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Monitoring the Coefficient of Variation using a Variable Sampling Interval Control Chart

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Abstract: The coefficient of variation CV is a quality characteristic that has several applications in applied statistics and is receiving increasing attention in quality control. Few papers have proposed control charts that monitor this normalized measure of dispersion. In this paper an adaptive Shewhart control chart implementing a VSI (Variable Sampling Interval) strategy is proposed to monitor the CV. A comparison is performed with a Shewhart FSR (Fixed Sampling Rate) chart for the CV. An example illustrates the use of these charts on real data gathered from a casting process.

Keywords: *coefficient of variation, variable sampling interval, average time to signal*
Collaborations :

1 Introduction

Quality is one of the most important consumer decision factors. Control charting techniques were extended to various sectors such as health, education, finance and various societal applications where the mean and the standard-deviation may not be constant all the time but the process is operating in-control. In this case, it is natural to explore the use of the coefficient of variation (CV, in short) γ which is a normalized measure of dispersion of a probability distribution that is defined as the ratio of the standard deviation σ to the mean μ. As pioneers in this field, [1] developed a control chart for monitoring the CV.Shewhart-type chart is applied. This makes this chart not very sensitive to small to moderate shifts. Adaptative strategies such as the variable sampling interval (VSI) charts and the variable sample size (VSS) charts are more advanced-type control charts to to overcome this problem. A very comprehensive survey about the design of adaptive charts was presented by [2].
The goal of this paper is to propose a CV Shewhart control chart using a VSI feature (from now on denoted as VSI-γ) and to evaluate its performance in terms of ATS (Average Time to Signal) and $SDTS$ (Standard Deviation Time to Signal).

2 The Variable Sampling Interval (VSI) CV Shewhart chart

[1] were the first to investigate the opportunity to monitor the coefficient of variation through a FSR Shewhart type chart, denoted as SH$-\gamma$ chart. The control limits $LCL_{\mathrm{SH}-\gamma}$ and $UCL_{\mathrm{SH}-\gamma}$ are equal to:

$$LCL_{\mathrm{SH}-\gamma} = F_{\hat{\gamma}}^{-1}\left(\frac{\alpha_0}{2}|n,\gamma_0\right), \quad (1)$$
$$UCL_{\mathrm{SH}-\gamma} = F_{\hat{\gamma}}^{-1}\left(1-\frac{\alpha_0}{2}|n,\gamma_0\right), \quad (2)$$

where $F_{\hat{\gamma}}^{-1}(\alpha|n,\gamma)$ is the inverse cumulative distribution function of $\hat{\gamma}$.
In this paper, we assume that the VSI$-\gamma$ chart only takes two sampling interval values h_S (S for Short) and h_L (L for Long) with $h_S < h_L$. Control limits are

$$LCL = \mu_0(\hat{\gamma}) - K\sigma_0(\hat{\gamma}), \quad (3)$$
$$UCL = \mu_0(\hat{\gamma}) + K\sigma_0(\hat{\gamma}), \quad (4)$$

as well as the warning limits

$$LWL = \mu_0(\hat{\gamma}) - W\sigma_0(\hat{\gamma}), \quad (5)$$
$$UWL = \mu_0(\hat{\gamma}) + W\sigma_0(\hat{\gamma}), \quad (6)$$

where $W > 0$ and $K \geq W$ are the warning and control limit parameters, respectively. The VSI strategy works as follows:

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- if $\hat{\gamma} \in [LWL, UWL]$, the process is declared "in-control" and the next sample is collected after a long sampling interval h_L.
- if $\hat{\gamma} \in [LCL, LWL] \cup [UWL, UCL]$, the process is also declared "in-control" but the next sample is collected after a short sampling interval h_S.
- if $\hat{\gamma} < LCL$ or $\hat{\gamma} > UCL$, the process is declared "out-of-control" and the potential assignable cause(s) must be found and removed.

The properties of a VSI type control chart are determined by the number of samples and the length of time until a signal is given. The number of samples before a signal is usually called the run length in the quality control literature, and the expected number of samples is called the average run length (ARL). With a fixed interval between samples, the ARL can be easily converted to the expected time to signal ATS by multiplying it by the fixed sampling interval h. Hence, the ARL can be thought of as the expected time to signal. With a VSI type chart, however, the time to signal is not a constant multiple of the number of samples to signal. It is necessary to keep track of both the number of samples to signal and the time to signal.

[3] defined the number of samples to signal N as the number of samples taken from the start of the process to the time that the chart signals and they defined the Average Number of Samples to Signal $ANSS = E(N)$ as the expected value of the number of samples to signal. They also defined the time to signal T as the time from the start of the process to the time when the chart signals. If $h_i = \{h_S, h_L\}$ is the last sampling interval used before the i^{th} sample, then :

$$T = \sum_{i=1}^{N} h_i \tag{7}$$

The Average Time to Signal $ATS = E(T)$ and the standard-deviation time to signal $SDTS = \sqrt{V(T)}$ are the expected value and the standard deviation of the time to signal, respectively.

Let p_S, p_L and q be the following probabilities:

$$\begin{aligned} p_L &= F_{\hat{\gamma}}(UWL|n, \gamma_1) - F_{\hat{\gamma}}(LWL|n, \gamma_1), & (8) \\ p_S &= F_{\hat{\gamma}}(UCL|n, \gamma_1) - F_{\hat{\gamma}}(LCL|n, \gamma_1) - p_L, & (9) \\ q &= 1 - p_S - p_L & (10) \end{aligned}$$

where $F_{\hat{\gamma}}(x|n, \gamma)$ is the cumulative distribution of $\hat{\gamma}$ and where $\gamma_1 = \tau\gamma_0$ is an out-of-control value for the CV.

The Average Sampling Interval $ASI = E(h_i)$.

$$ASI = E(h_i) = \frac{h_S p_S + h_L p_L}{1 - q} \tag{11}$$

and for the same reason, we also have

$$E(h_i^2) = \frac{h_S^2 p_S + h_L^2 p_L}{1 - q} \tag{12}$$

[3] derived the following expressions:

$$ATS = E(N)E(h_i) = \frac{h_S p_S + h_L p_L}{q(1 - q)} \tag{13}$$

and

$$\begin{aligned} SDTS &= \sqrt{E(N)V(h_i) + V(N)E^2(h_i)} & (14) \\ &= \sqrt{\frac{h_S^2 p_S + h_L^2 p_L}{q(1-q)} + \frac{(1-2q)(h_S p_S + h_L p_L)^2}{q^2(1-q)^2}}. & (15) \end{aligned}$$

3 Numerical analysis

Usually, we use the mean (ARL) and the standard deviation ($SDRL$) of the run length (RL) distribution to evaluate the performance of control charts. But for the VSI$-\gamma$ chart, we use the mean (ATS) and the standard deviation ($SDTS$) of the time to signal T. When the process is in-control, the average of T will be denoted as ATS_0 and the standard deviation of T will be denoted as $SDTS_0$. On the contrary, when a process is out-of-control, the average of T will be denoted as ATS_1 and the standard deviation of T will be denoted as $SDTS_1$. A control chart is considered better than its competitors if it has smaller ATS_1 value for a specific shift τ, when ATS_0 is the same for all the charts. For a FSR model, the ATS is a multiple of the ARL since the sampling interval h is fixed, i.e. $ATS^{(\mathrm{FSR})} = h \times ARL^{(\mathrm{FSR})}$. For a VSI model, due to the variation of the sampling interval, the equation should be changed to $ATS^{(\mathrm{VSI})} = ASI \times ARL^{(\mathrm{VSI})}$. The comparison of the statistical performances among different static charts must be conducted by forcing the same value for ARL_0. In the remainder of this paper, we will assume $ARL_0 = 370.4$.

Furthermore, the in-control average sampling interval ASI_0 of the VSI$-\gamma$ chart should be equal to h. Without loss of generality, in this paper we assume $h = 1$ t.u. (time unit) and, therefore, to make a fair comparison with other control charts, we assume that $ASI_0 = 1$ t.u. (this ensures $ATS_0 = ARL_0 = 370.4$). Consequently, the chart parameters h_S, h_L, W, K of a properly designed VSI$-\gamma$ chart must satisfy the following two equations :

$$\begin{aligned} ATS(n, h_S, h_L, W, K, \gamma_0, \tau = 1) &= ATS_0 = 370.4, & (16)\\ ASI(n, h_S, h_L, W, K, \gamma_0, \tau = 1) &= ASI_0 = 1. & (17)\end{aligned}$$

Analysis is done as follows; For every chart parameters (W, K) obtained from several couples of (h_S, h_L), we compute the out-of-control (ATS1, SDTS1,ASI1) of the VSI-γ chart for parameters n, γ_0 and τ. For the same combinations, we compute the out-of-control $(ARL1, SDRL1)$ of the $SH-\gamma$ chart knowing that in this case $h_S = h_L = 1$. Comparison between run length properties is done to choose the best control chart. The best performance is obtained by the VSI-γ chart for the couple $(h_S = 0.1, h_L = 4.0)$. The larger is the spread between h_S and h_L, the shorter is the out-of-control ATS assured by the investigated chart. This evidence stems from the fact that as the difference between h_S and h_L increases, W reduces: thereby, the warning region tends to cover a larger fraction of the control interval between UCL and LCL. We suggest selecting the couple (h_S, h_L) over a range as large as possible, compatibly with the technological constraint related to the rate of inspection and the need to collect a sufficiently large number of samples during the process run. Concerning the increasing case, as it was expected, whatever the values of n, γ_0 or τ, the ATS values of the VSI-γ chart are much smaller than the ones of the SH-γ chart, clearly demonstrating the out performance of the former over the latter.

4 An illustrative example

The proposed example considers actual data from a die casting hot chamber process kindly provided by a Tunisian company manufacturing zinc alloy (ZAMAK) parts for the sanitary sector. The quality characteristic X of interest is the weight (in grams) of scrap zinc alloy material to be removed between the molding process and the continuous plating surface treatment. A preliminary regression study, supervised by one of the authors in collaboration with the quality control department of the company, has demonstrated the presence of a constant proportionality $\sigma = \gamma \times \mu$ between the standard-deviation σ and the mean μ of the weight of scrap alloy The control limits of the SH$-\gamma$ chart are equal to

$$\begin{aligned} LCL_{\mathrm{SH}-\gamma} &= F_{\hat{\gamma}}^{-1}\left(\tfrac{0.0027}{2}|5, 0.01\right) = 0.0016,\\ UCL_{\mathrm{SH}-\gamma} &= F_{\hat{\gamma}}^{-1}\left(1 - \tfrac{0.0027}{2}|5, 0.01\right) = 0.0211.\end{aligned} \quad (18)$$

The SH-γ chart is plotted and it seems to confirm that the die casting hot chamber process is in-control. The process engineer has decided to implement a VSI-γ chart for the Phase II of the process with sampling intervals (in hours) $h_S = 0.3$, $h_L = 1.7$ and in-control $ASI_0 = 370.4$. For $n = 5$ and $\gamma_0 = 0.01$, the chart parameters W and K are equal to $W = 0.686$ and $K = 3.152$. We have $\mu_0(\hat{\gamma}) = 0.0094$, $\sigma_0(\hat{\gamma}) = 0.0034$ and the control and the warning limits of the VSI$-\gamma$ chart are

$$\begin{aligned} LCL &= 0.0094 - 3.152 \times 0.0034 = -0.0013\\ UCL &= 0.0094 + 3.152 \times 0.0034 = 0.0201\end{aligned} \quad (19)$$

And Warning Limits are

$$\begin{array}{l} LWL = 0.0094 - 0.686 \times 0.0034 = 0.0071 \\ UWL = 0.0094 + 0.686 \times 0.0034 = 0.0117 \end{array} \tag{20}$$

5 Conclusion

In this paper, a VSI-γ chart is proposed to monitor the CV. The ATS and $SDTS$ computation has been performed and a comparison with the SH-γ chart demonstrated the out performance of the VSI-γ chart over the SH-γ chart; A real implementation of the VSI-γ chart on data collected from a die casting hot chamber process has demonstrated its efficiency in the detection of out-of-control situations. Future research should be focused on extending the study to other adaptive schemes without and with run rules.

References

[1] C.W. Kang and M.S. Lee and Y.J. Seong and D.M. Hawkins, "A Control Chart for the Coefficient of Variation", Journal of Quality Technology. vol. 39, pp. 151–158, 2007.

[2] G. "Tagaras, A Survey of Recent Developments in the Design of Adaptive Control Charts", Journal of Quality Technology. vol. 30, pp. 212–223, 1998.

[3] M.R. Reynolds Jr. and J.C. Arnold, "Optimal One-Sided Shewhart Charts with Variable Sampling Intervals", Sequential Analysis. vol. 8, pp. 51–77, 1989.