Efficient Sovereign Debt Management: The role of History, Maturity, Buyback and Default*

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Abstract

This paper identifies the role of past history, buybacks and defaults in the context of constrained efficient sovereign borrowing. I derive a market economy in which a sovereign borrower trades non-contingent bonds of different maturities with a foreign lender. The borrower is relatively impatient and lacks commitment. I show that the market economy cannot implement the Planner's constrained efficient allocation through defaults but instead by changes in maturity and debt buybacks. Especially, when the borrower is sufficiently patient Markov strategies can implement the Planner's allocation in steady state. Otherwise, history-dependent strategies are required. Nevertheless, interpreting the impatient borrower as a shot-run player, small perturbations in the payoff of the market participants rule out any other strategies than Markov ones. In this case, the constrained efficient allocation represents an *ideal type* which can only be approximated by the market economy through history-invariant debt management policies.

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1 Introduction

The sovereign debt management of developing economies has three main features. First and foremost, those economies tend to default on their liabilities. Defaults last several years and are associated with output contractions as well as substantial debt reliefs, while default rates are usually above 2% on average.[1] Second, developing economies rely more extensively on short-term debt during debt crises. This comes from the increasing term premium which shifts the issuance towards shorter maturities.[2] Third, developing economies conduct debt buybacks to repurchase part of their debt on the market. Even though such events have become rare in the last few years, they are usually very costly and ineffective in reducing indebtedness.[3] The question that naturally arises is whether such debt management policies are efficient? This paper analyzes the role of each of the above policies in attaining or approximating the constrained efficient sovereign borrowing.

I consider an environment in which a foreign lender owns a production technology in a small open economy, provides the capital input and buys bonds issued by a sovereign borrower. Conversely, the sovereign borrower takes the decision on behalf of the small open economy, runs the production technology and issues non-contingent defaultable bonds of different maturities. In addition, domestic production is subject to persistent growth shocks. In this set-up, I introduce two frictions: the borrower is relatively impatient and cannot commit to repay the lender.

Relying on trigger strategies, I show that it is possible to sustain many different equilibria in this environment. I first focus on Markov equilibria to derive history-invariant debt management policies. I consider the case in which the lender inhibits the borrower's default incentives and the case in which it does not. I subsequently provide foundations for the use of Markov strategies. Particularly, interpreting the impatient borrower as a short-run player, I show that Markov strategies are in fact the only strategies robust to small and independent perturbations in payoffs. Secondly, I derive the history-dependent debt management policies emanating from an optimal contract. Such policies are constrained efficient and correspond to the best achievable outcome in this environment. Nevertheless, they rely on the strong assumption that both the borrower and the lender keep track of the entire history of past policies. Hence, except in some specific cases, market participants following Markov strategies will not be able to attain this outcome. That is why the constrained efficient allocation often represents an *ideal type* in the sense of Weber (2012).

[1] For default duration and haircuts, see Cruces and Trebesch (2013) and Asonuma and Trebesch (2016). For default rates, see Tomz and Wright (2007), Reinhart and Rogoff (2009) and Tomz and Wright (2013).

[2] See Arellano and Ramanarayanan (2012), Broner et al. (2013) and Bai et al. (2017).

[3] See Bulow and Rogoff (1988, 1991).

To analyze the history-invariant debt management policies, I consider two types of Markov equilibria. The first one is a version of Arellano and Ramanarayanan (2012) with growth shocks and endogenous default cost. The lender does not inhibit default incentives and prices bonds accordingly. The equilibrium features defaults and maturity shortening during debt crises as in the data. In the second Markov equilibrium, the lender explicitly prevents default incentives. Particularly, it introduces an endogenous borrowing limit which can be made state-contingent with the help of buyback programs. The borrowing limit prevents defaults while allowing for risk sharing as in Alvarez and Jermann (2000) rendering this equilibrium more efficient than the first one. I finally provide foundations for the use of Markov strategies. Interpreting the impatient borrower as a short-run player, I show that only Markov equilibria are robust to small and independent perturbations in payoffs (Bhaskar et al., 2012). The argument is that the lender conditions its actions on a past event only if the borrower does so and *vice versa*. Thus, conditioning on the past is only possible if both parties keep track of the entire history of play. As a result, there are two extremes: both parties either build on the entire history or do not rely on past history at all.

To analyze the history-dependent debt management policies, I derive the optimal contract between the borrower and the lender. The Planner accounts for a borrower's participation constraint and keeps track of the binding constraint through a co-state variable – i.e. the relative Pareto weight – which is sufficient statistics for the history of the play (Marcet and Marimon, 2019). The optimal contract attains the constrained efficient allocation. It features production distortion and state-contingent debt relief. Particularly, when the relative Pareto weight is sufficiently high, the punishment of autarky is a real threat. The contract can therefore sustain the productivity-maximizing level of capital. Otherwise, the threat of autarky fades. As a result, the Planner reduces the level of capital to relax the participation constraint. It never finds optimal to set capital to zero, though.

I implement the constrained efficient allocation in the sovereign debt market economy. Given that the Planner never distorts capital to zero, defaults – which imply market exclusion – cannot implement the constrained efficient allocations. Instead, the government adapts the maturity structure of its portfolio and conducts debt buybacks at some specific rate. The buyback implicitly introduces state contingency in the bond contract. It occurs at some very specific point in time implying that the price of long-term bonds evolves according to the likelihood of a buyback is in the future. This in turns generates the term structure necessary to mimic the state contingency in liabilities in the optimal contract.

Unlike Markov equilibria, the optimal contract builds on past history. Nevertheless, it admits a steady state with points of amnesia in which history becomes irrelevant for some time. Those points are located at the bounds of the ergodic set of the co-state variable and act as endogenous constraints. This means that history-dependent strategies are not always necessary in steady state. Particularly, when the borrower is sufficiently patient, the long-run constrained efficient allocation can be implemented by history-invariant debt management policies. This is however a knife-edge case. More generally, given that the evolution of the relative Pareto weight depends on the stochastic growth state, the number of periods necessary to hit one of the bounds can be (countably) infinitely large. That is, market participants cannot forgo the knowledge of the entire history of play.

The Markov equilibrium with default incentives is well suited for positive inquiries. I therefore calibrate the it to match moments of the Argentine economy over for the period 1970-2019. This country represents one of the major emerging economies and defaulted numerous time in the 1980s. The Markov equilibrium fits well the data and can replicate some non-targeted business cycle moments. Especially, it features defaults episodes in which indebtedness increases with respect to output and maturity shortens. Conversely, during restructurings, the level of debt remains substantial and the maturity lengthens.

I then compare the Markov equilibria with the optimal contract through various simulation exercises. The optimal contract benefits from a greater sophistication as the relative Pareto weights keep track of the binding constraints through times. This enables a more complex mechanism than the one implied by memoryless strategies. Consumption is less volatile and corresponds to a lower share of output, while investment is more volatile compared to the Markov equilibria. I find important welfare gains for the borrower to join the contract. I also show that the Markov equilibrium without default incentives is superior than the one with default incentives and can get closer to the constrained efficient allocation.

The paper combines elements of the literature on sovereign debt and defaults with elements of the literature about optimal contracts and their implementation. The literature on sovereign defaults assumes that markets are incomplete and agents follow Markov strategies (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; Arellano, 2008).[4] There, the borrower has access to only non-contingent claims and can obtain limited state contingency through defaults. My study is the closest to Arellano and Ramanarayanan (2012) given that I adopt two bonds with different maturities and to Mendoza and Yue (2012) given that the default cost is endogenous. I contribute to this literature in two ways. First, I show that the reliance on defaults to obtain state contingency is inefficient. Second, I provide foundations for the use of Markov strategies interpreting the standard assumption of impatient borrower as evidence of bounded memory and then applying the result of Bhaskar et al. (2012).

On a similar note, this paper relies on costly buybacks as a way to implement the con-

[4]See also Aguiar and Amador (2014) and Aguiar et al. (2016) for a detailed analysis of this class of models.

strained efficient allocation. It therefore relates to the seminal contribution of Bulow and Rogoff (1988, 1991) who document that buybacks are ineffective and suboptimal as they increase the recovery value per unit of bond and therefore increase the market price. Similarly, Aguiar et al. (2019) show that buybacks are costly as they shift the maturity structure and therefore affect the default risk. In opposition, Rotemberg (1991) shows that buybacks can be advantageous to all parties as they lower the bargaining costs.[5] My analysis goes in this direction as it emphasises the necessity and efficiency of buybacks.

The paper derives the optimal contract between a lender and a borrower and therefore relates to the seminal contributions of Kehoe and Levine (1993, 2001) and Thomas and Worrall (1994) who considered the case of limited enforcement, as well as Thomas and Worrall (1990), Atkeson (1991) and Kocherlakota (1996) who added moral hazard. My study accounts for limited enforcements similar to Aguiar et al. (2009).[6] Using the approach of Marcet and Marimon (2019) and growth shocks, my contribution is to compare the constrained efficient allocation with the Markov one. Especially, when and how history dependence matters.

After characterizing the constrained efficient allocation, I implement it in the market economy. Unlike Aguiar et al. (2019) and Müller et al. (2019), my implementation does not rely on Markov strategies. On the one hand, Aguiar et al. (2019) account for multiple maturities but consider a Planner's problem which does not take into consideration the legacy creditors in the objective function, whereas my Planner problem does. On the other hand, Müller et al. (2019) use preemptive restructurings and GDP-linked bonds to mimic the return of Arrow securities, while I rely on the term structure. An alternative to this approach is Dovis (2019) who develop an implementation through partial default and an active debt maturity management. He builds on Angeletos (2002) who shows that one can replicate the state-contingency of Arrow securities using non contingent bonds of different maturities. Unlike Alvarez and Jermann (2000) who assume a high sophistication of the financial market, his implementation relies on a high sophistication of the market participant's strategies building on the entire history of play. My implementation is the closest to Dovis (2019) with the exception that I rely on debt buybacks without default or face value reduction of the debt. Moreover, I compare my implementation with the Markov allocations.

The paper is organized as follows. I describe the economic environment in Section 2. Subsequently, I introduce the market economy and derive the set of sustainable equilibria in Section 3. Thereafter, I present the history-invariant debt management policies in Section 4. I then derive the history-dependent debt management policies in Section 5. The calibration and quantitative analyses are in Section 6. Finally, I conclude in Section 7.

[5] I thank Ugo Panizza for suggesting this reference.

[6] See Quadrini (2004) and Dovis (2019) for a version of this model with moral hazard.

2 Environment

Consider a small open economy over infinite discrete time $t = \{0, 1, \dots\}$ with a single homogenous good. The small open economy is populated by a benevolent government and a large number of homogenous households which own domestic firms, while a foreign lender invests in the small open economy.[7]

The risk neutral lender discounts the future at rate $\frac{1}{1+r}$. It equips the small open economy with a production technology, $F(\hat{k}, l)$, to produce goods and provides the capital input, $\hat{k}$, at price p, which depreciates at rate $\delta = 1$. Domestic households provide the labor input, l. In addition, the lender buys bonds issued by the government.

The representative domestic household discounts the future at rate $\beta < \frac{1}{1+r}$. Preference over consumption is represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(\hat{c}_t),$$

where $\hat{c}_t$ corresponds to the consumption at time t. The instantaneous utility function takes the CRRA form, $u(\hat{c}_t) = \frac{\hat{c}_t^{1-\sigma}}{1-\sigma}$, where $\sigma \in (0, 1]$ is the coefficient of relative risk aversion.

The government is benevolent and takes the decision on behalf of the small open economy. It can tax the payment of capital made by the domestic firms at rate $\tau_t \in [0, 1]$. Thus, the household's after-tax income is given by

$$\hat{y}(z_t, p_t, \hat{k}_t, \tau_t) := z_t F(\hat{k}_t, l_t) - p_t(1 - \tau_t)\hat{k}_t.$$

Domestic households are endowed with one unit of labor in each period.[8] I therefore denote $f(\hat{k}) \equiv F(\hat{k}, 1)$. The production technology has constant returns to scale, is continuous, increasing, $f_{\hat{k}}(\hat{k}) > 0$, concave, $f_{\hat{k}\hat{k}}(\hat{k}) < 0$, satisfies the Inada condition, $\lim_{\hat{k} \to 0} f_{\hat{k}}(\hat{k}) = \infty$, and $f(0) > 0$. The fact that $f(\hat{k})$ is concave implies that the production technology displays decreasing returns to scale with respect to $\hat{k}$. This means that there exists a level $\hat{k}^*(z)$ which maximizes the net production such that $z f_{\hat{k}}(\hat{k}^*(z)) = p(1 - \tau)$.

Domestic production is subject to a growth shock z_t. The law of motion is given by $z_t = g_t z_{t-1}$ where g_t represents the growth rate at time t. Growth takes value on the discrete set $G \equiv \{g_L, g_H\}$ with $0 < g_L < 1 < g_H$ and follows a Markov chain of order one with $\pi(g_{t+1}|g_t)$ corresponding to the probability of drawing g_{t+1} at date $t + 1$ conditional on drawing g_t at t. I further assume that shocks are persistent meaning that $\pi(g|g) > 0.5$.

[7] The present environment is similar to the one of Quadrini (2004), Aguiar et al. (2009) and Dovis (2019).

[8] As in Aguiar et al. (2009) and Dovis (2019), I combine the income of households and government together. Households provide labor inelastically and receive lump sum transfers from the government.

Finally, to ensure the economy converges to a meaningful steady state, $\beta(1+r)g^{-\sigma} \leq 1$ for all $g \in G$ and $\left(\frac{g_H}{g_L}\right)^{-\sigma} > \beta(1+r)$.

The sovereign government issues bonds with two different maturities. On the one hand, it has access to a one-period – i.e. short-term – bond, $\hat{b}_{st}$, with unit price q_{st}. On the other hand, there is a perpetual – i.e. long-term – bond, $\hat{b}_{lt}$, with unit price q_{lt}, which pays a coupon of one every period (Hatchondo and Martinez, 2009). I denote debt as a negative asset meaning that $\hat{b}_j < 0$ denotes a debt, while $\hat{b}_j > 0$ denotes an asset for all $j \in \{st, lt\}$. The financial market is incomplete as bonds do not discriminate the returns across g.

The borrower can conduct *official* buybacks on the consol at price $q_{lt}^{bb} = \frac{1}{r^{bb}}$ with $r^{bb} \geq r$.[9] Conversely, it can conduct *unofficial* buybacks in which it retires part of its debt at the market price without prior notice. In both cases, the prospective value of long-term debt is such that $\hat{b}'_{lt} \geq \hat{b}_{lt}$. *Official* buybacks are more costly but are necessary to implement the constrained efficient allocation.

I assume that the borrower cannot commit to repay the lender. If the government defaults, it loses access to the the capital and bond market. It subsequently consumes $\hat{c}(z) = zf(0) > 0$ but cain regain access to the market with a fixed probability λ. The default cost is therefore endogenous as it entirely relates to markets access. Collecting all the assumptions made so far, I obtain the following.

Assumption 1 (General Settings)**.** *The risk neutral lender discounts at rate* $\frac{1}{1+r}$*. The risk averse sovereign has a utility function* $u(\hat{c}_t) = \frac{\hat{c}_t^{1-\sigma}}{1-\sigma}$ *with* $\sigma \in (0,1]$ *and discounts at rate* $\beta < \frac{1}{1+r}$*. The growth shock* $g \in G \equiv \{g_L, g_H\}$ *follows an Markovian process of order one with* $0 < g_L < 1 < g_H$*,* $\pi(g|g) > 0.5$*,* $\beta(1+r)g^{-\sigma} \leq 1$ *for all* $g \in G$ *and* $\left(\frac{g_H}{g_L}\right)^{-\sigma} > \beta(1+r)$*. The foreign production technology is continuous, increasing, concave, satisfies the Inada condition,* $\lim_{\hat{k}\to 0} f_{\hat{k}}(\hat{k}) = \infty$*, and* $f(0) > 0$*. Capital depreciates at rate* $\delta = 1$*.*

The timing of actions is the following. In $t = 0$, the lender installs the technology in the sovereign country for a given $\hat{k}_0$. At the beginning of each period $t \geq 0$, given the level of $\hat{k}_t$, the shock, g_t, realizes, domestic production takes place, capital depreciates and the government determines the debt repayment – including potential *official* buybacks – and the tax τ_t. Conditional on repaying, the government issues debt, $\hat{b}_{st,t+1}$ and $\hat{b}_{lt,t+1}$, while the lender provides the capital, $\hat{k}_{t+1}$, used in the next period.

I present in the main text the detrended version of the model. I detrend the different variables by dividing them by z_{t-1}. For instance, I denote c_t as the detrended form of $\hat{c}_t$ such that $c_t = \frac{\hat{c}_t}{z_{t-1}}$. Note that given the assumed utility form, $u(\hat{c}_t) = z_{t-1}^{1-\sigma} u(c_t)$.

[9] The fact that *official* buybacks are settled at a rate above the risk-free price is consistent with the evidence that buybacks are costly for sovereign borrowers. See Bulow and Rogoff (1988, 1991).

3 The Market Economy

In this section, I define the set of sustainable equilibrium outcome in the market economy following the approach of Abreu (1988) and Chari and Kehoe (1990). The state space accounts for the entire history of play. Using the reversion to the worst equilibrium, it is possible to sustain many different equilibrium outcomes.

3.1 The Government's problem

Define $D_t \in \{0, 1\}$ as the government's default policy at time t. If $D_t = 0$, the government repays, while if $D_t = 1$, it defaults. Similarly, define $M_t \in \{0, 1\}$ as the government's *official* buyback policy at time t. If $M_t = 1$, the government buys its debt back, while if $M_t = 0$, it does not.

In addition, define the set of government's choices as $\mathcal{G}_t = \{D_t, M_t, b_{st,t+1}, b_{lt,t+1}, \tau_t\}$ and the government's strategy as σ_b. Furthermore, let $h^t = (h^{t-1}, g_t, p_t, k_t, \mathcal{G}_t)$ denote the history up to time t taking the initial debt $b_{st,0}$ and $b_{lt,0}$ as well as capital k_0 as given. Due to the specific timing of actions, further define the history of the lender and the borrower as $h_l^t = (h^{t-1}, g_t, p_t, k_t, \mathcal{G}_t)$ and $h_b^t = (h^{t-1}, g_t, p_t, k_t)$, respectively.

In the case in which the government decides to repay (i.e. $D_t = 0$), it determines its consumption and prospective borrowing given the realization of the history (h_b^t, g_t). In the case of no *official* buyback (i.e. $M_t = 0$), the budget constraint in detrended form reads

$$c_t + q_{st}(h_b^t, \mathcal{G}_t)g_t b_{st,t+1} + q_{lt}(h_b^t, \mathcal{G}_t)(g_t b_{lt,t+1} - b_{st,t}) = y(g_t, p_t, k_t, \tau_t) + b_{st,t} + b_{lt,t}.$$

The government repays the debt according to the current maturity and issues new debt. Conversely, in the case of an *official* buyback (i.e. $M_t = 1$), budget constraint is given by

$$c_t + q_{st}(h_b^t, \mathcal{G}_t)g_t b_{st,t+1} + q_{lt}(h_b^t, \mathcal{G}_t)g_t b_{lt,t+1} = y(g_t, p_t, k_t, \tau_t) + b_{st,t} + b_{lt,t}(1 + q_{lt}^{bb}).$$

The government retires the non-maturing long-term bond, $b_{lt,t}$, at $q_{lt}^{bb} = \frac{1}{r^{bb}}$ with $r^{bb} \geq r$ and issues a new debt such that $b_{t+1} \geq b_t$. Conversely, if the government decides to default (i.e. $D_t = 1$), it gets excluded from the markets and consumes the after-tax income

$$c_t = y(g_t, p_t, k_t, \tau_t).$$

The outstanding debt will be restructured with probability λ. Upon restructuring, the government can regain access to the market. In this case,

$$c_t + q_{st}(h_b^t, \mathcal{G}_t) g_t b_{st,t+1} + q_{lt}(h_b^t, \mathcal{G}_t) g_t b_{lt,t+1} = y(g_t, k_t, \tau_t) + W_t \left[1 + \frac{1+r}{r}\right].$$

Once the government paid the recovery value $W_t[1 + \frac{1+r}{r}]$, there is no remaining liabilities left. Note however that, in default, restructuring is not a choice variable; the government has the opportunity to restructure with probability λ each period.

After any history (h_b^t, g_t), the optimal strategy of the government, σ_b, is the solution of

$$W^b(h_b^t, g_t) = \max_{\mathcal{G}_t = \{b_{st,t+1}, b_{lt,t+1}, \tau_t, D_t, M_t\}} u(c_t) + \beta g_t^{1-\sigma} \mathbb{E}\left[W^b(h_b^{t+1}, g_{t+1}) \middle| h_b^t, \mathcal{G}_t\right], \tag{1}$$

subject to the budget constraint.

3.2 Sustainable equilibria

This subsection aims at defining and characterising the set of sustainable equilibria. The lender is competitive meaning that in expectations it makes zero profit. The price of one unit of bond can therefore be separated into two parts: the return when the government decides to repay and the recovery value when the government defaults. The price per unit of bond of maturity j for all $j \in \{st, lt\}$ is given by,

$$q_j(h^t) = \mathbb{E}\left[(1 - D(h^{t+1}))q_j^P(h^{t+1}) + D(h^{t+1})q_j^D(h^{t+1}) \middle| h_b^t, \mathcal{G}_t\right]. \tag{2}$$

If the government decides to default, the recovery value for all $j \in \{st, lt\}$ is

$$q_j^D(h^t) = \frac{1}{1+r}\mathbb{E}\left[(1-\lambda)q_j^D(h^{t+1}) + \lambda \frac{W_t \frac{1+r}{1+r-\mathbb{I}_{j=lt}}}{b_{j,t}} \middle| h_b^t, \mathcal{G}_t\right],$$

where $\mathbb{I}_{j=lt}$ is an indicator function taking value one if $j = lt$ and zero otherwise. If the government restructures its debt, the lender receives $\frac{W_t \frac{1+r}{1+r-\zeta\mathbb{I}_{j=lt}}}{b_{j,t}}$ per unit of bond issued. Conversely, if it does not restructure, the government does not disburse anything now, but in present value it pays $q_j^D(h^{t+1})$.

In case of repayment, the price depends on the maturity structure and the *official* buyback decision. For the one-period bond,

$$q_{st}^P(h^t) = \frac{1}{1+r},$$

while for the long-term bond,

$$q_{lt}^P(h^t) = \frac{1}{1+r}\mathbb{E}\left[1 + (1 - M(h^{t+1}))q_{lt}(h^{t+1}) + M(h^{t+1})q_{lt}^{bb}\middle|h_b^t, \mathcal{G}_t\right],$$

where $q_{lt}^{bb} = \frac{1}{r^{bb}}$ with $r^{bb} \geq r$ is the *official* buyback rate. Having properly determined the price, I can define the a sustainable equilibrium in the market economy.

Definition 1 (Sustainable Equilibrium)**.** *Given $\{b_{j,0}\}_{j\in\{st,lt\}}$ and k_0, a sustainable equilibrium in this environment consists of*

- *Strategy for the government, σ_b.*
- *Policy for the firm's capital, k.*
- *Price schedule for capital, p, and for bonds, (q_{st}, q_{lt}).*

such that

1. *Taking p, q_{st} and q_{lt} as given, σ_b is the solution to (1).*
2. *Taking p as given, the choice of capital by domestic firms is the solution to*

$$\max_{k_{t+1}} \mathbb{E}\left[\frac{u_c(c(h_b^{t+1}, g_{t+1})}{\mathbb{E}[u_c(c(h_b^{t+1}, g_{t+1}))]}(g_{t+1}f(k_{t+1}) - p(h_p^{t+1})(1-\tau_{t+1})k_{t+1})\middle|h_l^{t+1}\right]. \tag{3}$$

3. *Taking σ_b as given, the price of capital is consistent with*

$$\max_{k_{t+1}} \mathbb{E}\left[p(1-\tau(h_b^{t+1}, g_{t+1}))k_{t+1} - k_{t+1}\middle|h_l^{t+1}\right]. \tag{4}$$

4. *The price of bonds satisfy (2).*

Following the approach of Abreu (1988) and Chari and Kehoe (1990), I characterize the set of outcomes that can be sustained in equilibrium using reversion to the worst equilibrium. The following lemma shows that permanent autarky is the worst equilibrium outcome.

Lemma 1 (Worst Equilibrium Outcome)**.** *In this environment, the worst possible outcome is permanent autarky which can be supported as an equilibrium.*

Proof. See Appendix D □

Keeping track of the entire history of play, I can sustain many different equilibrium outcome relying on trigger strategies. I now focus on two specific types of equilibria: Markov equilibria which do not build on past histories and the optimal contract which extensively relies on past histories and is constrained efficient.

4 History-Invariant Debt Management

In this section, I derive the history-invariant debt management policies. I rely on Markov equilibria defined as a sustainable equilibria in which strategies are time independent and conditioned on the current shock, debt portfolio, capital price, and stock of capital. I consider the optimal debt management with and without default incentives. I finally provide foundation for the use of Markov strategies by the market participants.

4.1 Markov equilibrium with default incentives

Markov equilibria rely on memoryless strategies conditioned on the state which only encodes payoff-relevant information. The state space for a Markov equilibrium corresponds to $\Omega \equiv (g, p, k, b_{st}, b_{lt})$. All Markov equilibria are sustainable equilibria as they simply restrict the information set to the state $\Omega_t \subset h^t$ for any $t > 0$. However, the opposite is not true.

The Markov equilibrium with default is a version of Arellano and Ramanarayanan (2012) with growth shock and endogenous default cost. The government's overall beginning of the period value is given by

$$V(\Omega) = \max_{D \in \{0,1\}} \Big\{(1-D)V^P(\Omega) + DV^D(\Omega)\Big\}, \tag{5}$$

where V^P and V^D correspond to the value of repaying the debt and defaulting, respectively. Under repayment, the government chooses whether to officially buyback its current debt portfolio. Thus

$$V^P(\Omega) = \max_{M \in \{0,1\}} \Big\{(1-M)V^{NB}(\Omega) + MV^B(\Omega)\Big\}, \tag{6}$$

where V^B and V^{NB} are the values under buyback and no buyback, respectively. If the government decides to buy back its current debt portfolio,

$$\begin{aligned} V^B(\Omega) = \max_{\tau, b'_{st}, b'_{lt}} \; & u(c) + \beta g^{1-\sigma} \mathbb{E}_{g'|g}\Big[V(\Omega')\Big] \\ \text{s.t.} \quad & c + q_{st}(g, p', k', b'_{st}, b'_{lt})gb'_{st} + q_{lt}(g, p', k', b'_{st}, b'_{lt})gb'_{lt} = y(g, p, k, \tau) + b_{st} + b_{lt}(1 + q^{bb}_{lt}), \\ & b'_{lt} \geq b_{lt}. \end{aligned}$$

Under no *official* buyback, the government simply repays the debt according to the maturity structure and issues new bonds.

$$
\begin{aligned}
V^{NB}(\Omega) = \max_{\tau, b'_{st}, b'_{lt}} \; & u(c) + \beta g^{1-\sigma} \mathbb{E}_{g'|g}\Big[V(\Omega')\Big] \\
\text{s.t.} \quad & c + q_{st}(g, p', k', b'_{st}, b'_{lt}) g b'_{st} + q_{lt}(g, p', k', b'_{st}, b'_{lt})(g b'_{lt} - b_{lt}) = y(g, p, k, \tau) + b_{st} + b_{lt}.
\end{aligned}
$$

Under default, the government is excluded from the capital and bond market. It subsequently needs to restructure its debt. The value under default is given by

$$
V^D(g) = \max_{\tau} u(y(g, p, k, \tau)) + \beta g^{1-\sigma} \mathbb{E}_{g'|g}\Big[V^R(g')\Big], \tag{7}
$$

where V^R is the value under renegotiation. The government has the opportunity to restructure each period with probability λ,

$$
V^R(g) = (1-\lambda) V^D(g) + \lambda V^E(g), \tag{8}
$$

where the value upon reentry is given by

$$
\begin{aligned}
V^{E}(g) = \max_{\tau, b'_{st}, b'_{lt}} \; & u(c) + \beta g^{1-\sigma} \mathbb{E}_{g'|g}\Big[V(\Omega')\Big] \\
\text{s.t.} \quad & c + q_{st}(g, p', k', b'_{st}, b'_{lt}) g b'_{st} + q_{lt}(g, p', k', b'_{st}, b'_{lt}) g b'_{lt} = y(g, \infty, 0, \tau) + W\left[1 + \frac{1+r}{r}\right].
\end{aligned}
$$

The pricing equations and the equilibrium definitions are presented in Appendix A. Given this, what is the optimal level of capital the lender is willing to provide? The government is tempted to tax capital only in the case of default. It therefore sets $\tau = 1$ if $D(g, \Omega) = 1$ as it loses capital market access in the next period. However, it sets $\tau = 0$ if $D(\Omega) = 0$ as any tax on capital would directly reduce the small open economy's output. In a given state Ω, the lender's optimal policy is therefore to provide k' such that $\mathbb{E}_{g'|g} z f_k(k') = 1$ if $D(\Omega) = 0$ for all g and such that $\mathbb{E}_{g'|g} g f_k(k') = \frac{1}{\pi(g'|g)}$ if default happens in the complement of g'.

In terms of debt management, this equilibrium concept is the closest to what is observed in developing economies. On the one hand, defaults arise when growth is low (Arellano and Ramanarayanan, 2012). This is true even in the case of endogenous default cost as the level of capital is the lowest when g_L realizes. However, unlike defaults, *official* buybacks do not arise on equilibrium path. The borrower does not internalize the impact of *official* debt repurchase on the price. *Official* buybacks therefore never occur as they only incur costs to the borrower. On the other hand, maturity shortens during debt crises. The repayment of long-term debt is laddered through multiple periods which implies a greater default risk than the short-term debt. As a result, close to default, the long-term debt price drastically shifts which encourages the borrower to rely more on shorter maturities (Arellano and Ramanarayanan, 2012).

In Section 5, I use the value under default as the optimal contract's outside option. Thus, the Markov equilibrium with default incentives gives the reference point for comparing the different equilibrium concepts.

4.2 Markov equilibrium without default incentives

I now consider the Markov equilibrium without default incentives. The aim is to introduce an endogenous borrowing constraint that ensures no default on equilibrium path. I consider two types of constraints: a non-contingent and a contingent one. The former is defined as

$$b'_{st} + b'_{lt} \geq - \min_{g' \in G} \Big\{ -(b'_{st} + b'_{lt}) : V^P(\Omega') = V^D(g') \Big\}. \tag{9}$$

This limit ensures that the government is never able to accumulate a level of debt for which a default would be optimal. This is what Zhang (1997) defines as a no-default borrowing constraint.

The above constraint is relatively simple, though. Especially, it does not allow much risk sharing. With the market structure at hand, it is possible to create some state contingency. However, since agents will rely on Markov strategies, I cannot condition *official* buybacks on the history of play. Instead, I consider a simple rule stating that such debt repurchase occur only when $g = g_H$. Formally,

$$M(\Omega) = \begin{cases} 1 & \text{if} \quad g = g_H. \\ 0 & \text{else} \end{cases}$$

Given this, the long-term debt becomes a pseudo Arrow security as it pays out more in the high growth state than in the low growth state. With this, I can define an endogenous borrowing limit in the form of

$$\begin{aligned} b'_{st} + b'_{lt}[1 + q_{lt}(g, p', k', b''_{st}, b''_{lt})] &\geq \mathcal{A}(g_L), \\ b'_{st} + b'_{lt}[1 + q^{bb}_{lt}] &\geq \mathcal{A}(g_H), \end{aligned} \tag{10}$$

where the borrowing limit is defined such that

$$V^P(g_L, k', b'_{st}, b'_{lt}) = V^D(g_L) \quad \text{for all} \quad b'_{st} + b'_{lt}[1 + q_{lt}(g, p', k', b''_{st}, b''_{lt})] = \mathcal{A}(g_L),$$

$$V^P(g_H, k', b'_{st}, b_{lt}) = V^D(g_H) \quad \text{for all} \quad b'_{st} + b'_{lt}[1 + q^{bb}_{lt}] = \mathcal{A}(g_H).$$

As one can see, the long-term bond is state contingent, while the short-term bond plays the role of a regular non-contingent risk-free bond. The definition of the endogenous borrowing limit is the one of Alvarez and Jermann (2000). It plays the same role as (9) with the difference that it allows for some risk sharing between g_l and g_H.

The buyback policy implies that $b'_{lt} \geq b_{lt}$ whenever $g = g_H$. Hence, long-term debt holdings reduce as growth is high. Conversely, there are no direct restrictions on the holdings on short-term debt except that (10) has to hold. There are therefore no direct policy dictating the changes in maturity. As one will see in Section 5, this debt management is similar to the one attaining the long-run constrained efficient allocation when the borrower is sufficiently patient.

Provided that the lender has commitment, it will be the agent capable of implementing and enforcing such the above borrowing constraints. In the case of non-contingent borrowing limit, the lender simply provides bonds as long as (9) holds. The same holds true for the state-contingent borrowing limit with the additional requirement that the borrower implements the appropriate buyback policy. If the borrower decides not to buyback its long-term debt when g_H realizes, the lender can decide to set $k' = 0$ for every subsequent periods. This threat together with (10) ensures that the borrower will follow the above buyback policy.

Finally, in equilibrium, the bond price is independent of the debt portfolio choice. No defaults occur on equilibrium path – provided that (10) holds – and buybacks are state contingent.

4.3 Foundation of Markov equilibria

In light of section 3, the Markov equilibrium is a relatively unsophisticated equilibrium concept as it does not build on past history. However, interpreting the borrower as a short-run player and the lender a long-run player, I show that small perturbations in the payoff of both market participants rule out any equilibria except Markov ones. In other words, under bounded memory of at least one of the market participants, Markov equilibria are the only equilibria surviving small and independent idiosyncratic utility shocks.

Given that $\beta(1 + r) < 1$, the borrower is relatively more impatient than the lender. This assumption is standard in the literature on sovereign debt. Not only it is necessary to obtain empirically plausible debt ratios, it is also a property inherited from the general equilibrium analysis and the martingale convergence theorem. In addition, impatience has implications in terms of political economy. It is reduced form for the fact that governments are subject to re-elections and might lose office with positive probability (Alesina and Tabellini, 1990). In this situation, the borrower can be interpreted as a short-term player whose recollection of past actions is bounded. In other words, the memory of the borrower's memory goes back to a certain number of periods $T < \infty$.

Besides the boundedness of the borrower's memory, I introduce a small perturbation in the existing environment. Following Bhaskar et al. (2012) and Angeletos and Lian (2021), in each period t, a utility shock $\epsilon\varrho_{b,t}$ and $\epsilon\varrho_{l,t}$ with $\epsilon > 0$ is drawn for the borrower and the lender, respectively. It has compact support $P_i \subset \mathbb{R}^{|A_i|}$ with absolutely continuous density $\mu_{P_i} > 0$ where $|A_i|$ is the cardinality of the choice set of market participant $i \in \{b, l\}$. Moreover, it is independently distributed across market participants, histories and other shocks. If the market participant $i \in \{b, l\}$ chooses a particular action, say a, its utility is augmented by $\epsilon\varrho^a_{l,t}$. For instance, the instantaneous utility of the borrower taking action a (e.g. default) is given by $u(c_t) + \epsilon\varrho^a_{b,t}$. Finally, utility shock $\epsilon\varrho_{i,t}$ is privately observed by market participant $i \in \{b, l\}$.

Assumption 2 (Perturbed Bounded Memory)**.** *The borrower's memory is bounded to $T \in (0, \infty)$ periods in the past. Moreover, in each t, a utility shock $\epsilon\varrho_{i,t}$ with $\epsilon > 0$ is drawn from the compact support $P_i \subset \mathbb{R}^{|A_i|}$ with absolutely continuous density $\mu_{P_i} > 0$ for each $i \in \{b, l\}$. The utility shock is additive, privately observed and depend on the action taken.*

Given this functional form, the case considered in Section 3 corresponds to $\epsilon = 0$ and $T = \infty$. The case of perturbed bounded memory boils down to what happens when $\epsilon > 0$ but arbitrarily small and $T < \infty$. It means that (1) the government does not recall all the past history and (2) market participants have imperfect knowledge of the other participant's fundamentals. The presence of the privately observed shock – albeit small and independent – coupled with the short-sightedness of the government prevent both participants to rely on past history. This causes all non-Markov equilibria to unravel.

Proposition 1 (Foundation of Markov equilibria)**.** *Under Assumption 2, all non-Markov equilibria unravel.*

Proof. See Appendix D $\square$

The rationale behind that result follows Bhaskar (1998) and Bhaskar et al. (2012). Suppose the lender conditions its action at time t on a payoff-irrelevant past event, then the borrower must also condition on this past event. In equilibrium such conditioning is only possible if both the lender and the borrower keep track of the entire history of play. In other words, the knowledge of past values of ϱ_b is necessary for the lender to appropriately reward the borrower not to deviate. Similarly, the borrower needs to know the previous values of ϱ_l to appropriately respond. Thus, all history-dependent equilibria unravel as soon as those utility shocks are unobserved. As a result, I am left with two extremes: both parties either build on the entire (infinite) history of play or do not build on past histories at all.

5 History-Dependent Debt Management

This section presents the optimal debt management policies when market participants can build on past history. I first derive the optimal contract by means of the Lagrangian approach (Marcet and Marimon, 2019) and subsequently characterize the underlying constrained efficient allocation before implementing it the the market economy.

5.1 Optimal contract in sequential form

In what follows I derive the sequential formulation of the optimal contract which has to account for limited enforcements in repayment. I therefore incorporate a participation constraint to ensures that the sovereign does not break the contract.

The participation constraint of the sovereign deals with the fact that the sovereign can always break the contract and opt for autarky (Thomas and Worrall, 1994). Denoting g^t as the history of realized value of g at time t, it must hold that for all t and g^t

$$\sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t|g_0)\mathcal{G}_t(g_0)u(c(g^t)) \geq u(g_0 f(k(g^{-1}))) + \beta g_0^{1-\sigma} V^D(g_0), \tag{11}$$

with $\mathcal{G}_t(g_0) \equiv (g_0 g_1 \cdot \ldots \cdot g_{t-1})^{1-\sigma} = \prod_{i=0}^{t-1} g_i^{1-\sigma}$. If the sovereign breaks the contract, the borrower is sent to autarky for some time but can regain access to the market with probability λ and resume the Markov equilibrium with default incentives. $V^D(g_0)$ therefore corresponds to the value of default in the Markov equilibrium given by equation (7). As a result, the participation constraint ensures that the sovereign's value of remaining in the contract is at least as large as the value of opting out.

Given the above constraint, the optimal contract between the borrower and the lender in sequential form is the result of

$$\max_{\{k(g^t),c(g^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \mu_{b,0} \sum_{g^t} \pi(g^t|g_0)\mathcal{G}_t(g_0)u(c(g^t)) + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \mu_{l,0} \sum_{g^t} \pi(g^t|g_0) \prod_{i=0}^{t-1} g_i T(g^t)$$

$$\text{s.t.} \quad \sum_{j=t}^{\infty} \beta^{j-t} \sum_{g^j} \pi(g^j|g_t) \mathcal{G}_j(g_t) u(c(g^j)) \geq u(g_t f(k(g^{t-1}))) + \beta g_t^{1-\sigma} V^D(g_t),$$

$$T(g^t) = g_t f(k(g^{t-1})) - c(g^t) - k(g^{t-1}), \ \forall g_t, g^t, t$$

with $\mu_{b,0}$ and $\mu_{l,0}$ given.

The given weights $\mu_{b,0}$ and $\mu_{l,0}$ are the initial non-negative Pareto weights assigned by the Planner to the borrower and the lender, respectively.

The Planner allocates capital and consumption to maximize the lender's and the borrower's weighted utility subject to the resource constraint and the participation constraint. The above maximization problem combines the utility function $u(\cdot)$ with the production function $f(\cdot)$ and therefore might not be convex. Following, Aguiar et al. (2009), I assume the following

Assumption 3 (Convexity)**.** *Define the optimal level of capital k^* such that $f_k(k^*) = 1$ and $h := \mathbb{E}[g]f(k) - k$ for $k \in [0, k^*]$ with $h^* = \mathbb{E}[g]f(k^*) - k^*$. Let $A(h)$ denote the inverse mapping from $[0, h^*]$ to $[0, k^*]$ such that $k = A(h)$. For all g, $u(gf(k(h)))$ is convex in h for $h \in [0, h^*]$.*

This assumption consists of two parts. First, it substitutes the choice variable from k to h. Second, it limits the concavity of both the instantaneous utility and the production technology so that the participation constraint is convex in h. As a result, there is no need for randomization whenever the curvature of $u(\cdot)$ and $f(\cdot)$ is not too pronounced.

5.2 Optimal contract in recursive form

I now derive the recursive formulation of the above maximization problem. Following Marcet and Marimon (2019), I rewrite the sequential problem as a saddle-point Lagrangian problem,

$$\begin{aligned}
\mathcal{SP} \quad \min_{\{\gamma(g^t)\}_{t=0}^{\infty}} \max_{\{k(g^t), c(g^t)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t|g_0) \mu_{b,t}(g^t) \mathcal{G}_t(g_0) u(c(g^t)) \\
& + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \sum_{g^t} \pi(g^t|g_0) \mu_{l,t}(g^t) \prod_{i=0}^{t-1} g_i T(g^{t-1}, g_{i,t}) \\
& + \sum_{t=0}^{\infty} \beta^t \sum_{g^t} \pi(g^t|g_0) \gamma(g^t) \Big[\mathcal{G}_t(g_0) u(c(g^t)) - (u(g_t f(k(g^{t-1}))) + \beta g_t^{1-\sigma} V^D(g_t)) \Big] \\
\text{s.t.} \quad & T(g^{t-1}, g_t) = g_t f(k(g^{t-1})) - c(g^{t-1}, g_t) - k(g^{t-1}), \ \forall g_t, g^t, t \\
& \mu_{b,t+1}(g^t) = \mu_{b,t}(g^t) + \gamma(g^t),
\end{aligned}$$

$$
\mu_{l,t+1}(g^t) = \mu_{l,t}(g^t), \ \forall g_t, g^t, t
$$
$$
\text{with } \mu_{b,0}(g_0) \equiv \mu_{b,0} \text{ and } \mu_{l,0}(g_0) \equiv \mu_{l,0} \text{ given.}
$$

In this formulation, $\beta^t \pi(g^t|g_0) \mathcal{G}_t(g_0) \gamma(g^t)$ is the Lagrange multipliers attached to the participation constraint of the sovereign at time t. The above formulation of the problem defines two new co-state variables, $\mu_{b,t}(g^t)$ and $\mu_{l,t}(g^t)$, which are the temporary non-negative Pareto weights the Planner attributes to the borrower and the lender, respectively. These variables are initialized at the original Pareto weights and subsequently become time-variant.

Following Ábrahám et al. (2019), I further simplify the above problem by normalizing the Pareto weights. More precisely, I express them as

$$
\nu(g^t) = \frac{\gamma(g^t)}{\mu_{b,t}(g^t)}.
$$

Given this, I define the relative Pareto weights for borrower as

$$
x_t(g^t) = \frac{\mu_{b,t}(g^t)}{\mu_{l,t}(g^t)},
$$

Given the non-negativity and boundedness of the Lagrange multipliers, $x \in X \equiv [\underline{x}, \overline{x}]$ with $\underline{x} \geq 0$ and $\overline{x} < \infty$.[10] Defining $\eta \equiv \beta(1+r) \leq 1$, the law of motion of the different relative Pareto weight is given by

$$
x_{t+1}(g^t) = (1 + \nu(g^t)) g_{t-1}^{-\sigma} \eta x_t \text{ with } x_0 = \frac{\mu_{b,0}}{\mu_{l,0}}
$$

With this normalization, $\nu(g^t)$ represents the multiplier attached to the participation constraint. Following Marcet and Marimon (2019), the state vector for the problem reduces to (g, x) and the Saddle-Point Functional Equation is given by

$$
\begin{aligned}
FV(g_-, x) = \mathcal{SP} \min_{\{\nu(g)\}} \max_{\{k(g), c(g)\}} & \sum_g \pi(g|g_-) \left[x u(c(g)) + T(g, g_-) + \frac{1}{1+r} g FV(g, x') \right] \quad (12) \\
& + \sum_g \nu(g) x \left[u(c(g)) - \big(u(gf(k(g_-))) + \beta g^{1-\sigma} V^D(g) \big) \right] \\
\text{s.t.} \quad & T(g, g_-) = gf(k(g_-)) - c(g) - k, \ \forall g, g_- \\
& x'(g) = (1 + \nu(g)) g_-^{-\sigma} \eta x. \quad (13)
\end{aligned}
$$

[10] I later show that setting $\underline{x} \geq 0$ this is without loss of generality as the continuation of an efficient allocation is itself efficient.

The value function takes the form of

$$\begin{aligned}
FV(g_-, x) &= xV^b(g_-, x) + V^l(g_-, x), \text{ with} \\
V^b(g_-, x) &= \mathbb{E}_{g|g_-}\left[u\left(c\right) + \beta g^{1-\sigma} V^b\left(g, x'\right)\right], \text{ and} \\
V^l(g_-, x) &= \mathbb{E}_{g|g_-}\left[T + \frac{1}{1+r} g V^l\left(g, x'\right)\right].
\end{aligned}$$

I obtain the optimal consumption and capital policies by taking the first-order conditions of problem (12) with respect to c

$$u_c(c(g)) = \frac{\pi(g|g_-)}{x(\pi(g|g_-) + \nu(g))},$$

and with respect to k

$$\sum_g \pi(g|g_-) g f_k(k) - 1 = \sum_g \nu(g) u_c(g f(k)) g f_k(k) x.$$

In terms of consumption, the binding participation constraint of the borrower (i.e. $\nu > 0$) induces an increase in consumption.

Regarding capital, the economy does not reach the production-maximiging level of capital $k^*(g)$ as long as the participation constraint binds. Furthermore, the more this constraint binds, the more distorted is capital.

Consumption is determined after the realization of g, while k is set before. Thus, the capital policy functions is denoted by $k(g_-, x)$. Correspondingly, the consumption policy function of the high and low types are denoted by $c^H(g_-, x)$ and $c^L(g_-, x)$, respectively. Similarly, I define the pair of prospective relative Pareto weights of the high and low types by $(x^H(g_-, x)^H(g_-, x))$ and $(x^L(g_-, x)^L(g_-, x))$, respectively.

5.3 Equilibrium properties

I determine the main properties of the contract. I first determine the Pareto frontier in this environment. I subsequently present how the contract provides risk sharing across states. I also derive the characteristics of the long-run contract. Finally, I show how the contract's allocation can be implemented as a sustainable equilibrium in the market economy. Additional characterization can be found in Appendix B.

Proposition 2 (Efficiency)**.** *Under Assumptions 1-3, for all $x \in X$, the utility possibility frontier is strictly increasing and the autarkic allocation is not optimal.*

Proof. See Appendix D □

The proposition states that the optimal contract is the best achievable outcome in this environment. Hence, the debt management policies that I later derive from the optimal contract are the efficient ones. Given this, I derive a metric measuring the distance between the constrained efficient allocation and any alternative allocation. Consider the value of the lender in any given allocation as a function of the shock and the value of the borrower as $\ddot{V}^l : G \times \ddot{V}^b \to \mathbb{R}$. I then define

$$\mathcal{F}(g_-) = \frac{\int \ddot{V}^l(g_-, \ddot{V}^b) d\ddot{V}^b}{\int_{\underline{x}}^{\overline{x}} V^l(g_-, x) dx}.$$

The metric $\mathcal{F}$ is bounded between 0 and 1 as the constrained efficient allocation is the best achievable outcome in this environment. A value of $\mathcal{F}$ near 1 indicates that an allocation is close to the constrained efficient benchmark, whereas a value close to 0 indicates the opposite.[11] In section 6, I use this metric to measure the distance between the Markov equilibria and the constrained efficient allocation and assess the efficiency of history-invariant debt management policies.

The following proposition defines the main properties of the constrained efficient allocation. The contract features production distortions and risk sharing.

Proposition 3 (Constrained Efficient Allocation)**.** *Under Assumptions 1 and 3, the allocation is such that*

I. (Production). *There exists a level of relative Pareto weight x^* such that $k(z, x) = k^*$ for all $x \geq x^*$ and $z \in Z$. Conversely, for all $z \in Z$ and $x, \tilde{x} \in X$ with $x > \tilde{x}$, $0 < k(z, \tilde{x}) < k(z, x) < k^*$.*

II. (Risk-Sharing). *$c^L(z_-, x) \leq c^H(z_-, x)$ and $x^{L\prime}(z_-, x) \leq x^{H\prime}(z_-, x)$ for all (x, z_-) with equality when $x \geq x^*$.*

III. (Liabilities). *$b^L(g_-, x) < b^H(g_-, x)$ for all (g_-, x) with $b^i(g_-, x) \equiv T + \frac{1}{1+r} g_i \mathbb{E}_{g'|g_i} V^l(g', g_i, x')$ for all $i \in \{L, H\}$.*

Proof. See Appendix D □

Part I of the above proposition states that the production-maximizing level of of capital $k^*(g)$ such that $g f_k(k^*) = 1$ is attained only if the relative Pareto weight, x, is above a certain

[11] The metric $\mathcal{F}$ is based on the same concept as the Gini coefficient which measures the distance between the Lorenz curve and the equity line.

threshold. Capital distortion is a consequence of binding participation constraint (11). As this constraint depends on the level of capital in the economy, the Planner finds optimal to reduce k below $k^*(g)$ to relax it and to provide risk sharing to the extent possible. The Planner continues to decrease k as long as x decreases but never finds optimal to set $k = 0$. Capital is therefore pro-cyclical as it decreases when a bad shock hits the economy (Aguiar et al., 2009).

Part II states that the Planner always provides risk-sharing to the extent possible. Equalization of consumption is possible whenever the borrower's participation constraint ceases to bind. Otherwise, the Planner provides more consumption and a greater continuation value when the high growth state realizes.

Part III relates to the liabilities of the sovereign. In this environment, T corresponds to the borrower's current account balance. Hence, the net present value of the lender – after the realization of g – corresponds to the net foreign asset position in the contract.[12] A positive value $b(g, g_-, x)$ therefore indicates that the sovereign owes money to the lender and in that logic the greater is this value the greater is the sovereign liabilities towards the lender. The proposition states that the sovereign liabilities increase when g is high. This implies that the lender adopts a state-contingent policy as it provides debt relief in low growth times.

Having determined the constrained efficient allocation, I now characterize the steady state of the optimal contract. The long-term contract is characterized by an ergodic set of relative Pareto weights. The term ergodic refers to the fact that the relative Pareto weights in this set are aperiodic and recurrent with non-zero probability. In other words, the economy will move around the same set of relative Pareto weights over time and over histories.

Proposition 4 (Steady State)**.** *A steady state is defined by an ergodic set of relative Pareto weights $x \in [x^{lb}, x^{ub}] \subset X$. Under Assumptions 1-3, it holds that $x^{H\prime}(g_H, x^{ub}) = x^{ub}$ and $x^{L\prime}(g_L, x^{lb}) = x^{lb}$ and*

I. If $\eta g_L^{-\sigma} = 1$, then $x^{lb} = x^{ub} < x^*$.

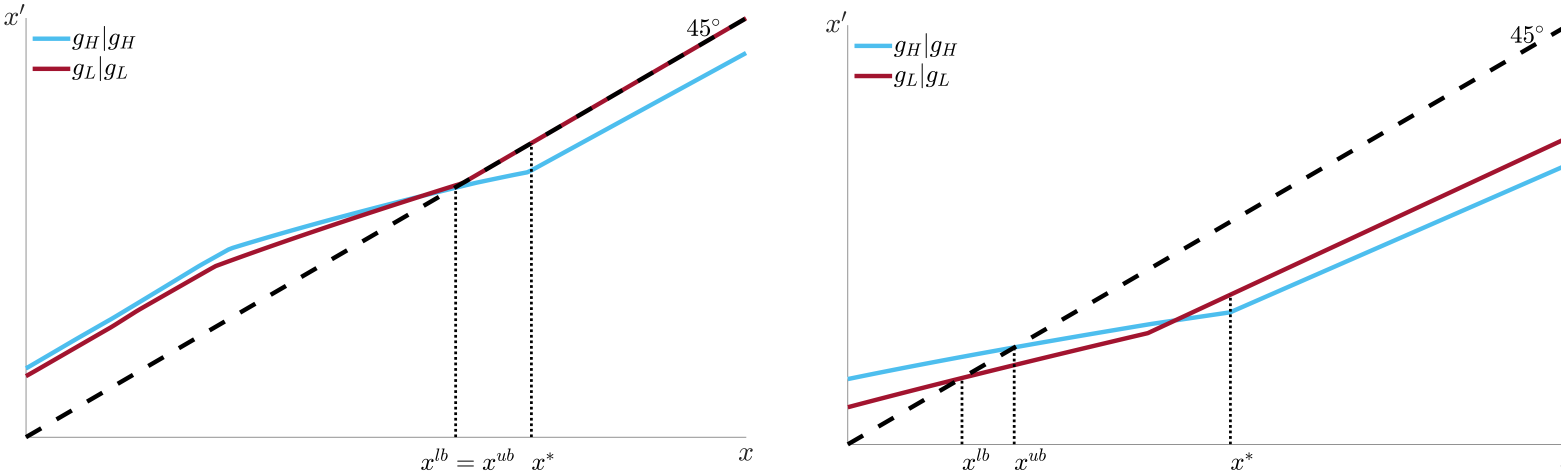
II. If $\eta g_L^{-\sigma} < 1$, then $x^{lb} < x^{ub} < x^*$.

Proof. See Appendix D □

The proposition states that whenever the borrower is sufficiently patient (i.e. $\eta g_L^{-\sigma} = 1$), the steady state does not display any dynamic. This means that there is no history dependence. Conversely, whenever the borrower is relatively impatient (i.e. $\eta g_L^{-\sigma} < 1$), the steady state displays history dependence. Note however that when the contract hits

[12]In the balance of payments statistics, this object corresponds to the net international investment position of a country.

one of the bounds, history becomes irrelevant. For instance, after a sufficiently long series of g_L (g_H), the contract hits x^{lb} (x^{ub}). It then stays there until g_H (g_L) realizes and that irrespective of the past history. Figure 1 illustrates each of the two steady states.



(a) $\eta g_L^{-\sigma} = 1$ (b) $\eta g_L^{-\sigma} < 1$

Figure 1: Steady State Dynamic

Nevertheless, amnesia is not deterministic when $\eta g_L^{-\sigma} < 1$. Given that the realization of g is stochastic, one cannot say *ex ante* how many periods are necessary to hit one of the bonds. Especially, given that no state is absorbing, the time required to hit one of the bounds is potentially (countably) infinite. Hence, even in steady state, the market participant cannot forgo knowledge of the entire history of play.

5.4 Equilibrium implementation

Having derived and characterized the constrained efficient allocation, I am now interested in implementing the contract as a sustainable equilibrium in the sovereign debt market economy presented in Section 3. Obviously, to replicate the constrained efficient allocation, the following conditions have to be met.

Lemma 2 (Implementation Requirements)**.** *Given $\{b_{j,0}\}_{j \in st,lt}$ and k_0, an equilibrium allocation $\{k(g^t), D(g^t), H(g^t), M(g^t), c(g^t), \tau(g^t), b_{st}(g^t), b_{lt}(g^t)\}$ with prices $\{q_{st}(g^t), q_{lt}(g^t), p(g^t)\}$ implements a constrained efficient outcome if and only if it satisfies the resource constraint, the participation constraints, (11), together with the budget constraints, the bond pricing equation, (2), the capital policy, (3), and the capital pricing equation, (4).*

Proof. See Appendix D □

In light of tis lemma, I construct a sustainable equilibrium in the sovereign debt market economy that implements the constrained efficient allocation.

Proposition 5 (Implementation)**.** *Under Assumptions 1-3, given a constrained efficient allocation, a sustainable equilibrium exists that implements it.*

Proof. See Appendix D ☐

The implementation works as follows. The government conducts *official* buybacks when the economy hits the upper bound of the ergodic set. As this bound is reached after a sufficiently long series of high growth shocks, this buyback policy generates a specific term structure in which high growth shocks are related to relatively larger long-term bond prices than low growth shocks, while short-term bond price remains unchanged. Given this, I can equalize the value of debt in the contract, $b(g_{-}, g, x)$, with the value of the debt in the market economy, $b_{st} + b_{lt}[1 + q_{lt}(g,x')]$, for each g. As I have two growth states and two maturities, this gives a system of two equations with two unknowns which has a unique solution given the specified term structure.

The implementation following Proposition 5 has many features in terms of past history, maturity, default and debt buyback. I discuss each of those points separately:

– *History dependence.*

  The implementation rests on the assumption that the economy knows the sequence $\{x_t\}_{t=0}^{\infty}$. This is a remarkable task as the path of relative Pareto weights is complex. In the case in which $\eta g_L^{-\sigma} = 1$, the steady state is degenerate as given by Proposition 4 part I. There it is possible to implement the long-run constrained efficient allocation by means of Markov strategies with the appropriate debt threshold and buybacks schedule as shown in Section 4. However, In the case in which $\eta g_L^{-\sigma} < 1$, the steady state is dynamic as given by Proposition 4 part II. Thus, the constrained efficient allocation represents an *ideal type* – in the sense of Weber (2012) – from the point of view of the market participants following Markov strategies.

– *No default.*

  The implementation does not rely on defaults. As previously shown, the Planner never finds optimal to distort capital to zero. This means that there is no proper market exclusion, even when the contract hits the borrower's participation constraint. It is therefore not possible to interpret the borrower's binding constraint as a default *stricto sensu* in my environment. Other studies however do. For instance, Müller et al. (2019) and Restrepo-Echavarria (2019) view the borrower's binding constraint as a form of preemptive restructuring which does not trigger market exclusion. Nonetheless, Asonuma and Trebesch (2016) show that even preemptive restructurings are followed by some periods of market exclusion.

– *Costly buybacks.*

I consider that the government conducts *official* buybacks when it hits the upper bound of the ergodic set – i.e. $x = x^{ub}$. Moreover, buybacks are costly as $r^{bb} \geq r$. In light of this, could *official* buybacks occur in the lower bound of the ergodic set – i.e. $x = x^{lb}$ – at a discount? The answer is negative. To reach the lower point of amnesia, the relative Pareto weight needs to decrease which means the the government's indebtedness increases. This goes against the idea of a debt repurchase which aims at reducing indebtedness. In addition, buybacks at discount rates are a form of default. As discussed previously, the point in which $x = x^{lb}$ cannot be interpreted as such since $k(g, x) > 0$ for all (g, x).

Hence, costly buybacks are necessary to implement the constrained efficient allocation. My argument goes against Bulow and Rogoff (1988, 1991) who show that buybacks are a boondoggle as the government ends up paying more than it reduces the stock of outstanding debt. Nevertheless, in my analysis, it is true that *official* buybacks are completely ineffective in reducing the government's indebtedness. In fact, when $x = x^{ub}$, $b'_{lt} = b_{lt}$ which means that indebtedness remains unchanged.

– *Maturity structure.*

Relying on the implementation of the constrained efficient allocation, I can analyze the evolution of the maturity structure in the contract. From (D.1) and (D.2),

$$\bar{b}_{st}(g_-, x) = \frac{b(g_-, g_H, x)[1 + q_{lt}] - b(g_-, g_L, x)[1 + q_{lt}^{bb}]}{(q_{lt} - q_{lt}^{bb})},$$
$$\bar{b}_{lt}(g_-, x) = -\frac{b(g_-, g_H, x) - b(g_-, g_L, x)}{(q_{lt} - q_{lt}^{bb})},$$

From Proposition 3 Part III, it holds that $b(g_-, g_H, x) > b(g_-, g_L, x)$ meaning that $b_{lt} < 0$ by application of Lemma D.5. Moreover, if the distance between $b(g_-, g_H, x)$ and $b(g_-, g_L, x)$ is not too large compared to the wedge between q_{lt} and q_{lt}^{bb}, $b_{st} < 0$ as well.

In addition, $-\bar{b}_{lt}$ is high when (a) $b(g_-, g_H, x) - b(g_-, g_L, x)$ is large or (b) $q_{lt}^{bb} - q_{lt}$ is small. Close to the lower bound of the ergodic set (b) is typically very large as buybacks are very unlikely in the near future. The opposite holds true in the vicinity of the upper bound of the ergodic set. Regarding (a), the distance is expected to get larger as the relative Pareto weight decreases given that the participation constraint stops binding at x^*.

– *Endogenous borrowing limits.*

  The points of amnesia (i.e. the bounds of the ergodic set) implicitly define endogenous borrowing and lending limits. On the one hand, when the borrower hits x^{lb}, borrowing remains the same every period as long as g_L realizes. Similarly, when the borrower hits x^{ub}, no additional lending takes place as long as g_H realizes. The borrowing limit attached to x^{lb} ensures that the borrower has no incentive to default. On the other hand, the lending limit attached to x^{ub} naturally arises due to impatience. In my analysis, this bound is also the place in which the borrower conduct buybacks which in turns generates the spread necessary to mimic the state contingency of the optimal contract.

– *Negative spread.*

  Given the buyback and the absence of defaults, the long-term bond spread is negative. This is a feature that one finds in other implementations such as the one of Alvarez and Jermann (2000). The mechanism at work is different, though. In my case, the negative spread enables to mimic the state-contingency in the contract provided that defaults do not arise on equilibrium path. In the case of Alvarez and Jermann (2000), the negative spread restricts the trade of state-contingent securities in a two-sided limited enforcement problem when the participation constraint of the lender binds.

6 Quantitative Analysis

This section starts with the calibration of the Markov equilibrium with default incentives to Argentine. Subsequently, it assesses the fit of the model to the data and compares the Markov equilibria with the constrained efficient allocation. Particularly, it gauges the goodness of Markov equilibria in approximating the constrained efficient debt management policies. Finally, it compares the implementation of Section 5 with an alternative one.

6.1 Calibration

The Markov equilibrium with default incentives enables a positive analysis, while the constrained efficient allocation provides a normative benchmark. I therefore calibrate the

Markov equilibrium with default incentives to match some specific moments of the Argentine economy over the period 1970-2019. Table 1 summarizes each parameter.

Table 1: Calibration

Parameter	Value	Description	Targeted Moment
A. Based on Literature			
σ	1.00	Risk aversion	
B. Direct Measure from the Data			
$\pi(g_H \mid g_H)$	0.58	Probability staying high state	GDP per capita growth
$\pi(g_L \mid g_L)$	0.62	Probability staying low state	
g_H	1.06	High state	
g_L	0.96	Low state	
$1-\alpha$	0.63	Labor share	Labor income share
r	0.03	Risk-free rate	10-year real US treasury rate
C. Based on Model solution			
β	0.75	Discount factor	Debt-to-GDP ratio
ϕ	1.40	CES production	Investment-to-GDP ratio
W	-0.04	Recovery value of debt	Average Spread
λ	0.08	Probability re-accessing market	Default rate

As exposed in Section 2, the utility function takes the CRRA form with a coefficient of relative risk aversion of σ. From the growth literature, I adopt $\sigma = 1$. In addition, in accordance with Assumption 1, the production function is CES

$$F(k,l) = \left[\alpha k^{\frac{\phi-1}{\phi}} + (1-\alpha) l^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}}.$$

where α represents the capital share and ϕ the CES parameter. The value of $1-\alpha$ is set to the average labor income share in Argentine estimated by Frankema (2010) and is close the standard value of 0.7 adopted in the literature on emerging economies (Mendoza and Yue, 2012; Restrepo-Echavarria, 2019). The CES parameter is $\phi = 1.4$ to match the share of investment in GDP and is within the range of admissible values in the business cycle literature on emerging economies.

The two-state Markov chain comes from the estimation of Ayres et al. (2019) for Argentina from 1980 to 2017 based on GDP per capita growth. I set the discount factor to $\beta = 0.75$ to match the average external debt-to-GDP ratio of 29.8%. This discount rate corresponds to a quarterly discounting above 0.93. Besides this, the risk-free rate is $r = 0.03$ which corresponds to the average 10-year treasury rate minus PCE inflation (Dvorkin et al., 2021).

According to Reinhart and Rogoff (2009) and Cruces and Trebesch (2013), Argentina defaulted 8 times between 1816 and 2022 which gives a 3.9% default rate. This is larger than the usual 3% considered in the literature (Tomz and Wright, 2007, 2013). The probability of re-accessing the market is 0.07 to match the Argentine default rate, while the repayment in restructuring, W is determined such that the model matches the average (Embig) spread of 8.9%.

Finally, there is one parameter left to calibrate: the buyback rate r^{bb}. However, as the government in the Markov equilibrium with default incentives does not internalize the impact of buybacks on the bond price, no buyback happens on equilibrium path. I therefore set $r^{bb} = 1\%$ to avoid getting a net asset position in the optimal contract.

6.2 Numerical results

This subsection presents the result of the calibration. It gauges the fit of the model with respect to the data for both targeted and non targeted moments. It also compares the outcome of the Markov allocation with default incentives (MA), without default incentives (MAND) and the constrained efficient allocation (CEA) together.

The left-hand side of Table 2 present the fit of the MA with respect to the Argentine economy in terms of targeted moments. It also reports the result of the CEA and the MAND. As one can see, the MA replicates relatively well the main features of the Argentine economy in terms of consumption, investment, spreads, defaults and indebtedness.

Table 2: Targeted and Non-Targeted Moments

A. Targeted Moments				
Variable	Argentina	MA	MAND	CEA
i/y	14.50	15.23	28.56	21.78
b/y	43.70	47.97	11.85	54.31
Spread	8.92	9.24	-1.53	-1.50
Default rate	3.88	3.30	0	0
B. Non-Targeted Moments				
Variable	Argentina	MA	MAND	CEA
c/y	75.80	102.80	46.36	52.82
tb/y	1.60	-2.82	53.64	25.41
$\sigma(c)/\sigma(y)$	1.16	1.44	11.23	0.66
$\sigma(i)/\sigma(y)$	3.09	0.24	0.19	2.13
$\sigma(tb/y)/\sigma(y)$	1.03	5.08	4.23	0.85
$\sigma(spread)/\sigma(y)$	5.02	0.78	0.00	0.01
$\rho(c, y)$	0.91	0.04	0.94	0.26
$\rho(i, y)$	0.89	0.94	0.11	0.48
$\rho(tb/y, y)$	-0.53	0.53	-0.93	0.52
$\rho(spread, y)$	-0.68	0.38	-0.49	-0.08

Note: For the volatilities and correlation statistics, I filter the simulated data through the HP filter with a smoothness parameter of 1600.

The right hand-side of Table 2 present the fit of the MA in terms of non-targeted moments. In general the MA is related to a too large (low) volatility of consumption (investment).

The trade balance in the MA admits a deficit and is pro-cyclical unlike the data. The model also cannot match the volatility of spread observed in the data. It however generates an empirically plausible correlation between investment and output and consumption volatility.

Table 3: Debt Structure

	Mean b (percent of y)	Mean b in g_H (percent of y)	Mean b in default (percent of y)	Mean b in restructuring (percent of y)
Argentina	43.7	40.2	51.8	51.0
MA	48.0	46.5	62.4	26.7
MAND	11.9	13.0	-	-
CEA	54.3	54.3	-	-
	Mean b_{st}/b (percent)	Mean b_{st}/b in g_H (percent)	Mean b_{st}/b in default (percent)	Mean b_{st}/b in restructuring (percent)
Argentina	18.9	18.2	20.2	14.8
MA	44.8	41.8	46.3	47.1
MAND	90.1	91.3	-	-
CEA	66.5	66.5	-	-
	$\rho(b_{st}, y)$	$\rho(b_{st}, spread)$	$\rho(b_{lt}, y)$	$\rho(b_{lt}, spread)$
Argentina	-0.04	0.30	0.03	-0.01
MA	-0.69	0.06	-0.55	-0.13
MAND	-0.98	0.00	0.00	0.00
CEA	-0.40	0.03	0.40	-0.19

The CEA achieves better risk-sharing than the MA. Consumption corresponds to a lower share of output and is less volatile. Investment corresponds to a lower share of output, correlates less with output and is more volatile. Finally, the bond spread is negative given that defaults do not arise on equilibrium path and *official* buybacks exceed the risk-free rate. The same holds true for the MAND with the exception of a greater (lower) consumption (investment) volatility than in the MA.

Table 3 depicts the underlying debt structure of the Markov equilibrium and the CEA. Two points deserve to be noted. First, the MA replicates well the data as maturity shortens during debt crises, while indebtedness relative to GDP increases. Second, during a restructuring, the maturity lengthens and the level of debt does not substantially decreases. In opposition, in both the CEA and the MAND, maturity shortens or remains stable in the high growth state. Nonetheless, while in the CEA, indebtedness decreases when g_H, realizes, the opposite holds true in the MAND.

6.3 Policy functions

In this section, I present the different policy functions relevant for my analysis. I start with the CEA and then shift to the MA and MAND. The main focus is to see how debt evolves between the different equilibria.

Figure [2](#) depicts the main policy functions related to the the optimal contract. The law of motion of the relative Pareto weight is consistent with the fact that $\eta g_L^{-\sigma} < 1$. In that logic, the steady state of the contract is not degenerate and is located around $x = 0.22$ and $x = 0.3$. Besides this, capital is distorted for low values of x. However, as soon as x is sufficiently high the productivity-maximizing level is reached. Nevertheless, such values of x are outside the steady state and therefore capital remains distorted in the long-run. Regarding borrowing, when growth is low, the borrower accumulates a substantial amount of debt. Moreover, short-term debt holdings reduce when x decreases, while the opposite is true for the long-term debt.

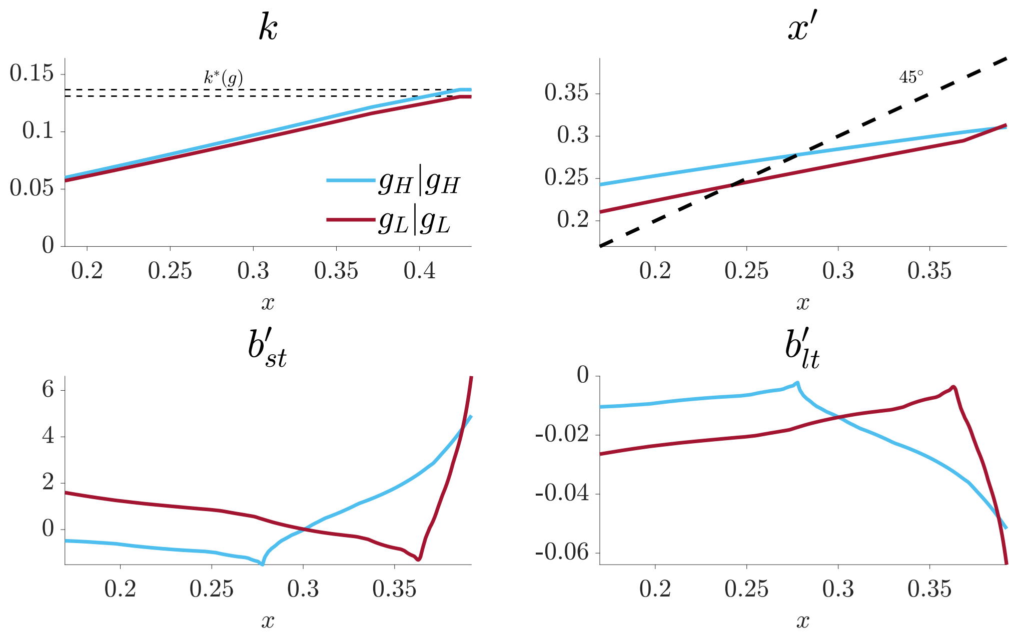
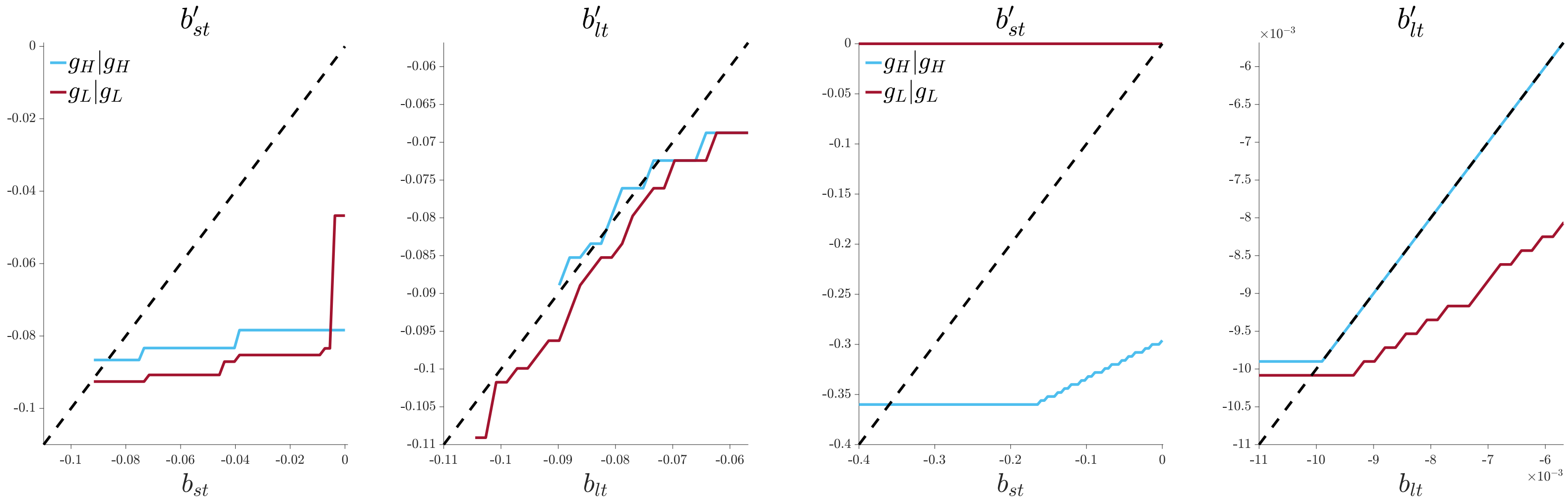


Figure 2: Policy and Value Functions of the CEA

I can now compare the bond policy of the CEA with the ones related to the Markov equilibria. Figure [3](#) depict the bond policy functions for the MA and the MAND. Starting with the MA, the borrower increases its indebtedness in the low growth state as in the CEA. However, it increases both its holdings of short-term and long-term debt. One can even say that the later increases more than the former consistent with the empirical evidence on developing economies. In the high growth state, short term debt continues to rise up to a certain point, while the long-term debt generally decreases.

Turing to the MAND, one notices the borrowing limit which takes the form of an horizontal line. This indicates that the borrower cannot borrow above a certain level. When growth is low, the borrower issues more long-term debt and does not change its holdings of short-term debt. However, when growth is high, the buyback program forces the borrower



(a) MA (b) MAND

Figure 3: Bond Policies of the Markov Equilibria

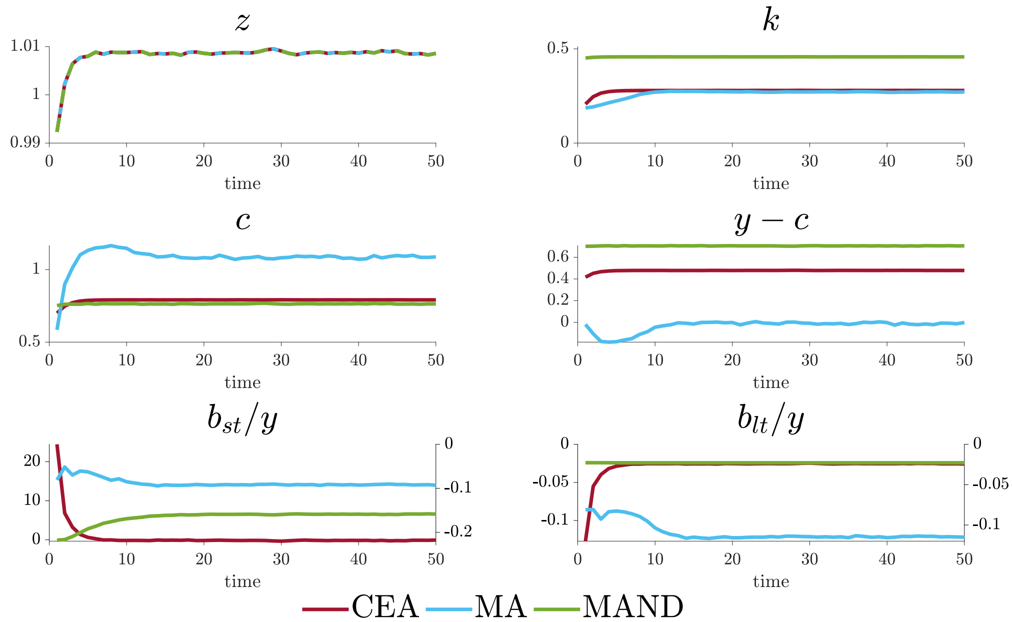
to maintain its holdings of long-term debt unchanged. As a result, the borrower adapts its short-term debt position. Note that the level of long-term debt is lower in the MAND than in the MA due to the costly buyback program.

6.4 Equilibria comparison

In this subsection, I explore in more details the differences between the Markov equilibria and the CEA. For this purpose, I conduct three main exercises. First, I construct impulse response functions following a stark negative shock in the economy. Second, I look at the dynamic of a specific shock path in both the Markov equilibria and the CEA. Finally, I compute welfare gains with respect to the MA.

Figure 4 depicts the impulse response functions resulting from a stark negative shock on selected key variables. The responses are computed as the mean of 10,000 independent shock histories starting with the lowest shock as well as initial debt holdings and relative Pareto weights drawn from the ergodic set. The blue line represents the MA, the green line the MAND and the red line the CEA. Notice that the relevant axis for the debt figures are on the right-hand sides for the Markov equilibria and the left-hand side for the CEA.

We see that at the outbreak of the shock's realization, capital is distorted in both the MA and the CEA – albeit to a lesser extent in the latter – but not in the MAND. Consumption drops as some economies in the MA repudiate their debt obligations. The hump-shaped pattern of consumption observed in the MA is due to the fact debt restructuring and the underlying regain of access to the market. Moreover, the level of indebtedness on impact



Note: For the debt figures, the relevant axis are on the right-hand sides for the MA and the MAND and the left-hand side for the CEA.

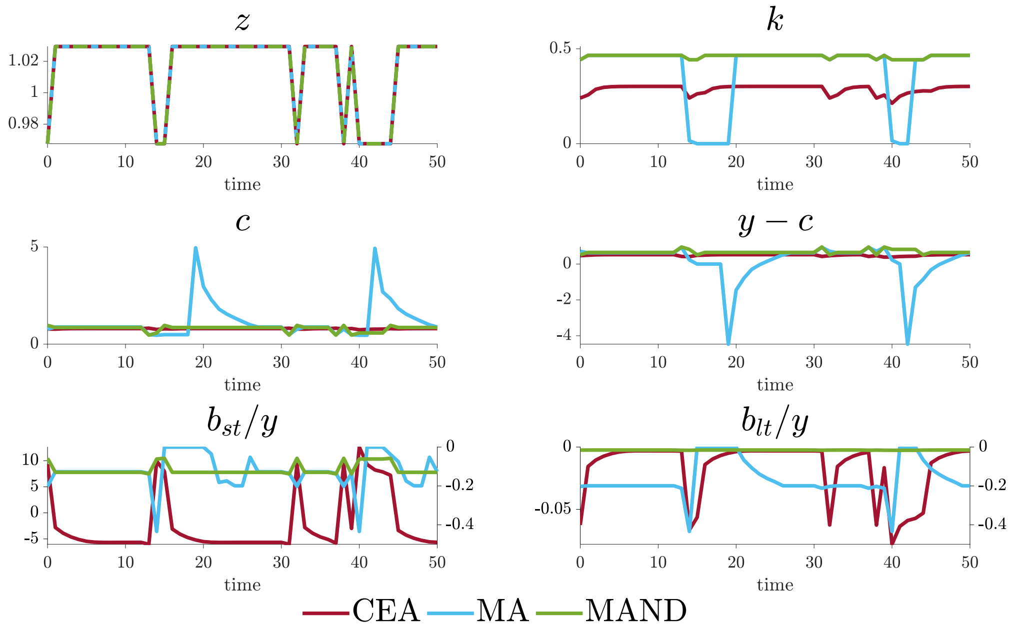
Figure 4: Impulse Response Functions to a Negative g Shock

in the MA reduces as many economies fall in default. This is however not the case for the CEA and the MAND which can both increase the indebtedness. Maturity shortens at the outbreak of the bad shock's realization in the MA as well as in the MAND and lengthens the CEA. The Markov equilibria rely mostly on short-term debt, while the CEA use both in opposite directions.

The impulse response functions give an idea of the long-run dynamic of the economy. However, it does not tell how the economy reacts in the short run especially when there is a transition between two values of g. Thus, I simulate the economy and generate one history of shocks for 400 periods. To avoid that the initial conditions blur the results, the first 350 periods are discarded. Again, the blue line represents the MA, the green line the MAND and the red line the CEA. Moreover, the relevant axis for the debt figures are on the right-hand sides for the Markov equilibria and the left-hand side for the CEA.

Figure 5 depicts the simulation results. One observes that, in the MA, the economy defaults in the transition from g_H to g_L. This causes market exclusion and therefore $k = 0$. Consumption largely drops and jumps once the borrower can re-access to the market. The volatility of consumption is therefore very high. In opposition, there are no defaults in the CEA and the MAND. In the transition from g_H to g_L, the borrower adapts the maturity of the debt and increases its indebtedness. Especially, one sees that the level of short-term debt follow the same movement in the MAND and the CEA – albeit with different magnitudes.

The movements in debt holdings are the most pronounced for the CEA and are substantially different than the movements implied by the MA.



Note: For the debt figures, the relevant axis are on the right-hand sides for the MA and the MAND and the left-hand side for the CEA.

Figure 5: Simulation of a Typical Path

Having identified the main difference between the MA, the MAND and the CEA, I can now conduct a welfare analysis. Table 4 presents the welfare gains of the CEA and the MAND in consumption equivalent terms with respect to the Markov equilibrium. Welfare gains are computed through the simulation of 10,000 independent shock histories starting with the lowest shock as well as initial debt holdings and relative Pareto weights drawn from the ergodic set.

Table 4: Welfare Analysis

Allocation	Welfare gains (percent)	Capital distortion (percent)	$\mathcal{F}$ (percent)
MA	–	41.4	14.7
MAND	7.5	0.0	75.6
CEA	20.1	39.1	100.0

As one can see, the CEA and the MAND imply substantial welfare gains compared to the MA. The losses come mainly from the occurrence of defaults. There capital is largely distorted compared to the CEA and the MAND. Setting capital to zero is extremely costly for both the lender and the borrower given the Inada conditions on the production function. In that logic, capital distortions are weaker in the CEA and inexistent in the MAND.

The CEA is superior than the MAND. It yields higher welfare gains despite some capital distortion. Moreover, if one looks at the metric $\mathscr{F}$ derived in Section 5, one sees that the MA is relatively far from the CEA and the MAND can get the economy closer to it. The metric $\mathscr{F}$ is important as it relates to the entire value of the debt contract and not only on the steady state unlike the welfare gains computed above.

6.5 Alternative implementation

In this subsection, I present an alternative way of implementing the optimal contract. Most notably, I follow, the approach of Alvarez and Jermann (2000) which relies on state-contingent securities and endogenous borrowing limits. Those limits ensure that the participation constraint is satisfied in equilibrium. They are however not sufficient to account for private information. Thus, so that the incentive compatibility constraint holds, I add taxes on the issuance of state-contingent assets as in Ábrahám et al. (2021).

Table 5: Alternative Implementation

	Benchmark	Alternative
b/y	54.31	35.97
Spread	-1.50	0.00
$\sigma(b/y)/\sigma(y)$	332.20	0.23
$\sigma(spread)/\sigma(y)$	0.01	0.00
$\rho(b/y, y)$	-0.41	0.87
$\rho(spread, y)$	-0.08	0.00

The main difference between the two implementations is that the benchmark case relies on changes in bond prices to mimic the state-contingency in the optimal contract, while the alternative case relies on changes in asset holdings provided that securities are state-contingent.

Appendix C present the details of the alternative implementation. I only highlight here the asset structure of the economy. At the start of a period, the borrower holds a perpetual debt a. The borrower can trade Z state contingent securities $a'(g')$ with a unit price of $q(g', a'(g')|g)$. The portfolio $a'(g')$ can be decomposed into a common bond $\bar{a}'$ that is independent of the next period state, traded at the implicit bond price $q(g, a') \equiv \sum_{g'|g} q(g', a'(g')|g)$, and an insurance portfolio of Z Arrow-type securities $\hat{a}'(g')$. Thus, $a'(g') = \bar{a}' + \hat{a}(g')$.

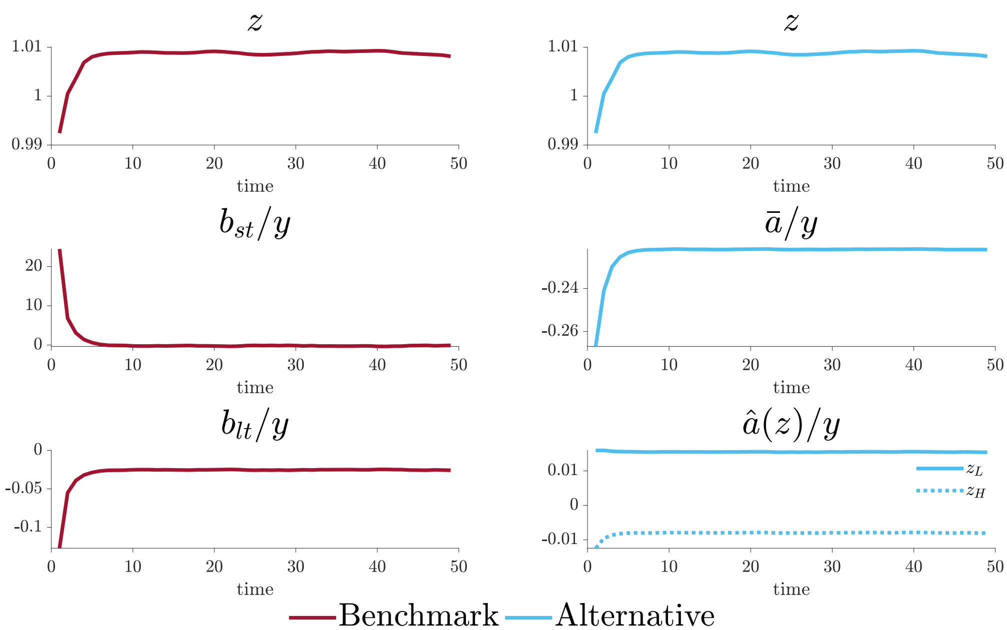


Figure 6: Impulse Response Functions to a Negative g Shock

Table 5 presents the main difference between the two implementations. The benchmark case is related to a larger level of indebtedness and a larger volatility of the debt ratio. This comes from the fact that bonds are non-contingent. Thus, large movements in debt holdings are necessary to replicate the state contingency in the contract. Besides this, the benchmark implementation displays negative spread owing to *official* buybacks, while the alternative case displays no spread at all. As explained before, the reason behind this is that the alternative implementation does not rely on changes in prices to mimic the state-contingency of the contract. Rather it relies on trade of state-contingent securities.

Similar to the previous section, I construct impulse responses to see how the two implementation work. Figure 6 depicts the responses in red for the benchmark implementation and in blue for the alternative one. The Arrow-like securities do most of the job in the alternative case, while the benchmark implementation needs to adapt both the long-term and the short-term debt at the same time.

Turning to the simulation, one can see that the level of short-term debt in the benchmark case closely follows the pattern of bonds in the alternative case. The magnitude of change in the former is nonetheless larger than in the latter. In terms of Arrow-like securities, $\hat{a}(g_L)$ closely follows the evolution of $\bar{a}$, while $\hat{a}(g_H)$ follows the opposite direction. The evolution of $\hat{a}(g_H)$ is therefore closely mimicking the evolution of b_{lt}. Given that $\hat{a}(g)$ is state contingent, the alternative implementation needs to change the debt portfolio with lower magnitude.

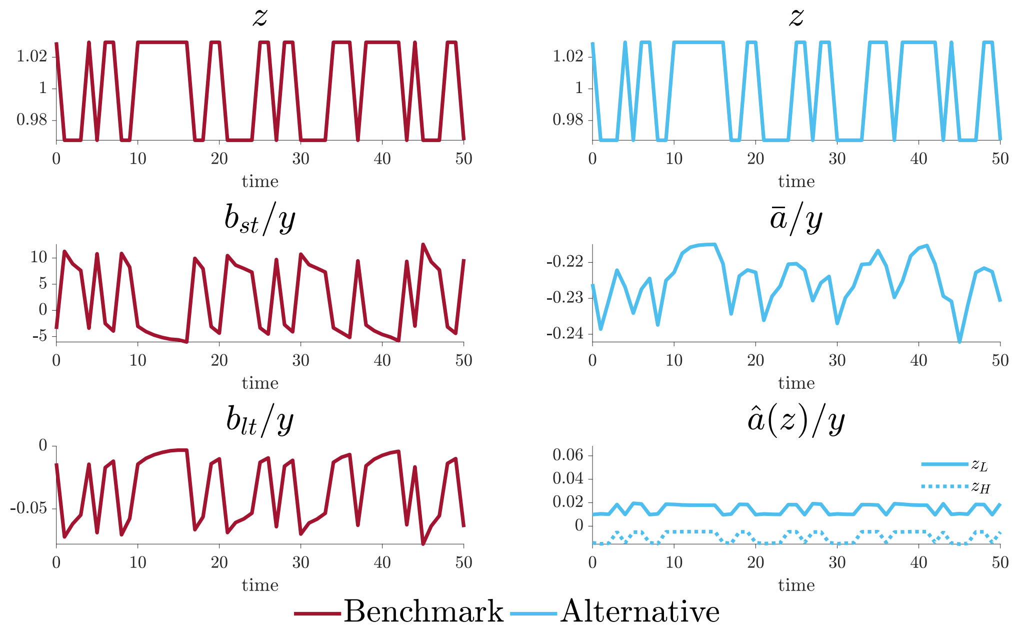


Figure 7: Simulation of a Typical Path

7 Conclusion

With complete markets, sovereign countries can fully diversify idiosyncratic risk. However, factual economies are often unable to perform such diversification. The literature on sovereign debt and default has therefore introduced incomplete markets to limit the extent of risk sharing. Assuming out incomplete markets is problematic, though. As the source of market frictions is not identified, potential policy analyses remain limited if not inconclusive.

This paper rationalizes the limitations in state contingency of financial contracts through impatience and limited enforcements. It derives the constrained efficient allocation emanating from an optimal contract to deduce the optimal sovereign debt management policy. The bottom line is that the reliance on debt repudiation on equilibrium path is inefficient. Instead, changes in maturity and debt buyback can implement the constrained efficient allocation. The implementation nevertheless requires highly sophisticated agents capable of building on past history. I show that less sophisticated agents would in fact rely on Markov strategies. Given this, I derive history-invariant debt management policies from the optimal contract and assess their efficiency.

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Appendix

A Price in Markov Equilibrium

The price of one unit of bond of maturity j for all $j \in \{st, lt\}$ is given by

$$q_j(g, p', k', b'_{st}, b'_{lt}) = \mathbb{E}_{g'|g}\Big[(1 - D(\Omega'))q_j^P(g', p', k'b'_{st}, b'_{lt}) + D(\Omega')q_j^D(g', p', k'b'_{st}, b'_{lt})\Big], \quad \text{(A.1)}$$

where recovery value given by

$$q_j^D(g, p', k', b'_{st}, b'_{lt}) = \frac{1}{1+r}\mathbb{E}_{g'|g}\left[(1-\lambda)q_j^D(g', p', k', b'_{st}, b'_{lt}) + \lambda\frac{W\frac{1+r}{1+r-\mathbb{I}_{j=lt}}}{b'_j}\right],$$

where $\mathbb{I}_{j=lt}$ is an indicator function taking value one if $j = lt$ and zero otherwise. In case of repayment, the price depends on the maturity, the repayment growth and the buyback decision.

$$q_{st}^P(g, p', k', b'_{st}, b'_{lt}) = \frac{1}{1+r},$$

$$q_{lt}^P(g, p', k', b'_{st}, b'_{lt}) = \frac{1}{1+r}\mathbb{E}_{g'|g}\Big[1 + (1 - M(\Omega'))q_{lt}(g', p', k', b''_{st}, b''_{lt}) + M(\Omega')q_{lt}^{bb}\Big].$$

Given this, a Markov equilibrium can be defined as

Definition A.2 (Markov Perfect Equilibrium)**.** *In this environment, a Markov equilibrium consists of a set of prices, $\{p' = p(g', b'_{st}, b'_{lt}), q_{st}(g, p', k', b'_{st}, b'_{lt}), q_{lt}(g, p', k', b'_{st}, b'_{lt})\}$, a set of policy functions $\mathcal{G}(\Omega) = \{D(\Omega), H(\Omega), M(\Omega), b_{st}(\Omega), b_{lt}(\Omega), \tau(\Omega)\}$ such that, at every possible state Ω,*

1. *Taking p, q_{st} and q_{st} as given, $\mathcal{G}(\Omega)$ solves the government's problem (5)-(8).*

2. *Taking p as given, the choice of capital by domestic firms is the solution to*

$$\max_{k'} \ \mathbb{E}_{g'|g}\left[\frac{u_c(c(\Omega'))}{\mathbb{E}_{g'|g}[u_c(c(\Omega'))]}(g'f(k') - p(g', b'_{st}, b'_{lt})k')\right].$$

3. *Taking $\mathcal{G}(\Omega)$ as given, the price of capital is consistent with*

$$\max_{k'} \ \mathbb{E}_{g'|g}\Big[p(g', b'_{st}, b'_{lt})(1 - \tau(\Omega'))k' - k'\Big].$$

4. *The price of bonds satisfy (A.1)*

B Further Characterization

The following lemma derives the inverse Euler Equation which gives the consumption dynamic in the contract.

Lemma B.1 (Inverse Euler Equation)**.** *Under Assumptions 1-3, the inverse Euler equation for a given $g_i \in G$ reads*

$$\mathbb{E}_{g'|g}\Big[\frac{\pi(g'|g)}{u_c(c(g'))(\pi(g'|g)+\nu(g'))}\Big] = \eta(1+\nu(g))\left[\frac{\pi(g|g_-)}{u_c(c(g))(\pi(g|g_-)+\nu(g))}\right].$$

Proof. See Appendix D □

If the participation constraint of the borrower ever binds, I obtain that for all (g, x),

$$\frac{1}{u_c(c(g))} \geq \mathbb{E}_{g'|g}\left[\frac{1}{u_c(c(g'))}\right]$$

with strict inequality when $\eta g_l^{-\sigma} < 1$. In this case, the inverse Euler Equation is a positive supermartingale. Immiseration is a consequence of the theorem stating that supermartingale converge almost surely to $-\infty$. Alternatively, when $\eta g_l^{-\sigma} = 1$, consumption remains constant. Under limited commitment of the borrower (i.e. $\nu(g) \geq 0$), one obtains a left bounded positive submartingale. The borrower's participation constraints therefore sets an upper bound on the supermartingale and prevents immiseration.

C Alternative Implementation

In what follows, I propose an alternative implementation as the one derived in the main text. More precisely, I implement the approach of Alvarez and Jermann (2000) using trade in state-contingent assets and endogenous borrowing limits.

The capital market is the same as in the main text. The lender provides k at price p and the borrower can decide to tax the repayment of capital to the lender at rate τ.

The structure of the financial market is the following. At the start of a period, the borrower holds a perpetual debt a. The borrower can trade Z state contingent securities $a'(g')$ with a unit price of $q(g', a'(g')|g)$. The portfolio $a'(g')$ can be decomposed into a common bond $\bar{a}'$ that is independent of the next period state, traded at the implicit bond price $q(g, a') \equiv \sum_{g'|g} q(g', a'(g')|g)$, and an insurance portfolio of Z Arrow-type securities

$\hat{a}'(g')$. Thus we have that $a'(g') = \bar{a}' + \hat{a}(g')$ with

$$\bar{a}' = \frac{\sum_{g'|g} q(g', a'(g')|g) a'(g')}{q(g, a')} \quad \text{and} \quad \sum_{g'|g} q(g', a'(g')|g)\hat{a}'(g') = 0.$$

The last equation represents the market clearing condition of the Arrow-type securities. The Borrower's problem therefore reads

$$W^b(g, k, a) = \max_{\{c, k', \{a'(g')\}_{g' \in g}\}} u(c) + \beta g^{1-\sigma} \mathbb{E}_{g'|g} \left[W^b(g', k', a'(g')) \right] \tag{C.1}$$

$$\text{s.t. } c + \sum_{g'|g} q(g', a'(g')|g)(ga'(g') - a) \leq gf(k) - p(1-\tau)k + a$$

$$\bar{a}' + \hat{a}(g') \geq \mathcal{A}(g', k'), \tag{C.2}$$

where $\mathcal{A}(g', k')$ represents the endogenous borrowing limit and is defined such that

$$W(g', \mathcal{A}(g', k')) = u(gf(k')) + \beta g^{1-\sigma} \mathbb{E}_{g''|g'} V^D(g''). \tag{C.3}$$

The lender's problem is static. I nonetheless express it in recursive form.

$$W^l(g, k_l, a_l) = \max_{\{c_l, k_l', \{a_l'(g')\}_{g' \in g}\}} c_l + \frac{1}{1+r} g \mathbb{E}_{g'|g} \left[W^l(g', k', a_l'(g')) \right] \tag{C.4}$$

$$\text{s.t. } c_l + \sum_{g'|g} q(g', a_l'(g')|g)(ga_l'(g') - a_l) \leq p(1-\tau)k_l - k_l + a_l.$$

Given this environment, I can determine a recursive competitive equilibrium in the following terms.

Definition C.3 (Recursive Competitive Equilibrium (RCE)). *A* recursive competitive equilibrium (RCE) *is a sequence of prices $q(g', a'(g')|g)$, value functions, $W^b(g, k, a)$ and $W^l(g, k, a)$, endogenous borrowing limits, $\mathcal{A}(g', k')$, as well as policy functions for (i) consumption, $c(g, k, a)$ and $c(g, k, a)$, (ii) capital, $k' = k(g, a)$ and $k_l' = k_l(g, a)$ as well as (iii) asset holdings $a'(g') = A(g', g, k, a)$ and $a_l'(g') = A_l(g', g, k, a)$ such that,*

1. *Given value functions for the outside value options of the borrower, $u(gf(k)) + \beta \mathbb{E}_{g'|g} V^D(g')$ as well as asset prices $q(g', a'(g')|g)$,*

    (a) the policy functions $c(g, k, a)$ and $A(g', g, a)$, together with the value function $W^b(g, k, a)$, solve the borrower problem (C.1) with the endogenous limit, $\mathcal{A}(g')$.

    (b) the policy functions $c_l(g, k_l, a_l)$ and $A_l(g', g, k_l, a_l)$, together with the value function $W^l(g, k_l, a_l)$, solve the lender's problem (C.4) and

2. The asset market clears, $a'(g') + a_l'(g') = 0$ for all $g' \in Z$.

3. The product and capital markets clear, $c(g,k,a) + c_l(g,k_l,a_l) = gf(k)$ with $k = k_l$.

For the borrower's problem, taking the first-order conditions with respect to consumption, capital and assets, one obtains

$$\begin{aligned} u_c(c) &= \mu_{BC}^l(g,k,a), \\ 0 &= \beta \mathbb{E}_{g'|g} u_c(c')(gf_k(k') - p(1-\tau)), \\ q(g',a(g')|g) &= \beta g^{1-\sigma}\pi(g'|g)g^{-\sigma}\frac{u_c(c')}{u_c(c)}[1 + \sum_{g''|g} q(g'',a''(g'')|g')] + \frac{\mu_{EBL}(k'a'(g'),g')}{u_c(c')}, \end{aligned}$$

where μ_{BC}^l and μ_{EBL} are the Lagrange multipliers attached to the budget constraint and the endogenous borrowing limit, respectively. Especially, $\mu_{EBL}(k',a'(g'),g') \geq 0$ with $\mu_{EBL}(k',a'(g'),g') = 0$ if $a'(g') > \mathcal{A}(g',k')$.

Conversely, taking the first-order conditions with respect to consumption, capital and assets of the lender's problem

$$\begin{aligned} 1 &= \mu_{BC}^l(g,k_l,a_l), \\ 1 &= p(1-\tau), \\ q(g',a(g')|g) &= \frac{1}{1+r}\pi(g'|g)(1 + \sum_{g''|g} q(g'',a''(g'')|g')). \end{aligned}$$

Following Krueger et al. (2008), the price is determined by the agent whose constraint is not binding. Therefore the price is determined by

$$q(g',a(g')|g) = \pi(g'|g)(1 + \sum_{g''|g} q(g'',a''(g'')|g')) \max\left\{\beta g^{-\sigma}\frac{u_c(c(g',k',a'(g'))}{u_c(c(g,k,a))}, \frac{1}{1+r}\right\}. \quad \text{(C.5)}$$

I now show how the constrained efficient allocation can be implemented as a RCE with state contingent-assets and taxes as well as endogenous borrowing limits.

Proposition C.1 (Alternative Implementation)**.** *Given initial conditions $\{g_0, x_0\}$, a constrained efficient allocation can be implemented as a competitive equilibrium with state contingent-assets and taxes as well as endogenous borrowing limits.*

Proof. Following Alvarez and Jermann (2000) we prove the proposition by construction.

First, define the asset price as

$$q(g', x'|g) = \frac{\pi(g'|g)}{1+r}\left[1 + \sum_{g''|g'} q(g'', x''|g')\right] \max\left\{g^{-\sigma}\frac{u'(c(g', x', b'))}{u'(c(g, x, b))}\eta, 1\right\}.$$

Second, iterating over the budget constraint of the borrower and applying the transversality condition gives

$$a(g^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(g^{t+j}, x(g^{t+j}), \tilde{x}(g^{t+j})|g^t)[c(g^{t+j}, x(g^{t+j})) \tag{C.6}$$

$$- Y(g^{t+j}, x(g^{t+j}))], \tag{C.7}$$

where, $Y(g^t, x(g^t)) = g_t f(k(g^{t-1}, x(g^t))) - k(g^{t-1}, x(g^t))$ for all t and g^t. Similarly, iterating over the budget constraint of the lender leads to

$$a_l(g^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(g^{t+j}, x(g^{t+j})|g^t) c_l(g^{t+j}, x(g^{t+j})) \tag{C.8}$$

$$= \mathbb{E}_t \sum_{j=0}^{\infty} Q(g^{t+j}, x(g^{t+j}), \tilde{x}(g^{t+j})|g^t)[Y(g^{t+j}, x(g^{t+j}))$$

$$- c(g^{t+j}, x(g^{t+j}))]$$

$$= - a(g^t).$$

The market clearing condition implies that $a_l(g^t) + a(g^t) = 0$ for all t and g^t.

To ensure that the capital level is the same as in the constrained efficient allocation, I set the capital tax rate and the level of intermediate price according to

$$\mathbb{E}_{g|g_-} u_c(c(g_-, g, x)) = \frac{1}{1 - \tau(g_-, k_-, a_-)} = p(g_-, k_-, a_-).$$

I now need to establish the correspondence between the initial conditions, x_0, in the Fund contract and the initial conditions in the recursive competitive equilibrium, $(a_0, a_{l,0})$. Given (C.6) and (C.8) evaluated at $t = 0$, one can determine $\bar{a}(g_0, k_0, a_0)$ using the budget constraint

$$c(g_0, a_0, b_0) + q(g_0, a_1)(\bar{a}' - a_0) + \sum_{g_1|g_0} q(g_1, a_1(g_1)|g_0)\hat{a}'(g_1) + \sum_{g_1|g_0} q(g_1, a_1(g_1)|g_0)a'(g_1)$$

$$\leq g_0 f(k) - p(1-\tau)k + a_0.$$

and the fact that $\sum_{g_1|g_0} q(g_1, a_1(g_1)|g_0)\hat{a}'(g_1) = 0$. Once, $\bar{a}(g_0, a_0, b_0)$ is determined, one can find the holdings of Arrow-type securities $\hat{a}'(g', g_0, a_0, b_0)$ for all $g' \in \Theta$. We can then retrieve the entire portfolio recursively for $t > 0$.

Third, define the endogenous borrowing limits such that

$$\mathcal{A}(g, k) = a(g, \underline{x}(g, b), b).$$

This definition implies that $a'(g', g, k, a) \geq \mathcal{A}(g', k')$. Hence, the constructed asset holdings satisfy the competitive equilibrium constraints.

Fourth, to ensure optimality of the policy functions by setting

$$\mu^b(g, k, a) = \frac{\pi(g|g_-)}{x(\pi(g|g_-) + \nu(g))}$$

Hence, since $c(g, x)$ satisfies the optimality conditions in the Planner's problem, it is also optimally determined in the competitive equilibrium. For the lenders, $c_l(g, x)$ is optimal if the asset portfolio is optimally determined. For this observe that

$$\begin{aligned} q(g', a'(g')|g) &= \frac{1}{1+r}\pi(g'|g)g^{-\sigma}\frac{u'(c(g', a'(g'), b'))}{u'(c(g, k, a))}\eta\left[1 + \sum_{g''|g'} q(g'', a''(g'')|g')\right] \\ &> \frac{1}{1+r}\pi(g'|g)\left[1 + \sum_{g''|g'} q(g'', a''(g'')|g')\right] \\ &\quad \text{if } a'(g', g, k, a) = \mathcal{A}(g', k'). \end{aligned}$$

Hence the portfolio is optimally determined. It then directly follows that $W^b(g, k, a) = V^b(g, x)$ and $W^l(g, k, a) = V^l(g, x)$.

We therefore obtain a one-to-one map between x and a for a given g. More precisely, $c(g, k, a) = c(g, x)$, $c_l(g, k, a) = T(g, x)$ and $k(g, k, a) = k(g, x)$. Moreover the endogenous limits binds uniquely and exclusively when the participation constraints of the borrower binds. $\square$

D Proofs

Proof of Lemma 1.

The value of permanent autarky is given by

$$v_a(g_t) = \sum_{j=t}^{\infty} \beta^{j-t} \sum_{g^j} \pi(g^j|g_t) u(g_j f(0)),$$

as the lender sets $k = 0$ in case of default. Permanent autarky is the worst equilibrium outcome as the borrower could always be better off with $k = \epsilon$ for small $\epsilon > 0$ given the Inada conditions on the production function.

Permanent autarky is an equilibrium of the market economy. Suppose that the lender believes that $\tau_t = 1$ as well as that $D_t = 1$ for all t. Then, it sets $k_t = 0$ and $q_{j,t} = 0$. Given this, the government finds optimal to choose $\tau_t = 1$ $D_t = 1$ for all t confirming the lender's beliefs. $\square$

Proof of Proposition 1.

I prove the proposition following Bhaskar et al. (2012). I first show that every equilibrium under Assumption 2 is essentially sequentially strict. I then prove that every essentially sequentially strict equilibrium is a Markov (perfect) equilibrium.

I start the proof with some definitions. Given the information structure, I split the histories into two categories: public and private. Public histories are the ones defined in Section 3 – that is h_b^t and h_l^t. Private histories of the borrower and the lender at time t are the ones tracking the utility shocks – that is $p_b^t = (p_b^{t-1}, \varrho_{b,t})$ and $p_l^t = (p_l^{t-1}, \varrho_{l,t})$, respectively. Finally, I denote he entire history of the play including the privately observed utility shocks by $\hat{h}^t$.

In addition, I denote σ_b and σ_l as the strategy profile of the borrower and the lender, respectively. Besides this, A_i corresponds to the countable set of actions with typical element a_i for market participant $i \in \{b, l\}$. For instance, actions taken by the borrower relate to borrowing, defaults, buybacks and taxation. Moreover, $W^b(\sigma_b, \sigma_l|h_b^t, p_b^t)$ and $W^l(\sigma_b, \sigma_l|h_l^t, p_l^t)$ represent respectively the value of the borrower and the lender from the strategy profile (σ_b, σ_l) at the relevant histories.

Given that each market participant has some private information regarding their payoff, they need to form beliefs about the unobserved utility shock of the other participants. Denote the belief of agent $i \in \{b, l\}$ over the entire history $\hat{h}^t$ as $\omega_i^{(h_i^t, p_i^t)}$. I follow Bhaskar et al. (2012) and put the least structure possible on such beliefs. They simply need to be independent of the private payoff shocks and put zero weight to events that history $\hat{h}^t$ is inconsistent with h_i^t. With this, I define the equilibrium concept as

Definition D.4 (Sequential Best Response)**.** *A strategy σ_i is a sequential best response to (σ_{-i}, ω_i), if for each history (h_i^t, p_i^t) and each alternative strategy $\tilde{\sigma}_i$*

$$\int W^i(\sigma_b, \sigma_l|\hat{h}^t) d\omega_i^{(h_i^t)}(\hat{h}^t) \geq \int W^i(\tilde{\sigma}_b, \sigma_l|h_b^t) d\omega_i^{(h_i^t)}(\hat{h}^t).$$

Strategy σ_i is a sequential best response to σ_{-i} if strategy σ_i is a sequential best response (σ_{-i}, ω_i) for some ω_i.

Given the information structure, there is no general solution concept which can be used here. That is why, Bhaskar et al. (2012) appeal to the very weak concept of sequential optimality. Nonetheless, a profile of mutual sequential best response for the borrower and the lender represents a perfect Bayesian equilibrium.

The other concept defined by the aforementioned authors is the current shock strategy which relies at most on the current value of the private shock. Formally

Definition D.5 (Current Shock Strategy)**.** *A strategy σ_i is a current shock strategy, if for any public history (h_i^t, p_i^t) and for any two histories, p_i^t and $\tilde{p}_i^t$, both finishing with the same ϱ_i, then for almost all ϱ_i*

$$\sigma_i(h_i^t, p_i^t) = \sigma(h_i^t, \tilde{p}_i^t).$$

The next lemma links Definitions D.4 and D.5. In words, any sequential response relies at most on the current value of the private shock. As a result the history of past private shocks becomes irrelevant.

Lemma D.2 (Sequential Strictness and Current Shock Strategy)**.** *If σ_i is a sequential best response to σ_{-i}, then σ_i is a current shock strategy.*

Proof. Consider a market participant i with history (h_i^t, p_i^t). The expected continuation value from choosing a certain action a_i under the strategy profile σ is given by

$$W^i(a_i, \sigma_{-i}, \omega_i|h_i^t, p_i^t) = \mathbb{E}_{g'|g} \int \int \max_{\sigma_i} W^i(\sigma_i, \sigma_{-i}|a_i, g', \varrho_i', \hat{h}^t) d\mu_{P_i}(\varrho_i') d\omega_i^{(h_i^t)}(\hat{h}^t).$$

Since σ_{-i} and $\omega_i^{(h_i^t, p_i^t)}$ do not depend on the private history, the value $W^i(a, \sigma_{-i}, \omega_i|h_i^t, p_i^t)$ is also independent of private history. Furthermore, since the density of ϱ_i is absolutely continuous, the market participant i can only be indifferent between two actions on a zero measure of the support. For different values of ϱ_i, the action is unique and independent of the past values of the shock. $\square$

Given that beliefs on the history of past private shock do not matter, I can suppress the dependence on the beliefs and the private shock realization in the value function. Thus, the expected continuation value from choosing a certain action a_i under the strategy profile σ is given by

$$W^i(a_i, \sigma_{-i}|h_i^t) = \int \mathbb{E}_{g'|g} \max_{\sigma_i} W^i(\sigma_i, \sigma_{-i}|a_i, g', \varrho_i', h_i^t) d\mu_{P_i}(\varrho_i').$$

I then arrive to the first step of the proof. Given that beliefs over private histories are irrelevant for optimality, every perfect Bayesian equilibrium (i.e. a profile of mutual sequential best responses) satisfying Assumption 2 are essentially sequentially strict.

Lemma D.3 (Sequential Best Response and Perfect Bayesian Equilibrium)**.** *Every perfect Bayesian equilibrium satisfying Assumption 2 is essentially sequentially strict.*

Proof. I need to show that for any period, history and for almost all values of the private shock, the optimal action is unique. I consider the case of the borrower first. The borrower's value from action a_b after the realization of ϱ_b is given by

$$W^b(a_b, \varrho_b, \sigma_b|h_b^t) = u(a_b, g) + \epsilon \varrho_b^a + \beta g^{1-\sigma} \mathbb{E}_{g'|g} W^i(\sigma_b|a_b, g', h_b^t).$$

Suppose two actions a_b and $\tilde{a}_b$, the equality $W^b(a_b, \varrho_b, \sigma_b|h_b^t) = W^b(\tilde{a}_b, \varrho_b, \sigma_b|h_b^t)$ implies that

$$\epsilon(\varrho_b^{a_b} - \varrho_b^{\tilde{a}-b}) = u(a_b, g) - u(\tilde{a}_b, g) + \beta g^{1-\sigma} \mathbb{E}_{g'|g} \left[W^b(\sigma_b|a_b, g', h_b^t) - W^b(\sigma_b|\tilde{a}_b, g', h_b^t) \right].$$

The set of actions is countable, whereas the set of values of private shocks for which a market participant can be indifferent has measure zero. Hence, for almost all values of ϱ_i, the set of maximizing actions must be a singleton, and the profile is essentially sequentially strict. The proof naturally extends to the case of the lender and is therefore omitted. □

Now that we have that all equilibria satisfying Assumption 2 are essentially sequentially strict, I simply need to show that sequentially strict equilibria are Markov equilibria.

Lemma D.4 (Sequential Strictness and Markov Equilibrium)**.** *Every essentially sequentially strict perfect Bayesian equilibrium is a Markov perfect equilibrium.*

Proof. Consider an a t period history h^t. As shown previously, the private history matters, so the focus is on public history. Under Assumption 2, the borrower's behavior will not depend on h^t anymore from $t + T + 1$ periods onward given that its memory is bounded to T periods back. This means that the lender's value will not depend on h^t from $t + T + 1$

periods. As a result, if the lender strategy is sequentially strict, then h^t becomes irrelevant from $t+T+1$ periods.

What happens in period $t+T$? This represents the last period in which strategies could be conditioned on h^t. However, at that time, the borrower's maximization problem is independent of h^t. In addition, sequential strictness implies that the maximizing action is a singleton. Applying this argument recursively completes the proof. $\square$

I have therefore shown that, under the assumption of bounded memory of the borrower, small perturbations in the payoff of the market participants unravel all equilibria except Markov ones. $\square$

Proof of Proposition 2.

From the first-order conditions on consumption, c is increasing in x'. Hence, so does the value of the borrower. In opposition, with a greater c, the value of the lender decreases.

One further shows that the autarkic allocation is not optimal. The proof follows Aguiar et al. (2009). Consider a version of the optimal contract in which the outside option corresponds to the value of permanent autarky is given by

$$v_a(g_t) = \sum_{j=t}^{\infty} \beta^{j-t} \sum_{g^j} \pi(g^j|g_t) u(g_j f(0)).$$

In autarky, $k=0$ and $u(c(g,\underline{\underline{x}})) = u(gf(0))$ for all g. Consider that one increases h by Δh and $u(c(g,\underline{\underline{x}}))$ by $\theta u_c(gf(0))\Delta h$ where

$$\theta = \frac{u_c(g_L f(0))}{u_c(g_L f(0)) + \frac{\beta}{1-\beta}\mathbb{E}u_c(gf(0))} < 1.$$

One defined θ such that the borrower's participation constraint holds. To see this, note that the increase of h increases the borrower's outside option by $u_c(gf(0))\Delta h$ as it can benefit from the additional level of capital before going to autarky forever. However, if the sovereign does not choose autarky, its value increases by $\theta(u_c(gf(0)) + \frac{\beta}{1-\beta}\mathbb{E}u_c(gf(0)))\Delta h \geq u_c(gf(0))\Delta h$ by definition of θ. Hence the borrower's participation constraint is satisfied. Furthermore, the value of the lender changes by

$$\Delta h \frac{1}{1+r}\left(1 - \mathbb{E}\left[\frac{u_c(gf(0))}{u_c(c(g,\underline{\underline{x}}))}\right]\theta\right) = \Delta h \frac{1}{1+r}(1-\theta) > 0.$$

As a result, the autarkic allocation is not optimal. Notice as well that the law of motion of the relative Pareto weight is given by

$$x'(g) = (1 + \nu(g))\eta g_{-}^{-\sigma} x.$$

As one can see, the only source of immiseration for x' is the borrower's relative impatience. The participation constraint can only increase x' over time. Hence, any continuation of an efficient allocation is itself efficient. In other words, what I showed above is that there exits a region of *ex post* inefficiencies in the vicinity of v_a in which the value of both the lender and the sovereign can be increased. However, from the law of motion (13), the binding constraint of the borrower only increases the value of the relative Pareto weight. This together with the fact that the initial value of the contract is efficient ensures that the region of inefficiencies is never attained. □

Proof of Proposition 3.

- Part I

  The optimal level of capital is given by

  $$\sum_g \pi(g|g_-)gf_k(k) - 1 = \sum_g \pi(g|g_-)\nu(g)u_c(gf(k))gf_k(k)x.$$

  As one can see, as soon as the participation constraint does not bind (i.e. $\nu(g) = 0$ for all g), the contract can attain the production-maximizing level of capital $k^*(g)$ such that $gf_k(k^*) = 1$. As soon as this condition is not met, $k < k^*(g)$. Thus, define x^* such that

  $$u(c(g_L, x^*) + \beta \mathbb{E}_{g'|g} V^b(g', x^*) = u(gf(k^*(g))) + \beta V^D(g).$$

  By the above definiton, if $x < x^*$, capital is distorted, while if $x \geq x^*$, capital is at the production-maximizing level.

  Observe that ν is the multiplier attached to the sovereign's participation constraint. Hence, when this constraint binds, $\nu > 0$, whereas $\nu = 0$ when it does not. In that logic, the larger is ν the more binding is the constraint.

  Assume by contradiction that for $x_1 < x_2$ one has that $k(g, x_1) \geq k(g, x_2)$ for all $g \in G$.

Using the first-order condition with respect to capital, one has

$$x = \frac{f_k(k) - 1}{\sum_g \pi(g|g_-)\nu(g)u_c(gf(k))f_k(k)}.$$

Given that $x_1 < x_2$,

$$\frac{f_k(k(g, x_1)) - 1}{\sum_g \pi(g|g_-)\nu(g)u_c(gf(k(g, x_1)))f_k(k(g, x_1))} < \frac{f_k(k(g, x_2)) - 1}{\sum_g \pi(g|g_-)\nu(g)u_c(gf(k(g, x_2)))f_k(k(g, x_1))}$$

With the assumption that $k(g, x_1) \geq k(g, x_2)$, the above inequality is satisfied only if $\nu(g, x_1) > \nu(g, x_2)$. This is a contradiction as a lower level of capital should relax the sovereign's participation constraint and not the opposite.

The fact that that $k(g, x) > 0$ for all (g, x) follows directly from Proposition 2 which shows that the autarkic allocation is not optimal.□

– Part II

The law of motion of the relative Pareto weight is given by $x'(g) = (1 + \nu(g))g_-^{-\sigma}\eta x$, while the first-order condition on consumption reads $u_c(c(g)) = \frac{\pi(g|g_-)}{x(\pi(g|g_-)+\nu(g))}$.

Given the first-order condition, $c^L(z_-, x) \leq c^H(z_-, x)$ only when $\nu(g_L) \leq \nu(g_H)$. Assume by contradiction that $\nu(g_L) > \nu(g_H)$. This implies that $c^L(z_-, x) > c^H(z_-, x)$ and $x^{L\prime}(z_-, x) > x^{H\prime}(z_-, x)$. Especially, consider the case in which $\nu(g_L) > \nu(g_H) = 0$. In this case,

$$u(c(g_H)) + \beta g_H^{1-\sigma} V^b(g', g_H, x^{H\prime}) > u(g_H f(k(g_-))) + \beta g_H^{1-\sigma} V^D(g_H),$$
$$u(c(g_L)) + \beta g_L^{1-\sigma} V^b(g', g_L, x^{L\prime}) = u(g_L f(k(g_-))) + \beta g_L^{1-\sigma} V^D(g_L).$$

Given that $g_H > g_L$, $u(g_H f(k(g_-))) > u(g_L f(k(g_-)))$ and $V^D(g_H) > V^D(g_L)$. For $\sigma \leq 1$, this implies that

$$u(c(g_H)) + \beta g_H^{1-\sigma} V^b(g', g_H, x^{H\prime}) > u(c(g_L)) + \beta g_L^{1-\sigma} V^b(g', g_L, x^{L\prime}).$$

which contradicts the fact that $c^L(z_-, x) > c^H(z_-, x)$ and $x^{L\prime}(z_-, x) > x^{H\prime}(z_-, x)$. Hence, $\nu(g_L) \leq \nu(g_H)$ which gives $c^L(z_-, x) \leq c^H(z_-, x)$ and $x^{L\prime}(z_-, x) \leq x^{H\prime}(z_-, x)$ as desired. □

– Part III

The value of liabilities in the optimal contract for all $i \in \{L, H\}$ is given by

$$b^i(g_-, x) \equiv g_i f(k(g_-)) - k(g_-) - c^i(g_-, x) + \frac{1}{1+r} g_i \mathbb{E}_{g'|g_i} V^l(g', g_i, x').$$

Assume by contradiction that for a given x it holds that $b^H(g_-, x) < b^L(g_-, x)$. For $x \geq x^*$, one directly reaches a contradiction as $c^L(z_-, x) = c^H(z_-, x)$ which implies that $b^H(g_-, x) > b^L(g_-, x)$. For $x < x^*$, the participation constraint of the borrower is binding in at least on growth state. That is there is a wedge between the consumption allocations and the borrower gets more in the high growth state. Nonetheless this wedge is only explained by the different incentive to default and does not exhaust the surplus difference between two states for the lender. □

Proof of Proposition 4.

Two assumptions are here crucial: $\beta(1+r)g^{-\sigma} \leq 1$ for all $g \in G$ ensures that the law of motion of the relative Pareto weight eventually crosses the 45° degree line and $\left(\frac{g_H}{g_L}\right)^{-\sigma} > \beta(1+r)$ ensures that the law of motion of the relative Pareto weight for g_L crosses the 45° degree line before the one for g_H. If the former assumption is not met, the relative Pareto weight never ceases to increase. Conversely, if the second assumption is not met, the high growth state is related to a lower relative Pareto weight in steady state than the low growth state. Now, recall the law of motion of the relative Pareto weight

$$x'(g) = (1 + \nu(g))g_-^{-\sigma}\eta x.$$

The motion of the relative Pareto weight is dictated by the relative impatience, $\eta g^{-\sigma}$, and the binding participation constraint, ν. I consider two cases. On the one hand, if $\eta g_L^{-\sigma} < 1$, the relative Pareto weight increases only if $\nu(g) > 0$ is sufficiently large to overcome impatience. As we know, when $x \geq x^*$, $\nu(g, x) = 0$ meaning that impatience eventually dominates the limited commitment issue. On the other hand, if $\eta g_L^{-\sigma} = 1$ immiseration due to impatience does not exist in the low growth state and the relative Pareto weight remains constant there. In the high growth state, however, $\eta g_H^{-\sigma} < 1$ and the previous argument applies.

Assumption 1 states that $\eta g^{-\sigma} \leq 1$ for all g. However since $g_H > 1 > g_L$, having $\eta g_H^{-\sigma} = 1$ would imply $\eta g_L^{-\sigma} > 1$ for $\sigma \in (0, 1]$. Hence, $\eta g_H^{-\sigma} < 1$, always. As a result, x^* is never reached in the steady state as when the high growth state realizes, $x^{H\prime}(g_-, x)$ decreases and $x^{L\prime}(g_-, x) \leq x^{H\prime}(g_-, x)$.

When $\eta g_L^{-\sigma} = 1$, the upper bound of the ergodic set coincides with the lower bound. As show in Proposition 3 $x^{L\prime}(g_-, x) \leq x^{H\prime}(g_-, x)$ meaning that the law of motion of the relative Pareto weight in the low growth state crosses the 45° line before the one of the high growth state. This coupled with the fact that $x^{L\prime}(g_-, x)$ lies on the 45° when $\nu(g_L, x) = 0$ leads to $x^{ub} = x^{lb}$. Conversely, when $\eta g_L^{-\sigma} < 1$, impatience immiserates the relative Pareto in the low growth state implying that $x^{ub} > x^{lb}$.

To show the existence of the ergodic set, one shows that the dynamic of the contract satisfies the conditions given by Stokey et al. (1989, Theorem 12.12). Set $\ddot{x}$ as the midpoint of $[x^{lb}, x^{ub}]$ and define the transition function $Q : [x^{lb}, x^{ub}] \times \mathcal{X}([x^{lb}, x^{ub}]) \to \mathbb{R}$ as

$$Q(x, G) = \sum_{g'|g} \pi(g'|g) \mathbb{I}\{x' \in G\}$$

One wants to show is that $\ddot{x}$ is a mixing point such that for $M \geq 1$ and $\epsilon > 0$ one has that $Q(x^{lb}, [x, x^{ub}])^M \geq \epsilon$ and $Q(x^{ub}, [x^{lb}, x])^M \geq \epsilon$. Starting at x^{ub}, for a sufficiently long but finite series of g_L, the relative Pareto weight transit to x^{lb} (either through impatience or because $x^{lb} = x^{ub}$). Hence for some $M < \infty$, $Q(x^{ub}, [x^{lb}, \ddot{x}])^M \geq \pi(g_L|g_L)^M > 0$. Moreover, starting at x^{lb}, after drawing $M < \infty$ g_H, the relative Pareto weight transit to x^{ub} (either through the binding constraint or because $x^{lb} = x^{ub}$) meaning that $Q(x^{lb}, [\ddot{x}, x^{ub}])^M \geq \pi(g_H|g_H)^M > 0$. Setting $\epsilon = \min\{\pi(g_L)^M, \pi(g_H)^M\}$ makes $\ddot{x}$ a mixing point and the above theorem applies. □

Proof of Lemma 2.

Necessity:

> The pricing equation, (2), as well as the capital choice and price conditions, (3) and (4) follow directly from the competitive equilibrium's definition. The budget constraints in the repayment and default states is required by feasibility. The participation constraint, (11), ensure that neither the lender nor the government has an incentive to break the contract and end up in permanent autarky.

Sufficiency:

> Let's rely on trigger strategy (Abreu, 1988). That is, each player is punished by the worst outcome of the game (i.e. permanent autarky which is an equilibrium as shown above) if he or she decides to deviate. Since the outcome satisfies (2), (3) and (4), it is optimal. Also as it satisfies the different budget constraints it is feasible. Finally, no deviations from play is profitable given that (11) holds.

□

Proof of Proposition 5.

The proof of this proposition is by construction. Similar to Dovis (2019), I express the policy functions of the implemented contract as a function of the relative Pareto weights, x and $\tilde{x}$, and the growth state, g. Formally, define

$$\begin{aligned}
&\bar{\tau}, \bar{p} : Z \times X \times \tilde{X} \to \mathbb{R}, \\
&\bar{D}, \bar{H}, \bar{M} : Z \times Z \times X \times \tilde{X} \to \{0, 1\}, \\
&\bar{q}_{st}, \bar{q}_{tl}, \bar{b}_{st}, \bar{b}_{lt} : Z \times Z \times X \times \tilde{X} \to \mathbb{R}.
\end{aligned}$$

Notice that the tax policy only depends on g_- and not g as I want to replicate the constrained efficient allocation through the maturity structure of the debt.

Given the timing of actions, the price schedules and bond policies depend on the prospective relative Pareto weights after the growth shock realizes. Those objects can therefore be rewritten as

$$\begin{aligned}
\bar{b}_j(g_-, g, x) &= \bar{b}_j(g_-, x'(g_-, g, x)), \\
\bar{q}_j(g_-, g, x) &= \bar{q}_j(g_-, x'(g_-, g, x)) \quad \text{for all } j \in \{st, lt\}.
\end{aligned}$$

I first determine the default and restructuring policies. Subsequently, I compute the underlying prices. I then define the portfolio of bonds to match the total value of debt $b(g_-, g, x)$ implied by the constrained efficient allocation. Finally, I determine the optimal tax rate from the optimality conditions of the lender and the domestic firms.

Autarky is never optimal in the contract. Hence, the government never enters into default. That is $\bar{D}(g_-, g, x) = 0$ implying that $\bar{H}(g_-, g, x) = 0$ for all (g_-, g, x). The government will therefore rely on changes in the maturity structure and *official* buyback as in the Markov equilibrium. I assume that *official* buybacks arise only if the economy hits the upper bound of the ergodic set,

$$\bar{M}(g_-, g, x) = \begin{cases} 1 & \text{if} \quad g = g_H \text{ and } x = x^{ub} \\ 0 & \text{else} \end{cases}$$

Given the above policies, the short-term bond price equates the risk-free price,

$$\bar{q}_{st}(g_-, x) = \frac{1}{1+r},$$

while the long-term bond price,

$$\bar{q}_{lt}(g_-, x) = \begin{cases} \frac{1}{1+r} q_{lt}^{bb} & \text{if} \quad x = x^{ub}. \\ \frac{1}{1+r} \mathbb{E}_{g'|g}[1 + \bar{q}_{lt}(g, x')] & \text{else} \end{cases}$$

Note further that, the long-term bond price has the following properties.

Lemma D.5 (Bond Price)**.** *Under Assumption 1, with $q_{lt}^{bb} = \frac{1}{r^{bb}}$ with $r^{bb} \geq r$, the long-term bond price is the unique fixed point of $\bar{q}_{lt}$, is decreasing and is such that*

$$\bar{q}_{lt}(g_-, x'(g_-, g_H, g_-, x)) > \bar{q}_{lt}(g_-, x'(g_-, g_L, x)) > \frac{1}{r}.$$

Proof. See Appendix D □

Having properly determined the different price schedules, I can now determine the bond holdings and the maturity in order to match the total value of the debt implied by the constrained efficient allocation. Particularly, it must hold that when $x \geq x^{ub}$ and $\tilde{x} = \underline{x}$,

$$-b(g_-, g_H, x) = \bar{b}_{st}(g_-, x) + \bar{b}_{lt}(g_-, x)[1 + q_{lt}^{bb}], \tag{D.1}$$
$$-b(g_-, g_L, x) = \bar{b}_{st}(g_-, x) + \bar{b}_{lt}(g_-, x)[1 + \bar{q}_{lt}(g_-, x'(g_-, g_L, x))]. \tag{D.2}$$

Otherwise, the relationship is given by

$$-b(g_-, g_H, x) = \bar{b}_{st}(g_-, x) + \bar{b}_{lt}(g_-, x)[1 + \bar{q}_{lt}(g_-, x'(g_-, g_H, x))],$$
$$-b(g_-, g_L, x) = \bar{b}_{st}(g_-, x) + \bar{b}_{lt}(g_-, x)[1 + \bar{q}_{lt}(g_-, x'(g_-, g_L, x))].$$

This is a system of 2 equations with 2 unknowns for which Lemma D.5 ensures a unique solution. The maturity structure of the bond portfolio is therefore properly determined.

To complete the proof, I determine the optimal tax rate and the level of intermediate price. From the optimality conditions of the domestic firms and the lender I get

$$\mathbb{E}_{g|g_-}\left(\frac{u_c(c(g_-, g, x))}{\mathbb{E}_{g|g_-}[u_c(c(g_-, g, x))]}\right) = \frac{1}{1 - \bar{\tau}(g_-, x)} = \bar{p}(g_-, x). \tag{D.3}$$

Hence, the constrained efficient allocation can be replicated with the above policies for default, maturity swap, and bond holdings. The optimality conditions of the lender and the domestic firms are satisfied as well as the price schedules.

This concludes the proof as the market allocation satisfied the necessary and sufficient conditions provided in Lemma 2. Especially, I used the budget constraints to determine the optimal bond holdings given the prices computed according to (2). The tax level is set to match the conditions (3) and (4). Finally, the resource constraint and (11) are satisfied as the constrained efficient allocation meet those requirements. □

Proof of Lemma D.5.

Recall that the long-term bond price is given by

$$q_{lt}(g,x) = \frac{1}{1+r}\mathbb{E}_{g'|g}\left[(1-D(g',x))\left\{1+(1-M(g',x))q_{lt}(g',x)+M(g',x)q_{lt}^{bb}\right\}+D(g',x)q_{lt}^{D}(g',x)\right],$$

Given that $D(g',x)=0$ for all (g',x) and $M(g',x)=1$ if $g'=g_H$ as well as $x=x^{ub}$ and $M(g',x)=0$ otherwise. From Proposition 4, g_H and $x=x^{ub}$ arises with strictly positive probability for any (g,x),

$$\frac{1}{r^{bb}} > q_{lt}(g,x) > \frac{1}{r}.$$

Define Q_{lt} as the space of bounded functions $q_{lt}:[0,\bar{\bar{x}}]\to[0,\frac{1}{r^{bb}}]$ and $\mathbb{T}:Q_{lt}\to Q_{lt}$ as

$$\mathbb{T}q_{lt}(g,x) = \begin{cases} \frac{1}{1+r}q_{lt}^{bb} & \text{if } x = x^{ub} \\ \frac{1}{1+r}\sum_{i=1}^{N}\pi(g_i)[1+q_{lt}(g',x')] & \text{else} \end{cases}$$

By the Blackwell sufficient conditions $\mathbb{T}$ is a contraction mapping. As a result, there exists a unique fixed point to $\mathbb{T}$, $\bar{q}_{lt}$ which is increasing as $\mathbb{T}$ maps increasing functions into increasing functions. This implies that $q_{lt}(x'(g_H,x)) \geq q_{lt}(x'(g_L,x))$ as $x'(g_H,x) > x'(g_L,x)$ for all x in the above specified domain. Assume now that there exists a x such that $q_{lt}(x'(g_H,x)) = q_{lt}(x'(g_L,x))$. This requires that $x, x'(g_H,x), x'(g_L,x)$ belongs to a subset $[x_t, x_{t+1}]$ where q_{lt} stays constant. Hence, for any $\ddot{x}\in[x_t,x_{t+1}]$, it must be that $x'(g_H,\ddot{x}), x'(g_L,\ddot{x})\in[x_t,x_{t+1}]$ which is a contradiction as $x'(g_H,x_{t+1}) > x_{t+1}$ owing to the incentive compatibility constraint. Therefore it must be that $q_{lt}(x'(g_H,x)) > q_{lt}(x'(g_L,x))$. □

Proof of Lemma B.1.

The law of motion of the relative Pareto weight is given by

$$x'(g) = (1 + \nu(g)) g_-^{-\sigma} \eta x.$$

and the level of consumption by

$$u_c(c(g)) = \frac{\pi(g|g_-)}{x(\pi(g|g_-) + \nu(g))}.$$

Isolating x leads to

$$x = \frac{\pi(g|g_-)}{u_c(c(g))(\pi(g|g_-) + \nu(g))}. \tag{D.4}$$

Plugging this back into the law of motion gives

$$x'(g) = (1 + \nu(g)) g_-^{-\sigma} \eta \frac{\pi(g|g_-)}{u_c(c(g))(\pi(g|g_-) + \nu(g))}.$$

Replacing $x'(g)$ by with the forward equivalent of (D.4) gives

$$\frac{\pi(g'|g)}{u_c(c(g'))(\pi(g'|g) + \nu(g'))} = (1 + \nu(g)) g_-^{-\sigma} \eta \frac{\pi(g|g_-)}{u_c(c(g))(\pi(g|g_-) + \nu(g))}.$$

Taking expectations on both sides,

$$\mathbb{E}_{g'|g} \left[\frac{\pi(g'|g)}{u_c(c(g'))(\pi(g'|g) + \nu(g'))} \right] = (1 + \nu(g)) g_-^{-\sigma} \eta \frac{\pi(g|g_-)}{u_c(c(g))(\pi(g|g_-) + \nu(g))},$$

which gives the inverse Euler equation. □