Asian MathSci League, Inc (AMSLI)

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Grade 9 Day 7

Binomial Theorem

I. Binomial Theorem

In expanding a binomial expression $(x+y)^n$, the common method used is the Pascal's triangle. However, this method can quickly become time consuming as n grows. For example, try getting $(x+y)^{12}$.

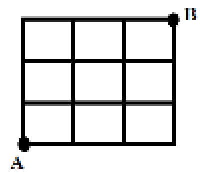
In this case, we can use the binomial theorem instead. The binomial theorem uses combinatorial properties to express the binomial expansion into a sum of terms of the form $x^a y^b$ where $a, b \geq 0$ and $a+b=n$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Examples 1 and 2 show the equivalence of the Pascal's triangle and the binomial theorem in a visual manner.

The use of the combinatorial properties can be extended to trinomial expressions (see Exercise 3).

Example 1
How many ways can one travel from town A to town B if one can only go up or right?



Example 2
Find the coefficient of x^3y^3 in the expansion of $(x+y)^6$.

Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

II. Some Exercises on Binomial Theorem

1. Find the value of the following:
   a. where $n \geq 0$, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n$

   b. where $n \geq 0$, $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = (1-1)^n = 0 \text{ when } n > 0; \text{ when } n = 0, \binom{0}{0} = 1$

   c. where $n \geq 1$, $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1} n \binom{n}{n}$
   $= n\left[\binom{n-1}{0} - \binom{n-1}{1} + \binom{n-1}{2} - \cdots + (-1)^{n-1}\binom{n-1}{n-1}\right] = 0 \text{ when } n > 1, \text{otherwise}, 1$

   d. where $n \geq 0$, $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$

   e. *where* $n \geq 1$, $\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2$

2. Find the expansion of
   a. $(x-y)^6$
   b. $(3x+2y)^4$
   c. $\left(x+\frac{1}{x}\right)^4$
   d. $(x^2+1)^5$

3. The expansion of $(x+2y)^{20}$ contains two terms with the same coefficient, $Kx^a y^b$ and $Kx^{a+1}y^{b-1}$. Find a.

Prepared by Ezra Templonuevo

4. Find the greatest coefficient of $(2a - 3b)^{18}$

5. Find the ratio of the largest and the smallest coefficients of the expansion of $(2x - y)^7$

6. Show that $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$

III. Additional Exercises

1. Find the expansion of

   a. $(a + b + 1)^3$ b. $(2a - 3b + c)^3$ c. $\left(x + \frac{1}{x} + 1\right)^4$

2. *(1998 Chinese Mathematical Olympiad)* Determine all positive integers $n \geq 3$ such that 2^{2000} is divisible by $1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$

3. *(1999 Hungarian-Israel Binational Mathematical Competition)* Prove or disprove the following claim: For any positive integer k, there exists a positive integer $n > 1$ such that the binomial coefficient $\binom{n}{i}$ is divisible by k for any $1 \leq i \leq n - 1$.

4. *(1999 IMO Selection Test: Romania)* Show that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

   is the sum of two consecutive perfect squares.

5. *(2000 APMO)* Let n, k be positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k (n-k)^{n-k}} < \frac{n!}{k!\,(n-k)!} < \frac{n^n}{k^k (n-k)^{n-k}}$$