -\tan \alpha$	$\tan (\pi/2 - \alpha) = \cot \alpha$	$\tan (\alpha \pm \pi) = \tan \alpha$

Example 2 (1988 AMC) If $\sin x = 3 \cos x$, what is $\sin x \cos x$

Inverse trigonometric functions

The inverse trigonometric functions allow you to get the value of the angle if you know the value of the sine, cosine or whatever trigonometric function is available. The most common convention is to name inverse trigonometric functions using an arc-prefix: $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$, etc. The notations $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$, etc., are often used as well and this convention complies with the notation of an inverse function.

Some of the inverse functions of the six trigonometric functions above are defined as follows:

- $y = x$ if and only if $x = \sin y$ where $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$
- $y = x$ if and only if $x = \cos y$ where $x \in [-1, 1]$; $y \in [0, \pi]$.
- $y = x$ if and only if $x = \tan y$ where $x \in \mathbb{R}$; $y \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$
- $y = x$ if and only if $x = \cot y$ where $x \in \mathbb{R}$; $y \in (0; \pi)$.
- $y = x$ if and only if $x = \sec y$ where $x \in (-\infty, -1] \cup [1, +\infty)$; $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.
- $y = x$ if and only if $x = \csc y$ where $x \in \mathbb{R}$; $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

Example 3 Give the principal values of the following: a. $(-\frac{\sqrt{3}}{2})$ b. (-1) c. (1)

Trigonometric Equations and Identities

Trigonometric equations are called equations involving trigonometric functions. To solve an equation, ideally solve for a single trigonometric function. You may use identities, factoring, or the quadratic formula. Once you have an equation for a single trigonometric function, solve for the angle. You will then use the definition of the inverse trigonometric functions for this step. Example 5a can be done at this point.

A trigonometric identity is an equation involving any of the six trigonometric functions that holds for any real number for which each member of the equation is defined. The following table gives the basic trigonometric identities which are useful to simplify a given expression. NOTE: You are not expected to prove all these identities. Choose some which are easy (double angle from sum identity, half angle from double angle)

Sum Identities: $\sin \sin (A+B)=\sin \sin A \cos \cos B+\cos \cos A \sin s$ $\cos \cos (A+B)=\cos \cos A \cos \cos B-\sin \sin A \sin s$ $\tan \tan (A+B)=\frac{\tan \tan A+\tan \tan B}{1-\tan \tan A \tan \tan B}$	Double-Angle Identities $\sin 2 A=2 \sin A \cos A$ $\cos 2 A=\cos ^2 A-\sin ^2 A=1-2 \sin ^2 A=2 \cos ^2 A-1$ $\tan 2 A=\frac{2 \tan A}{1-\tan ^2 A}$
Difference Identities: $\sin \sin (A-B)=\sin \sin A \cos \cos B-\cos \cos A \sin s$ $\cos \cos (A-B)=\cos \cos A \cos \cos B+\sin \sin A \sin s$ $\tan \tan (A+B)=\frac{\tan \tan A-\tan \tan B}{1+\tan \tan A \tan \tan B}$	Half-Angle Identities $\sin \frac{A}{2}=\pm \sqrt{\frac{1-\cos A}{2}} ; \cos \frac{A}{2}=\pm \sqrt{\frac{1+\cos A}{2}}$ $\tan \frac{A}{2}=\pm \sqrt{\frac{1-\cos A}{1+\cos A}}=\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$ The choice \pm depends on what quadrant $A / 2$ is.
Products of Sines and Cosines $2 \sin A \cos B=[\sin (A+B)+\sin (A-B)]$ $2 \cos A \sin B=[\sin (A+B)-\sin (A-B)]$ $2 \cos A \cos B=[\cos (A+B)+\cos (A-B)]$ $2 \sin A \sin B=[\cos (A+B)-\cos (A-B)]$	Sum and Difference of Sines and Cosines $\sin A+\sin B=2 \sin \left(\frac{A+B}{2}\right) \cos \left(\frac{A-B}{2}\right)$ $\sin A-\sin B=2 \cos \left(\frac{A+B}{2}\right) \sin \left(\frac{A-B}{2}\right)$ $\cos \cos A+\cos \cos B=2 \cos \cos \left(\frac{A+B}{2}\right) \cos \cos \left(\frac{A-B}{2}\right)$ $\cos \cos A-\cos \cos B=-2 \sin \sin \left(\frac{A+B}{2}\right) \sin \sin \left(\frac{A-B}{2}\right)$

Example 5: Solve the following trigonometric equations over the interval.

a. $x-\sin \sin x=\frac{1}{4}$ b. $\cot \cot x+\csc \csc x=\sqrt{3}$

Example 6: Find the exact value of $\cos 100^{\circ} \cos 10^{\circ}+\sin 100^{\circ} \sin 10^{\circ}$

Example 7: Simplify: a. $\frac{\sin \sin \left(45^{\circ}+A\right)}{\cos \cos A+\sin \sin A}$ b. $\cot \cot \left(135^{\circ}+A\right) \cot \cot \left(135^{\circ}-A\right)$

Example 8: Simplify to a single trigonometric function: a. $\frac{7 A}{2}-\frac{7 A}{2}$ b. $\frac{(4 A)-1}{2 \cot \cot (4 A)}$

Example 9: If $\cos 2 \theta=-\frac{3}{5}, \theta$ is an angle in the 2nd quadrant, find the exact value of $\sin \theta, \cos \theta, \cot \theta$.

Example 10: Simplify the following into a single trigonometric function:

a. $\sqrt{\frac{40^{\circ}}{2}}$ b. $\sqrt{\frac{1-\cos \cos 550^{\circ}}{1+\cos \cos 550^{\circ}}}$ c. $\frac{\sin \sin \frac{A}{2}}{1+\cos \cos \frac{A}{2}}$

Example 11: Prove the following identity: $\left(\frac{\sin \sin 2 A}{1+\cos \cos 2 A}\right)\left(\frac{\cos \cos A}{1+\cos \cos 2 A}\right)=\tan \tan \frac{A}{2}$.

Example 12: Prove that: $\frac{\sin \sin A-\sin \sin B}{\sin \sin A+\sin \sin B}=\tan \tan \frac{A-B}{4} \cot \cot \frac{A+B}{4}$

Exercise A

- Find the value of $\cos \cos 570^\circ \sin \sin 150^\circ + \sin \sin (-330^\circ) \cos \cos (-390^\circ)$.
- Find $\sin \sin x \cos \cos x$ if $\frac{1+\tan \tan x}{1-\tan \tan x} = 3 + 2\sqrt{2}$
- Solve for the real values of x satisfying the equation:
 - $2 \tan \tan x \cos \cos 2x - 2\sqrt{3} \cos \cos 2x - \sqrt{3} \tan \tan x + 3 = 0$
 - $x - 3 \sec \sec x + 3 = 0, 0 \leq x < 2\pi$
- Simplify: a. $\sin \sin \left(\frac{\pi}{3} + A\right) \sin \sin \left(\frac{\pi}{3} - A\right) - \cos \cos \left(\frac{\pi}{4} + A\right) \cos \cos \left(\frac{\pi}{4} - A\right)$
b. $\cos \cos 25^\circ - \sin \sin 55^\circ + \sin \sin 25^\circ - \sin \sin 35^\circ$ as a function of $b = \sin \sin 5^\circ$
- Show that $(\tan \tan \alpha - 1)^2 + (1 - \cot \cot \alpha)^2 = (\sec \sec \alpha - \csc \csc \alpha)^2$.
- Prove the following:
 - $\cos \cos (2x - y) \cos \cos y - \sin \sin (2x - y) \sin \sin y = x - x$
 - $\sin \sin (A + B) \sin \sin (A - B) = A - B$
 - $\sin \sin 3A = -4A + 3 \sin \sin A$

Exercise B

- Solve for x : a. $(-\sqrt{2}) = x - \frac{\pi}{5}$ b. $x + \sqrt{3} = x - \frac{2\pi}{3}$
- Find $f(\beta)$ if $f(a) = \frac{\sin (5\pi - a) \cos (6\pi - a) \tan \left(\frac{7\pi - a}{2}\right)}{\sin (-2\pi - a) \tan \left(-\frac{\pi}{2} - a\right)}$ if $\cos \cos \beta - \frac{7\pi}{2} = \frac{1}{3}$ and β is in the third quadrant.
- Evaluate: $\left(\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)\right)$.
- Evaluate the sum $\frac{\pi}{8} + \frac{3\pi}{8} + \frac{5\pi}{8} + \frac{7\pi}{8}$. (Math League 1983)
- Solve the following equation: $\sin \sin 5x + \sin \sin 3x = 8 \sin \sin x \cos \cos x \cos \cos 2x$
- Prove the following:
 - $\frac{1}{\tan \tan 3A + \tan \tan A} - \frac{1}{\cot \cot 3A + \cot \cot A} = \cot \cot 4A$
 - $\frac{\tan \tan (225^\circ + y)}{\tan \tan (225^\circ - y)} = \frac{1 + 2 \sin \sin y \cos \cos y}{1 - 2 \sin \sin y \cos \cos y}$
 - $\frac{\cos \cos 2A}{1 + \sin \sin 2A} = \tan \tan \left(\frac{\pi}{4} - A\right)$.

Exercise C

- Evaluate: $\cos \cos 36^\circ \cos \cos 108^\circ$. (Math League 1987)
- The lengths of the sides of a convex quadrilateral ABCD are 5, 6, 7 and x . If $\sin \sin A = \sin \sin B = \sin \sin C = \sin \sin D$, find all possible values of x . (Math League 1984)
- (2011 Purple Comet Meet) Let x be a real number in the interval $\left(0, \frac{\pi}{2}\right)$ such that $\frac{1}{\sin x \cos x} + 2 \cot 2x = \frac{1}{2}$. Evaluate $\frac{1}{\sin x \cos x} - 2 \cot 2x$
- (2006 Mathirang Mathibay) Given: $\sin A = \frac{15}{17}$ and A is in the first quadrant; $\sin \cos B = -\frac{4}{5}$, where the terminal point of $-B$ is in the third quadrant. Find $\sin (A + B)$.
- Find all degree measures of all positive acute angles such that $x + x + x + x + x + x = 31$. (Math League 1990)
- Evaluate $\cos 10^\circ \cos 50^\circ \cos 70^\circ + \sin 10^\circ \sin 50^\circ \sin 70^\circ$.