Efficient Sovereign Debt Management in Emerging Economies*

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Abstract

This paper assesses the sovereign debt management of emerging economies. I consider a market economy in which a sovereign borrower trades non-contingent bonds of different maturities with two foreign lenders. The strategic interaction between the lenders generates two Markov equilibria. The first one features buybacks at a premium and no defaults, while the second one predicts the opposite. The two equilibria can be Pareto ranked. In particular, the market economy can implement the Planner's constrained efficient allocation through changes in maturity and premium-bearing buybacks. Defaults cannot substitute for such buybacks. In addition, the lenders must enforce the payment of the premium. Nevertheless, such enforcement might fail and the implementation with it. I relate my findings to the experience of Argentina and Brazil since 1995. I estimate that the Brazilian government paid an average premium of 6% at issuance on its buyback operations. The model predicts that the debt management of Brazil is not constrained efficient yet more efficient than the one of Argentina.

Keywords: sovereign debt, default, maturity, buyback, Markov, multiplicity, constrained Pareto efficiency

JEL Classification: C73, D52, E61, F34, F41, G15, H63

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1 Introduction

Emerging economies have adopted different forms of debt management. While it is true that the better part of these economies rely more extensively on short-term debt during debt crises,[1] they do not have the same experience towards default and buyback. For instance, some countries have repeatedly defaulted on their liabilities, whereas others have not or ceased to do so.[2] The same holds true for buybacks. Some countries avoid buybacks as this is relatively costly and perceived as ineffective, while others regularly conduct such operations.[3] In light of this, the question that arises is twofold. What can explain these opposite debt management? And which of these debt management is more efficient? I analyze the role of the lenders in generating multiple equilibria and the role of maturity, buyback and default in implementing the constrained (Pareto) efficient allocation.

In general, the literature on sovereign debt has focused on the borrower's decision making. Differences between countries come from differences in terms of preferences, default costs or productivity shocks. I complement this view by arguing that part of these differences can also be explained by the strategic interaction between the lenders.

In terms of maturity, the literature on fiscal policy with commitment suggests to trade non-contingent bonds of different maturities to replicate the return of Arrow securities.[4] The portfolio of bonds emanating from this approach is however empirically implausible.[5] To reconcile the model's prediction with the data, the literature has introduced different frictions such as limited commitment and trade constraints.[6] I pursue this path by introducing limited enforcement in both repayment and buyback.

In terms of default and buyback, the literature on sovereign debt argues that it might be optimal to conduct the former as this provides a source of state contingency, while the latter is suboptimal as it only benefits the lenders.[7] I argue the opposite. A default generates deadweight losses which impact both the borrower and the lenders. Hence, it is Pareto improving to avoid such an event. In opposition, buybacks at a premium can generate state contingency without causing the aforementioned deadweight losses. As the bond price

[1] See Arellano and Ramanarayanan (2012), Broner et al. (2013), Perez (2017) and Bai et al. (2017).

[2] See Reinhart and Rogoff (2009), Cruces and Trebesch (2013) and Tomz and Wright (2007, 2013).

[3] See Bulow and Rogoff (1988, 1991) and Medeiros et al. (2007).

[4] See Kreps (1982), Angeletos (2002) and Barro (2003).

[5] Buera and Nicolini (2004) and Faraglia et al. (2010) show that the borrower ought to *sell* long-term bonds and *buy* short-term bonds in the magnitude of several multiples of GDP.

[6] Debortoli et al. (2017) introduce limited commitment in fiscal policy, Faraglia et al. (2019) limit the extent of debt repurchase and reissuance and Kiiashko (2022) adds limited commitment in repayment.

[7] See notably Grossman and Van Huyck (1988), Adam and Grill (2017), Roettger (2019) and Hatchondo et al. (2020a) for defaults as a source of risk sharing and Bulow and Rogoff (1988, 1991) and Cohen and Verdier (1995) for buybacks as suboptimal policy.

incorporates such premium, it is possible to generate state-contingent capital losses and gains with the appropriate buyback policy. As a result, the optimal sovereign debt management consists of no default and occasional costly buybacks.

I consider a small open economy populated by two foreign lenders and one sovereign borrower. The lenders are risk neutral and trade non-contingent bonds of different maturities with the borrower. Only one of the two lenders holds the outstanding debt every period. I call it the *legacy* lender. The other lender is the *new* lender. The borrower is risk averse, impatient, receives a stochastic endowment and cannot commit to repay the lenders.

I first characterize the market economy and show that there are multiple Markov equilibria. The multiplicity originates from strategic interactions between the two lenders. Seeking to maximize the value of outstanding debt, the *legacy* lender is willing to avoid dilution and supports premium-bearing buybacks. Holding no outstanding debt, the *new* lender is indifferent. Hence, the *new* lender can either collude with the *legacy* lender or satisfy the borrower's demand for bonds. In the former case, there is a Markov equilibrium without default in which premium-bearing buybacks can occur on equilibrium path. In particular, the collusion enables the lenders to enforce a no-default constraint and under certain conditions the payment of the premium. The second equilibrium corresponds to a Markov equilibrium with default in which premium-bearing buybacks never occur. It is equivalent to the equilibrium in the canonical incomplete markets model with default and multiple maturities characterized by Arellano and Ramanarayanan (2012).

I then compare the allocation in the market economy with the one of a Planner. The constrained efficient allocation features state-contingent debt relief. Particularly, the borrower records capital gains in low endowment states and capital losses otherwise. In addition, the Planner rules out autarky whenever there are strictly positive rents to be shared between the borrower and the lenders. This means that defaults – which imply markets exclusion – cannot implement the constrained efficient allocation. Instead, the borrower conducts buybacks at a premium. Such buybacks implicitly introduce state contingency in the bond contract as the bond price incorporates the premium paid. They occur in high endowment states implying that the price of long-term bonds increases after the realization of high endowment shocks. This in turn increases the value of outstanding long-term debt resulting in capital losses for the borrower. The opposite happens after the realization of low endowment shocks. Thus, buybacks at a premium can generate the capital losses and gains necessary to mimic the state contingency in the Planner's allocation.

The market economy usually fails to implement the constrained efficient allocation, though. The reason is that premium-bearing buybacks need to be enforced by the lenders. More precisely, when colluding, the lenders can threaten the borrower with a sudden stop for the non payment of the premium. This threat is credible only if the borrower does not issue assets and buybacks are not too costly. Yet, to replicate the Planner's allocation, the borrower needs to hold short-term assets unless the buyback premium is sufficiently large. Particularly, I find that the market economy fails to implement the Planner's allocation under empirically plausible buyback premia. Note that, when colluding, the two lenders restrict the indebtedness of the borrower to avert default. Hence, while the collusion is sufficient to avoid default, it is not to ensure premium-bearing buybacks.

I relate my findings to the experience of Argentina and Brazil from 1995 to 2019. On the one hand, Brazil has not defaulted since its last restructuring in 1994. Moreover, it started a comprehensive buyback program in 2006. These buybacks had the peculiarity that they entailed a premium. On the other hand, Argentina defaulted in 2001 and got excluded from the international financial markets until 2006. Furthermore, it has not conducted buybacks over the period considered.[8]

I estimate the premium paid by the Brazilian government during buyback operations. My approach consists of reconstructing the stream of cashflows of the bonds involved in buyback operations. In particular, I compare the present value of a bond with and without buyback at the issue yield to maturity. To ensure that such yield does not anticipate future buybacks, I restrict my attention to bonds issued prior to the start of the Brazilian buyback program. I find that the premium averages 6%. It is mainly due to the fact that bonds were issued below the par value and bought back above the par value. The premium displays some state contingency as it positively correlates with the output and the level of indebtedness.

I calibrate the Markov equilibrium with default to match moments of the Argentine economy over the period 1995-2019. The calibrated model fits the data well despite having only two endowment states. More importantly, using the calibration for Argentina, I find that the Markov equilibrium without default is quantitatively close to Brazil. I therefore interpret Brazil as the counterfactual of Argentina with costly buybacks and without defaults. This supports my claim that part of the difference between Argentina and Brazil can be attributed to the lenders.

In line with the literature on fiscal policy with commitment, I find that the implementation of the constrained efficient allocation generates unrealistic debt portfolios unlike Markov equilibria. More broadly, this implies that the Markov equilibrium without default is not constrained efficient. This is because buybacks that implement the constrained efficient allocation arise in parts of the state space in which the lenders cannot enforce them. Endogenizing the buyback premium with a Nash bargaining protocol does not solve this issue.

[8]Note that, in 2023, Argentina conducted some buybacks at a discount (i.e. below par). Such operations are risky as they can be interpreted as a default in the form of distressed debt exchange by rating agencies.

I compare the Markov equilibria with the implementation of the constrained efficient allocation through various simulation exercises. Relying on costly buybacks instead of defaults implies important welfare gains for both the borrower and the lenders. In that logic, the Markov equilibrium without default is quantitatively the closest to the constrained efficient allocation. Notably, I develop a metric measuring the distance to the Pareto frontier. If 1 corresponds to an allocation on the Pareto frontier, the Markov allocation with and without default have a value of 0.54 and 0.62, respectively. This indicates substantial room for policy interventions in both cases.

The paper is organized as follows. I review the literature in Section 2. The economic environment is in Section 3. I present the Markov and the constrained efficient debt management in Sections 4 and 5, respectively. The quantitative analysis is in Section 6. Section 7 concludes. The appendix contains additional results, data sources and proofs.

2 Literature Review

The paper combines elements of the literature on sovereign defaults and buybacks with elements of the literature about optimal contracts and their implementation.

The literature on sovereign defaults assumes that the borrower has access to only non-contingent claims and can obtain limited state contingency through defaults (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; Arellano, 2008).[9] My study is the closest to Arellano and Ramanarayanan (2012) and Niepelt (2014) given that I adopt two bonds with different maturities and to Kovrijnykh and Szentes (2007) as I consider two strategic lenders. My study contributes to this literature in two ways. First, it shows that the reliance on defaults to obtain state contingency is inefficient. Second, I focus on the lender's side of the decision problem, while the literature has mainly analyzed the borrower's decision problem. In particular, I show that strategic complementarities from the side of the lender can overcome the borrower's limited commitment under certain circumstances.

On a similar note, this paper relies on costly buybacks as a way to implement the constrained efficient allocation. It therefore relates to the seminal contribution of Bulow and Rogoff (1988, 1991) who document that buybacks are suboptimal as they increase the recovery value per unit of bond and therefore fail to reduce indebtedness. In light of this, Cohen and Verdier (1995) show that buybacks are effective only if they remain secret. Similarly, Aguiar et al. (2019) find that buybacks reduce welfare as they shift the maturity structure and therefore affect the default risk. In opposition, Rotemberg (1991) shows that buybacks

[9] See also Aguiar and Amador (2014), Aguiar et al. (2016) and Aguiar and Amador (2021).

can be advantageous to all parties as they lower the bargaining costs. Moreover, Acharya and Diwan (1993) argue that buybacks provide a positive signal about the borrower's willingness to repay. Finally, Kovrijnykh and Szentes (2007) find that buybacks are socially efficient despite the fact that only lenders benefit from it *ex post*. My analysis goes in this direction as it emphasizes the efficiency of buybacks as a source of risk sharing between the borrower and the lenders. The main difference is that I analyze buybacks above the par value, while the aforementioned studies focus on buybacks below the par value.

Showing multiplicity in the Markov equilibrium, this study also relates to the work of Calvo (1988), Cole and Kehoe (2000), Conesa and Kehoe (2017), Lorenzoni and Werning (2019) and Aguiar et al. (2022).[10] Unlike Cordella and Powell (2021), I relate the enforcement of a no-default borrowing limit to strategic decisions of the lenders rather than commitment. This generates multiple equilibria different than in Alvarez and Jermann (2000) and in Kirpalani (2017). Especially, multiplicity comes from the lenders' exposure towards defaults and costly buybacks in the presence of long-term debt. Similar to Aguiar and Amador (2020), I uncover a borrowing and a saving equilibrium.

The paper derives an optimal contract between foreign lenders and a borrower and therefore relates to the seminal contributions of Kehoe and Levine (1993, 2001) and Thomas and Worrall (1994). My study accounts for limited commitment similar to Aguiar et al. (2009) and is close to Kehoe and Perri (2002) and Restrepo-Echavarria (2019) as it relies on the approach of Marcet and Marimon (2019) with the difference that I implement the contract in a market economy.

The paper therefore addresses the literature on optimal contract's implementation. Note that I discuss the following studies in more details in Appendix A. Unlike Aguiar et al. (2019) and Müller et al. (2019), the Second Welfare Theorem may fail in my environment. On the one hand, Aguiar et al. (2019) account for multiple maturities but consider a Planner's problem which has no participation constraint, unlike my Planner problem. On the other hand, Müller et al. (2019) use preemptive restructurings and GDP-linked bonds, whereas I rely on the maturity structure. An alternative to this approach is Dovis (2019) who develop an implementation through partial defaults and an active debt maturity management. He builds on Angeletos (2002) and Buera and Nicolini (2004) who show that one can replicate the state-contingency of Arrow securities using non contingent bonds of different maturities.[11] My implementation is the closest to Dovis (2019) with the difference that I rely on Markov strategies rather than trigger strategies. Moreover, my implementation builds on premium-bearing buybacks in good times instead of defaults in bad times.

[10] See also Ayres et al. (2018), Stangebye (2020), Galli (2021) and Corsetti and Maeng (2021).

[11] See also the references in footnotes 4, 5 and 6.

3 Environment

Consider a small open economy over infinite discrete time $t = \{0, 1, \dots\}$ that receives a stochastic endowment. The government of the economy (i.e. the borrower) trades bonds with two foreign lenders.

The stochastic endowment takes value on the discrete set $Y \equiv \{y_L, y_H\}$ with $0 < y_L < y_H$ and is independent and identically distributed with $\pi(y_{t+1})$ corresponding to the probability of drawing y_{t+1} at date $t+1$.

Foreign lenders are risk neutral, strategic and break even in expectations. They discount the future at rate $\frac{1}{1+r}$ with r being the exogenous risk-free rate. The borrower discounts the future at rate $\beta \leq \frac{1}{1+r}$. Preference over consumption is represented by $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ where c_t corresponds to the consumption at time t. The instantaneous utility function takes the CRRA form with a coefficient of relative risk aversion of $\vartheta > 0$, i.e. $u(c_t) = \frac{c_t^{1-\vartheta}}{1-\vartheta}$.

The government is benevolent and takes the decision on behalf of the small open economy. It has access to bond contracts with two different maturities: short-term and long-term. The short-term bond $b_{st,t+1}$ is a one-period bond with unit price $q_{st,t}$ which pays a coupon of one next period. The long-term bond $b_{lt,t+1} \leq 0$ is a consol with unit price $q_{lt,t}$ which pays a coupon of one every period. I denote debt as a negative asset meaning that $b_j < 0$ is a debt, while $b_j > 0$ is an asset for all $j \in \{st, lt\}$. The borrower can hold short-term assets but not long-term assets.

A bond auction determines the new issuance of bond contracts. Similar to Kovrijnykh and Szentes (2007), the two lenders simultaneously offer a couple $(b_{st,t+1}, b_{lt,t+1})$ during the auction in t and the borrower chooses among the two offers. Importantly, the lenders' offer cannot force the borrower to save or to borrow more than it desires to.[12] For the long-term bond, all outstanding bonds have to be repurchased before issuing new ones. This implies that only one of the two lenders holds the legacy claim each period. I call this lender the *legacy* lender and the other the *new* lender.

The bond contract specifies the conditions for default and buyback. A default corresponds to a missed coupon payment which triggers permanent exclusion from the bond market. There is no recovery value. Conversely, a buyback is defined as any new long-term debt issuance below the initial amount outstanding, i.e. $b_{lt,t+1} \geq b_{lt,t}$. There are two types of buybacks: *official* and *unofficial*. In the latter, the borrower repurchases the outstanding

[12] This assumption intends to capture the main characteristics of a debt auction. First, when borrowing, the borrower needs not accept all submitted bids. Second, the submitted bids may not exhaust the borrower's demand for funds. Third, when saving, the borrower submits bids to the lenders who decide whether to accept them.

debt at $q_{lt,t}$.[13] In the former, it repurchase the outstanding debt at $q^{bb}_{lt,t} = q_{lt,t} + \chi$ where $\chi > 0$ is the buyback premium. I endogenize the buyback premium in Appendix B.

I consider Markov equilibria. That is I restrict my attention to the payoff-relevant space which corresponds to $\Omega_t \equiv (s_t, b_{st,t}, b_{lt,t})$. I use the indicator function $\varpi_0 \in \{0, 1\}$ for the equilibrium selection so the exogenous shock space is $s_t \equiv (y_t, \varpi_0)$.

The borrower cannot commit to pay the lenders. In particular, it cannot commit to pay the coupon due every period and to conduct *official* instead of *unofficial* buybacks. The timing of actions is the following. At the beginning of each period $t \geq 0$, y_t realizes and the borrower decides whether to default given $(b_{st,t}, b_{lt,t})$. Conditional on no default, the borrower decides whether to conduct an official buyback and subsequently enters the bond auction with the two lenders to determine $(b_{st,t+1}, b_{lt,t+1})$. The two lenders can therefore condition their offer on whether an official buyback took place.

4 Markov Debt Management

This section defines the borrower's and lenders' problems, shows that there are multiple Markov equilibria and characterizes these equilibria.

4.1 The market economy

Let $\Omega \equiv (s, b_{st}, b_{lt})$ be the payoff-relevant space. Given this, the borrower's overall beginning of the period value is

$$W^b(\Omega) = \max_{D\in\{0,1\}} \Big\{(1-D)V^P(\Omega) + DV^D(y)\Big\}, \tag{1}$$

where V^P and V^D correspond to the value of repayment and default, respectively. Under default, the borrower receives the value of permanent autarky $V^D(y) = u(y) + \beta\frac{\mathbb{E}[u(y')]}{1-\beta}$. Under repayment, the borrower chooses whether to conduct *official* buybacks. Thus

$$V^P(\Omega) = \max_{M\in\{0,1\}} \Big\{(1-M)V^{NB}(\Omega) + MV^B(\Omega)\Big\}, \tag{2}$$

where V^B and V^{NB} are the values under *official* buyback and no *official* buyback, respectively. The former corresponds to

$$V^B(\Omega) = \max_{b'_{st}, b'_{lt} \geq b_{lt}} u(c) + \beta\mathbb{E}\left[W^b(\Omega')\right] \tag{3}$$

[13] *Unofficial* buybacks are a version of what Cohen and Verdier (1995) call a "secret buyback".

$$\text{s.t.} \quad c + q_{st}(s, b'_{st}, b'_{lt})b'_{st} + q_{lt}(s, b'_{st}, b'_{lt})(b'_{lt} - b_{lt}) = y + b_{st} + b_{lt}(1+\chi). \tag{4}$$

The borrower pays the premium χ and issues new long-term debt such that $b_{lt,t+1} \geq b_{lt,t}$. Conversely, under no *official* buyback,

$$V^{NB}(\Omega) = \max_{b'_{st}, b'_{lt}} u(c) + \beta \mathbb{E}\left[W^b(\Omega')\right] \tag{5}$$

$$\text{s.t.} \quad c + q_{st}(s, b'_{st}, b'_{lt})b'_{st} + q_{lt}(s, b'_{st}, b'_{lt})(b'_{lt} - b_{lt}) = y + b_{st} + b_{lt}. \tag{6}$$

There is no restriction on the issue of long-term debt meaning that the borrower can potentially conduct *unofficial* buybacks. The value of the *legacy* lender is given by

$$W^l_{\text{legacy}}(\Omega) = \max_{(b'_{st}, b'_{lt}) \in \Gamma(\Omega)} -\Big[b_{st} + b_{lt}(1 + q_{lt}(s, b'_{st}, b'_{lt}) + M(\Omega)\chi)\Big](1 - D(\Omega)). \tag{7}$$

The continuation value is equal to zero given the break-even assumption. In addition, as the lenders cannot force the borrower to borrow or save, the lenders' offer is restricted to

$$\Gamma(\Omega) = \left\{(b'_{st}, b'_{lt}) : b'_{st} \in [\min(0, B_{st}(\Omega)), \max(0, B_{st}(\Omega))] \ \wedge \ b'_{lt} \in [b_{lt}, \max(b_{lt}, B_{lt}(\Omega))]\right\},$$

where $B_{st}(\Omega)$ and $B_{lt}(\Omega)$ are respectively the short-term and long-term bond policy functions that solve (2).

Given the break-even assumption and the fact that the *legacy* lender holds (b_{st}, b_{lt}), the value of the *new* lender is $W^l_{\text{new}}(\Omega) = 0$ for all Ω. Hence, the the *new* lender's value is independent of the state of the economy. The value of the two lenders together is then $W^l(\Omega) = W^l_{\text{legacy}}(\Omega) + W^l_{\text{new}}(\Omega) = W^l_{\text{legacy}}(\Omega)$.

4.2 Multiplicity of equilibria

Having derived the borrower's and lenders' problem, I derive the bond prices. I then show that there are multiple equilibria due to the *new* lender's indifference. Proofs can be found in Appendix H.

Both lenders break even meaning that in expectations they make zero profit. The price of one unit of bond is given by

$$q_{lt}(s, b'_{st}, b'_{lt}) = \frac{1}{1+r}\mathbb{E}\Big[(1 - D(\Omega'))\Big\{1 + (1 - M(\Omega'))q_{lt}(s', b''_{st}, b''_{lt}) + M(\Omega')q^{bb}_{lt}(s', b''_{st}, b''_{lt})\Big\}\Big], \tag{8}$$

$$q_{st}(s, b'_{st}, b'_{lt}) = \frac{1}{1+r}\mathbb{E}\Big[(1 - D(\Omega'))\Big].$$

The multiplicity of equilibria originates from the interaction between the *legacy* and the *new* lender. On the one hand, the former lender is unwilling to let the borrower dilute legacy debt as this directly reduces its payoff. Similarly, it is supportive of *official* buybacks as the payment of the premium χ increases its payoff.[14] On the other hand, the *new* lender is indifferent to both dilution and *official* buybacks as it does not hold any outstanding debt. The break-even assumption further ensures zero continuation payoff.[15]

Proposition 1 (Legacy Debt)**.** *The legacy lender is unwilling to dilute legacy long-term debt claims and is willing to have official buybacks. The new lender is always indifferent.*

The *new* lender's indifference implies multiple equilibria. In particular, it can offer debt contracts that either satisfy the borrower's problem in (2) or that satisfy the *legacy* lender's problem in (7). The indicator function, ϖ_0, takes value one in the former case and zero in the latter. Thus, I impose that the *new* lender chooses which side to take in period 0. This enables me to nest the equilibrium of Arellano and Ramanarayanan (2012) as shown next.

4.3 Properties of equilibria

There exist two equilibria: one without default where *official* buybacks can occur and one with default where *official* buybacks never occur on equilibrium path. This is because the *new* lender's indifference has implications on whether borrowing is risky and on whether buybacks entail a premium. Additional results can be found in Appendix D.

Regarding risky borrowing, define the no-default borrowing limit as

$$\mathcal{B} = \max_{s'}\{(b'_{st}, b'_{lt}) : V^P(\Omega') = V^D(y')\}. \tag{9}$$

Following Zhang (1997), the borrower has no incentive to default if it does not accumulate debt beyond $\mathcal{B}$. Moreover, the *legacy* lender never offers $(b'_{st}, b'_{lt}) < \mathcal{B}$ given Proposition 1. Thus, if the *new* lender decides to satisfy the *legacy* lender's problem, the borrower never enters the risky borrowing region. In opposition, if it decides to satisfy the borrower's problem, the borrower can enter the risky borrowing region if willing to do so. As a result, the collusion between the two lenders generates a no-default borrowing limit which would not exist otherwise.

[14] This is true only in the presence of long-term debt. Without long-term debt, there is neither dilution nor buyback meaning that the *legacy* lender is as indifferent as the *new* lender.

[15] Note that the break even assumption with zero recovery value rules out any monopoly power of the *legacy* lender close to the default decision. My environment therefore does not feature the default cycles characterized by Kovrijnykh and Szentes (2007).

Regarding *official* buybacks, the behavior of the *new* lender is also crucial for enforcement. The borrower cannot commit to conduct *official* buybacks as it can avoid the payment of χ through *unofficial* buybacks.[16] As a result,

Proposition 2 (Official Buyback Aversion)**.** *The borrower cannot commit to conduct official buybacks as it always strictly prefers unofficial buybacks for the same offer* (b'_{st}, b'_{lt}).

Thus, if the *new* lender decides to satisfy the borrower's problem, *official* buybacks never occur on equilibrium path. In opposition, if the *new* lender decides to satisfy the *legacy* lender's problem, *official* buybacks may occur on equilibrium path. Given the timing of actions, the two lenders can condition their offer on M. In particular, they can offer an unfavorable debt contract if $M = 0$ to force the borrower to pick $M = 1$. As $(b'_{st}, b'_{lt}) \in \Gamma(\Omega)$, the worst they can offer is a sudden stop debt contract consisting of $b'_{st} \geq 0$ and $b'_{lt} \geq b_{lt}$. This does not necessarily mean that an *official* buyback occurs every period.[17]

Lemma 1 (Official Buyback Enforcement)**.** *When the two lenders offer* $(b'_{st}, b'_{lt}) \geq (0, b_{lt})$ *under no official buyback, the borrower conducts an official buyback if it does not want to save (i.e.* $b'_{st} < 0$*) and either* $-b_{st} > 0$ *is sufficiently large or* $-b_{lt}\chi$ *is not too large.*

The lemma states that an *official* buyback is preferable to a sudden stop when the borrower does not possess any assets and *official* buybacks are not too costly. A direct corollary of Lemma 1 is that the state space Ω can be separated in two zones:

1. The enforcement zone:

   In this zone, the borrower is worse off with a sudden stop contract rather than paying the premium χ. Hence, the borrower picks $M = 1$ as of Lemma 1. In other words, the *official* buyback takes place.

2. The impunity zone:

   In this zone, the borrower is always worse off paying the premium χ. The *official* buybacks therefore does not happen as of Proposition 2.

Hence, the enforcement zone is the only part of the state space in which an *official* buyback occurs. Under rational expectations, in the enforcement zone, the two lenders offer a sudden stop contract if the borrower chooses $M = 0$, while they offer a contract satisfying

[16] An *official* buyback is a reverse default as it corresponds to an overpayment, whereas a default is an underpayment of liabilities. While underpayments are sanctioned by markets exclusion, there is no direct reward after overpayments.

[17] Without the constraint $\Gamma(\Omega)$, the two lenders could force the borrower to save or to borrow whenever $M = 0$ leading to a perfect monopoly power over *official* buybacks.

(3) subject to $(b'_{st}, b'_{lt}) \geq \mathcal{B}$ if the borrower chooses $M = 1$. The sudden stop therefore remains off equilibrium path. In the impunity zone, no *official* buybacks can be enforced. The two lenders offer the same contract satisfying (5) subject to $(b'_{st}, b'_{lt}) \geq \mathcal{B}$. Hence, as before, the collusion between the two lenders generates a buyback enforcement constraint which would not exist otherwise.

Given the above multiplicities, there are two Markov equilibria. The first one is a Markov equilibrium without default in which *official* buybacks can occur on equilibrium path. It happens when the *new* lender offers debt contracts that satisfy the *legacy* lender's problem in (7) (i.e. $\varpi_0 = 0$). The second one is a Markov equilibrium with default in which *official* buybacks never occur. It happens when the *new* lender offers debt contracts that satisfy borrower's problem in (2) (i.e. $\varpi_0 = 1$).

The Markov equilibrium with default is the one characterized by Arellano and Ramanarayanan (2012). On the one hand, defaults arise on equilibrium path and especially when the endowment is low. On the other hand, maturity shortens during debt crises. The repayment of long-term debt is laddered through multiple periods which implies that the claim of the *legacy* lender can be diluted. The long-term bond therefore admits a greater risk premium than the short-term bond. As a result, close to default, the long-term debt price drastically drops which encourages shorter maturity.[18] Finally, *official* buybacks never occur. Thus, if the borrower is willing to default, this corresponds to a borrowing equilibrium as defined by Aguiar and Amador (2020).

The Markov equilibrium without default predicts the opposite of what the previous Markov equilibrium does. Defaults do not arise on equilibrium path, while *official* buybacks can. Moreover whether maturity shortens in the low endowment state depends on the location of the enforcement zone. This corresponds to a saving equilibrium as defined by Aguiar and Amador (2020). The difference is that the lenders enforce the saving equilibrium directly through a non-default constraint. Hence, the existence of such equilibrium solely depends on ϖ_0.

5 Constrained Efficient Debt Management

This section presents the constrained efficient debt management. I first derive the Planner problem and subsequently characterize the constrained efficient allocation before implementing it the market economy.

[18] As shown by Niepelt (2014), this result is also a consequence of the fact that a default implicates the entire long-term and short-term debt. It would not arise if the borrower would only default on the maturing portion of the long-term debt. See also Perez (2017).

5.1 The Planner's problem

The constrained efficient allocation is the outcome of a problem in which a Planner allocates consumption to maximize the lenders' and the borrower's weighted utility subject to a participation constraint.[19] The participation constraint accounts for limited commitment in repayment (Thomas and Worrall, 1994). Denoting y^t as the history of realized value of y at time t, it must hold that for all t and y^t

$$\sum_{j=t}^{\infty} \beta^{j-t} \sum_{y^j} \pi(y^j) u(c(y^j)) \geq V^D(y_t). \tag{10}$$

This condition ensures that the borrower's value is at least as large as the value of opting out. Given this, the Planner's maximization problem in sequential form reads

$$\begin{aligned} \max_{\{c(y^t)\}_{t=0}^{\infty}} \mu_{b,0} \sum_{t=0}^{\infty} \beta^t \sum_{y^t} \pi(y^t) u(c(y^t)) + \mu_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \sum_{y^t} \pi(y^t) \left[y_t - c(y^t)\right] \\ \text{s.t.} \quad (10) \text{ for all } y^t, t \text{ with } (\mu_{b,0}, \mu_{l,0}) \geq 0 \text{ given.} \end{aligned} \tag{11}$$

The given weights $\mu_{b,0}$ and $\mu_{l,0}$ are the initial non-negative Pareto weights assigned by the Planner to the borrower and the lenders, respectively. Following Marcet and Marimon (2019), I reformulate (11) as a saddle-point Lagrangian problem,

$$\begin{aligned} \mathcal{SP} \quad \min_{\{\gamma(y^t)\}_{t=0}^{\infty}} \max_{\{c(y^t)\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \sum_{y^t} \pi(y^t) \mu_{b,t}(y^t) u(c(y^t)) + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \sum_{y^t} \pi(y^t) \mu_{l,t}(y^t) \left[y_t - c(y^t)\right] \\ & + \sum_{t=0}^{\infty} \beta^t \sum_{y^t} \pi(y^t) \gamma(y^t) \Big[\sum_{j=t}^{\infty} \beta^{j-t} \sum_{y^j} \pi(y^j) u(c(y^j)) - V^D(y_t) \Big] \\ & \mu_{b,t+1}(y^t) = \mu_{b,t}(y^t) + \gamma(y^t) \text{ and } \mu_{l,t+1}(y^t) = \mu_{l,t}(y^t) \text{ for all } y^t, t \\ & \text{with } \mu_{b,0}(y_0) \equiv \mu_{b,0} \text{ and } \mu_{l,0}(y_0) \equiv \mu_{l,0} \text{ given.} \end{aligned}$$

In this formulation, $\gamma(y^t)$ denotes the Lagrange multiplier attached to the participation constraint at time t. As the value of the borrower appears in both the Planner's objective function and the participation constraint, there is a direct link between $\mu_{b,t}(y^t)$ and $\gamma(y^t)$. More precisely, the borrower's Pareto weight evolves according to $\mu_{b,t+1}(y^t) = \mu_{b,t}(y^t) + \gamma(y^t)$, while the lenders' Pareto weight, $\mu_{l,t}(y^t)$, remains constant.

Following Marcet and Marimon (2019), the saddle-point Lagrangian problem is homoge-

[19] As the *new* lender always has a payoff of zero, it is without loss of generality to consider the two lenders together with the same weight in the Planner problem.

nous of degree one in $(\mu_{b,t}(y^t), \mu_{l,t}(y^t))$. I can therefore redefine the maximization problem over $(x_t(y^t), 1)$ where $x_t(y^t) = \frac{\mu_{b,t}(y^t)}{\mu_{l,t}(y^t)}$ corresponds to the relative Pareto weight – i.e. the Pareto weight attributed to the borrower relative to the lenders. Given that $(\mu_{b,0}, \mu_{l,0}) \geq 0$ and $\gamma(y^t) \geq 0$ for all t, $x \in X \equiv [\underline{x}, \overline{x}]$ with $\underline{x} \geq 0$ and $\overline{x} \leq \infty$. Moreover,

$$x_{t+1}(y^t) = (1 + \nu(y^t))\eta x_t \quad \text{with} \quad x_0 = \frac{\mu_{b,0}}{\mu_{l,0}}, \tag{12}$$

where $\eta \equiv \beta(1+r) \leq 1$ corresponds to the borrower's impatience relative to the lenders' and $\nu(y^t) \equiv \frac{\gamma(y^t)}{\mu_{b,t}(y^t)}$ represents the normalized multiplier attached to the participation constraint. The state space for the problem therefore reduces to (y, x) and the Saddle-Point Functional Equation is given by

$$\begin{aligned} PV(y,x) = \mathcal{SP} \min_{\nu(y)} \max_{c} \; & x\Big[(1+\nu(y))u(c) - \nu(y)V^D(y)\Big] \\ & + y - c + \frac{1}{1+r}\sum_{y'} \pi(y')PV(y', x') \\ \text{s.t.} \quad & x'(y) = (1+\nu(y))\eta x \;\forall y. \end{aligned} \tag{13}$$

The value function takes the form of $PV(y,x) = xV^b(y,x) + V^l(y,x)$ with $V^b(y,x) = u\left(c\right) + \beta\mathbb{E}\left[V^b\left(y', x'\right)\right]$ being the value of the borrower and $V^l(y,x) = y - c + \frac{1}{1+r}\mathbb{E}\left[V^l\left(y', x'\right)\right]$ being the value of the two lenders together. I obtain the optimal consumption by taking the first-order conditions in (13)

$$u_c(c) = \frac{1}{x(1+\nu(y))}. \tag{14}$$

The binding participation constraint of the borrower (i.e. $\nu(y) > 0$) induces an increase in consumption. In the next subsection, I formalize this argument in Proposition 3.

5.2 Properties of the Planner's allocation

I characterize the main properties of the Planner's allocation in terms of Pareto frontier and risk sharing. Additional characterization can be found in Appendix D.

I start with the definition of a threshold values for the relative Pareto weight $x_D(y)$ which is such that $V^b(y, x_D(y)) = V^D(y)$. In words, $x_D(y)$ is the weight at which the participation constraint binds in y. Given this, the following proposition highlights the main properties of the constrained efficient allocation.

Proposition 3 (Constrained Efficient Allocation)**.**

I. (Efficiency). *$V^l(y,x)$ is strictly decreasing, while $V^b(y,x)$ is strictly increasing in $x \in \tilde{X} \equiv [x_D(y_L), \overline{x}]$ for all $y \in Y$ and $x_D(y_H) > x_D(y_L)$.*

II. (Risk-Sharing). *$c(y_L,x) < c(y_H,x)$ and $x'(y_L,x) < x'(y_H,x)$ for all $x \leq x_D(y_H)$ and $c(y_L,x) = c(y_H,x)$ and $x'(y_L,x) = x'(y_H,x)$ otherwise.*

III. (Liabilities). *$V^l(y_L,x) < V^l(y_H,x)$ for all $x \in \tilde{X}$.*

IV. (Autarky). *$c(y, x_D(y)) \neq y$ for all $y \in Y$ unless $\beta \to 0$, $\vartheta \to 0$, $(y_H - y_L) \to 0$ or $\min_y \pi(y) \to 0$.*

Part I states that the allocation is constrained efficient. Accounting for the participation constraint, it is not possible to make one of the contracting parties better off without making the other worse off.

Part II states that the Planner always provides risk sharing to the extent possible. Equalization of consumption is possible whenever the borrower's participation constraint ceases to bind in all endowment states. Otherwise, the Planner provides more consumption and a greater continuation value when the high endowment state realizes.

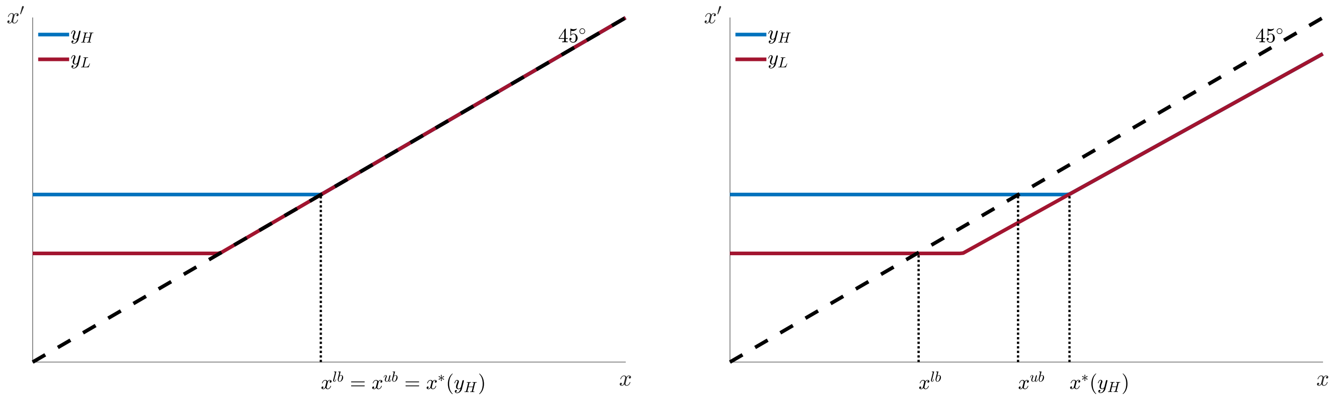
Part III relates to the liabilities of the borrower. In this environment, the value of the two lenders represents the net foreign asset position in the contract. A positive value of $V^l(y,x)$ indicates the extent to which the borrower is indebted. The proposition states that the liabilities increase when y is high. This implies that the Planner adopts a state-contingent policy as it provides debt relief in low endowment states. This state contingency will be replicated through *official* buybacks in the market economy.

Part IV states that autarky is not optimal unless the following parameters are sufficiently close to zero: the discount factor, the coefficient of relative risk aversion, the variance of the endowment shock or the minimum of the transition vector. In general, the Planner rules out the autarkic allocation whenever there are strictly positive rents to be shared among the contracting parties. That is why I make the following interiority assumption.

Assumption 1 (Interiority)**.** *For all $y^t, t \geq 0$, there is a sequence $\{\tilde{c}(y^t)\}$ satisfying*

$$\sum_{j=t}^{\infty} \beta^{j-t} \sum_{y^j} \pi(y^j) u(\tilde{c}(y^j)) - V^D(y_t) > 0.$$

Assumption 1 not only rules out autarky as a feasible allocation, it also ensures the uniform boundedness of the Lagrange multipliers. As shown in Appendix D, this guarantees existence and uniqueness of the Planner allocation.[20]



(a) Patience ($\eta = 1$)

(b) Impatience ($\eta < 1$)

Note: The figure depicts the law of motion of the relative Pareto weight in the case of a patient (i.e. $\eta = 1$) and impatient (i.e. $\eta < 1$) borrower. The blue line corresponds to the law of motion in y_H and the red line to the law of motion in y_L. The black dotted line represents the 45° line. x^{lb} and x^{ub} correspond to the lower and upper bounds of the ergodic set, respectively. $x_D(y)$ corresponds to the weight at which the participation constraint binds in y.

Figure 1: Steady State Dynamic

Having determined the constrained efficient allocation, I now show that the long-term allocation is characterized by an ergodic set of relative Pareto weights.[21]

Proposition 4 (Steady State). *A steady state is defined by an ergodic set of relative Pareto weights $x \in [x^{lb}, x^{ub}] \subset \tilde{X}$ such that $x'(y_H, x^{ub}) = x^{ub}$, $x'(y_L, x^{lb}) = x^{lb}$ and*

I. If $\eta = 1$, then $x^{lb} = x^{ub} = x_D(y_H)$.

II. If $\eta < 1$, then $x^{lb} < x^{ub} < x_D(y_H)$.

The proposition states that whenever the borrower is patient (i.e. $\eta = 1$), the steady state does not display any dynamic. Conversely, whenever the borrower is relatively impatient (i.e. $\eta < 1$), the steady state is dynamic. This dynamic is however bounded below by x^{lb} and above by x^{ub}. For instance, after a sufficiently long series of y_L (y_H), the economy hits x^{lb} (x^{ub}). It then stays there until y_H (y_L) realizes and that irrespective of the past realizations of the shock. Figure 1 illustrates each of the two types of steady states.

5.3 Implementation of the Planner's allocation

Having derived and characterized the constrained efficient allocation, I construct a Markov equilibrium that implements the constrained efficient allocation in the market economy. That is, I show the conditions for the Second Welfare Theorem to hold.

[20]See also Proposition 4.10 in Alvarez and Jermann (2000).

[21]The term ergodic refers to the fact that the relative Pareto weights in this set are aperiodic and recurrent with non-zero probability.

Proposition 5 (Implementation)**.** *Under Assumption 1, a Markov equilibrium with a no-default borrowing limit and a buyback enforcement constraint implements the constrained efficient allocation.*

Proof. The proof of this proposition is by construction. I first determine the default and *official* buyback policies necessary to implement the constrained efficient allocation. I subsequently derive the underlying portfolio of bonds. Finally, I set the conditions to enforce the default and buyback policies.

Similar to Dovis (2019), I express the policy functions of the implemented Planner's allocation as a function of (y, x). Formally, define $\bar{D}, \bar{M} : Y \times X \to \{0, 1\}$ and $\bar{q}_{st}, \bar{q}_{tl}, \bar{b}_{st}, \bar{b}_{lt} : Y \times X \to \mathbb{R}$. Given the timing of actions, the bond policies and the price schedules can be rewritten as $\bar{b}_j(y, x) = \bar{b}_j(x'(y, x))$ and $\bar{q}_j(y, x) = \bar{q}_j(x'(y, x))$ for all $j \in \{st, lt\}$.

To generate the appropriate state contingent debt relief, I assume that *official* buybacks arise when the economy hits the upper bound of the ergodic set,

$$\bar{M}(y, x) = \begin{cases} 1 & \text{if} \quad y = y_H \text{ and } x = x^{ub} \\ 0 & \text{else} \end{cases} \tag{15}$$

In addition, given Assumption 1, autarky is never optimal meaning that $\bar{D}(y, x) = 0$ for all (y, x). Given these two policies and (8), the bond prices are given by

$$\bar{q}_{st}(x) = \frac{1}{1 + r} \quad \text{and} \quad \bar{q}_{lt}(x) = \frac{1}{1 + r}\mathbb{E}[1 + \bar{q}_{lt}(x')].$$

Given the buyback policy (15), the long-term bond price is state contingent and increases in the realisation of y_H as shown in the following lemma.

Lemma 2 (Long-Term Bond Price)**.** *With $\chi > 0$ and $M(y_H, x) = 1$ for at least one $x \in [x^{lb}, x^{ub}]$, the long-term bond price is the unique fixed point of $\bar{q}_{lt}$, is decreasing and*

$$\frac{1 + \chi}{r} > \bar{q}_{lt}(x'(y_H, x)) \geq \bar{q}_{lt}(x'(y_L, x)) > \frac{1}{r},$$

with strict inequality when $\eta < 1$.

Having properly determined the prices, I can determine the bond portfolio necessary to match the constrained efficient allocation. Particularly, it must hold that when $x = x^{ub}$,

$$-V^l(y_H, x) = \bar{b}_{st}(x) + \bar{b}_{lt}(x)[1 + \bar{q}_{lt}(x'(y_H, x)) + \chi], \tag{16}$$

$$-V^l(y_L, x) = \bar{b}_{st}(x) + \bar{b}_{lt}(x)[1 + \bar{q}_{lt}(x'(y_L, x))]. \tag{17}$$

Note that x^{ub} is effectively a point of (weak) *official* buyback as $\bar{b}_{lt}(x'(y_H, x^{ub})) = \bar{b}_{lt}(x^{ub})$ from Proposition 4. Conversely, for $x \neq x^{ub}$, the relationship is given by

$$-V^l(y_H, x) = \bar{b}_{st}(x) + \bar{b}_{lt}(x)[1 + \bar{q}_{lt}(x'(y_H, x))],$$
$$-V^l(y_L, x) = \bar{b}_{st}(x) + \bar{b}_{lt}(x)[1 + \bar{q}_{lt}(x'(y_L, x))].$$

This is a system of 2 equations with 2 unknowns for which Lemma 2 ensures that there exists a unique solution. The bond portfolio is therefore determined.

Regarding the buyback policy (15), I introduce an enforcement constraint such that for all $y \in Y$ and for all $x \in \tilde{X}$

$$\bar{b}_{st}(x) + \bar{b}_{lt}(x)[1 + \bar{q}_{lt}(x'(y, x)) + \bar{M}(y, x)\chi] \leq \mathcal{E}(x). \tag{18}$$

where $\mathcal{E}(x) = \bar{b}_{st}(x^{ub}) + \bar{b}_{lt}(x^{ub})[1 + \bar{q}_{lt}(x^{ub}) + \chi]$ if $x \leq x^{ub}$ and $\mathcal{E}(x) = \bar{b}_{st}(\bar{x}) + \bar{b}_{lt}(\bar{x})[1 + \bar{q}_{lt}(\bar{x})]$ else. This forces the borrower to conduct an *official* buyback uniquely and exclusively in (y_H, x^{ub}) as $\mathcal{E}(\bar{x}) > \bar{b}_{st}(x^{ub}) + \bar{b}_{lt}(x^{ub})[1 + \bar{q}_{lt}(x^{ub})] > \mathcal{E}(x^{ub})$ as, by Proposition 3, $V^l(y, x)$ is strictly decreasing in x and strictly increasing in y.

Following Alvarez and Jermann (2000), to ensure that the participation constraint (10) holds, I introduce a no-default borrowing limit such that for all $y' \in Y$ and for all $x' \in \tilde{X}$

$$\bar{b}_{st}(x') + \bar{b}_{lt}(x')[1 + \bar{q}_{lt}(x''(y', x'))] \geq \mathcal{G}(y'), \tag{19}$$

where $\mathcal{G}(y') = \bar{b}_{st}(x_D(y')) + \bar{b}_{lt}(x_D(y'))[1 + \bar{q}_{lt}(x_D(y'))]$ which by definition of $x_D(y')$ ensures that for all $y' \in Y$ and for all $x' \in \tilde{X}$

$$V^b(y', x') \geq V^D(y').$$

The buyback policy (15) ensures the sate contingency of $\mathcal{G}(y')$. Notice the difference in timing: (18) refers to the current state (y, x), while (19) refers to the future state (y', x').

This concludes the proof. I used the budget constraints in (4) and in (6) to determine the optimal bond holdings given the prices computed according to (8). Moreover, the participation constraint (10) holds and *official* buybacks are enforceable.

□

The implementation works as follows. The borrower conducts *official* buybacks when the economy hits the upper bound of the ergodic set (i.e. $x = x^{ub}$). As this bound is reached after a sufficiently long series of high endowment shocks (Proposition 4), this buyback policy generates a specific term structure in which the high endowment shock is related to relatively larger long-term bond prices the low endowment shock, while the short-term bond price remains unchanged. This state contingency in the price enables to form a portfolio of bonds that implements the constrained efficient allocation.

Lemma 3 (Official Buyback and No Default)**.** *Under Assumption 1, the implementation features official buybacks on equilibrium path and no default. However, official buybacks cannot occur when y_L realizes.*

Lemma 3 is made of two parts. First, the implementation does not rely on defaults. Under Assumption 1, the Planner rules out autarky. This means that there is no proper markets exclusion in any region of the Planner's allocation. It is therefore not possible to interpret the borrower's binding constraint as a default in my environment.[22]

Second, *official* buybacks generate the capital losses and gains necessary to mimic the state contingency in liabilities of the Planner's allocation. In particular, as *official* buybacks involve a premium χ, they arise in the high endowment state. Unlike defaults, they are an efficient source of risk sharing. Defaults entail costs for both the lenders and the borrower, while *official* buybacks are solely costly for the latter. A default is therefore not renegotiation proof as both contracting parties would be strictly better off avoiding this event *ex post*.

A corollary of Lemma 3 is that the long-term bond spread is negative.[23] On the one hand, in the absence of default, there is no positive spread. On the other hand, *official* buybacks entail a premium χ implying that the long-term bond price exceeds the risk-free price.

Another corollary is that whether maturity shortens in the low endowment state depends on the exact parameters of the model. On the one hand, there is a substitution effect which pushes the maturity towards the long end in y_L. In particular, every successive realization of y_L makes q_{lt} less sensitive to changes in y. This is because the lenders anticipate that *official* buybacks are less likely to occur. More long-term debt is therefore required to replicate the state contingency in the Planner's allocation. On the other hand, there is an income effect which increases the total indebtedness in y_L. In steady state $x'(y_L) \leq x$ as shown in Proposition 4. By Proposition 3 Part I, this implies that the value of the lenders increases as y_L realizes. Thus, more short-term and long-term debt are needed to replicate the liabilities in the Planner's allocation.

[22] Müller et al. (2019) and Restrepo-Echavarria (2019) interpret the borrower's binding constraint as a form of preemptive restructuring which does not trigger markets exclusion. Nonetheless, Asonuma and Trebesch (2016) show that even preemptive restructurings are followed by markets exclusion in the data.

[23] This is a feature that one finds in other implementations such as the ones of Ábrahám et al. (2019) and Liu et al. (2020). The mechanism is different though. The negative spread restricts the trade of state-contingent securities in a two-sided limited commitment problem when the participation constraint of the risk-neutral lenders binds.

Besides this, given the borrower's limited commitment, the implementation requires a no-default and a buyback enforcement constraint. The former ensures that the borrower has no incentive to enter autarky, while the latter guarantees that *official* buybacks take place where they should. As we have seen in Section 4, these constraints endogenously appear in the Markov equilibrium without default (i.e. $\varpi = 0$). There is a clear analogy between the no-default borrowing limit in (9) and in (19). Also, the buyback enforcement constraint (18) relates to the enforcement zone in the market economy. However, the following lemma shows that *official* buybacks that implement the constrained efficient allocation are not generally located in this zone.

Lemma 4 (Limited Buyback Enforcement)**.** *Under Assumption 1, in the implementation of the constrained efficient allocation, the point of official buyback is not necessarily located in the enforcement zone.*

Lemma 1 states that *official* buybacks are enforceable in a Markov equilibrium when there is no short-term assets and *official* buybacks are not too costly. However, Lemma 4 shows that, in the implementation of the constrained efficient allocation, at the point of *official* buyback, the borrower needs to hold short-term assets unless the buyback premium is sufficiently large. Hence, *official* buybacks are not automatically enforceable in the market economy. Especially in the next section, I show quantitatively that that the market economy fails to implement the Planner's allocation under empirically plausible buyback premia.

A consequence of Lemma 4 is that the Second Welfare Theorem does not necessarily hold. From a general standpoint, this can be explained by the fact that neither the borrower nor the lenders share the objective function of the Planner. On the one hand, the borrower lacks commitment. On the other hand, the *legacy* lender is aligned with the Planner's view on defaults but is biased towards *official* buybacks as it does not internalize the need of state contingency of the borrower.

In Appendix C, I explore alternatives to *official* buybacks. Empirically, such alternatives do not exist or remain underdeveloped. Moreover, they raise similar enforcement issues as *official* buybacks. That is why I do not consider them in the main analysis.

6 Quantitative Analysis

This section first estimates the premium paid by Brazil during buyback operations. It then calibrates the Markov equilibrium with default to Argentina. It finally contrasts the two Markov equilibria with the constrained efficient allocation. The details on the data used in this section can be found in Appendix F.

6.1 Premium Estimation

Brazil defaulted last in the 1980s and regained access to the international market after the implementation of the Brady Plan in 1994.[24] In 2006, the Brazilian government started the Early Redemption Program which aimed at correcting the average maturity of the foreign-currency debt and reducing the potential refinancing risk. Repurchases were conducted by the Brazilian National Treasury either directly on the secondary market or indirectly through tender offers.

Figure 2 depicts the amount bought back by the Brazilian government between 2006 and 2018 for bonds denominated in USD and issued prior to 2006. The black bar represents the face value, while the grey bar corresponds to the financial value. In the period considered, buybacks amounted a total of 21 USD billion in face value and 28 USD billion in financial value.[25] Financial value buybacks were the largest in 2007 with 7 USD billion, in 2006 with 6 USD billion and in 2010 with 4 USD billion. The same holds true for face value buybacks with a total amount of 5 USD billion in 2007, 4 USD billion in 2006 and 3 USD billion in 2010. In addition, buybacks were the largest for bonds with a relatively long residual maturity. For instance, buybacks for bonds due in 20 years or more amounted to 10 USD billion in face value and 14 USD billion in financial value.

(a) 2006-2010 (b) 2011-2018

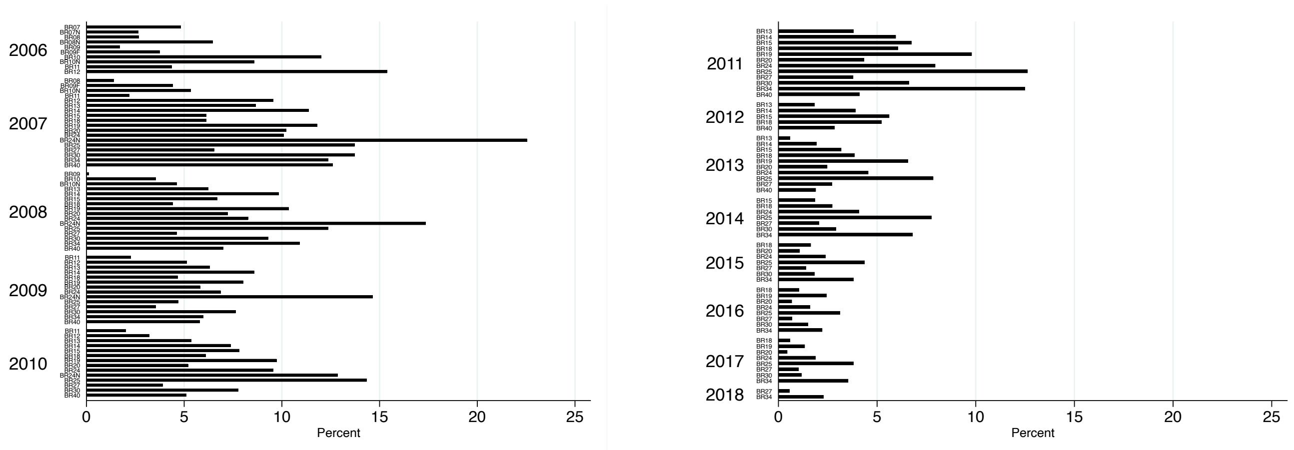
Note: The figure depicts the buyback amount by year and by bonds. All bonds are denominated in USD and issued prior to 2006. The Financial value corresponds to the amount required for payment of the securities redeemed, while the face value in blue corresponds to the value of debt in the national statistics.

Figure 2: Buyback Amount by Bond and Year

[24]The Brady Plan is an extensive debt restructuring program aimed at resolving the numerous of sovereign debt defaults in 1980s especially in Latin America. In general, see Buera and Nicolini (2021) for the economic history of Argentina and Ayres et al. (2021) for Brazil.

[25]Note that buybacks for bonds issued from 2006 onwards amounts to an additional 2 USD billion in both face and financial values.

In the model, the buyback premium reflects the capital transfer that operates between the borrower and the lenders during a buyback transaction. A direct comparison between the face value and the financial value of the bond imperfectly estimates this transfer. A bond consists of a loan contract which specifies a sequence of coupon payments and a principal repayment at maturity. A borrower issues bonds on the primary market at a given price. In the absence of buyback, the borrower receives the value of the bond at the primary-market price and transfers to the lenders the coupons and the principal when due. In the presence of a buyback, the borrower transfers the coupons due up to the buyback and the secondary-market price of the bond during the buyback. I therefore estimate the buyback premium by comparing the cashflow stream of a bond with and without buyback.



(a) 2006-2010 (b) 2011-2018

Note: The figure depicts the buyback premium by year and by bond. The buyback premium is computed according to (20).

Figure 3: Buyback Premium by Bond and Year

Formally, consider a bond with a face value b, a coupon rate κ, a yield to maturity i and a maturity T. The present value of the coupons is given by $FVC(T, \kappa, b; i) = \sum_{t=1}^{T} \frac{b\kappa}{(1+i)^t}$. The present value of the face value corresponds to $FVP(T, b; i) = \frac{b}{(1+i)^T}$. Given this, in the absence of buybacks, the financial value of a bond for a given i is

$$FV^{NB}(T, \kappa, b; i) = FVC(T, \kappa, b; i) + FVP(T, b; i).$$

Evaluated at the issue yield to maturity, this equation gives the value the lenders get if they hold the bond from the issuance to the maturity.

In the case of a buyback, the borrower repurchases the bond prior to maturity, say $t_B < T$. The bond is repurchased on the secondary market at a financial value $B = FVC(T - t_B, \kappa, b; i_B) + FVP(T, b; i_B)$ where i_B is the yield to maturity at t_B. Thus, in the presence of buybacks, the financial value of a bond for a given i is

$$FV^B(t_B, \kappa, b, B; i) = FVC(t_B, \kappa, b; i) + FVP(t_B, B; i).$$

The yield to maturity represents the internal rate of return of the bond. Hence, whenever $FV^B(t_B, \kappa, b, B; i) > FV^{NB}(T, \kappa, b; i)$ for a given i, the borrower pays a premium in the buyback operation. In the opposite case, there is a discount.

Table 1: Buyback Amount, Yield and Premium by Bond

Bond	Maturity at Issuance (Year)	Coupon Rate (Percent)	Total Issuance (USD Billion)	Average i_I (Percent)	Total Buyback (USD Billion)	Average i_B (Percent)	Average Premium (Percent)
BR07	4	10.00	1.00	10.68	0.50	3.56	4.83
BR09F	5	6.98	0.75	7.33	0.31	4.72	4.19
BR08N	6	11.50	1.25	11.74	0.65	5.47	6.45
BR07N	7	11.25	1.50	12.43	0.39	5.47	2.66
BR10N	7	9.25	1.50	9.45	0.63	5.75	5.60
BR10	8	12.00	1.00	12.38	0.46	6.39	7.78
BR11	8	10.00	1.25	7.49	0.65	5.70	2.49
BR08	10	9.38	1.25	9.42	0.25	5.17	2.03
BR09	10	14.50	2.00	14.57	0.95	7.32	0.92
BR12	10	11.00	1.25	12.60	0.47	6.48	7.60
BR13	10	10.25	1.25	10.58	0.59	4.91	4.62
BR14	10	10.50	1.25	10.80	0.78	5.06	6.68
BR15	10	7.88	2.10	7.90	1.03	4.65	5.36
BR18	13	8.00	4.51	8.00	2.89	5.96	4.28
BR19	15	8.88	1.50	9.15	0.90	5.82	7.69
BR20	20	12.75	1.00	13.27	0.53	7.12	4.68
BR25	20	8.75	2.25	8.90	1.26	6.08	8.99
BR24N	21	8.88	0.83	12.59	0.10	6.46	18.75
BR24	23	8.88	2.15	9.67	0.94	6.18	5.67
BR27	30	10.13	3.51	10.30	1.38	6.57	3.43
BR30	30	12.25	1.60	13.15	0.65	6.74	6.77
BR34	30	8.25	2.50	8.75	0.86	6.54	7.00
BR40	40	11.00	5.16	13.73	3.84	8.13	5.55

Note: The table depicts the main statistics of bonds bought back by the Brazilian government between 2006 and 2018. All bonds are denominated in USD and issued prior to 2006. For each bond the table gives the maturity at issuance, the coupon rate, the total amount issued, the issue yield to maturity, the total amount bought back, the buyback yield to maturity and the average premium computed according to (20).

As the borrower issues bonds on the primary market and potentially buy them back on the secondary market, there are two yields to maturity to consider. The first one is the issue yield to maturity, i_I, which gives the bond's expected total return at issuance. The second one is i_B necessary to compute B. The buyback premium at issuance is therefore

$$\frac{FV^B(t_B, \kappa, b, B, i_I) - FV^{NB}(T, \kappa, b, i_I)}{FV^{NB}(T, \kappa, b, i_I)} \tag{20}$$

Note that I can use the yield to maturity at issuance only if the lenders do not expect buybacks to happen in the future. That is why I only consider bonds issued prior to the start of the Early Redemption Program.

Figure 3 depicts the buyback premium in percent by year and by bond. At the issuance, the premium averages 5.79%. The maximum and the minimum premium amounts to 22.54% and 0.13%, respectively. Two points deserved to be noted. First, the buyback premium is always strictly positive. This means that the Brazilian government never had any discount on its buybacks. Second, there is no direct link between the face or the financial value of the buyback and the premium. In addition, one observes in Table 1 that the issue yield to maturity is almost always larger than the coupon rate. However, the opposite is true for the buyback yield to maturity. This means that bonds were issued below the par value but bought back above the par value. This explains the presence of a buyback premium. Second, the Brazilian government bought back on average 45% of the initial issuance. Finally, all bonds involved in the buyback had a maturity of at least 4 years and at most 40 years at issuance.

My estimation strategy differs from the one of Bulow and Rogoff (1988) in the case of Bolivia. The two autors take the difference in the value of outstanding debt in the secondary market before and after the buyback. They then compare this difference with the face value of the buyback. This comparison omits the distinction between the primary and the secondary market. It measures the premium as if the debt was issued in the secondary market prior to the buyback.

Table 2: Regression Analysis

	(1) Premium	(2) Premium	(3) Premium
Real GDP Cycle	33.87***		46.80***
	[11.16]		[11.17]
External Debt to GDP		0.03***	0.05***
		[0.01]	[0.01]
Year FE	Yes	Yes	Yes
Bond FE	Yes	Yes	Yes
Observations	295	295	295
R^2 adjusted	0.82	0.82	0.83

Note: *** $p < .01$, ** $p < .05$, * $p < .10$. Robust standard errors in brackets.

Finally, I test whether the buyback evolves alongside the economic condition of Brazil. Table 2 depicts the outcome of a regression of the buyback premium on the level of GDP and the level of external debt at the quarterly frequency. The premium positively and significantly correlates with both output and indebtedness. It therefore seems that there is some degree of state-contingency in the premium. In particular, the regression suggests that χ increases in y and $-(b_{st} + b_{lt})$.

6.2 Calibration and Outcome

I calibrate the Markov equilibrium with default as it corresponds to the workhorse model in the literature on sovereign defaults. The calibration aims at matching some specific moments of the Argentine economy over the period 1995-2019. To properly compare the two Markov equilibria in terms of welfare, I assume that Argentina and Brazil are identical in terms of economic fundamentals and use the same calibration to solve the two Markov equilibria and the constrained efficient allocation. The only difference lies in the specification the *new* lender's offer (i.e. whether $\varpi_0 = 1$ or $\varpi_0 = 0$).

When calibrating the model, I relax the following assumptions. First, the endowment shock follows a Markov chain of order one with $\pi(y'|y)$ corresponding to the probability of drawing y' tomorrow conditional on drawing y today. Second, upon default, the borrower can re-access the market with probability $\lambda \in [0, 1]$. Third, there is an output penalty when defaulting denoted by $y^D \leq y$.

Table 3: Calibration

Parameter	Value	Description	Targeted Moment
A. Based on Literature			
ϑ	2.00	Relative risk aversion	
r^f	0.01	Risk-free rate	
B. Direct Measure from the Data			
$\pi(y_H\|y_H)$	0.95	Probability staying high state	
$\pi(y_L\|y_L)$	0.63	Probability staying low state	Real GDP cycle
y_L	0.89	Productivity in low state	
y_H	1.02	Productivity in high state	
λ	0.271	Probability re-accessing market	Default duration
r^e	0.04	Excess return	US excess return on debt
C. Based on Model Solution			
β	0.906	Discount factor	Debt-to-GDP ratio
φ	0.30	Scale parameter	Volatility of consumption relative to output
ψ	0.9158	Default penalty	Average spread
χ	0.057	*Official* buyback premium	Premium at issuance

Table 3 summarizes each parameter. I set $\vartheta = 2$ as it is standard in the real business cycle literature and select a discount factor of 0.906 to match the average public sector external debt-to-GDP ratio of 28.71%. In addition, I estimate the Markov transition matrix by means of a Markov-switching AR(1) process with two states. For this, I use data on the log real GDP of Argentina from 1995 to 2019 HP filtered with a smoothing parameter of 100. I then rescale the shock vector by multiplying it by φ to match the relative volatility of consumption.

Regarding the exogenous rate r, I consider that $r = r^f + r^e$ where r^f represents the risk-free rate and r^e corresponds to the lenders' excess return. This means that the lenders borrow at r^f and lend at $r > r^f$. This has two purposes. First, it better captures the potential risk premium US investors demand on emerging market bonds. I therefore set $r^e = 0.0434$ consistent with the US excess return on debt instruments estimated by Gourinchas et al. (2017) and $r^f = 0.01$ as it is standard in the literature. Second, as the spread is calculated with respect to the risk-free rate, modelling an excess return enables to correct the negative spread which has little empirical support for the countries under study.

Given that the default of Argentina in 2001 lasted 3.7 years, I set $\lambda = 0.271$. In addition, I consider an asymmetric default penalty as in Arellano (2008). Formally, upon default the endowment is given by $y^D = \min\{y, \psi \mathbb{E}(y)\}$. I set ψ to match the average (EMBI) spread of 8.6%. Finally, for the *official* buyback premium, I set χ to match the average buyback premium at issuance estimated for the Brazilian Early Redemption program. In the model, the premium at issuance corresponds to the wedge between the bond price q_{lt} and the return with neither default nor buyback (i.e. $\frac{1}{r}$).

Note that a buyback is such that $b'_l \geq b_l$. Hence, when $\varpi_0 = 0$, depending on the location of the enforcement zone, the steady state may feature no *official* buyback as b'_l may eventually be zero. To avoid that, I assumes that all *official* buybacks are weak (i.e. $b'_l = b_l$). This implies multiple steady states which depend on the initial $b_{l,0}$. I therefore select $b_{l,0}$ to get as close as possible to the public sector external debt-to-GDP ratio of Brazil.

Table 4: Targeted and Non-Targeted Moments

A. Targeted Moments					
Variable	Argentina	MA	Brazil	MAND	CEA
Average $-b/y$	28.71	27.82	10.12	9.52	-982.30
Average spread	8.59	9.02	4.97	4.19	4.23
$\sigma(c)/\sigma(y)$	1.08	1.16	1.07	1.01	0.30
Average issue premium	-	-	5.89	5.89	4.45
B. Non-Targeted Moments					
Variable	Argentina	MA	Brazil	MAND	CEA
$\sigma(tb/y)/\sigma(y)$	0.50	0.58	0.48	0.01	2.17
$\sigma(spread)$	0.05	0.01	0.03	0.00	0.01
$\text{corr}(c, y)$	0.96	0.87	0.94	0.99	0.92
$\text{corr}(tb/y, y)$	-0.82	-0.03	-0.58	-0.99	0.98
$\text{corr}(spread, y)$	-0.47	0.12	-0.33	0.00	-0.85

Note: The variable $\sigma(\cdot)$ denotes the volatility and tb/y denotes the trade balance over output. For the volatilities and correlation statistics, I filter the simulated data – except the spread – through the HP filter with a smoothness parameter of 100. MA refers to the Markov equilibrium with default, MAND to the Markov equilibrium without default and CEA to the constrained efficient allocation.

Table 4 compares the outcome of the Markov allocation with default (MA), without default (MAND) and the constrained efficient allocation (CEA) together. The upper part of the table presents the fit of the MA with respect to the Argentine economy in terms of targeted moments. As one can see, the MA replicates relatively well the main features of the Argentine economy in terms of consumption, spreads and indebtedness.

The lower part of Table 4 presents the fit of the MA in terms of non-targeted business-cycle moments. The MA generates a volatility of the trade balance close to the data. In addition, the trade balance is counter-cyclical. However, the model cannot reproduce the high volatility of the spread observed in the data. It also generates a positive correlation between spread and output as default arise in the transition from y_H to y_L and not after a sufficiently long series of y_L.

Turning to the MAND, Table 4 presents many similarities with Brazil. Brazil has not defaulted since the end of the 1980s, whereas Argentina defaulted in 2001 and got excluded from the international capital market until 2006. Second, as already shown, Brazil conducted an official buyback program since 2006. Third, in terms of economic fundamentals, Brazil records a lower average debt ratio and a lower average spread than Argentina for the period 1995 to 2019. The MAND is capable of matching most of the main moments of the Brazilian economy despite the fact that none of them where directly targeted. It therefore seems that, through the lens of the model, Brazil can be interpreted as the counterfactual of Argentina with buybacks and without default in the period 1995-2019. More importantly, this supports my claim that part of the difference between Argentina and Brazil can be attributed to the lenders.

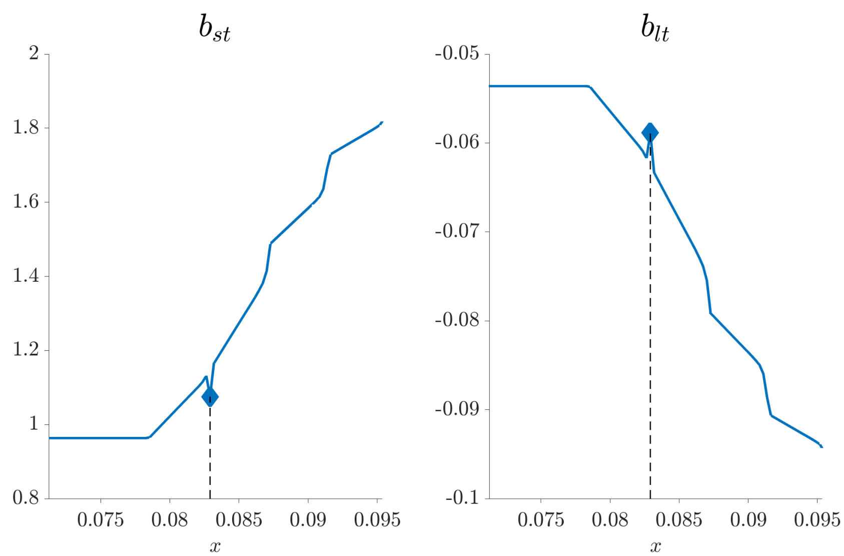
Relatedly, in Appendix E, I adapt the calibration of the MAND to Brazil. In particular, I directly estimate the endowment process from the log real GDP of Brazil. Similar to Arellano and Ramanarayanan (2012), the variance of the shock is lower for Brazil than for Argentina. In addition, the discount factor is higher, while the default cost is larger for Brazil. Nevertheless, the dynamic in steady state remains close to the one depicted here – i.e. the steady state is not dynamic.

Looking at the last column of Table 4, one directly observes that the CEA is associated to different moments than the two Markov equilibria. This means that none of the Markov allocation is constrained efficient. In addition, the implementations of the CEA predicts empirically implausible average indebtedness. In fact, the borrower holds a net asset position. Such prediction is well known in the literature on fiscal policy under commitment as highlighted by notably Buera and Nicolini (2004) and Faraglia et al. (2010). Even though I consider an alternative environment with limited commitment, the bond portfolio implementing the CEA remains at odds with the data.

6.3 Buyback and State Contingency

I investigate why the MAND is not constrained efficient. There are two elements. First, the calibrated buyback premium is too low for buybacks to be located in the enforcement zone. Second, the endogenization of χ through a Nash bargaining does not solve the problem.

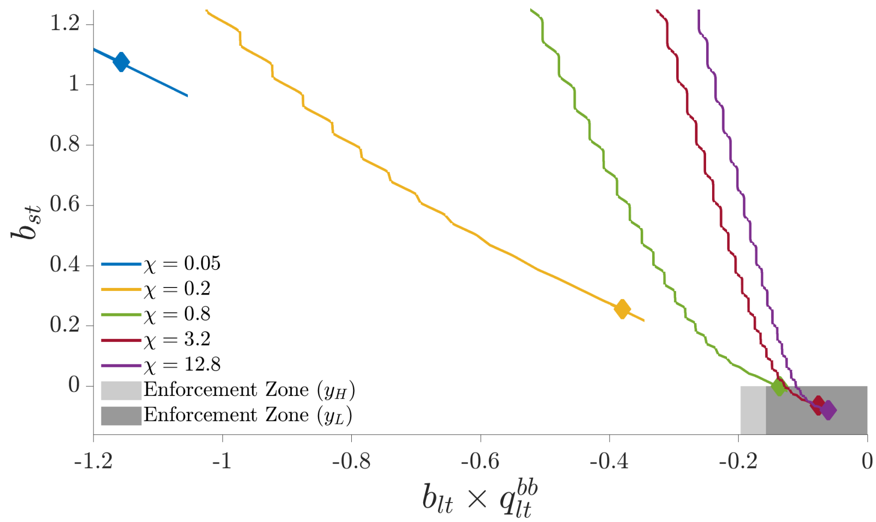
Figure 4 depicts the holdings of short-term and long-term debt that implement the CEA. The diamond marker represents the relative Pareto weight at which the *official* buyback occurs – i.e. $x = x^{ub}$. When x is low, the borrower accumulates more short-term debt and less long-term debt. In opposition, when x gets larger, the opposite is true. Furthermore, the borrower holds short-term assets – especially when *official* buybacks occur. This means that the point of *official* buyback is outside the enforcement zone which explains the fact that the MAND is not constrained efficient.



Note: The figure holdings of short-term and long-term debt that implement the CEA as a function of the relative Pareto weight. The diamond marker represents the point of *official* buyback.

Figure 4: Bond Holdings in the CEA

Figure 5 depicts the holdings of short-term and long-term debt that implement the CEA for different values of χ. Again, the diamond marker represents the point of *official* buyback. The grey region corresponds to the enforcement zone. To obtain short-term debt holdings when *official* buybacks occur, the value of χ should be larger than the calibrated one. The rationale behind this is that with a larger premium, the long-term bond price is more sensitive to the realization of y. As a result, more short-term debt and less long-term debt are required to replicate the state contingency in the Planner's allocation. As one can see, it is possible that the borrower holds short-term debt – and not asset – by multplying χ by more than 13 relative to the calibration benchmark. This also brings the point of *official* buyback in the enforcement zone.[26] Hence, the market economy fails to implement the Planner's allocation under empirically plausible *official* buyback premia in emerging economies.



Note: The figure depicts the holdings of short-term and long-term debt (multiplied by the buyback price) that implement the CEA for different values of χ. The grey area depicts the enforcement zone and the diamond marker represents the point of *official* buyback.

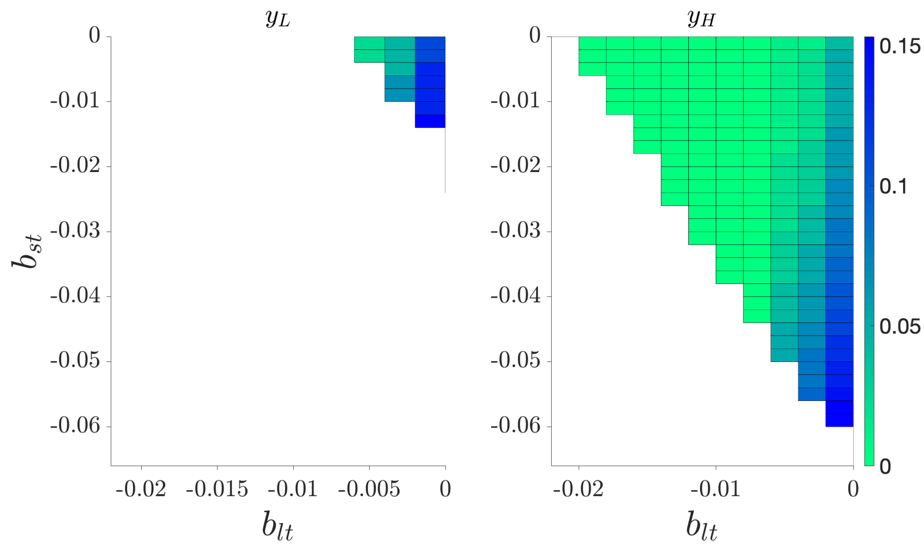
Figure 5: Enforcement Zone and χ

Thus, with respect to the literature on fiscal policy under commitment and the findings of Buera and Nicolini (2004) and Faraglia et al. (2010), I reconcile the model's prediction with the data by arguing that the lender lacks the enforcement power to implement the CEA. Under an empirically plausible cost of *official* buyback, the economy can only approximate – as opposed to replicate – the returns of Arrow securities with non-contingent bonds of multiple maturities. Nevertheless, such approximation is consistent with the sovereign debt management of emerging economies as shown previously.

In the model χ is constant. However, Table 2 suggests some degree of state contingency in the data. To model this, I endogenize the buyback premium by means of a Nash bargaining protocol. The details of the protocol can be found in Appendix B. I set the bargaining power of the lender to match the average buyback premium estimated for the Brazilian Early Redemption program. I obtain a value of 0.55 which is below the one of 0.28 found by Yue (2010) in the case of default renegotiation.

Figure 6 depicts the outcome of the Nash bargaining protocol. One observes three elements. First, the premium is larger when y is lower. This is because the threat of sudden

[26]Note that for $\chi = 3.2$ and $\chi = 12.8$, the point of *official* buyback is also within the enforcement zone. However, holdings of debt in the vicinity of this point are also within the zone meaning that *official* buybacks would occur more often than the Planner desires to



Note: The figure depicts the premium χ as the outcome of a Nash bargaining protocol for specific values of y, b_{st} and b_{lt}. The white region is the part of the state space in which the no-default constraint does not hold.

Figure 6: Nash Bargaining and χ

stop is more credible when endowment is low. Second, for the same reason, the higher is the level of short-term debt, the higher is the premium. Third, the buyback premium increases when the level of long-term debt reduces. This is because the debt service cost is larger with more long-term debt.

The outcome of the Nash bargaining is problematic for the following reasons. First, the largest premium the protocol can generate is still related to short-term assets in the implementation of the CEA. Second, the largest premium arises in y_L with a relatively high (low) stock of short-term (long-term) debt. However, looking at Figure 4, *official* buybacks arise in y_H with a relatively low (high) level of short-term (long-term) debt. Hence, a Nash bargaining protocol does not enable the economy to get closer to the constrained efficient benchmark.

6.4 Welfare

I explore the differences in welfare for the Markov equilibria and the CEA. First, I compute welfare gains with respect to the MA. Second, I measure the distance of each equilibria from the Pareto frontier.

Table 5 depicts the welfare gains of the CEA and the MAND in consumption equivalent terms with respect to the MA for both the borrower and the lenders. Welfare gains are computed through the simulation of 5,000 independent shock histories starting with initial

Table 5: Steady State Welfare Analysis

State	Borrower welfare gains (percent) MAND	Borrower welfare gains (percent) CEA	Lenders welfare gains (percent) MAND	Lenders welfare gains (percent) CEA	$\mathcal{F}(g)$ (percent) MA	$\mathcal{F}(g)$ (percent) MAND	$\mathcal{F}(g)$ (percent) CEA
y_H	0.067	0.069	2.569	4.462	56.6	63.6	100.0
y_L	0.046	0.050	4.373	7.260	32.5	47.7	100.0
average	0.064	0.067	2.769	4.772	53.9	61.9	100.0

Note: The table presents the welfare gains in consumption equivalent relative to the MA. See Appendix G for details on the computation. MA refers to the Markov equilibrium with default, MAND to the Markov equilibrium without default and CEA to the constrained efficient allocation.

debt holdings and relative Pareto weights drawn from the ergodic set. The details of the welfare computation are presented in Appendix G.

As one can see, the CEA and the MAND imply substantial welfare gains compared to the MA, on average for both the lenders and the borrower. The CEA leads to the largest welfare gains in all states for all market participants. Those are more pronounced when y_H realizes. Hatchondo et al. (2020a) find similar results when comparing the MA with a Ramsey plan.

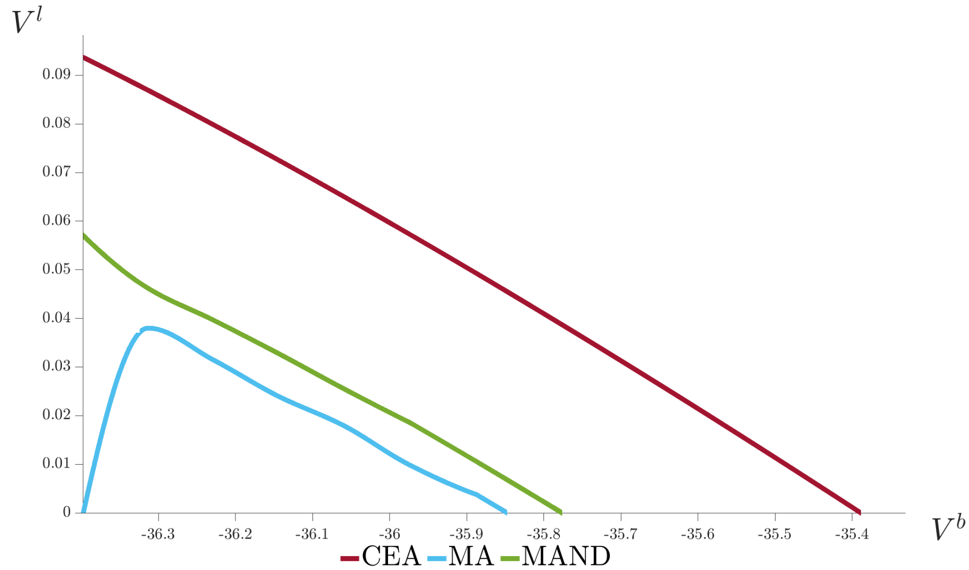
Table 6: Borrower Welfare Decomposition

State	MAND State contingency (percent)	MAND Cost of default (percent)	CEA State contingency (percent)	CEA Cost of default (percent)
y_H	99.75	0.25	97.19	2.81
y_L	98.68	1.32	92.56	7.44
average	99.63	0.37	96.68	3.32

Note: The table presents the decomposition of the borrower's welfare gains in two factors: cost of default and state-contingency. The gains related to state contingency come from the computation of the MAND without *official* buybacks. The residual welfare gains is attributed to the cost of default. MA refers to the Markov equilibrium with default, MAND to the Markov equilibrium without default and CEA to the constrained efficient allocation.

In Table 6, I decompose the borrower's welfare gains of the MAND and the CEA by calculating the percentage of gains that can be attributed to the following two factors: cost of default and state-contingency. I isolate those two factors in the following way. To compute the gains related to state contingency, I compute the MAND without *official* buybacks. The residual welfare gains can then be attributed to the cost of default. Doing so I find that, in the MAND, 99% of the welfare gains come from the state contingency on average and the remaining part can be attributed to the cost of default. In the CEA, I find that 97% of the gains come from state contingency and the rest comes from the cost of default.

Besides the welfare gains, I can compute the distance with respect to the Pareto frontier.



Note: The figure depicts the utility possibility frontiers related to the MA, the MAND and the CEA. Those frontiers express the value of the lenders V^l as a function of the value of the borrower V^b. The CEA is in red, the MA in blue and the MAND in green. MA refers to the Markov equilibrium with default, MAND to the Markov equilibrium without default and CEA to the constrained efficient allocation.

Figure 7: Distance to Pareto Frontier in y_L

For this purpose, I derive a metric measuring the distance between the constrained efficient allocation and any alternative allocation. For a given state, it is always possible to map the value of the lenders with the value of the borrower. I can therefore express the value of the lenders in any Markov equilibrium as a function of the borrower's value, say $V^b \in \mathcal{V} \equiv [V^D(y_L), \frac{u(c(y_H,\bar{x}))}{1-\beta}]$, instead of (b_{st}, b_{lt}), i.e. $\ddot{V}^l : Y \times \mathcal{V} \to \mathbb{R}$. I then define

$$\mathcal{F}(y) = \frac{\int_{\mathcal{V}} \ddot{V}^l(y,v)dv}{\int_{\underline{x}}^{\bar{x}} V^l(y,x)dx}.$$

The metric $\mathcal{F}(y)$ measures the distance between the Markov allocation and the CEA. Given Proposition 3 Part I, it is bounded between 0 and 1. A value of $\mathcal{F}$ near 1 indicates that an allocation is close to the constrained efficient benchmark, whereas a value close to 0 indicates the opposite.[27]

Figure 7 depicts the different frontiers: in red the Pareto frontier and in blue and green the utility possibility frontier related to the MA and the MAND, respectively. Defaults in the MA produce an upward sloping part of the frontier in which both the borrower and the lenders can be made better off. Neither the CEA nor the MAND display such upward slope.

[27]The metric $\mathcal{F}(y)$ is based on the same concept as the Gini coefficient which measures the distance between the Lorenz curve and the equity line.

This shows the inefficiency of default (see Fudenberg et al. 1990).

Looking at the metric $\mathcal{F}(y)$ in the last column of Table 5, the MAND is superior to the MA but not to the CEA. More precisely, the MA is relatively far from the CEA and the MAND can get the economy closer to it. On average, the MA and the MAND have a value of 0.54 and 0.62, respectively. The MAND therefore provides a better approximation of the CEA than the MA. Nevertheless, the MAND remains far from the CEA meaning that there is still room for policy intervention. The metric $\mathcal{F}(y)$ is important as it relates to the entire value of the debt contract (i.e. the combined value for the borrower and the lenders) and not only on the steady state unlike the welfare gains computed above.

7 Conclusion

This paper derives the constrained efficient allocation emanating from a central planning problem to deduce optimal sovereign debt management policies. The bottom line is that the reliance on defaults on equilibrium path is inefficient. Instead, changes in maturity and costly debt buybacks can implement the constrained efficient allocation. Nevertheless, the implementation requires an appropriate enforcement power of the lenders. Collusion between the lenders is sufficient to avoid default but is not to ensure premium-bearing buybacks. I relate this result to the experience of Argentina and Brazil since 1995.

This study stresses the fact that the strategic interaction of the lenders is key. The literature on sovereign debt and default has focused on the borrower's side. However, it is possible to explain a variety of alternative dynamics in equilibrium by looking at the lenders and the way they interact.

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Appendix

A Discussion on Alternative Implementations

This section discusses the relationship between the implementation presented in Section 5 and the main alternatives that exist in the literature.

Dovis (2019) considers an environment similar to the one presented in Section 3 with the only difference that $y = zf(m)$ where m is some intermediary good and z is a privately-observed shock to the production technology f. He derives an optimal contract subject to a participation and an incentive compatibility constraint to account for limited commitment and adverse selection, respectively. He subsequently decentralizes the aforementioned contract through partial defaults and an active debt maturity management. The main difference with my study is twofold. First, he explicitly uses defaults – instead of costly buybacks – to implement the constrained efficient allocation. This is because the combination of limited commitment and adverse selection generates a region of *ex post* inefficiencies in which the Planner implements the autarkic allocation.[28] As I only consider limited commitment, this region does not exist in my analysis – as shown in Proposition 3 Parts I and IV . Thus, my implementation works in the environment of Dovis (2019), while the opposite is not true.[29] Second, Dovis (2019) relies on sustainable equilibria, whereas I focus on Markov equilibria. As I cannot use trigger strategies, my implementation explicitly requires the presence of a no-default and a buyback enforcement constraint.

Besides this, Alvarez and Jermann (2000) propose a way to implement the allocation derived in Kehoe and Levine (1993) through Arrow securities and endogenous borrowing limits. My implementation relies on changes in the term premium to mimic the state-contingency in the Planner's allocation, while their implementation relies on changes in security holdings provided that securities are state-contingent. Thus, the main difference with my analysis is that the two authors assume a greater financial sophistication as securities are state contingent, while I generally need high enforcement power of the lenders to implement the appropriate default and buyback policies.

The study of Müller et al. (2019) considers a small open economy with a stochastic default cost and two endowment states: recession and normal time. The authors assume a financial market formed by two securities: a one-period non-contingent defaultable bond and a state-contingent bond which pays out only in normal time (i.e. GDP-linked bond). The authors

[28] Using different environments, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007) and Yared (2010) also characterize a region of *ex-post* inefficiencies in optimal contracts.

[29] In general, the implementation in Dovis (2019), does not apply to renegotiation-proof contracts, while mine applies to contracts with or without *ex post* inefficiencies.

additionally assume that the borrower lacks commitment only in recession and renegotiation upon default is endogenous. This coupled with the aforementioned market structure, enables the two bonds to act as Arrow securities. In other words, the defaultable bond is recession contingent and spans the different stochastic default costs through renegotiation, while the contingent bond spans the good state which is free from default risk. Hence, the bonds act as proper Arrow securities.

The last study that I would like to discuss is the one of Aguiar et al. (2019) who consider a small open economy with a stochastic default cost and two endowment states as in Müller et al. (2019). The authors assume a continuum of maturities. They show the equivalence between the Markov equilibrium and the constrained efficient equilibrium. The Planner's problem is nonetheless peculiar as it does not take into consideration the legacy creditors in the surplus maximization. In other words, the Planner problem is sequential and only accounts for the current creditors, taking as given the inherited debt level. Furthermore, there is no participation constraint of the borrower. That is, the Planner cannot prevent the occurrence of defaults on equilibrium path. Hence, in the absence of a participation constraint – i.e. a forward-looking constraint – the Planner needs not build on past history. This combined with the disregard of legacy creditors directly leads to the Markov equilibrium in the spirit of Eaton and Gersovitz (1981).

B Foundations for Costly Debt Buybacks

In what follows, I endogenize the cost of *official* buybacks in two ways: a Nash bargaining protocol and a signalling game.

Before that, I present the mechanism of Bulow and Rogoff (1988, 1991) and highlight why this does not suit my framework. The two authors show that a buyback increases the value of debt as the recovery value is divided among fewer creditors. To see this, consider a Markov equilibrium in which $\mathbb{E}D(\Omega') > 0$. In addition, suppose that there is a re-access probability λ and a fixed recovery value of w after default. The bond price can therefore be separated into two parts: the return when the borrower decides to repay and the recovery value when the borrower defaults.

$$q_{lt}(s, b'_{st}, b'_{lt}) = \mathbb{E}\Bigg[(1 - D(\Omega'))q^P_{lt}(s', b'_{st}, b'_{lt}) + D(\Omega')q^D_{lt}(s', b'_{st}, b'_{lt})\Bigg],$$

where the recovery value is given by

$$q^D_{lt}(s', b'_{st}, b'_{lt}) = \frac{1}{1+r}\Bigg[(1 - \lambda)q^D_{lt}(s', b'_{st}, b'_{lt}) + \lambda\frac{w}{b'_{lt}}\Bigg],$$

and the repayment value reads

$$q_{lt}^P(s', b'_{st}, b'_{lt}) = \frac{1}{1+r}\Big[1 + (1 - M(\Omega'))q_{lt}(s', b''_{st}, b''_{lt}) + M(\Omega')q_{lt}^{bb}(s', b'_{st}, b'_{lt})\Big].$$

The buyback premium naturally emerges from the bond price as w is constant. When $-b'_{lt}$ increases, q_{lt}^D decreases which implies that q_{lt} decreases given the strictly positive default probability. This is what the literature calls dilution. With a buyback the opposite happens as $-b'_{lt}$ decreases. There is a *reverse* dilution which increases q_{lt}^D and therefore q_{lt}.

This mechanism however works as long as there is a strictly positive default probability. If defaults never arise on equilibrium path, the long-term bond price remains constant. Moreover, this mechanism can only rationalize buybacks at a discount (i.e. below par) on the secondary market. For instance, it cannot explain the case of Brazil which bought back its debt at a premium as shown in Section 6.

B.1 Nash Bargaining

I introduce a Nash bargaining protocol in the Markov equilibrium. I assume below that $\varpi_0 = 0$ meaning that the *new* lender makes an offer that satisfies the *legacy* lender's problem. When $\varpi_0 = 1$, the outcome is the same as the one presented in the main text.

Given the constraint $\Gamma(\Omega)$, the threat point is that the borrower is not able to roll over its debt in the current period if the *official* buyback does not take place. In such circumstance, the borrower's value is given by

$$\begin{aligned}
\bar{V}^{NB}(\Omega) = &\max_{b'_{st}, b'_{lt}} u(c) + \beta\mathbb{E}\Big[W^b(\Omega')\Big] \\
&\text{s.t.} \quad c + q_{st}(s, b'_{st}, b'_{lt})b'_{st} + q_{lt}(s, b'_{st}, b'_{lt})(b'_{lt} - b_{lt}) = y + b_{st} + b_{lt}, \\
&\qquad\; b'_{lt} \geq b_{lt}, \\
&\qquad\; b'_{st} \geq 0.
\end{aligned}$$

Notice that the borrower can issue short-term assets. For any *official* buyback premium χ, I define the surplus of the borrower as

$$\Delta^b(\Omega; \chi) = V^B(\Omega; \chi) - \max\Big\{\bar{V}^{NB}(\Omega), V^D(y)\Big\}.$$

The borrower's surplus corresponds to the difference between the value of conducting the *official* buyback and the value of rejecting it and suffering the underlying sudden stop.

To define the surplus of the *legacy* lender, I first need to derive the *legacy* lender's value under *official* buyback, under no *official* buyback and under default. The former reads

$$W^l_{\text{legacy},B}(\Omega;\chi) = \max_{(b'_{st},b'_{lt})\in\Gamma(\Omega)} -\left[b_{st} + b_{lt}(1 + q_{lt}(s, b'_{st}, b'_{lt}) + \chi)\right]$$

while under no *official* buyback

$$\bar{W}^l_{\text{legacy},NB}(\Omega) = \max_{(b'_{st},b'_{lt})\in\Gamma(\Omega)} -\left[b_{st} + b_{lt}(1 + q_{lt}(s, b'_{st}, b'_{lt}))\right]$$

and finally, under default we have $W^l_{\text{legacy},NB}(s, 0, 0) = 0$. The surplus of the lenders corresponds to the difference in the value under *official* buyback and no *official* buyback

$$\Delta^l_{\text{legacy}}(\Omega;\chi) = W^l_{\text{legacy},B}(\Omega;\chi) - \Big[(1 - D(\Omega))\bar{W}^l_{\text{legacy},NB}(\Omega) + D(\Omega)W^l_{\text{legacy},NB}(s, 0, 0)\Big].$$

If the *legacy* lender has all the bargaining power, then it could extract a large *official* buyback premium (i.e. $\chi \to \infty$). In opposition, if the borrower has all the bargaining power, it can conduct *official* buybacks at low cost (i.e. $\chi \to 0$). To consider the case in between those two extremes, I assume that the lenders have a bargaining power of $\zeta \in [0, 1]$ and the borrower of $1 - \zeta$. In Ω, the *official* buyback premium $\chi(\Omega)$ is the solution to

$$\begin{aligned}
\chi(\Omega) = \arg\max\nolimits_{\tilde{\chi}\geq 0} & \left[\Delta^l_{\text{legacy}}(\Omega;\tilde{\chi})^{\zeta} + \Delta^b(\Omega;\tilde{\chi})^{1-\zeta}\right] \\
\text{s.t.} \quad & \Delta^l_{\text{legacy}}(\Omega;\tilde{\chi}) \geq 0, \\
& \Delta^b(\Omega;\tilde{\chi}) \geq 0.
\end{aligned}$$

I can now establish one important property of the Nash baragining outcome. The buyback premium decreases with the amount of long-term debt, and there is no premium for debt levels smaller than a given threshold which depends on the endowment and the level of short-term debt.

Lemma B.1 (State-Contingent Premium)**.** *For a bargaining power ζ, there exists a threshold $\bar{B}(y, b_{st}) < 0$ such that the equilibrium buyback premium satisfies:*

$$\chi(\Omega) = \begin{cases} \frac{\bar{B}(y)-b_{st}}{b_{lt}} - 1 & \textit{if } b_{lt} > \bar{B}(y, b_{st}) \\ 0 & \textit{else} \end{cases}$$

The buyback premium is thus state contingent. In addition, in light of Lemma 1, this means that the buyback premium gets larger as the threat of the sudden stop grows. Particularly, the premium is larger when the level of short-term debt is large or the endowment is low.

B.2 Signalling in Markov Equilibrium

Besides the Nash bargaining protocol, I can rationalize costly *official* buybacks with a signalling game. For this purpose, consider that y is privately observed by the borrower. The lenders must therefore form beliefs on y – i.e. the borrower's type.

To be an equilibrium, beliefs have to be consistent with the market participants' strategies and, given the beliefs, each market participant's strategy must be optimal. A belief system for the lenders, $\mathcal{H}(b_{st}, b_{lt})$, specifies the a probability distribution over Y,

$$\mathcal{H}(b_{st}, b_{lt}) = Pr(y = y_H | b_{st}, b_{lt}).$$

The lenders rely on the debt repayment, say S, as signal for the borrower's type. I therefore construct a separating equilibrium in which the borrower signals its type through debt repayment as in Cole et al. (1995) and Phan (2017a,b). The timing of actions is the following. First, y realizes and is privately observed by the borrower which then decides how much debt to repay, S. Conditional on the repayment, the lenders offer a bond price schedule $q_{st}(S, b'_{st}, b'_{lt,H})$ and $q_{lt}(S, b'_{st}, b'_{lt})$ for the short-term bond, b'_{st}, and the long-term bond, b'_{lt}, respectively.[30]

The repayment signal works as follows. If the repayment is sufficiently large, then the lenders believe that y_H realized. In opposition, a low repayment signals that y_L realized. The signal therefore fully reveals the shock. However, to be an equilibrium, the low type should not be willing to choose a high repayment and *vice versa*.

I assume the following. If the borrower draws y_H, it chooses to conduct an *official* buyback. In opposition, if it draws y_L, there is neither an *official* buyback nor a default. Hence, the repayment of the high type for a given (b_{st}, b_{lt}) is

$$S_H(b_{st}, b_{lt}) = b_{st} + b_{lt}(1 + q^{bb}_{lt}),$$

and for the low type,

$$S_L(b_{st}, b_{lt}) = b_{st} + b_{lt}(1 + q_{lt}).$$

With $\chi > 0$, it directly follows that $S_L(b_{st}, b_{lt}) > S_H(b_{st}, b_{lt})$ for all (b_{st}, b_{lt}). Thus, costly

[30]Note that similar to the timing in Eaton and Gersovitz (1981), it is implicitly assumed that the borrower can commit to repayment decision made before the debt auction. See Cole and Kehoe (2000) and Ayres et al. (2018) for more details.

official buybacks are necessary to signal types in the absence of defaults. Whenever the lenders receive a repayment lower than $-S_H(b_{st}, b_{lt})$, it believes that y_L realized. Obviously, those beliefs are consistent only if the high type has no incentive to repay according to the low type and *vice versa*. Thus, it must hold that for any (b_{st}, b_{lt}),

$$
\begin{aligned}
&u(y_H + S_H(b_{st}, b_{lt}) - q_{lt}(S_H, b'_{st,H}, b'_{lt,H})b'_{lt,H} - q_{st}(S_H, b'_{st,H}, b'_{lt,H})b'_{st,H}) = \\
&u(y_H + S_L(k, b_{st}, b_{lt}) - q_{lt}(S_L, b'_{st,L}, b'_{lt,L})b'_{lt,L} - q_{st}(S_L, b'_{st,L}, b'_{lt,L})b'_{st,L}),
\end{aligned} \tag{B.1}
$$

where $b'_{j,i}$ denotes the bond choice of maturity $j \in \{st, lt\}$ of reported borrower's type $i \in \{L, H\}$. Equation (B.1) makes the high type indifferent between paying S_H or S_L. Moreover, it implies that

$$
\begin{aligned}
&u(y_L + S_H(b_{st}, b_{lt}) - q_{lt}(S_H, b'_{st,H}, b'_{lt,H})b'_{lt,H} - q_{st}(S_H, b'_{st,H}, b'_{lt,H})b'_{st,H}) = \\
&u(y_L + S_L(k, b_{st}, b_{lt}) - q_{lt}(S_L, b'_{st,L}, b'_{lt,L})b'_{lt,L} - q_{st}(S_L, b'_{st,L}, b'_{lt,L})b'_{st,L}).
\end{aligned}
$$

As a result, if the the high type indifferent between paying S_H or S_L, the low type has no incentive to pay S_H instead of S_L. Thus, if (B.1) holds, the beliefs are updated according to

$$
\mathcal{H}(b_{st}, b_{lt}) = \begin{cases} 1 & \text{if} \quad S \leq S_H(b_{st}, b_{lt}) \\ 0 & \text{else} \end{cases}
$$

By Proposition 3 in Phan (2017b), the set of strategies and beliefs presented in this subsection constitutes a separating Markov equilibrium.

We see that from the definition of S_H and S_L, the *official* buyback premium χ ought to be strictly larger than zero for the signal to be informative.[31] On the other hand, as $\chi \to \infty$, $S_H(b_{st}, b_{lt}) \to \infty$. By (B.1) this would imply that the low type has to accumulate an infinite amount of assets. There is therefore a cap on how large χ can be and – similar to Lemma 1 – the lower is the *official* buyback premium, the easier is (B.1) satisfied in a given state Ω.

C Alternatives to Official Buybacks

In this section, I provide alternatives to *official* buybacks: "excusable" defaults, variable-coupon bonds and variable-maturity bonds.

First, Grossman and Van Huyck (1988) develop the concept of "excusable" defaults.[32]

[31] Otherwise, instead of *official* buybacks, the borrower would need to rely on defaults to signal its type as in Cole et al. (1995) and Phan (2017a,b).

[32] See also the recent proposal on contingent convertible bonds by Hatchondo et al. (2022).

The idea is that defaults which are on the path of play agreed by all market participants are not punished. In other words, the debt contract specifies *ex ante* the circumstances in which the borrower is allowed to repudiate its debt without suffering from markets exclusion. Given this, if defaults were "excusable", then the borrower's binding constraint – i.e. $x = x^{lb}$ – could be interpreted as a default. The issue is that the borrower might be willing to repudiate debt more often than what the debt contract specifies. To deal with this, one can use endogenous borrowing limits as in the main analysis. Nevertheless, the concept of "excusable" defaults has little empirical relevance. The closest policy that has been implemented to this date is a sovereign debt standstill analyzed by Hatchondo et al. (2020b) with the only difference that there is no arrears accumulation in "excusable" defaults. In addition, Mateos-Planas et al. (2023) show that if the borrower were to choose the conditions for "excusable" defaults, such events would be extremely rare if not inexistent.

Second, the long-term debt can have variable coupon as in Faraglia et al. (2019) and Aguiar et al. (2021). Particularly, assume that the coupon payment is a choice variable, say $\kappa \in [0, 1]$, for the borrower. Obviously, the variability of the coupon is a covenant in the debt contract. In other words, changes in coupon are agreed by the contracting parties *ex ante* and do not pertain to a contract renegotiation – e.g. an outright default in case of reduced coupon payment. With such debt contract, it is possible to implement the constrained efficient allocation in two ways: the borrower sets a standard coupon payment $\tilde{\kappa}$ and either increases it to $\bar{\kappa} > \tilde{\kappa}$ when $x = x^{ub}$ or decreases it to $\underline{\kappa} < \tilde{\kappa}$ when $x = x^{lb}$. In the former case, a variant of Proposition 2 applies as the borrower is not willing to pay a larger coupon payment. Hence, the same enforcement issue arises as with *official* buybacks. In opposition, in the case of reduced coupon payment, the borrower might be tempted to reduce the coupon payment more frequently than the Planner would. Thus, the lenders would also need to supervise the coupon policy.

Lastly, bonds can have variable maturities. That is, the maturity of outstanding short-term (long-term) debt can be lengthened (shortened). Similar to variable-coupon bonds, this is a feature which should be explicitly mentioned in the debt contract. To implement the constrained efficient allocation, the borrower ought to either lengthen the maturity of short-term debt when $x = x^{ub}$ or shorten the maturity of long-term debt when $x = x^{lb}$. Implicitly, by shortening the maturity, the borrower pays less coupons than it initially promised. In other words, the claim of legacy creditors is reduced. The opposite happens in the case of maturity lengthening. Thus, similar to variable-coupon bonds, maturity lengthening would need to be enforced, while maturity shortening should be closely supervised to avoid lowering legacy creditors' claim too frequently.

D Further Theory Developments

This section is composed of two parts. The first one comprises additional theory developments for the Markov equilibrium. The second one is about the Planner's allocation.

D.1 Markov equilibrium

Define $Z(b_{st}, b_{lt}) = \mathbb{E}[W^b(\Omega')]$ as the expected continuation value. In addition, the default set is given by

$$d(b_{st}, b_{lt}) = \min\Big\{\{y : V^P(y, b_{st}, b_{lt}) \geq V^D(y)\} \cup \{y_H\}\Big\}.$$

The following proposition shows monotonicity of the main policy functions and prices when $\varpi_0 = 1$.

Proposition D.1 (Monotonicity)**.** *When $\varpi_0 = 1$,*

1. *$V^P(y, b_{st}, b_{lt})$ is strictly increasing in y and strictly increasing in b_{st} and b_{lt}.*
2. *$Z(b_{st}, b_{lt})$ is non-decreasing in b_{st} and b_{lt}.*
3. *$d(b_{st}, b_{lt})$ is non-decreasing in b_{st} and b_{lt}.*
4. *$B_{st}(y, b_{st}, b_{lt})$ is non-decreasing in y and non-decreasing in b_{st} and b_{lt}. The same holds true for $B_{lt}(y, b_{st}, b_{lt})$ if $q_{lt}(b'_{st}, b'_{lt})$ is non-decreasing in b'_{st} and b'_{lt}.*
5. *There exist $q_{st}(y, b_{st}, b_{lt})$ and $q_{lt}(y, b_{st}, b_{lt})$ that are non-decreasing in b_{st} and b_{lt}.*

It is difficult to establish similar results for the case in which $\varpi_0 = 0$. The reason is that monotonicities shown in Proposition D.1 can easily be reverted depending on the location of the enforcement zone.

D.2 Planner allocation

The following proposition shows existence and uniqueness of the constrained efficient allocation under the assumption of interiority.

Proposition D.2 (Existence and Uniqueness)**.** *Under Assumption 1, there exists a unique allocation with initial condition (y_0, x_0).*

Having shown existence and uniqueness of the contract allocation, the following lemma derives the inverse Euler Equation which gives the consumption dynamic in the Planner's allocation.

Lemma D.2 (Inverse Euler Equation)**.** *The inverse Euler equation for a given $y \in Y$ reads*

$$\mathbb{E}\left[\frac{1}{u_c(c(y'))(1+\nu(y'))}\right] = \eta \frac{1}{u_c(c(y))},$$

If the participation constraint never binds, I obtain that for all (y, x),

$$\frac{1}{u_c(c(y))} \geq \mathbb{E}\left[\frac{1}{u_c(c(y'))}\right]$$

with strict inequality when $\eta < 1$. In this case, the inverse Euler Equation is a positive super-martingale. Immiseration is a consequence of Doob's theorem stating that such super-martingales converge almost surely. With $\eta < 1$, the inverse of the marginal utility of consumption converges to 0. Under limited commitment of the borrower (i.e. $\nu(y) \geq 0$), one obtains a left bounded positive submartingale. The borrower's participation constraints therefore sets an upper bound on the supermartingale and prevents immiseration. Alternatively, when $\eta = 1$ consumption remains constant whenever $\nu(y) = 0$.

Table D.1: Alternative Calibration

Parameter	Value	Description	Targeted Moment
A. Based on Literature			
ϑ	2.00	Relative risk aversion	
r^f	0.01	Risk-free rate	
B. Direct Measure from the Data			
$\pi(y_H\|y_H)$	0.79	Probability staying high state	Real GDP cycle
$\pi(y_L\|y_L)$	0.86	Probability staying low state	
y_L	0.98	Productivity in low state	
y_H	1.03	Productivity in high state	
λ	0.069	Probability re-accessing market	Default duration (p95)
r^e	0.04	Excess return	US excess return on debt
C. Based on Model Solution			
β	0.915	Discount factor	Debt-to-GDP ratio
ψ	0.986	Default penalty	Volatility consumption and output
χ	0.056	*Official* buyback premium	Premium at issuance
D. Based on Main Calibration			
φ	0.30	Scale parameter	

E Further Quantitative Developments

In the main calibration, I consider that Argentina and Brazil are subject to the same shock process. In particular, I estimate the process from the log real GDP of Argentina. I now consider a shock process estimated from the log real GDP of Brazil and adapt the calibration of the MAND accordingly.

Table E.2: Targeted and Non-Targeted Moments – Alternative Calibration

A. Targeted Moments			
Variable	Brazil	MAND	CEA
Average $-b/y$	10.12	8.43	-553.70
$\sigma(c)/\sigma(y)$	1.07	1.01	0.17
Average issue premium	5.89	5.89	1.81
B. Non-Targeted Moments			
Variable	Brazil	MAND	CEA
Average spread	4.97	4.19	4.29
$\sigma(tb/y)/\sigma(y)$	0.48	0.01	2.40
$\sigma(spread)$	0.03	0.00	0.04
corr(c, y)	0.94	0.99	0.99
corr$(tb/y, y)$	-0.58	-0.99	0.99
corr$(spread, y)$	-0.33	0.00	-0.81

Note: The variable $\sigma(\cdot)$ denotes the volatility and tb/y denotes the trade balance over output. For the volatilities and correlation statistics, I filter the simulated data – except the spread – through the HP filter with a smoothness parameter of 100. MAND to the Markov equilibrium without default and CEA to the constrained efficient allocation.

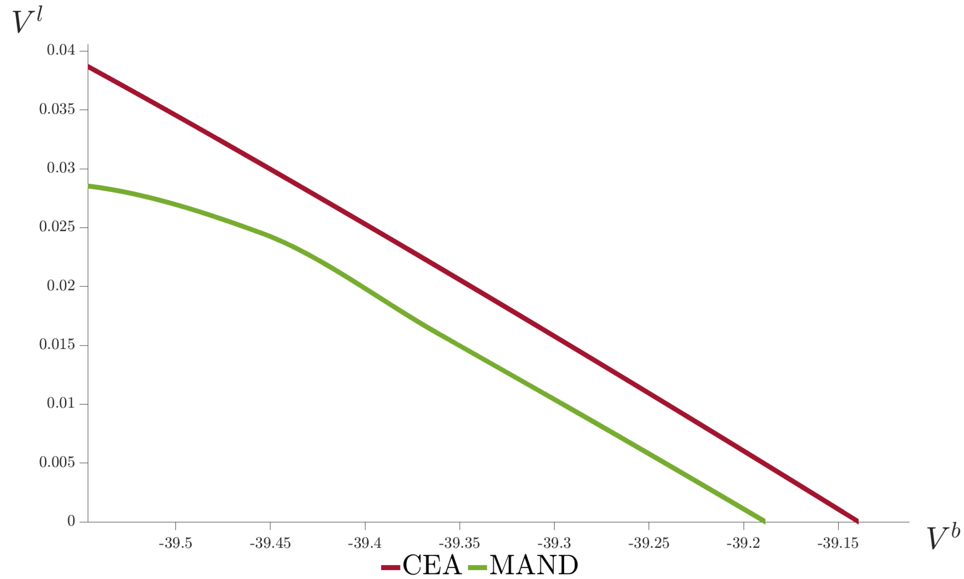
Table D.1 depicts the parameter used in the calibration for Brazil. To ensure comparability between the two calibrations, I take the same the scale parameter φ as in the main calibration. Moreover, to make the no-default borrowing limit loose enough, the probability to re-access the market λ is estimated from the 95th percentile of default duration in the data (i.e. 14.6 years).

Similar to the main calibration, I estimate the Markov transition matrix by means of a Markov-switching AR(1) process with two states. For this, I use data on the log real GDP of Brazil from 1995 to 2019 HP filtered with a smoothing parameter of 100. I also set β to match the debt to GDP ratio. Compared to the main calibration, the variance of the shock process is smaller and the borrower is more patient.

I then select the output penalty ψ and the premium parameter χ to match the relative volatility of consumption and the premium at issuance. Compared to the main calibration, the output penalty and the premium are both smaller.

Table E.2 depicts the outcome of the calibration. The MAND can match the targeted moments. Nevertheless, like the main calibration, it does not display much dynamic in steady state. The table also depicts the moments related to the CEA using the alternative calibration. As in the main calibration, the CEA diverges from the allocation in the MAND. In addition, the level of indebtedness related to the CEA is at odds with the data.

Note that I do not report the moments related to the MA as the alternative calibration does not generate defaults in steady state. The reason is the variance of the shock is too low. This, together with the fact that default-related parameters are difficult to calibrate in the MAND, speaks in favor of having the main calibration based on Argentina and the MA.



Note: The figure depicts the utility possibility frontiers related to the MA, the MAND and the CEA. Those frontiers express the value of the lenders V^l as a function of the value of the borrower V^b. The CEA is in red, the MA in blue and the MAND in green. MAND to the Markov equilibrium without default and CEA to the constrained efficient allocation.

Figure E.1: Distance to Pareto Frontier in y_L – Alternative Calibration

Finally, Figure E.1 depicts the distance to the Pareto frontier. The red line corresponds to the Pareto frontier, while the green line represents the frontier related to the MAND. I find that $\mathcal{F}(y_L) = 0.76$, $\mathcal{F}(y_L) = 0.83$ and $\mathbb{E}\mathcal{F}(y) = 0.79$ which means that the MAND is actually closer to the Pareto frontier than in the main calibration. However, the MAND remains not constrained efficient.

F Data

Table F.3 specifies the source of the data used in the analysis. For national accounts, I rely on the World Bank's World Development Indicators. To compute correlations and volatilities, I detrend the series using the HP filter with a smoothing parameter of 100.[33] Debt statistics come from the World Bank's Quarterly External Debt Statistics. Finally, spreads data come from the Global Financial Database where I exclude the years in which the country is in default.

[33]More precisely, I take the logarithm of output and consumption before detrending them. As the trade balance can be negative, I simply express it as a share of output and detrend it.

Table F.3: Data Sources and Definitions

Series	Sources	Unit
Output	World Development Indicators[a]	constant 2015 USD
Consumption	World Development Indicators[b]	constant 2015 USD
Exports	World Development Indicators[c]	constant 2015 USD
Imports	World Development Indicators[d]	constant 2015 USD
External debt	Quarterly External Debt Statistics[e]	current USD
Global bonds	Refinitiv Eikon Datastream[f]	current USD
Buybacks	Brazilian National Treasury[g]	current USD
EMBI+	Global Financial Database[h]	basis point
Default date	Asonuma and Trebesch (2016)[i]	month

[a] GDP (constant 2015 USD). Series code: NY.GDP.MKTP.KD.
[b] Final consumption expenditure (constant 2015 USD). Series code: NE.CON.TOTL.KD.
[c] Exports of goods and services (current USD). Series code: NE.EXP.GNFS.CD.
[d] Imports of goods and services (current USD). Series code: NE.IMP.GNFS.CD.
[e] General government gross external debt position for all maturities and all instruments (current USD). Series code: DT.DOD.DECT.CD.GG.AR.US.
[f] USD-denominated bonds issued by the government of Brazil. See Table F.4.
[g] Buyback by month and by bond in financial and face value. See Monthly Debt Report.
[h] Government bond spread. Series code: EMBPARGD and EMBPBRAD.
[i] Default date of individual restructurings. See Asonuma and Trebesch (2016).

Regarding the buyback there are two main sources. First, I use the Monthly Debt Report published by the Brazilian National Treasury to retrieve the amount bought back for each bond. Reports of buyback are bimonthly and specify the face and the financial value of each bond repurchased.[34] Note that three reports are missing: January-February 2009, September-October 2013 and January-February 2015. In addition, the 2006 report is annual with only the total financial value across the different bonds available. When missing, I estimate the financial value using the average yield to maturity for each bond.

Second, I re-construct the cashflow stream of each bond using Refinitiv Eikon Datastream. Table F.4 indicates the different bonds used in the analysis. All bonds are denominated in USD and issued prior to 2006. The table specifies the ISIN code, the coupon rate, the initial amount issued and the maturity.[35] I also have information about the yield to maturity from the issuance to the maturity. Note that there are other bonds involved in the Brazilian buyback program. Notably, I did not account for bonds denominated in foreign currency other than USD and bonds in USD issued after 2006.[36]

[34] Starting 2018, buybacks reports are monthly.

[35] All bonds have semi-annual coupons except for BR09F which pays coupon at a quarterly frequency.

[36] There were some bonds in EUR and JPY involved in the buyback program. However, they correspond to a negligible amount compared to the ones denominated in USD. In addition, buybacks for bonds issued from 2006 onwards amounts to an additional 2 USD billion in both face and financial values.

Table F.4: Bond

Bond	ISIN Code	Coupon (Percent)	Issue (USD Billion)	Issuance (Year)	Maturity (Year)
BR07	US105756AW05	10.00	1.00	2003	2007
BR07N	US105756AM23	11.25	1.50	2000	2007
BR08	US105756AG54	9.38	1.25	1998	2008
BR08N	US105756AU49	11.50	1.25	2002	2008
BR09	US105756AJ93	14.50	2.00	1999	2009
BR09F	US105756BC32	6.98	0.75	2004	2009
BR10	US105756AV22	12.00	1.00	2002	2010
BR10N	US105756BA75	9.25	1.50	2003	2010
BR11	US105756AY60	10.00	1.25	2003	2011
BR12	US105756AT75	11.00	1.25	2002	2012
BR13	US105756AX87	10.25	1.25	2003	2013
BR14	US105756BD15	10.50	1.25	2004	2014
BR15	US105756BG46	7.88	2.10	2005	2015
BR16	US105756BJ84	12.50	1.53	2005	2016
BR18	US105756BH29	8.00	4.51	2005	2018
BR19	US105756BE97	8.88	1.50	2004	2019
BR20	US105756AK66	12.75	1.00	2000	2020
BR24	US105756AR10	8.88	2.15	2001	2024
BR24N	US105756AZ36	8.88	0.83	2003	2024
BR25	US105756BF62	8.75	2.25	2005	2025
BR27	US105756AE07	10.13	3.51	1997	2027
BR30	US105756AL40	12.25	1.60	2000	2030
BR34	US105756BB58	8.25	2.50	2004	2034
BR40	US105756AP53	11.00	5.16	2000	2040

G Welfare Analysis

To compute the borrower's welfare, first define the borrower's value for a sequence of consumption $\{c(s^t)\}$ starting from an initial state at $t = 0$ as

$$W^b(\{c(s^t)\}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c(s^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c(s^t)^{1-\vartheta}}{1-\vartheta},$$

I denote the borrower's consumption allocation in the benchmark model by $\{c^b(s^t)\}$ and the consumption allocation in the alternative model by $\{c^a(s^t)\}$. In addition, I define the consumption-equivalent welfare gain of the alternative model with respect to the benchmark model by ι such that

$$W^b(\{(1+\iota)c^b(s^t)\}) = W^b(\{c^a(s^t)\}).$$

Given the functional form of the instantaneous utility one obtains

$$(1+\iota)^{1-\vartheta}\Big[W^b(c^b(s^t)\})\Big] = W^b(\{c^a(s^t)\}).$$

The borrower's welfare gain therefore boils down to

$$\iota = \left[\frac{W^b(\{c^a(s^t)\})}{W^b(\{c^b(s^t)\})}\right]^{\frac{1}{1-\vartheta}} - 1.$$

The lenders' welfare gains can be computed in the same way by setting $\vartheta = 0$ owing to the risk neutrality.

H Proofs

Proof of Proposition 1

Consider the value of the *legacy* lender when there is no default ever. In that case

$$W^l_{\text{legacy},ND}(\Omega) = -\Big[b_{st} + b_{lt}(1+\frac{1}{r})\Big].$$

In opposition, when there is dilution

$$W^l_{\text{legacy},D}(\Omega) = -\Big[b_{st} + b_{lt}(1+q_{lt}(y, B_{st}(\Omega), B_{lt}(\Omega))))\Big] < W^l_{\text{legacy},ND}(\Omega),$$

where the inequality comes from the fact that $q_{lt}(y, B_{st}(\Omega), B_{lt}(\Omega)) < \frac{1}{r}$. As a result, the *legacy* lender is never willing to dilute. Similarly, the value of the *legacy* lender when there is an *official* buyback is

$$W^l_{\text{legacy},B}(\Omega) = -\Big[b_{st} + b_{lt}(1+q_{lt}(y, B_{st}(\Omega), B_{lt}(\Omega))) + \chi)\Big],$$

while when there is no *official* buyback

$$W^l_{\text{legacy},NB}(\Omega) = -\Big[b_{st} + b_{lt}(1+q_{lt}(y, B_{st}(\Omega), B_{lt}(\Omega))))\Big] < W^l_{\text{legacy},B}(\Omega),$$

As a result, the *legacy* lender is always willing to have an *official* buyback. In opposition, the value of the *new* lender is

$$W^l_{\text{new}}(\Omega) = W^l_{\text{new},ND}(\Omega) = W^l_{\text{new},D}(\Omega) = W^l_{\text{new},B}(\Omega) = W^l_{\text{new},NB}(\Omega) = 0.$$

As a result, the *new* lender is indifferent to dilution and *official* buyback. □

Proof of Proposition 2

Assume by contradiction that in a given state Ω, the borrower wants to conduct an *official* buyback. That is, the borrower picks a pair (b'_{st}, b'_{lt}) such that

$$V^{NB}(\Omega) < V^{B}(\Omega).$$

The consumption under *official* buyback is given by

$$c^{B}(\Omega) = y + b_{st} + b_{lt}(1 + q_{lt}^{bb}) - q_{st}(y, b'_{st}, b'_{lt})b'_{st} - q_{lt}(y, b'_{st}, b'_{lt})b'_{lt},$$

and the expected continuation value by

$$\mathbb{E}\Big[W^{b}(y', b'_{st}, b'_{lt})\Big].$$

Now consider the alternative strategy of picking the same pair (b'_{st}, b'_{lt}) but conducting an *unofficial* buyback. In such circumstance, consumption is given by

$$c^{NB}(\Omega) = y + b_{st} + b_{lt} - q_{st}(y, b'_{st}, b'_{lt})b'_{st} - q_{lt}(y, b'_{st}, b'_{lt})(b'_{lt} - b_{lt}).$$

It is clear from the *official* buyback premium that $c^{NB}(\Omega) > c^{B}(\Omega)$. Moreover, as the borrower chooses the same (b'_{st}, b'_{lt}), the continuation value is the same as before. Hence,

$$V^{NB}(\Omega) = u(c^{NB}(\Omega)) + \beta\mathbb{E}\Big[W^{b}(y', b'_{st}, b'_{lt})\Big] > u(c^{B}(\Omega)) + \beta\mathbb{E}\Big[W^{b}(y', b'_{st}, b'_{lt})\Big] = V^{B}(\Omega),$$

which contradicts the fact that an *official* buyback is ever optimal. □

Proof of Lemma 1

- If $b'_{st} \geq 0$

  Suppose by contradiction that the lenders can enforce *official* buybacks in a state Ω such that $B_{st}(\Omega) \geq 0$. Formally, in the case of an *official* buyback, the borrower chooses $B_{st}(\Omega) \geq 0$ and $B_{lt}(\Omega) \geq b_{lt}$ to maximize its utility.

  Now consider the case in which the borrower does not conduct the *official* buyback but mimics the debt choice in the case of *official* buyback. This is possible as $b'_{st} \geq 0$ and $b'_{lt} \geq b_{lt}$. The contradiction is immediate as per Proposition 2. Thus, *official* buybacks are not enforceable in case of short-term asset issuance.

– If $b'_{st} < 0$

Consider a state Ω in which, $B_{st}(\Omega) < 0$ and $B_{lt}(\Omega) = b_{lt}$. Given this, from Proposition D.1.2, we have that for all s', $W^b(s', B_{st}(\Omega), B_{lt}(\Omega)) \leq W^b(s', 0, B_{lt}(\Omega))$. In words, the continuation value under the no-roll-over punishment is weakly larger than the continuation value under an *official* buyback.

I consider two cases. First, if the level of short-term liabilities is large, then the incapacity to issue new short-term debt can lead to a default. It is then preferable for the borrower to pay the premium χ if $V^B(y, b_{st}, b_{lt}) > V^{NB}(y, b_{st}, b_{lt}) = V^D(y)$.

Second, given that the continuation value under the punishment is weakly larger, for the *official* buyback to be preferred, consumption ought to be lower under the punishment. Formally

$$\begin{aligned} -q_{st}(y, b'_{st}, b'_{lt})b'_{st} - q_{lt}(y, b'_{st}, b'_{lt})(b'_{lt} - b_{lt}) + \chi b_{lt} \geq & - q_{lt}(y, 0, b'_{lt})(b'_{lt} - b_{lt}) \\ \geq & - q_{lt}(y, b'_{st}, b'_{lt})(b'_{lt} - b_{lt}), \end{aligned}$$

where the second inequality follows from Proposition D.1.5. This implies that

$$\chi b_{lt} \geq q_{st}(y, b'_{st}, b'_{lt})b'_{st}$$

Thus, if $-\chi b_{lt}$ is sufficiently large, it is possible to have $V^B(\Omega) > \bar{V}^{NB}(\Omega)$ ensuring the enforcement of *official* buybacks.

□

Proof of Proposition 3

– Part I

The law of motion of the relative Pareto weight is given by $x'(y) = (1 + \nu(y))\eta x$, while the first-order condition on consumption reads $u_c(c(y)) = \frac{1}{(1+\nu(y))x} = \frac{\eta}{x'(y)}$.

Consider the interval $[x_D(y_L), \overline{x}]$. From the law of motion of the relative Pareto weight, x' is strictly increasing in x. From the first-order conditions on consumption, c is strictly increasing in x. Hence, so does the value of the borrower. In opposition, with a greater c (or equivalently a greater x), the instantaneous payoff of the lenders, $y - c$, decreases. That is the lenders' value is strictly decreasing in x.

Moreover, as $V^D(y_H) > V^D(y_L)$ and the value of the borrower is strictly increasing in x, it must be that $x_D(y_H) > x_D(y_L)$.

– Part II

Observe that, given the first-order condition, $c(y_L, x) \leq c(y_H, x)$ only when $\nu(y_L) \leq \nu(y_H)$. Assume by contradiction that $\nu(y_L) > \nu(y_H)$. This implies that $c(y_L, x) > c(y_H, x)$ and $x'(y_L, x) > x'(y_H, x)$. Given this, from Part I, $V^b(y', x'(y_L, x)) > V^b(y', x'(y_H, x))$. As a result.

$$u(c(y_L, x)) + \beta \mathbb{E} V^b(y', x'(y_L, x)) > u(c(y_H, x)) + \beta \mathbb{E} V^b(y', x'(y_H, x)).$$

Moreover as $\nu(y_L) > \nu(y_H) \geq 0$

$$\begin{aligned} u(c(y_H, x)) + \beta \mathbb{E} V^b(y', x'(y_H, x)) &\geq V^D(y_H), \\ u(c(y_L, x)) + \beta \mathbb{E} V^b(y', x'(y_L, x)) &= V^D(y_L). \end{aligned}$$

This implies that $V^D(y_L) > V^D(y_H)$, a contradiction. Hence, $\nu(y_L) \leq \nu(y_H)$ which gives $c(y_L, x) \leq c(y_H, x)$ and $x(y_L, x) \leq x(y_H, x)$ as desired.

Especially, by definition, when $x \geq x_D(y_H)$, then $\nu(y) = 0$ for all y implying that $c(y_L, x) = c(y_H, x)$ and $x(y_L, x) = x(y_H, x)$. Otherwise, $c(y_L, x) < c(y_H, x)$ and $x(y_L, x) < x(y_H, x)$. $\square$

– Part III

This proof is a modified version of Thomas and Worrall (1990, Lemma 4). The value of liabilities in the optimal contract is given by

$$V^l(y, x) \equiv y - c(y, x) + \frac{1}{1+r} \mathbb{E} V^l(y', x'(y, x)).$$

Assume by contradiction that for a given x it holds that $V^l(y_H, x) \leq V^l(y_L, x)$. For $x \geq x_D(y_H)$, one directly reaches a contradiction as $c(y_L, x) = c(y_H, x)$ and $x(y_L, x) = x(y_H, x)$ – as shown in Part II – which implies that $V^l(y_H, x) > V^l(y_L, x)$.

For $x < x_D(y_H)$, consider the pooling allocation in which $u(\ddot{c}(y_H, x)) = u(\ddot{c}(y_L, x)) = u(c(y_H, x))$ and $\ddot{V}^b(y_H, x) = \ddot{V}^b(y_L, x) = V^b(y_H, x)$. Under this allocation, the participation constraint is trivially satisfied. This leads to

$$\ddot{V}^l(y_H, x) > \ddot{V}^l(y_L, x),$$

which is a direct contradiction. Hence, $V^l(y_H, x) \geq V^l(y_L, x)$. $\square$

– Part IV

The autarkic allocation is such that $c(y, x) = y$. Recall the binding participation constraint

$$u(c(y, x)) + \beta \mathbb{E} V^b(y', x') = u(y) + \beta \mathbb{E} V^D(y').$$

Consider the following perturbation: increase the consumption in y_L by ϵ and decrease it by $\epsilon \frac{\pi(y_L)}{\pi(y_H)}$ otherwise. Formally, $c(y_L, x) = y_L + \epsilon$ and $c(y_H, x) = y_H - \epsilon \frac{\pi(y_L)}{\pi(y_H)}$. As the perturbation is such that $\mathbb{E}c(y, x) = \mathbb{E}y$, $\mathbb{E}V^b(y', x') > \mathbb{E}V^D(y')$ by strict concavity. The perturbation therefore relaxes the participation constraint without incurring losses to the lenders which contradicts the fact that autarky is optimal.

The above argument however ceases to hold whenever $\sigma \to 0$, $\beta \to 0$, $\pi(y_L) \to 0$, $\pi(y_H) \to 0$ or $y_H \to y_L$. If $\sigma = 0$, there is no curvature in the utility function and therefore any gains for the borrower translates into a 1-to-1 loss for the lenders and *vice versa*. Hence, $c(y, x) = y$ is optimal. Similarly, when $\beta = 0$, we get

$$u(c(y, x)) = u(y) \quad \text{for all } y \in Y$$

The only way to satisfy this constraint in all y is to set $c(y, x) = y$. If $1 - \pi(y) = 0$,

$$u(c(y, x)) + \beta V^b(y', x') = \frac{u(y)}{1 - \beta}.$$

There is no uncertainty and therefore no need for risk sharing. One obtains the same result when $y_H = y_L$. $\square$

Proof of Proposition 4

Recall the law of motion of the relative Pareto weight

$$x'(y) = (1 + \nu(y))\eta x.$$

The motion of the relative Pareto weight is dictated by the relative impatience, η, and the binding participation constraint, ν. I consider two cases. On the one hand, if $\eta < 1$, the relative Pareto weight increases only if $\nu(y) > 0$ is sufficiently large to overcome impatience. By definition, when $x \geq x_D(y)$, $\nu(y, x) = 0$ meaning that impatience eventually dominates the limited commitment issue. On the other hand, if $\eta = 1$ immiseration due to impatience does not exist and the relative Pareto weight remains constant.

When $\eta = 1$, the upper bound of the ergodic set coincides with the lower bound. As shown in Proposition 3 Part II, $x'(y_L, x) \leq x'(y_H, x)$. Moreover, as $x_D(y_H) > x_D(y_L)$, $x^{ub} = x^{lb} = x_D(y_H)$. Conversely, when $\eta < 1$, impatience prevents the contract to reach $x_D(y_H)$ as $\nu(y_H, x_D(y_H)) = 0$. Hence, $x_D(y_H) > x^{ub}$. Moreover, $x'(y_L, x) < x'(y_H, x)$ when $x < x_D(y_H)$ implying that $x^{ub} > x^{lb}$. In other words, impatience immiserates the relative Pareto in the low endowment state implying that $x_D(y_H) > x^{ub} > x^{lb}$.

To show the existence of the ergodic set, one shows that the dynamic of the contract satisfies the conditions given by Stokey et al. (1989, Theorem 12.12). Set $\ddot{x}$ as the midpoint of $[x^{lb}, x^{ub}]$ and define the transition function $Q : [x^{lb}, x^{ub}] \times \mathcal{X}([x^{lb}, x^{ub}]) \to \mathbb{R}$ as

$$Q(x, G) = \sum_{g'|g} \pi(y'|g)\mathbb{I}\{x' \in G\}$$

One wants to show is that $\ddot{x}$ is a mixing point such that for $M \geq 1$ and $\epsilon > 0$ one has that $Q(x^{lb}, [x, x^{ub}])^M \geq \epsilon$ and $Q(x^{ub}, [x^{lb}, x])^M \geq \epsilon$. Starting at x^{ub}, for a sufficiently long but finite series of y_L, the relative Pareto weight transit to x^{lb} (either through impatience or because $x^{lb} = x^{ub}$). Hence for some $M < \infty$, $Q(x^{ub}, [x^{lb}, \ddot{x}])^M \geq \pi(y_L|y_L)^M > 0$. Moreover, starting at x^{lb}, after drawing $M < \infty$ y_H, the relative Pareto weight transit to x^{ub} (either through the binding constraint or because $x^{lb} = x^{ub}$) meaning that $Q(x^{lb}, [\ddot{x}, x^{ub}])^M \geq \pi(y_H|y_H)^M > 0$. Setting $\epsilon = \min\{\pi(y_L)^M, \pi(y_H)^M\}$ makes $\ddot{x}$ a mixing point and the above theorem applies. $\square$

Proof of Lemma 2

Recall that the long-term bond price is given by

$$\bar{q}_{lt}(y, x) = \frac{1}{1+r}\mathbb{E}\left[(1 - \bar{D}(y', x'))\left\{1 + (1 - \bar{M}(y', x'))\bar{q}_{lt}(y', x') + \bar{M}(y', x')q_{lt}^{bb}\right\}\right],$$

I consider that $\bar{D}(y', x') = 0$ for all (y', x') and $\bar{M}(y', x') = 1$ if $y' = y_L$ as well as $x' = x^{ub}$ and $\bar{M}(y', x') = 0$ otherwise. From Proposition 4, y_H and $x = x^{ub}$ arises with strictly positive probability for any (y, x),

$$\frac{1+\chi}{r} > \bar{q}_{lt}(y, x) > \frac{1}{r}.$$

Define Q_{lt} as the space of bounded functions $\bar{q}_{lt} : [\underline{x}, \bar{x}] \rightarrow [0, \frac{1+\chi}{r}]$ and $\mathbb{T} : Q_{lt} \rightarrow Q_{lt}$ as

$$\mathbb{T}\bar{q}_{lt}(y, x) = \frac{1}{1+r} \sum_{g'} \pi(y'|g)[1 + \bar{q}_{lt}(y', x')].$$

By the Blackwell sufficient conditions $\mathbb{T}$ is a contraction mapping. As a result, there exists a unique fixed point to $\mathbb{T}$, $\bar{q}_{lt}$ which is increasing as $\mathbb{T}$ maps increasing functions into increasing functions. This implies that $\bar{q}_{lt}(x'(y_H, x)) \geq \bar{q}_{lt}(x'(y_L, x))$ as $x'(y_H, x) > x'(y_L, x)$ for all x in the above specified domain. Assume now that there exists a x such that $\bar{q}_{lt}(x'(y_H, x)) = \bar{q}_{lt}(x'(y_L, x))$. This requires that $x'(y_H, x)$ and $x'(y_L, x)$ belongs to a subset $[x_t, x_{t+1}]$ where $\bar{q}_{lt}$ stays constant. Hence, for any $\ddot{x} \in [x_t, x_{t+1}]$, it must be that $x'(y_H, \ddot{x}), x'(y_L, \ddot{x}) \in [x_t, x_{t+1}]$ which is a contradiction as $x'(y_H, x_{t+1}) > x_{t+1}$ when $\eta < 1$. Therefore it must be that $\bar{q}_{lt}(x'(y_H, x)) > \bar{q}_{lt}(x'(y_L, x))$ when $\eta < 1$ and $\bar{q}_{lt}(x'(y_H, x)) \geq \bar{q}_{lt}(x'(y_L, x))$ otherwise. $\square$

Proof of Lemma 3

The fact that default never occurs is a direct corollary of Assumption 1 and Proposition 3 Part IV. Regarding buybacks, I consider that the borrower conducts *official* buybacks when it hits the upper bound of the ergodic set – i.e. $x = x^{ub}$ and $y = y_H$. I therefore need to consider two alternatives.

First, could *official* buybacks occur in the lower bound of the ergodic set – i.e. $x = x^{lb}$? The answer is negative. To reach the lower bound, the relative Pareto weight needs to decrease. More precisely, in steady state $x'(y_L) \leq x$ as shown in Proposition 4. This implies that the value of the lenders increases as y_L realizes. This goes against the idea of a debt repurchase which (weakly) reduces indebtedness. Furthermore, from Part III of Proposition 3, it holds that $V^l(y_H, x) > V^l(y_L, x)$ which implies debt relief in the low endowment state. However, the price of long-term debt would increases as y_L realizes (i.e. the reverse of Lemma 2) which goes against the idea of a debt relief. Hence, it is not possible to have an *official* buyback at any point related to the realization of y_L.

Second, could *official* buybacks occur before the contract hits the upper bound of the ergodic set – i.e. $x < x^{ub}$ and $y = y_H$? The answer is positive as the realization of y_H is associated with a debt decrease. However, one has to be careful that each *official* buyback should be such that $b'_{lt} \geq b_{lt}$. Moreover, note that if *official* buybacks happen at say $\tilde{x} < x^{ub}$ and $y = y_H$, then for all $x \in (\tilde{x}, x^{ub}]$ $\bar{q}_{lt}(x'(y_H, x)) \leq \bar{q}_{lt}(x'(y_L, x))$. $\square$

Proof of Lemma 4

From (16) and (17), the short and long-term holdings at the *official* buyback are respectively

$$\bar{b}_{st}(x) = \frac{V^l(y_H, x)[1 + \bar{q}_{lt}(x'(y_L, x))] - V^l(y_L, x)[1 + q_{lt}^{bb}]}{q_{lt}^{bb}(x'(y_H, x)) - \bar{q}_{lt}(x'(y_L, x))} < 0,$$
$$\bar{b}_{lt}(x) = -\frac{V^l(y_H, x) - V^l(y_L, x)}{q_{lt}^{bb}(x'(y_H, x)) - \bar{q}_{lt}(x'(y_L, x))} \lesseqgtr 0,$$

where $q_{lt}^{bb}(x'(y_H, x)) = q_{lt}(x'(y_H, x)) + \chi$. From Part III of Proposition 3, it holds that $V^l(y_H, x) > V^l(y_L, x)$ meaning that $b_{lt} < 0$. However, it is not guaranteed that $b_{st} < 0$. Particularly, b_{st} can be negative only if q_{lt}^{bb} is very large with respect to q_{lt}. Moreover, recall that the *official* buyback takes place in x^{ub} meaning that $b'_{st} = b_{st}$ and $b'_{lt} = b_{lt}$ by Proposition 4. As a result, Lemma 1 does not generally apply as b'_{st} is negative only when χ is sufficiently large. $\square$

Proof of Lemma B.1

The proof follows Yue (2010). Because $\Delta^l_{\text{legacy}}(\Omega; \chi)$ and $\Delta^b(\Omega; \tilde{\chi})$ are both functions of $b_{lt}\chi$, I can redefine the maximization problem as

$$\begin{aligned}
\chi(\Omega) = &\arg\max_{\tilde{\chi}\geq 0} \left[\hat{\Delta}^l_{\text{legacy}}(y, b_{st}; b_{lt}\tilde{\chi})^\zeta + \hat{\Delta}^b(y, b_{st}; b_{lt}\tilde{\chi})^{1-\zeta}\right] \\
&\text{s.t.} \quad \hat{\Delta}^l_{\text{legacy}}(y, b_{st}; b_{lt}\tilde{\chi}) \geq 0, \\
&\qquad\;\; \hat{\Delta}^b(y, b_{st}; b_{lt}\tilde{\chi}) \geq 0.
\end{aligned}$$

Hence, the solution of the Nash bargaining protocol is solely a function of (y, b_{st}) that I denote by $\bar{B}(y, b_{st})$. When $b_{lt} \leq \bar{B}(y, b_{st})$, the constraint $\chi \geq 0$ is not binding and $\chi = \frac{\bar{B}(y, b_{st})}{b_{lt}}$. Otherwise, the constraint is binding and $\chi = 0$. $\square$

Proof of Proposition D.1

1. Consider an endowment such that $\hat{y} > y$. In addition, define

$$R(b_{st}, b_{lt}; b'_{st}, b'_{lt}) = -b_{st} - b_{lt} + q_{st}(b'_{st}, b'_{lt})b'_{st} + q_{lt}(b'_{st}, b'_{lt})(b'_{lt} - b_{lt}).$$

Observe that the bond prices do not depend on y given that y is i.i.d. distributed. We then have that

$$V^P(\hat{y}, b_{st}, b_{lt}) = u\big(\hat{y} + R(b_{st}, b_{lt}; \hat{b}'_{st}, \hat{b}'_{lt})\big) + \beta Z(\hat{b}'_{st}, \hat{b}'_{lt})$$

$$
\begin{aligned}
&\geq u\big(\hat{y} + R(b_{st}, b_{lt}; b'_{st}, b'_{lt})\big) + \beta Z(b'_{st}, b'_{lt}) \\
&> u\big(y + R(b_{st}, b_{lt}; b'_{st}, b'_{lt})\big) + \beta Z(b'_{st}, b'_{lt}) \\
&= V^P(y, b_{st}, b_{lt}),
\end{aligned}
$$

where the first inequality follows from the definition of optimality, and the second from the strict monotonicity of $u(c)$. Now consider $\hat{b}_{st} < b_{st}$. We obtain that

$$
\begin{aligned}
V^P(y, b_{st}, b_{lt}) &= u\big(y + R(b_{st}, b_{lt}; b'_{st}, b'_{lt})\big) + \beta Z(b'_{st}, b'_{lt}) \\
&\geq u\big(y + R(b_{st}, b_{lt}; \hat{b}'_{st}, \hat{b}'_{lt})\big) + \beta Z(\hat{b}'_{st}, \hat{b}'_{lt}) \\
&> u\big(y + R(\hat{b}_{st}, b_{lt}; \hat{b}'_{st}, \hat{b}'_{lt})\big) + \beta Z(\hat{b}'_{st}, \hat{b}'_{lt}) \\
&= V^P(y, \hat{b}_{st}, b_{lt}),
\end{aligned}
$$

where the first inequality follows from the definition of optimality, and the second from the strict monotonicity of $u(c)$. The exact same argument holds for $\hat{b}_{lt} < b_{lt}$. $\square$

2. Recall the definition of $Z(b_{st}, b_{lt})$

$$
Z(b_{st}, b_{lt}) = \mathbb{E}\left[\max\left\{V^P(y, b_{st}, b_{lt}), V^D(y)\right\}\right].
$$

The proof is immediate from Proposition D.1.1.$\square$

3. Recall the definition of the default set

$$
d(b_{st}, b_{lt}) = \min\Big\{\{y : V^P(y, b_{st}, b_{lt}) \geq V^D(y)\} \cup \{y_H\}\Big\}.
$$

The proof is immediate from Proposition D.1.1. $\square$

4. Following Chatterjee and Eyigungor (2012), define the loss function of choosing $(b'_{st,0}, b'_{lt,0})$ instead of $(b'_{st,1}, b'_{lt,1})$ in state (y, b_{st}, b_{lt}) by

$$
\begin{aligned}
-\Delta(b'_{st,0}, b'_{st,1}, b'_{lt,0}, b'_{lt,1}|b_{lt}) =& b_{lt}\big[q_{lt}(b'_{st,1}, b'_{lt,1}) - q_{lt}(b'_{st,0}, b'_{lt,0})\big] + \\
& q_{st}(b'_{st,0}, b'_{lt,0})b'_{st,0} - q_{st}(b'_{st,1}, b'_{lt,1})b'_{lt,1} + \\
& q_{lt}(b'_{st,0}, b'_{lt,0})b'_{lt,0} - q_{lt}(b'_{st,1}, b'_{lt,1})b'_{lt,1}.
\end{aligned}
$$

Observe that Δ does not depend on y or b_{st}.

For the first part of the proposition, fix (y, b_{st}) and consider two debt levels $b_{lt,0} < 0$ and $b_{lt,1} < 0$ such that $b_{lt,1} < b_{lt,0} < 0$. Assume that in $b_{lt,0}$, the borrower optimally chooses $(b'_{st,0}, b'_{lt,0})$ and obtains a consumption level c_0. This means that for a $(\hat{b}'_{st}, \hat{b}'_{lt}) > (b'_{st,0}, b'_{lt,0})$ leading to a consumption level $\hat{c}$, we have by optimality that

$$u(c_0) + \beta Z(b'_{st,0}, b'_{lt,0}) \geq u(\hat{c}) + \beta Z(\hat{b}'_{st}, \hat{b}'_{lt}). \tag{H.1}$$

As $Z(b'_{st,0}, b'_{lt,0}) \leq Z(\hat{b}'_{st}, \hat{b}'_{lt})$ from D.1.2, it must be that $c_0 \geq \hat{c}$. Now observe that $\Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,0}) = c_0 - \hat{c} \geq 0$ and

$$\Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,0}) - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,1}) = -(b_{lt,0} - b_{lt,1})\big[q_{lt}(\hat{b}'_{st}, \hat{b}'_{lt}) - q(b'_{st,0}, b'_{lt,0})\big] \leq 0,$$

where the inequality comes form the fact that $b_{lt,1} < b_{lt,0} < 0$ and that $q_{lt}(b'_{st}, b'_{lt})$ is non decreasing in (b'_{st}, b'_{lt}). This means that the loss in $b_{lt,1}$ is at least as large as the loss in $b_{lt,0}$. With this define $\tilde{c}$ being the consumption level in state $b_{lt,1}$ choosing $(b'_{st,0}, b'_{lt,0})$. By the budget constraint, it directly follows that $\tilde{c} < c_0$ given that $b_{lt,1} < b_{lt,0} < 0$. Combining this with the strict concavity of $u(\cdot)$, we get

$$\begin{aligned}
u(\tilde{c}) - u(\tilde{c} - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,1})) &> u(c_0) - u(c_0 - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,1})) \\
&\geq u(c_0) - u(c_0 - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,0})) \\
&= u(c_0) - u(\hat{c}) \geq 0,
\end{aligned}$$

where the first inequality comes from $c_0 > \tilde{c}$, the second from $\Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,1}) \geq \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,0})$ and the third from the definition of $\Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,0})$. This means that the wedge in utility between $\tilde{c}$ and $\hat{c} = \tilde{c} - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,1})$ is larger than the wedge in utility between c_0 and $\hat{c} = c_0 - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b_{lt,0})$. By (H.1), this implies that

$$u(\tilde{c}) + \beta Z(b'_{st,0}, b'_{lt,0}) > u(\hat{c}) + \beta Z(\hat{b}'_{st}, \hat{b}'_{lt}).$$

Hence it cannot be that in $b_{lt,1}$, the optimal choice $-(b'_{st}, b'_{lt})$ is lower than $-(b'_{st,0}, b'_{lt,0})$. The bond policy function is therefore non increasing in b_{lt}.

Fix (b_{st}, b_{lt}) and consider two two levels $b_{st,0} < 0$ and $b_{st,1} < 0$ such that $b_{st,1} < b_{st,0} < 0$. As before, assume that in $b_{st,0}$, the borrower optimally chooses $(b'_{st,0}, b'_{lt,0})$ and obtains a consumption level c_0. Considering a $(\hat{b}'_{st}, \hat{b}'_{lt}) > (b'_{st,0}, b'_{lt,0})$ leading to a consumption level $\hat{c}$, we get the same argument around (H.1) as before. With this define $\tilde{c}$ being the consumption level in state $b_{st,1}$ choosing $(b'_{st,0}, b'_{lt,0})$. From the budget constraint, $\tilde{c} = c_0 + b_{st,1} - b_{st,0} < c_0$ as $b_{st,1} < b_{st,0} < 0$. Combining this with the strict concavity of $u(\cdot)$, we get

$$
\begin{aligned}
u(\tilde{c}) - u(\tilde{c} - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b) + b_{st,1} - b_{st,0}) &> u(c_0) - u(c_0 - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b) + b_{st,1} - b_{st,0}) \\
&> u(c_0) - u(c_0 - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b)) \\
&= u(c_0) - u(\hat{c}) \geq 0.
\end{aligned}
$$

The wedge in utility between $\tilde{c}$ and $\hat{c} = \tilde{c} - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b) + b_{st,1} - b_{st,0}$ is larger than the wedge in utility between c_0 and $\hat{c} = c_0 - \Delta(b'_{st,0}, \hat{b}'_{st}, b'_{lt,0}, \hat{b}'_{lt}|b)$. By (H.1), it cannot be that in $b_{st,1}$, the optimal choice b' is lower than $-(b'_{st,0}, b'_{lt,0})$. The bond policy function is therefore non increasing in b_{st}.

Finally, fix (b_{st}, b_{lt}) and consider two income levels $y_0 > 0$ and $y_1 > 0$ such that $0 < y_1 < y_0$. The argument is the same as for b_{st} as Δ does not depend on y. $\square$

5. Recall that the short-term bond price is given by

$$
q_{st}(b'_{st}, b'_{lt}) = \frac{1}{1+r} \mathbb{E}\left[1 - D(y', b'_{st}, b'_{lt})\right].
$$

The proof is immediate from Proposition D.1.3. For the case of long-term debt, following Mateos-Planas et al. (2022), consider the limit of finite horizon. The proof goes by backward induction. In the last period T, the borrower defaults for every $(b_{st}, b_{lt}) < 0$. Hence, the bond price in $T-1$ is given by

$$
q_{lt,T-1}(b_{st}, b_{lt}) = 0.
$$

From D.1.3, for any $0 \geq (b_{st,1}, b_{lt,1}) > (b_{st,2}, b_{lt,2})$,

$$
q_{lt,T-1}(b_{st,1}, b_{lt,1}) - q_{lt,T-1}(b_{st,2}, b_{lt,2}) = \frac{1}{1+r} \mathbb{E}\left[D_T(y', b_{st,2}, b_{lt,2}) - D_T(y', b_{st,1}, b_{lt,1})\right] \geq 0.
$$

Subsequently, in period $T-1$, we have that

$$
\begin{aligned}
&q_{lt,T-2}(b_{st,1}, b_{lt,1}) - q_{lt,T-2}(b_{st,2}, b_{lt,2}) = \\
&\qquad \frac{1}{1+r}\left[D_{T-1}(y', b_{st,2}, b_{lt,2}) - D_{T-1}(y', b_{st,1}, b_{lt,1})\right] \\
&\qquad + \frac{1}{1+r} \mathbb{E}\Big[(1 - D_{T-1}(y', b_{st,1}, b_{lt,1})) q_{lt,T-1}(B_{st,T-1}(y, b_{st,1}, b_{lt,1}), B_{lt,T-1}(y, b_{st,1}, b_{lt,1})) \\
&\qquad\qquad - (1 - D_{T-1}(y', b_{st,2}, b_{lt,2})) q_{lt,T-1}(B_{st,T-1}(y, b_{st,2}, b_{lt,2}), B_{lt,T-1}(y, b_{st,2}, b_{lt,2} \\
&\quad \geq \frac{1}{1+r}\left[D_{T-1}(y', b_{st,2}, b_{lt,2}) - D_{T-1}(y', b_{st,1}, b_{lt,1})\right]
\end{aligned}
$$

$$+ \frac{1}{1+r}\mathbb{E}\Big[(1 - D_{T-1}(y', b_{st,2}, b_{lt,2}))\Big\{q_{lt,T-1}(B_{st,T-1}(y, b_{st,1}, b_{lt,1}), B_{lt,T-1}(y, b_{st,1}, b_{lt,1})) - q_{lt,T-1}(B_{st,T-1}(y, b_{st,2}, b_{lt,2}), B_{lt,T-1}(y, b_{st,2}, b_{lt,2}))\Big\}$$

where the inequality comes from D.1.3. Relying on the same result we have that $\mathbb{E}[D_{T-1}(y', b_{st,2}, b_{lt,2}) - D_{T-1}(y', b_{st,1}, b_{lt,1})] \geq 0$. In addition, from D.1.4, $B_{j,T-1}(y, b_{st,1}, b_{lt,1}) \leq h_{T-1}(y, b_{st,2}, b_{lt,2})$ for all $j \in \{st, lt\}$ implying that the term within the expectation is weakly positive given that $q_{lt,T-1}(b_{st,1}, b_{lt,1}) - q_{lt,T-1}(b_{st,2}, b_{lt,2}) \geq 0$ for $0 > (b_{st,1}, b_{lt,1}) > (b_{st,2}, b_{lt,2})$. Hence,

$$q_{lt,T-2}(b_{st,1}, b_{lt,1}) - q_{lt,T-2}(b_{st,2}, b_{lt,2}) \geq 0.$$

Repeating the argument until $t = 0$ completes the proof.

□

Proof of Proposition D.2

Existence and uniqueness follow from Theorem 3 in Marcet and Marimon (2019). The two authors make the following assumptions: A1 a well defined Markov chain process for y, A2 continuity in $\{c\}$ and measurability in y, A3 non-empty feasible sets, A4 uniform boundedness, A5 convex technologies, A6 concavity for the lenders and strict concavity for the borrower, and a strict interiority condition. Assumption A1, A2, A5 and A6 are trivially met given my environment. Since feasible c is bounded, payoffs functions are bounded as well. This combined with the fact that the outside options are also bounded ensure that A4 is met. Whether A3 is satisfied depends on the initial condition (y_0, x_0). Assumption 1 ensures feasibility and that the strict interiority condition is satisfied.

It should be noted that Theorem 3 in Marcet and Marimon (2019) is the recursive, saddle-point, representation corresponding to the original contract problem (11). To obtain the recursive formulation of the contract, I have normalized the co-state variable. I relied on the the homogeneity of degree one in (μ_b, μ_l) to redefine the contracting problem using x – i.e. effectively $(x, 1)$ – as a co-state variable. Given this and the fact that multipliers are uniformly bounded, the theorem applies. That is, if I define the set of of feasible Lagrange multipliers by $L = \{(\mu_b, \mu_l) \in \mathbb{R}^2_+\}$ and the set of feasible consumption by $A = \{c \in \mathbb{R}_+\}$, the correspondence $SP : A \times L \to A \times L$ mapping non-empty, convex, and compact sets to themselves, is non-empty, convex-valued, and upper hemicontinuous. I can therefore apply Kakutani's fixed point theorem and existence immediately follows.

Marcet and Marimon (2019) additionally show that the the saddle point functional equation (13) is a contraction mapping. Thus, given the concavity assumptions of $u(\cdot)$, the allocation is unique. □

Proof of Lemma D.2

The law of motion of the relative Pareto weight is given by

$$x'(y) = (1 + \nu(y))\eta x.$$

and the level of consumption by

$$u_c(c(y)) = \frac{1}{x(1 + \nu(y))}.$$

Isolating x leads to

$$x = \frac{1}{u_c(c(y))(1 + \nu(y))}. \tag{H.2}$$

Plugging this back into the law of motion gives

$$x'(y) = (1 + \nu(y))\eta \frac{1}{u_c(c(y))(1 + \nu(y))}.$$

Replacing $x'(y)$ by with the forward equivalent of (H.2) gives

$$\frac{1}{u_c(c(y'))(1 + \nu(y'))} = \eta \frac{1}{u_c(c(y))}.$$

Taking expectations on both sides,

$$\mathbb{E}\left[\frac{1}{u_c(c(y'))(1 + \nu(y'))}\right] = \eta \frac{1}{u_c(c(y))},$$

which gives the inverse Euler equation. □