PRES LUNAM
Ecole Doctorale STIM
Université du Maine

Spécialité : Mathématiques
Laboratoire : Laboratoire Manceau de Mathématiques
Equipe : Faculté des Sciences et Techniques

BSDEs and Applications in Stochastic Differential Games

MU Rui

Mél : rui.mu.etu@univ-lemans.fr

Abstract: In this article, we mainly discussed the non-zero sum stochastic differential game problem and its associated backward stochastic differential equations (BSDEs for short) with the generator which is continuous and stochastic linear growth. We dealed with the case of a polynomial growth terminal condition. It generalized the case in S.Hamadene (1997) [1]. We give the L^p solution of BSDEs and meanwhile the equilibrium point of the game problem under some appropriate conditions.

Keywords: *Backward stochastic differential equation, stochastic differential game.*

1 Setting of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and σ be the function defined as:

$$\begin{aligned} \sigma : [0,T] \times \mathbf{R}^m &\longrightarrow \mathbf{R}^{m\times m} \\ (t,x) &\longmapsto \sigma(t,x) \end{aligned}$$

and which satisfies the following assumptions:

Assumptions (A1) σ is uniformly lipschitz w.r.t x, bounded, non-degenerate and invertible.

Next let $(t,x) \in [0,T] \times \mathbf{R}^m$. Under Assumptions (a)-(d), we know there exists a process $(X_s^{t,x})_{s\le T}$ that satisfies the following stochastic differential equation (see e.g. Karatzas and Shreve, 1991 [2]):

$$X_s^{t,x} = x + \int_t^s \sigma(u, X_u^{t,x})dB_u, \quad \forall s \in [t,T] \text{ and } X_s^{t,x} = x \text{ for } s \in [0,t]. \tag{1.1}$$

Next let us denote by U_1 and U_2 two compact metric spaces and let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) be the set of $\mathcal{P}$-measurable processes $u = (u_t)_{t\le T}$ (resp. $v = (v_t)_{t\le T}$) with values in U_1 (resp. U_2) ; we denote by $\mathcal{M}$ the set $\mathcal{M}_1 \times \mathcal{M}_2$.

Let f be a Borelian function from $[0,T] \times \mathbf{R}^m \times U_1 \times U_2$ into $\mathbf{R}^m$ and for $i = 1,2$ let h_i and g^i be Borelian functions from $[0,T] \times \mathbf{R}^m \times U_1 \times U_2$ (resp. $\mathbf{R}^m$) into $\mathbf{R}^+$ which satisfy:

Assumptions (A2)

(i) for any $(t,x) \in [0,T] \times \mathbf{R}^m$, $(u,v) \mapsto f(t,x,u,v)$ is continuous on $U_1 \times U_2$. Moreover f is of linear growth w.r.t x, i.e. there exists a constant C_4 such that $|f(t,x,u,v)| \le C_4(1+|x|), \forall (t,x,u,v) \in [0,T] \times \mathbf{R}^m \times U_1 \times U_2$.

(ii) for any $(t,x) \in [0,T] \times \mathbf{R}^m$, $(u,v) \mapsto h_i(t,x,u,v)$ is continuous on $U_1 \times U_2$, $i = 1,2$. Moreover h_i is of polynomial growth w.r.t x, for $i = 1,2$. i.e. there exists a constant C_5 and $\gamma > 1$ such that $|h_i(t,x,u,v)| \le C_5(1+|x|^\gamma), \forall (t,x,u,v) \in [0,T] \times \mathbf{R}^m \times U_1 \times U_2$.

(iii) g^i, $i = 1,2$, is of polynomial growth with respect to x, i.e. there exists a constant C_6 and $\gamma > 1$ such that $\left|g^i(x)\right| \le C_6(1+|x|^\gamma), \forall x \in \mathbf{R}^m$, for i=1, 2. □

For $(u,v)\in\mathcal{M}$, let $\mathbf{P}^{(u,v)}$ be the measure on $(\Omega,\mathcal{F})$ whose density function is defined as follows:

$$\frac{d\mathbf{P}^{(u,v)}}{dP}=\zeta_T\left(\sigma^{-1}(s,X_s^{t,x})f(s,X_s^{t,x},u_s,v_s)\right), \tag{1.2}$$

where

$$\forall s\leq T, \zeta_s(\eta):=e^{\int_0^s\eta_r dB_r-\frac{1}{2}\int_0^s|\eta_r|^2dr} \tag{1.3}$$

Thanks to Assumptions (A1) and (A2)-(i) on σ and f, we can infer that $\mathbf{P}^{(u,v)}$ is a probability on $(\Omega,\mathcal{F})$. Then by Girsanov's theorem (Girsanov, 1960 [3], pp.285-301), the process $B^{(u,v)}:=(B_s-\int_0^s\sigma^{-1}(r,X_r^{t,x})f(r,X_r^{t,x},u_r,v_r)dr)_{s\leq T}$ is a $(\mathcal{F}_s,\mathbf{P}^{(u,v)})$-Brownian motion and $(X_s^{t,x})_{s\leq T}$ satisfies the following stochastic differential equation,

$$dX_s^{t,x}=f(s,X_s^{t,x},u_s,v_s)ds+\sigma(s,X_s^{t,x})dB_s^{(u,v)},\quad \text{for } s\in[t,T] \text{ and } X_s^{t,x}=x \text{ for } s<t. \tag{1.4}$$

In general, the process $(X_s^{t,x})_{s\leq T}$ is not adapted with respect to the filtration generated by the Brownian motion $(B_s^{(u,v)})_{s\leq T}$, therefore $(X_s^{t,x})_{s\leq T}$ is called a weak solution for the SDE (1.4).

Next, for $i=1,2$, we define the payoffs of the players by:

$$J^i(u,v)=\mathbf{E}^{(u,v)}[\int_0^T h_i(s,X_s^{t,x},u_s,v_s)ds+g^i(X_T^{t,x})], \tag{1.5}$$

where $\mathbf{E}^{(u,v)}(\cdot)$ is the expectation under the probability $\mathbf{P}^{(u,v)}$.

Our problem is to find an admissible control (u^*,v^*) such that

$$J^1(u^*,v^*)\leq J^1(u,v^*) \text{ and } J^2(u^*,v^*)\leq J^2(u^*,v) \quad \text{for any } (u,v)\in\mathcal{M}. \tag{1.6}$$

The control (u^*,v^*) is called a Nash equilibrium point for the nonzero-sum stochastic differential game.

Next for $i=1,2$, we define the Hamiltonian functions of the game $H_i:[0,T]\times\mathbf{R}^{2m}\times U_1\times U_2\to\mathbf{R}$, by:

$$H_i(t,x,p,u,v)=p\sigma^{-1}(t,x)f(t,x,u,v)+h_i(t,x,u,v) \tag{1.7}$$

and we introduce the following assumption (A3) called the generalized Isaacs'condition.

Assumption (A3) (i) There exist two borelian applications u_1^*, u_2^* defined on $[0,T]\times\mathbf{R}^{3m}$, valued respectively in U_1 and U_2 such that for any $(t,x,p,q,u,v)\in[0,T]\times\mathbf{R}^{3m}\times U_1\times U_2$, we have:

$$H_1^*(t,x,p,q)=H_1(t,x,p,u_1^*(t,x,p,q),u_2^*(t,x,p,q))\leq H_1(t,x,p,u,u_2^*(t,x,p,q))$$

and

$$H_2^*(t,x,p,q)=H_2(t,x,q,u_1^*(t,x,p,q),u_2^*(t,x,p,q))\leq H_2(t,x,q,u_1^*(t,x,p,q),v).$$

(ii) the mapping $(p,q)\in\mathbf{R}^{2m}\longmapsto(H_1^*,H_2^*)(t,x,p,q)\in\mathbf{R}$ is continuous for any fixed (t,x). $\square$

In order to show that the game has a Nash equilibrium point, it is enough to show that its associated BSDE, which is multi-dimensional and of continuous generator, has a solution.

2 Relation Between Non-zero sum differential game problem and BSDEs.

Let $(t,x) \in [0,T] \times \mathbf{R}^m$ and $(\theta_s^{t,x})_{s \le T}$ be the solution of the following forward stochastic differential equation:

$$\begin{cases} d\theta_s = b(s,\theta_s)ds + \sigma(s,\theta_s)dB_s, & s \in [t,T]; \\ \theta_s = x, & s \in [0,t] \end{cases} \tag{2.1}$$

where $\sigma : [0,T] \times \mathbf{R}^m \to \mathbf{R}^{m\times d}$ satisfies Assumption (A1) and b: $[0,T] \times \mathbf{R}^m \to \mathbf{R}^m$ is a measurable function which satisfies the following assumption:

Assumption (A4): The function b is uniformly Lipschitz w.r.t x and of linear growth, i.e., there exists a constant C_7 such that:

$$\forall t \in [0,T],\ \forall\ x,x' \in \mathbf{R}^m,\ |b(t,x) - b(t,x')| \le C_7\,|x - x'| \text{ and } |b(t,x)| \le C_8(1+|x|).$$

It is well-known that, under (A1) and (A4), the stochastic process $(\theta_s^{t,x})_{s\le T}$ satisfies the following estimation, see for example (Karatzas, I. 1988 [4] pp.306)

$$\forall\ q \in [1,\infty), \qquad E\left[\left(\sup_{s\le T}|\theta_s^{t,x}|\right)^{2q}\right] \le C(1+|x|^{2q}).\quad \Box \tag{2.2}$$

Theorem 2.1. *Let us assume:*

(i) Assumption (A3) is fulfilled ;

(ii) There exist two deterministic with polynomial growth functions $u^i(t,x)$, $i = 1,2$, and two pairs of processes (W^i, Z^i), $i = 1,2$, such that:

(a) For $i = 1,2$, $\mathbf{P}$-a.s. $W_s^i = u^i(s, X_s^{0,x})$, $\forall s \le T$, and $Z^i(\omega)$ is dt-square integrable ;

(b) For any $s \le T$,

$$\begin{cases} -dW_s^i = H_i\left(s, X_s^{0,x}, Z_s^i, u_1^*\left(s, X_s^{0,x}, Z_s^1, Z_s^2\right), u_2^*\left(s, X_s^{0,x}, Z_s^1, Z_s^2\right)\right)ds - Z_s^i dB_s, \\ W_T^i = g^i\left(X_T^{0,x}\right). \end{cases} \tag{2.3}$$

Then the admissible control $(u^*(s, X_s^{0,x}, Z_s^1, Z_s^2), v^*(s, X_s^{0,x}, Z_s^1, Z_s^2))_{s\le T}$ is a Nash equilibrium point for the game.

3 Existence of solutions for markovian BSDE

Theorem 3.1. *Let $x \in \mathbf{R}^m$ be fixed. Then under Assumptions (A1), (A2) and (A3), there exist:*

(i) Two pairs of $\mathcal{P}$-measurable processes $(W_s^i, Z_s^i)_{s\le T}$, $i = 1,2$, such that: $\forall i \in \{1,2\}$,

$$\begin{cases} \mathbf{P} - a.s.,\ Z^i(\omega) \text{ is } dt - \text{ square integrable ;} \\ -\,dW_s^i = H_i\left(s, X_s^{0,x}, Z_s^i, u_1^*\left(s, X_s^{0,x}, Z_s^1, Z_s^2\right), u_2^*\left(s, X_s^{0,x}, Z_s^1, Z_s^2\right)\right)ds - Z_s^i dB_s,\ s \le T\ ; \\ W_T^i = g^i(X_T^{0,x}). \end{cases} \tag{3.1}$$

(ii) Two measurable deterministic with polynomial growth functions u^i, $i = 1,2$, from $[0,T] \times \mathbf{R}^m$ into $\mathbf{R}$ such that:

$$\forall i = 1,2,\ W_s^i = u^i(s, X_s^{0,x}),\ \forall s \in [0,T].$$

Proof. We introduce the mainly mind here. It will be divided into several steps. We first construct a sequence of BSDEs which have solutions according to S. Hamadène 1997 [1] (Theorem 27.2). Then we provide a priori estimates of the solutions. Then, we proof that the sequences of solutions are convergent. Finally, we verify that the limits of the sequences are exactly the solutions of the BSDE. □

References

[1] J.-P. Lepeltier S. Hamadène and S. Peng. BSDEs with continuous coefficients and stochastic differential games. *Pitman Research Notes in Mathematics Series*, pages 115–128, 1997.

[2] I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Verlag, 1991.

[3] IV Girsanov. On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability and its Applications*, 5:285, 1960.

[4] I. Karatzas. S. shreve (1988). brownian motion and stochastic calculus. *Graduate Texts in Mathematics*, 113.