Covariance matrix computation

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The goal of this document is to provide a very detailed computation of the covariance matrices of CMB power spectra. We describe the algorithm implemented in *pspy* to compute them.

1 Covariance of a spin 0 x spin 0 power spectrum on the sphere

Let's start with a simple data model, the observed temperature field at frequency ν_1 is related to the true temperature field by

$$\tilde{T}^{\nu_1}(\hat{n}) = W_T^{\nu_1}(\hat{n}) \left(\int B^{\nu_1}(\hat{n}, \hat{n}') T^{\nu_1}(\hat{n}') d\hat{n}' + n^{\nu_1}(\hat{n}) \right) \tag{1}$$

Here $W_T^{\nu_1}$ is the window function, $B^{\nu_1}(\hat{n})$ is the beam of the instrument and $n^{\nu_1}(\hat{n})$ is the instrumental noise. In harmonic space, this expression becomes

$$\tilde{T}^{\nu_1}_{\ell m} = \sum_{\ell' m'} K^{\nu_1}_{\ell m, \ell' m'} (B^{\nu_1}_{\ell'} a^{\nu_1}_{\ell' m'} + n^{\nu_1}_{\ell' m'}) \tag{2}$$

With the coupling kernel

$$K^{\nu_1}_{\ell m, \ell' m'} = \int d\hat{n} Y_{\ell' m'}(\hat{n}) W_T^{\nu_1}(\hat{n}) Y^*_{\ell m}(\hat{n}). \tag{3}$$

An estimate of the power spectrum can be written

$$\langle \tilde{C}^{T_{\nu_1} T_{\nu_2}}_{\ell} \rangle = \sum_{\ell_1} M^{00, \nu_1 \nu_2}_{\ell \ell_1} \langle C^{T_{\nu_1} T_{\nu_2}}_{\ell_1} \rangle \tag{4}$$

Where $M^{\nu_1 \nu_2 00}_{\ell \ell_1}$ is the standard master mode coupling matrix for spin 0 $\times$ spin 0 spectra.

$$M^{\nu_1 \nu_2 00}_{\ell \ell_1} = \frac{2\ell_1 + 1}{4\pi} \sum_{\ell_3} (2\ell_3 + 1) \mathcal{W}^{\nu_1 \nu_2}_{\ell_3} \begin{pmatrix} \ell & \ell_1 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{5}$$

An unbiased estimator for the spectra can be formed using

$$\begin{aligned}
\hat{C}^{T_{\nu_1} T_{\nu_2}}_{b} &= \sum_{b'} (M^{\nu_1 \nu_2 00})^{-1}_{bb'} \tilde{C}^{T_{\nu_1} T_{\nu_2}}_{b} && (6) \\
\tilde{C}^{T_{\nu_1} T_{\nu_2}}_{b} &= P_{b\ell} \tilde{C}^{T_{\nu_1} T_{\nu_2}}_{\ell} && (7) \\
M^{\nu_1 \nu_2 00}_{bb'} &= P_{b\ell} M^{\nu_1 \nu_2 00}_{\ell \ell'} B^{\nu_1}_{\ell'} B^{\nu_2}_{\ell'} Q_{\ell' b'} && (8)
\end{aligned}$$

The variance of the binned unbiased estimator is then given by

$$\begin{aligned}
\Xi^{\nu_1 \times \nu_2, \nu_3 \gamma \times \nu_4 \eta}_{bb'} &= \langle (\hat{C}^{T_{\nu_1} T_{\nu_2}}_{b} - C^{T_{\nu_1} T_{\nu_2}}_{b})(\hat{C}^{T_{\nu_3} T_{\nu_4}}_{b'} - C^{T_{\nu_3} T_{\nu_4}}_{b'}) \rangle \\
&= (M^{\nu_1 \nu_2 00})^{-1}_{bb_1} \langle \tilde{C}^{T_{\nu_1} T_{\nu_2}}_{b_1} \tilde{C}^{T_{\nu_3} T_{\nu_4}}_{b_2} \rangle^t (M^{\nu_3 \nu_4 00})^{-1}_{b_2 b'} - C^{T_{\nu_1} T_{\nu_2}}_{b} C^{T_{\nu_3} T_{\nu_4}}_{b'} \\
&= (M^{\nu_1 \nu_2 00})^{-1}_{bb_1} P_{b_1 \ell_1} P_{b_2 \ell_2} \langle \tilde{C}^{T_{\nu_1} T_{\nu_2}}_{\ell_1} \tilde{C}^{T_{\nu_3} T_{\nu_4}}_{\ell_2} \rangle^t (M^{\nu_3 \nu_4 00})^{-1}_{b_2 b'} - C^{T_{\nu_1} T_{\nu_2}}_{b} C^{T_{\nu_3} T_{\nu_4}}_{b'} \\
&\approx (M^{\nu_1 \nu_2 00})^{-1}_{bb_1} P_{b_1 \ell_1} P_{b_2 \ell_2} \langle \Delta \tilde{C}^{T_{\nu_1} T_{\nu_2}}_{\ell_1} \Delta \tilde{C}^{T_{\nu_3} T_{\nu_4}}_{\ell_2} \rangle^t (M^{\nu_3 \nu_4 00})^{-1}_{b_2 b'} && (9)
\end{aligned}$$

We are left with computing:

$$\langle \Delta\tilde{C}^{T_{\nu_1}T_{\nu_2}}_{\ell_1}\Delta\tilde{C}^{T_{\nu_3}T_{\nu_4}}_{\ell_2}\rangle = f_{\ell_1\ell_2}\sum_{m_1m_2}\langle\tilde{T}^{\nu_1}_{\ell_1m_1}\tilde{T}^{\nu_3}_{\ell_2m_2}\rangle\langle\tilde{T}^{\nu_2*}_{\ell_1m_1}\tilde{T}^{\nu_4*}_{\ell_2m_2}\rangle + \langle\tilde{T}^{\nu_1}_{\ell_1m_1}\tilde{T}^{\nu_4*}_{\ell_2m_2}\rangle\langle\tilde{T}^{\nu_2}_{\ell_1m_1}\tilde{T}^{\nu_3*}_{\ell_2m_2}\rangle \quad (10)$$

$$f_{\ell_1\ell_2} = \frac{1}{2\ell_1+1}\frac{1}{2\ell_2+1} \quad (11)$$

We can expand the first term

$$T_1 = f_{\ell_1\ell_2}\sum_{m_1m_2}\langle\tilde{T}^{\nu_1}_{\ell_1m_1}\tilde{T}^{\nu_3}_{\ell_2m_2}\rangle\langle\tilde{T}^{\nu_2*}_{\ell_1m_1}\tilde{T}^{\nu_4*}_{\ell_2m_2}\rangle \quad (12)$$

$$= f_{\ell_1\ell_2}\sum_{m_1m_2}\sum_{\ell_3m_3}\sum_{\ell_4m_4}K^{\nu_1}_{\ell_1m_1,\ell_3m_3}K^{\nu_3}_{\ell_2m_2,\ell_3m_3}K^{\nu_2*}_{\ell_1m_1,\ell_4m_4}K^{\nu_4*}_{\ell_2m_2,\ell_4m_4}C^{T_{\nu_1}T_{\nu_3}}_{\ell_3}C^{T_{\nu_2}T_{\nu_4}}_{\ell_4} \quad (13)$$

We are basically stuck here and have to rely on harsh approximation to go further. We will remove the power spectra from the sum and replace their product by a symmetric version

$$T_1 = f_{\ell_1\ell_2}C^{T_{\nu_1}T_{\nu_3}}_{\ell_1\ell_2}C^{T_{\nu_2}T_{\nu_4}}_{\ell_1\ell_2}\sum_{m_1m_2}\sum_{\ell_3m_3}\sum_{\ell_4m_4}K^{\nu_1}_{\ell_1m_1,\ell_3m_3}K^{\nu_3}_{\ell_2m_2,\ell_3m_3}K^{\nu_2*}_{\ell_1m_1,\ell_4m_4}K^{\nu_4*}_{\ell_2m_2,\ell_4m_4} \quad (14)$$

$$C^{T_{\nu_1}T_{\nu_3}}_{\ell_1\ell_2} = \frac{C^{T_{\nu_1}T_{\nu_3}}_{\ell_1}+C^{T_{\nu_1}T_{\nu_3}}_{\ell_2}}{2} \quad (15)$$

Then we use

$$\sum_{\ell_3m_3}K^{\nu_1}_{\ell_1m_1,\ell_3m_3}K^{\nu_3*}_{\ell_2m_2,\ell_3m_3} = \sum_{\ell_3m_3}\int d\hat{n}_1Y_{\ell_3m_3}(\hat{n}_1)W^{\nu_1}_T(\hat{n}_1)Y^*_{\ell_1m_1}(\hat{n}_1)\int d\hat{n}_2Y^*_{\ell_3m_3}(\hat{n}_2)W^{\nu_3}_T(\hat{n}_2)Y_{\ell_2m_2}(\hat{n}_2)$$

$$= \int d\hat{n}_1Y_{\ell_2m_2}(\hat{n}_1)W^{\nu_1}_T(\hat{n}_1)W^{\nu_3}_T(\hat{n}_1)Y^*_{\ell_1m_1}(\hat{n}_1) = K^{\nu_1\nu_3}_{\ell_1m_1,\ell_2m_2} \quad (16)$$

The covariance matrix can finally be written

$$\langle \Delta\tilde{C}^{T_{\nu_1}T_{\nu_2}}_{\ell_1}\Delta\tilde{C}^{T_{\nu_3}T_{\nu_4}}_{\ell_2}\rangle \approx f_{\ell_1\ell_2}C^{T_{\nu_1}T_{\nu_3}}_{\ell_1\ell_2}C^{T_{\nu_2}T_{\nu_4}}_{\ell_1\ell_2}\sum_{m_1m_2}K^{\nu_1,\nu_3}_{\ell_1m_1,\ell_2m_2}K^{\nu_2,\nu_4}_{\ell_1m_1,\ell_2m_2} \quad (17)$$

$$+ f_{\ell_1\ell_2}C^{T_{\nu_1}T_{\nu_4}}_{\ell_1\ell_2}C^{T_{\nu_2}T_{\nu_3}}_{\ell_1\ell_2}\sum_{m_1m_2}K^{\nu_1,\nu_4}_{\ell_1m_1,\ell_2m_2}K^{\nu_2,\nu_3}_{\ell_1m_1,\ell_2m_2} \quad (18)$$

$$= C^{T_{\nu_1}T_{\nu_3}}_{\ell_1\ell_2}C^{T_{\nu_2}T_{\nu_4}}_{\ell_1\ell_2}M_{\ell_1\ell_2}(W^{\nu_1}_TW^{\nu_3}_T, W^{\nu_2}_TW^{\nu_4}_T)) \quad (19)$$

$$+ C^{T_{\nu_1}T_{\nu_4}}_{\ell_1\ell_2}C^{T_{\nu_2}T_{\nu_3}}_{\ell_1\ell_2}M_{\ell_1\ell_2}(W^{\nu_1}_TW^{\nu_4}_T, W^{\nu_2}_TW^{\nu_3}_T) \quad (20)$$