Asian MathSci League, Inc (AMSLI)

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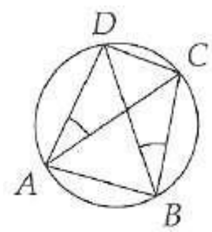
Grade 9 Day 8

Cyclic Quadrilaterals

Some of the beautiful problems in geometry appear when we merge different concepts that we have learned. In this case, we mix circles (and its properties on inscribed angles, power of a point), triangles (similarity and congruence), and special quadrilaterals.

Definition A set of points is **concyclic** if all the points lie on a circle. If the vertices of a quadrilateral lie on a circle, the quadrilateral is **cyclic**.

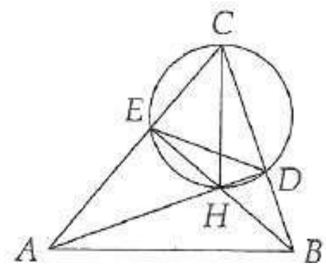
Theorem 1 A convex quadrilateral is cyclic if and only if two opposite angles are supplementary.



Example 1 Let $ABCD$ be a cyclic quadrilateral such that AB and DC intersect at X. Show that $\Delta XAD \sim \Delta XCB$.

Example 2 Prove that a trapezoid is cyclic if and only if it is isosceles.

Theorem 2 A convex quadrilateral is cyclic if and only if the angles subtended by any side at the other two vertices are equal.



Example 3 Let H be the orthocenter of ΔABC and D and E be the feet of the altitudes from A and B respectively. Show that $\angle DCH = \angle DEH$.

Theorem 3 (Ptolemy's theorem): A convex quadrilateral $ABCD$ is cyclic if and only if

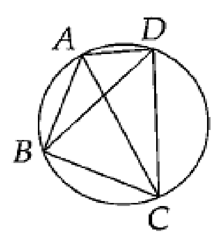
$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Example 4 Quadrilateral $ABCD$ with consecutive sides 8,15, and 12 is inscribed in a circle with circumference 17π. Given that AC is a diameter of the circle, what is the length of the other diagonal of the quadrilateral?

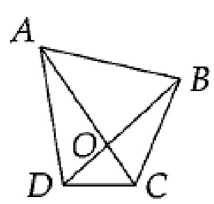
Exercises:

1. In an isosceles trapezoid, the length of each leg is 3 , the length of each diagonal is 7, and the length of the longer base is 8. Find the length of the shorter base. (ARML 1977)

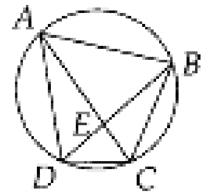
2. Consider cyclic quadrilateral $ABCD$ such that $AB = 6, BC = 7, CD = 8, DA = 9$. Find AC^2. (MAθ 1991)



3. In quadrilateral $ABCD$, we have $AB = 20, BC = 15, CD = 7, DA = 24$ and $AC = 25$. Let $\alpha = \angle ACB$ and $\beta = \angle ABD$. What is $tan\, tan\, (\alpha + \beta)$? (Mandelbrot #2)

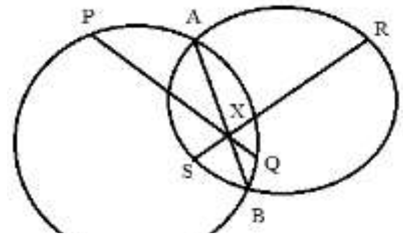


4. Prove that if the diagonals of $ABCD$ intersect at O and $AO \cdot CO = BO \cdot DO$, then $ABCD$ is cyclic.

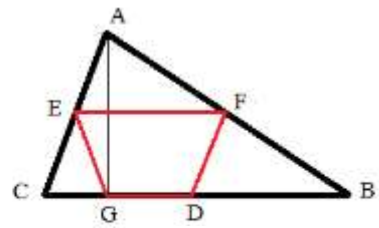


5. In cyclic quadrilateral $ABCD$ with diagonals intersecting at E, we have $AB = 5, BC = 10, BE = 7$ and $CD = 6$. Find EC.

6. Inscribed in a circle is a quadrilateral having sides of length 25, 39, 52, and 60 taken consecutively. What is the diameter of this circle? (AHSME 1972)

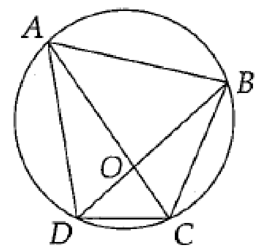


7. Let Γ_1, Γ_2 be two circles intersecting at points A, B. Let segments PQ in Γ_1 and RS in Γ_2 intersect at a point X in AB. Prove that $PRQS$ is cyclic.

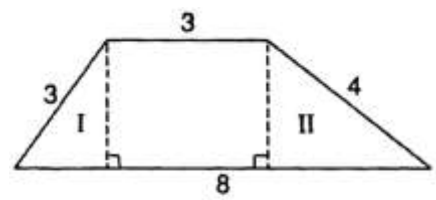


8. Show that the three midpoints of the sides of a triangle and the foot of an altitude of the triangle are concyclic.

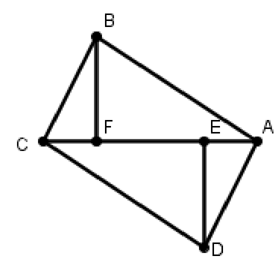
Olympiad section:



1. In a quadrilateral $ABCD$ with diagonals AC and BD intersecting at $O, BO = 4, OD = 6, AO = 8, OC = 3$ and $AB = 6$. Find AD. (AHSME 1967)

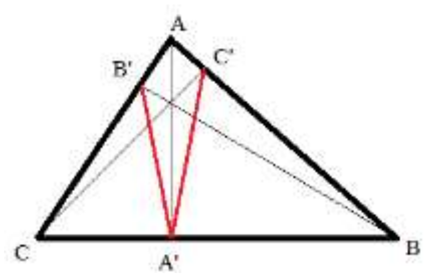


2. The sides of a quadrilateral are 3,3,4 and 8 (in some order). Two of its angles have equal sines but unequal cosines, yet the quadrilateral cannot be inscribed in a circle. Compute the area of the quadrilateral. (ARML 1986)

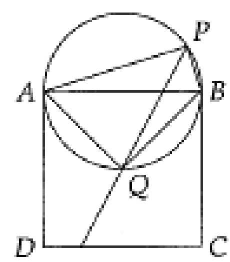


3. In the figure, $ABCD$ is a quadrilateral with right angles at A and C. Points E and F are on AC, and DE and BF are perpendicular to AC. IF $AE = 3, DE = 5, CE = 7$,find BF. (AHSME 1990)

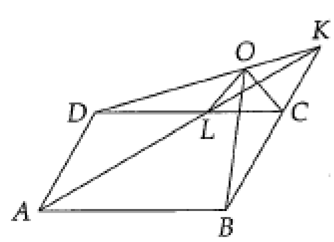
Challenge:



1. The triangle whose vertices are the feet of the three altitudes of ΔABC is called the orthic triangle of ABC. Show that the measures of the angles of the orthic triangle of an acute triangle ΔABC are $180 - 2A, 180 - 2B, 180 - 2C$(in degrees).



2. Side AB of square $ABCD$ is also the hypotenuse of right triangle ABP, with ABP lying outside $ABCD$. Prove that the angle bisector of $\angle APB$ bisects the area of $ABCD$. (M&IQ 1992)



3. A parallelogram $ABCD$ with acute andle $\angle BAD$ is given. The bisector of $\angle BAD$ intersects CD at point L, and the line BC at point K. Let O be the circumcenter of ΔLCK. Prove that $DBCO$ is inscribed in a circle. (Bulgaria 1993)

References:

[1] Art of Problem Solving 2 by R.Rusczyk and S. Lehoczky

[2] Mathematical Excursions by I. Garces and E. Bautista