Moral Hazard with Risk-Sharing and Safe Debt*

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Abstract

We study the design a Financial Stability Fund under different provisions of incentives. More precisely, we generalize the *flexible* moral hazard approach of Georgiadis et al. (2024) in dynamic models and contrast it with the *canonical* dynamic moral hazard model of Atkeson and Lucas (1992). The provision of incentives under flexible moral hazard does not rely on the realized outcome. This has two main consequences. On the one hand, the optimal contract features *bliss* as opposed to immiseration. On the other hand, flexible moral hazard does not disrupt risk-sharing. We bridge the flexible and the canonical approach by back-loading incentives. This allows for perfect risk-sharing for spans of time (e.g. until limited enforcement constraints bind) and is a source of welfare gains for the borrower. We also analyze a restricted version of the flexible moral hazard. This framework provides a level playing field to compare the different Fund contracts. Our simulations show that restricted-flexible and canonical-back-loaded contracts are distinct but similar in their performance.

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1 Introduction

In models of debt, or risk-sharing, moral hazard is usually a concern: with effort the risk-averse borrower can improve its risk-profile; however, effort is not contractible (e.g. non observable) and, therefore, lenders and insurers can only infer the risk-profile with the *ex-post* observable outcomes. When debt is a risky asset, or risk-sharing is imperfect, the market discipline of prices is a mechanism to curb moral hazard problems, but an imperfect one. Because, the closer it is to the efficient outcome of safe debt, or perfect risk-sharing, the weaker the price signal is, aggravating the moral hazard problem. Mechanism design, with *incentive compatibility* (IC) constraints, can overcome this problem. More precisely, we can find a constrained-efficient debt-and-insurance contract where debt safety and sustainability are preserved and there is an optimal balance between effort and imperfect risk-sharing. However, there is not a unique way to introduce IC constraints and, correspondingly, there is no unique constrained-efficient outcome. While IC constraints are not new in the debt and risk-sharing theoretical literature, as we refer below, the issue of how different forms of IC constraints interact with risk-sharing and limited enforcement constraints remains mostly open. This is the general focus of this paper.

The enquire is broader after the recent contribution of Georgiadis et al. (2024), since their *flexible moral hazard* approach introduces a qualitatively different formulation of IC constraints in need of a generalization to dynamic contract and, in particular, of a quantification in relation to – what we call, the 'canonical' – approach, pioneered by Holmstrom (1979) and in dynamic contracts by Atkeson and Lucas (1992). The generalization of the theory of flexible moral hazard and its contrast with the dynamic canonical moral hazard theory is the first contribution of this paper.

In brief, the canonical approach applies the general contracting enforcement principle of 'the carrot or the stick' to a limited observability (or limited contactability) IC problem. In a principal-agent relationship the agent is rewarded or punished, based on the observed (or contractable) performance, which in general means that the *ex-post* value of the contract varies accordingly (e.g. it may decrease even if the bad performance is just bad luck not lack of effort). When the contract is a risk-sharing contract between a risk-averse agent and a risk-neutral principal, the IC constraint disrupts the full risk-sharing that could be achieved with observable effort. In fact, in a dynamic context the disruption increases (as a submartingale); i.e. with an unbounded concave utility the *ex-post* value and of the agent decays (as a supermartingale) to *immiseration* (Atkeson and Lucas (1992)). In contrast, in the *flexible moral hazard* approach the general contracting applied is: 'reward the cost beyond the minimum performance' which does not disrupt risk-sharing as long as the agent's participation (limited enforcement) constraint – at the minimum performance – is not binding. As we show (part of the first contribution), in a dynamic contract set-up, if the agent is not too impatient, with respect to the patient principal, if there is no stop, the *ex-post* value of the agent increases (as a submartingale) to *bliss*. There is, however, another important (double) difference between the two approaches, making them not perfect substitutes, nor one a special case of the other. It can be seen as a 'different commitment-timing to the distribution' : in the canonical approach, the agent is pre-committed to one distribution, or a mix of distributions of an observable outcome as function of the effort (as in multi-armed bandit problems, where the arms are given and the agent chooses how much effort exercise in each one); while in the flexible moral hazard approach, the agent *flexibly* chooses among distributions (as having the choice of arms, each one with a different effort required). The difference is (double), because in the former the cost is in the effort exercised and in the latter in the marginal-cost of the distribution chosen. However, the mapping between one valuation and the other may not be straightforward. Nevertheless, the different IC constraints have a basic element in common: *first-order stochastic dominance* in exercising effort (canonical) and in choosing higher marginal-cost distributions (flexible).

Our ultimate interest is in the design of *constrained-efficient* contracts between, impatient and risk-averse, borrowing sovereign countries and a, patient and risk-neutral, lender (i.e. a Fund). Therefore, we develop the theory of *flexible moral hazard* and its comparison with the *canonical moral hazard* in a broader context. We consider economies where there are impatient risk-averse representative-agent countries and a more-patient risk-neutral Financial Stability Fund with long-term state-contingent contracts. The Fund has country-specific contracts that guarantee that the sovereign debt is safe and sustainable (i.e. there are no *ex-post* expected losses for the lender). If there is no moral hazard risk-sharing is perfect, except when *limited enforcement* (LE) constraints are binding. We build on Ábrahám et al. (2022), where defaultable sovereign debt is transformed into a safe Fund contract, which accounts for moral hazard. They assume that the Fund has an exclusivity contract (i.e. absorbs all sovereign debt of the country which, as it is common in debt models, corresponds to the country's current account). We exploit the fact that IC constraints are disruptions to perfect risk-sharing. Therefore, it is enough that the Fund has exclusivity in providing risk-sharing, the IC constraints are part of the Fund contract, as well as the limited enforcement constraints: of the borrower not defaulting and of the Fund-lender not incurring expected losses. As in Liu et al. (2020) the Fund it is only required to be able to absorb a *minimal amount of debt*.

However, the comparison of long-term state-contingent contracts with limited enforcement constraints with flexible MH and with canonical MH is interesting, but too crude: if limited enforcement constraints are not binding in the flexible moral hazard IC constraints *never* distort risk-sharing, while in the canonical moral hazard they *always* do. A second contribution of the paper is to show how canonical IC contracts can be designed with minimal disruptions to risk-sharing. In particular, we consider long-term Fund contracts as a sequence of subprograms, within subprograms full-risk sharing is preserved, but when one of the limited enforcement constraints binds the subprogram terminates and a new subprogram starts with the initial condition accounting for the performance of the previous sub-contract. That is, in the subprogram IC punishments and rewards are *back-loaded* to the start of the following subprogram. However, since the end of the subprogram is, endogenously, determined by one of the two constraints (the no-default of the borrowing country and the no-expected losses of the Fund) binds, the punishment-reward mechanism must satisfy the constraint. In the case of the no-default constraint being binding, to satisfy it, brings the back-loaded design closer to the flexible moral hazard design. We then bring the latter closer to the former by restricting the choice of distributions.

Our third contribution is our quantitative exploration of these different designs, using as benchmark – but adapting it to the flexible MH framework – the calibrations of Ábrahám et al. (2022) for the euro area stressed economies.

Given the focus of our analysis, we abstain from discussing the 'decentralized' of our contract designs. As it has already been said, they can be implemented in the economies studied in Liu et al. (2020) and Callegari et al. (2023), where the Fund only absorbs a *minimal* share of the sovereign debt, while maintaining the exclusivity of risk-sharing. In these economies, most sovereign debt, most of the time, is held by private lenders and the same decentralization works for our contracts with moral-hazard, which reinforces the role of the Fund in these economies.

2 Literature Review

The paper derives the optimal contract between a lender and a borrower therefore relates to the seminal contributions of Kehoe and Levine (1993, 2001) and Thomas and Worrall (1994) who considered the case of limited enforcement. The difference with our approach is that we consider two-sided limited enforcement, while the literature has focused on one-sided limited enforcement analyzing the borrower's perspective. We solve the optimal contract by means of the Lagrangian approach of Marcet and Marimon (2019) which has been widely used to account for limited enforcements (e.g. Kehoe and Perri (2002) and Ferrari et al. (2021)) and its combination with moral hazard (e.g. Ábrahám et al. (2019) and Simpson-Bell (2020)).

We develop an optimal contract combining limited enforcement and moral hazard constraints. Our analysis is close to Atkeson (1991) who – similar to Thomas and Worrall (1994) – studies lending contracts in international contexts. However, Atkeson (1991) models moral hazard with respect to consuming or investing the borrowed funds, while we focus on risk-reduction policies. Quadrini (2004) also combine moral hazard and limited enforcement to study when and how contracts are renegotiation-proof. Similarly, Dovis (2019) shows that the combination of moral hazard and limited enforcement can generate a region of *ex post* inefficiency. This is not the focus of our analysis as our contract is both *ex ante* and *ex post* efficient. In addition, Müller et al. (2019) study dynamic sovereign lending contracts with moral hazard, with respect to reform policy efforts, and limited enforcement. Their characterization of the constrained-efficient allocation is more stylised (normal times are an absorbing state) and focuses on one form of moral hazard only.

We contribute to the literature on moral hazard in dynamic macroeconomic models. Our work relates to the pioneer work Prescott and Townsend (1984) who show that, in a static economy with moral hazard and adverse selection problems, a constrained efficient allocation can be the allocation of competitive equilibrium if the space of contracts is restricted to satisfy the corresponding incentive compatibility constraints. We generalize the flexible moral hazard approach of Georgiadis et al. (2024) to a dynamic model. We then contrast the provision of incentives with the canonical dynamic moral hazard model of Atkeson and Lucas (1992). In the canonical model, moral hazard leads to immiseration. This comes from the incentive-compatible mechanism which rewards high types with larger future utility and low types with lower future utility. This mechanism also disrupts risk sharing owing to the lower future utility of low types. We show that the flexible moral hazard approach leads to the opposite of immiseration – what we call *bliss*. In addition, it does not disrupt risk-sharing as the provision of incentives does not rely on the realized outcome. We then propose two ways to minimize the disruption to risk sharing in the canonical model.

Our work more closely contributes to the recent literature on the design of an optimal stability Fund. Roch and Uhlig (2018), Liu et al. (2020) and Callegari et al. (2023) focus on the lender's side of the contract and therefore disregard moral hazard issues. In opposition, Dovis and Kirpalani (2023) account for moral hazard and show that the provision of effort is back-loaded. Similarly, Ábrahám et al. (2022) expressively accounts for moral hazard with both a theoretical and quantitative focus. We pursue this line of research by providing a more comprehensive analysis of moral hazard. In particular, we study how different provisions of incentives affect risk sharing both theoretically and quantitatively.

3 Environment

We introduce flexible moral hazard in the environment studied in Ábrahám et al. (2022). Consider an infinite-horizon small open economy with a single homogenous consumption good in discrete time. A benevolent government acts as a representative agent and takes decisions on behalf of the small open economy.

The economy can produce goods using a decreasing-returns labour technology $y = \theta f(n)$, where $f'(n) > 0$, $f''(n) < 0$, $n \in [0, 1]$ denotes labor and $\theta \in \Theta = \{\theta_1, \ldots, \theta_N\}$ with $\theta_i < \theta_{i+1}$ is a productivity shock. The shock follows a Markov chain of order one with $\pi(\theta'|\theta)$ corresponding to the probability of drawing θ' tomorrow conditional on drawing θ today.

The economy also needs to cover government expenditures g. For this, the government can generate any expenditure *distribution* with support in a compact subset G of $\mathbb{R}^+$. We define $\mathcal{M}$ as the set of Borel probability measures on G and the lowest as well as the highest possible expenditure as $\underline{g} = \min\{g \in G\} > 0$ and $\overline{g} = \max\{g \in G\}$, respectively. We denote by $\mu(\theta')$ the distribution of g' conditional on θ' and $\boldsymbol{\mu} = \{\mu(\theta'_1), \ldots, \mu(\theta'_N)\}$ the vector of all such conditional distributions. We assume that the expenditure distribution is not contractible and let the state at the beginning of a period to be $s \equiv \{\theta, \boldsymbol{\mu}\}$.

The government's discounts the future at the rate β, satisfying $\beta < 1/(1+r)$, where r is the risk-free world interest rate. The government's utility is additively separable in consumption, labor and the effort cost of producing. In particular, $U : \mathbb{R}^+ \times [0,1] \times \mathcal{M}^N \to \mathbb{R}$. So, if the government chooses a distribution vector $\boldsymbol{\mu}$, supplies labor n and receives consumption c then its payoff is $U(c, n, \boldsymbol{\mu}) \equiv u(c) + h(1-n) - \sum_{i=1}^{N} \pi(\theta_i|\theta) v(\mu_(\theta_i))$ where $1-n$ denotes leisure.

Regarding consumption and leisure, we make standard assumptions on preferences. For the production distribution, we additionally assume that the cost of producing is convex, Gateaux twice differentiable and that producing more cost more. We also normalize the Gateaux derivative to be zero at $\overline{g}$.

Assumption 1 (Monotonicity, Differentiability and Convexity) *The utility function from consumption, $u : \mathbb{R}^+ \to \mathbb{R}$, and leisure, $h : [0,1] \to \mathbb{R}$, are strictly increasing, strictly concave, continuous and differentiable. The utility function from effort, $v : \mathcal{M} \to \mathbb{R}$, is continuous and convex. Moreover, it is Gateaux twice differentiable where $v_\mu : G \to \mathbb{R}^+$ and $w_\mu : G \times G \to \mathbb{R}^+$ denote the first and the second Gateaux derivative, respectively. We normalize $v_\mu(\overline{g}) = 0$. Finally, if the distribution μ first-order stochastically dominates $\tilde{\mu}$ then $v(\mu) \geq v(\tilde{\mu})$.*

The borrowing country can manage its private and public debt liabilities with the help of a Financial Stability Fund (Fund), which acts as a benevolent risk-neutral planner who has access to the international capital markets at the risk-free rate.

The country can default on its liabilities. We assume that the country's outside option corresponds to the autarky value of the standard incomplete market model with default which is

$$V^D(s) = \max_{n,\boldsymbol{\mu}} \left\{ U(\theta^d f(n) - g, n, \boldsymbol{\mu}) + \beta \mathbb{E}_{\theta'|\theta} \left[\int \left[(1-\lambda) V^D(s') + \lambda J(s', 0) \right] \mu(\theta')(\mathrm{d}g') \right] \right\}, \tag{1}$$

where $\theta^d \leq \theta$ contains the penalty for defaulting and $\lambda \geq 0$ is the probability to re-access the private bond market. Furthermore, V^D corresponds to the value under financial autarky and J to the value of reintegrating the private bond market without the Fund. More precisely, $J(s, b) = \max_{D \in \{0,1\}} \{(1-D) V^P(s, b) + D V^D(s)\}$, with

$$\begin{aligned} V^P(s, b) = & \max_{\{c, n, \boldsymbol{\mu}, b'\}} \left\{ U(c, n, \boldsymbol{\mu}) + \beta \mathbb{E}_{\theta'|\theta} \left[\int \left[J(s', b') \right] \mu(\theta')(\mathrm{d}g') \right] \right\} \\ & \text{s.t. } c + g + q(s, b')(b' - \delta b) \leq \theta f(n) + (1 - \delta + \delta\kappa) b. \end{aligned}$$

In the private bond market, the government can borrow long-term defaultable bonds, b', at a unit price of $q_p(s, b')$. A fraction $1-\delta$ of each bond matures today and the remaining fraction δ is rolled-over and pays a coupon κ. Private lenders are competitive and the price of one unit of private bond is given by $q_p(s,b') = \frac{1}{1+1}\mathbb{E}_{\theta'|\theta}\int(1-D(s',b'))[1-\delta+\delta\kappa+\delta q_p(s',b'')]\mu(\theta')(\mathrm{d}g')$ where D is the default policy taking value one in case of default and zero otherwise.

We end this section by providing more structure to the underlying stochastic structure.

Definition 1 (Partial-order in s) *We say that $s \geq \tilde{s}$ if, and only if, $\theta \geq \tilde{\theta}$ and the distribution μ first-order stochastically dominates $\tilde{\mu}$.*

The following monotonicity assumption on the outside values, guarantees that they are well-behaved with respect to the exogenous shock and the endogeneous of μ, which in turn will allow us to make meaningful comparisons of different moral hazard regimes.

Assumption 2 (Monotonicity of $V^D(s)$ with respect to the s partial-order) *If $s \geq \tilde{s}$ then $V^D(s) \geq V^D(\tilde{s})$, with inequality if, in addition, $s \neq \tilde{s}$.*

Lemma 1 *If, in addition to $s \geq \tilde{s}$, $\mathbb{E}[\theta'|\theta] \geq \mathbb{E}[\theta'|\tilde{\theta}]$, then Assumption 2 is satisfied.*

Note that the additional condition guarantees that higher θ shocks are associated with relatively higher expected future shocks (which is compatible with mean-reversion), a property that stochastic dominance guarantees for the $\boldsymbol{\mu}$ process.

4 The Fund under Flexible Moral Hazard

The Fund contract chooses a state-contingent sequence of consumption, labor and expenditure distribution that maximises the life life-time utility of the borrower given some initial level of the borrower's debt. It seeks to provide risk-sharing between the contracting parties. However, limited enforcement and moral hazard frictions preclude perfect risk-sharing.

The optimal contract is self-enforcing through the presence of two limited-enforcement constraints. First, we assume that if the country ever defaults on the Fund contract, it will not be able to sign a new contract with the Fund and will enter the markets for defaultable long-term debt as a defaulter. The Fund contract, however, makes sure that the country never finds it optimal to renege the contract. Second, the contract also prevents the Fund from ever incurring undesired expected losses, i.e. undesired permanent transfers.

In addition, the contract also has an incentive compatibility constraint, since the distribution of g is non-contractible (i.e. it is private information, or a sovereign right of the country). Thus, the long term contract must provide sufficient incentives for the country to implement a (constrained) efficient distribution.

4.1 The Constraints

Given the aforementioned frictions, the Fund has to account for three differents constraints. The first one is the *limited enforcement constraint* for the borrower. For any $s^t, t \geq 0$, it should be that

$$\mathbb{E}_t\left[\sum_{j=t}^{\infty} \beta^{j-t} U(c(s^j), n(s^j), \boldsymbol{\mu}_{j+1})\right] \geq V^D(s_t). \tag{2}$$

The outside value for the borrower is denoted by $V^D(s_t)$ and is given by (1). For now, we just need to assume that $V^D(s)$ is bounded below, increasing in s, and that there is always room for risk-sharing, in spite of the limited enforcement constraints.[1] In principle, a diverse set of default scenarios can satisfy these requirements. Note also that the notation is implicit about the fact that expectations are conditional on the implemented distribution sequence and θ_t.

The second constraint is the *limited enforcement constraint* for the lender. For any $s^t, t \geq 0$, it should hold that

$$\mathbb{E}_t\left[\sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \left(\theta_j f(n(s^j)) - c(s^j) - g(s^j)\right)\right] \geq Z(s_t). \tag{3}$$

The finite outside option of the lender $Z(s_t) \leq 0$ measures the extent of *ex-post* redistribution the Fund is willing to tolerate. That is, if $Z(s_t) < 0$ the Fund is allowed to make a permanent loss in terms of lifetime expected net present value – i.e. the Fund can find better investment opportunities in the international financial market and if it does not renege it is because it has committed to sustaining $Z(s_t) < 0$. Clearly, the level of $Z(s_t)$ has an important impact on the amount of risk sharing in our environment and it can thus be interpreted as the extent of solidarity the Fund is willing to accept in state s, as in Tirole (2015).

Finally, the last constraint is the *incentive compatibility constraint* (ICC). Define $V^b(s^t) = \mathbb{E}_t[\sum_{j=0}^{\infty} \beta^j U(c(s^{t+j}), n(s^{t+j}), \boldsymbol{\mu}_{t+j+1})]$ as the value of the borrower at time t. For any $s^t, t \geq 0$ and a given consumption and leisure schedules $\{c(s^t), n(s^t)\}_{t=0}^{\infty}$, the optimal vector of distributions is

$$\boldsymbol{\mu}_{t+1} = \operatorname*{argmax}_{\tilde{\boldsymbol{\mu}}} \left\{ -\sum_{\theta^{t+1}|\theta^t} \pi(\theta^{t+1}|\theta^t) v(\tilde{\mu}(\theta^{t+1})) + \beta \sum_{\theta^{t+1}|\theta^t} \pi(\theta^{t+1}|\theta^t) \left[\int V^b(s^{t+1}) \tilde{\mu}(\theta^{t+1})(\mathrm{d}g^{t+1})\right] \right\},$$

which given Assumption 1 can be rewritten as

$$\boldsymbol{\mu}_{t+1} = \operatorname*{argmax}_{\tilde{\boldsymbol{\mu}}} \left\{ \sum_{\theta^{t+1}|\theta^t} \pi(\theta^{t+1}|\theta^t) \left[\int \left[\beta V^b(s^{t+1}) - v_{\mu_{t+1}}(g^{t+1})\right] \tilde{\mu}(\theta^{t+1})(\mathrm{d}g^{t+1})\right] \right\}.$$

[1] In particular, $V^D(\underline{s}) < V^D(\overline{s})$.

We then re-scale the maximization problem by stating the gains and cost of effort in relative terms to the *no-effort option*,

$$\boldsymbol{\mu}_{t+1} = \underset{\tilde{\boldsymbol{\mu}}}{\text{argmax}} \left\{ \sum_{\theta^{t+1}|\theta^t} \pi(\theta^{t+1}|\theta^t) \left[\int \left[\beta \left(V^b(s^{t+1}) - V_0^b(\theta^{t+1}) \right) - \left(v_{\mu_{t+1}}(g^{t+1}) - v_{\mu_{t+1}}(\overline{g}) \right) \right] \tilde{\mu}(\theta^{t+1})(\mathrm{d}g^{t+1}) \right] \right\}$$

where $V_0^b(\theta^{t+1}) = V^b(\{\theta^{t+1}, -\overline{g}\})$ and $v_{\mu_{t+1}}(\overline{g}) = 0$ by Assumption 1. Therefore the ICC is for any $s^{t+1}, t \geq 0$,

$$v_{\mu_{t+1}(\theta^{t+1})}(g^{t+1}) = \beta \left(V^b(s^{t+1}) - V_0^b(\theta^{t+1}) \right), \tag{4}$$

Note that $v_{\mu_{t+1}(\theta^{t+1})}(g^{t+1})$ is a Gateaux derivative which gives the derivative of the cost of production at the specific distribution $\mu_{t+1}(\theta^{t+1})$.

By defining the ICC in this way, we use the first-order approach. That is we replace the agent's full optimization problem with respect to $\tilde{\boldsymbol{\mu}}$ by its necessary first-order condition. Our approach builds on Georgiadis et al. (2024) which is more 'flexible' than the standard approach of Rogerson (1985). The government chooses directly a distribution and not simply a level of effort for a given distribution. That is the government can arbitrarily manipulate the relative likelihood of any collection of expenditures. As a result, there is no information in the realization of any g. The provision of incentive is therefore solely directed towards the compensation of the marginal cost of producing each expenditure. The next sections compare the outcome of the two approaches in more details.

4.2 The Long Term Contract

In its extensive form, the *Fund contract* specifies that in state $s^t = (s_0, \ldots, s_t)$, the country consumes $c(s^t)$, uses labour $n(s^t)$ and chooses the distribution vector $\boldsymbol{\mu}_{t+1}$, resulting in a transfer to the Fund of $\theta f(n(s^t)) - c(s^t) - g(s^t)$, implying that the country is effectively borrowing. With *two-sided limited enforcement* and *moral hazard*, an optimal Fund contract is a solution to the following Fund problem:

$$\max_{\{c(s^t), n(s^t), \boldsymbol{\mu}_{t+1}\}} \mathbb{E}_0 \left[\alpha_{b,0} \sum_{t=0}^{\infty} \beta^t U(c(s^t), n(s^t), \boldsymbol{\mu}_{t+1}) + \alpha_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left[\theta_t f(n(s^t)) - c(s^t) - g(s^t) \right] \right]$$
$$\text{s.t.} \quad (2), \ (3), \ \text{and} \ (4), \ \forall s^t, t \geq 0.$$

Note that $(\alpha_{b,0}, \alpha_{l,0})$ are the initial Pareto weights, which are key for our interpretation of the Fund contract as a lending contract. In particular, we show later that the initial relative Pareto weight determines uniquely the level of debt that the Fund takes over when the country joins.

Given (2), (3) and (4), we take the following interiority assumption to ensure the uniform boundedness of the Lagrange multipliers.

Assumption 3 (Interiority) *There is an* $\epsilon > 0$*, such that, for all* $s_0 \in S$ *there is a program* $\{\tilde{c}(s^t), \tilde{n}(s^t), \tilde{\boldsymbol{\mu}}_t\}_{t=0}^{\infty}$ *satisfying constraints* (2) *and* (3) *when, on the right-hand side,* $V^D(s_t)$ *and* $Z(s_t)$ *are replaced by* $V^D(s_t) + \epsilon$ *and* $Z(s_t) + \epsilon$*, respectively, and similarly, when in* (4) $\overline{v}_{\tilde{\mu}_{t+1}(\theta^{t+1})}(g^{t+1})$ *is replaced by* $\overline{v}_{\tilde{\mu}_{t+1}(\theta^{t+1})}(g^{t+1}) + \epsilon$ *and* $=$ *is replaced by* $\leq$*.*

For constraints (2) and (3), this assumption requires that, in spite of the *enforcement constraints*, there are strictly positive rents to be shared since otherwise there may not be a constrained-efficient risk-sharing contract. The last part of this assumption is satisfied if an interior distribution vector $\boldsymbol{\mu}_{t+1}$ is feasible.[2]

Following Marcet and Marimon (2019) and Mele (2011), we can rewrite the Fund contract problem as a saddle-point Lagrangian problem:

$$
\begin{aligned}
\text{SP} \min_{\{\gamma_b(s^t),\gamma_l(s^t),\boldsymbol{\xi}(s^{t+1})\}} \max_{\{c(s^t),n(s^t),\boldsymbol{\mu}_{t+1}\}} & \Bigg\{ \mathbb{E}_0 \Bigg[\sum_{t=0}^{\infty} \beta^t \Big(\alpha_{b,t}(s^t) U\left(c(s^t), n(s^t), \boldsymbol{\mu}_{t+1}\right) \\
& - \xi(s^{t+1}) \left(\beta V_0^b(\theta^{t+1}) + v_{\mu_{t+1}(\theta^{t+1})}(g^{t+1}) \right) \\
& + \gamma_b(s^t) \left[U(c(s^t), n(s^t), \boldsymbol{\mu}_{t+1}) - V^D(s_t) \right] \Big) \\
& + \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \Big(\alpha_{l,t+1}(s^t) c_l(s^t) - \gamma_l(s^t) \left[c_l(s^t) - Z(s^t) \right] \Big) \Bigg] \\
\text{s.t.} \qquad & c_l(s^t) = \theta_t f(n(s^t)) - c(s^t) - g(s^t) \\
& \alpha_{b,t+1}(s^{t+1}) = \alpha_{b,t}(s^t) + \gamma_b(s^t) + \xi(s^{t+1}), \\
& \alpha_{l,t+1}(s^t) = \alpha_{l,t}(s^t) + \gamma_l(s^t), \\
& \alpha_{b,0}(s^0) \equiv \alpha_{b,0}, \alpha_{l,0}(s^0) \equiv \alpha_{l,0} \text{ given,}
\end{aligned}
$$

where $\gamma_b(s^t)$, $\gamma_l(s^t)$ and $\xi(s^{t+1})$ are the Lagrange multipliers of the limited enforcement constraints in (2) and in (3), and the ICC in (4), respectively, in state s^{t+1}. The vector of multipliers attached to the ICC is then given by $\boldsymbol{\xi}(s^{t+1}) = \{\xi(\theta_1, -g^{t+1}), \ldots, \xi(\theta_N, -g^{t+1})\}$.

The above formulation of the problem defines two new co-state variables $\alpha_b(s^t)$ and $\alpha_l(s^t)$, which represent the temporary Pareto weights of the borrower and the lender respectively. These variables are initialized at the original Pareto weights and they become time-variant because of the limited commitment and moral hazard frictions. In particular, a binding participation constraint of the borrower (lender) will imply a higher welfare weight of the borrower (lender) so that it does

[2]Note that these assumption can easily be satisfied since there are potential positive gains from risk-sharing (and borrowing an lending) in a contract between a risk-averse agent and a risk-neutral agent as long as there is a sufficiently high penalty for default. In other words, for β high enough, the risk sharing gains are strictly positive if $Z = 0$. The interiority of the distribution can be guaranteed if full risk sharing is not the only feasible allocation and appropriate Inada conditions are imposed on the cost $v(\mu)$.

not leave the contract. In addition, the moral hazard friction implies that the co-state variable of the borrower will increase as $\xi(s^{t+1}) \geq 0$.

It turns out that, in the previous problem, only relative Pareto weights, defined as $x_t(s^t) \equiv \alpha_{l,t}(s^t)/\alpha_{b,t}(s^t)$, matter for the allocations, and this allows us to reduce the dimensionality of the co-state vector and write the problem recursively by using a convenient normalization. Let $\eta \equiv \beta(1+r) < 1$. We normalize the multipliers as follows:

$$\nu_b(s^t) = \frac{\gamma_b(s^t)}{\alpha_{b,t}(s^t)},\ \nu_l(s^t) = \frac{\gamma_l(s^t)}{\alpha_{l,t}(s^t)} \text{ and } \varrho(s^{t+1}) = \frac{\xi(s^{t+1})}{\alpha_{b,t}(s^t)}. \tag{5}$$

The *Saddle-Point Functional Equation* (**SPFE**) — i.e. the saddle-point version of Bellman's equation — is given by

$$\begin{aligned} FV(x,s) = \text{SP} \min_{\{\nu_b,\nu_l,\boldsymbol{\varrho}'\}} \max_{\{c,n,\tilde{\boldsymbol{\mu}}'\}} & \Big\{ x\Big[(1+\nu_b)\left(u(c)+h(1-n)\right) - \nu_b V^D(s)\Big] \qquad (6)\\ & + \Big[(1+\nu_l)[\theta f(n) - c - g] - \nu_l Z(s)\Big] \\ & + \mathbb{E}\Big[\int \Big[\frac{1+\nu_l}{1+r} FV(x'(s'),s') - x(1+\nu_b) v_{\mu'(\theta')}(g') - x\varrho(s')\left(v_{\tilde{\mu}'(\theta')}(g') + V_0^b(x',\theta')\right)\Big] \tilde{\mu}'(\theta')(\mathrm{d}g')|\theta\Big] \\ \text{s.t.} \quad & x'(s') \equiv \overline{x}'(s) + \hat{x}'(s') = \left[\frac{1+\nu_b}{1+\nu_l} + \frac{\varrho'(s')}{1+\nu_l}\right]\eta x, \qquad (7) \end{aligned}$$

where $\boldsymbol{\varrho}'(s') \equiv \{\varrho_{\theta_1}(s'), \ldots, \varrho_{\theta_N}(s')\}$ corresponds to the vector of multipliers attached to the ICC. The Fund's value functions can be decomposed as follows

$$FV(x,s) = xV^b(x,s) + V^l(x,s) \text{ with} \tag{8}$$

$$V^l(x,s) = \theta f(n) - c - g + \frac{1}{1+r}\mathbb{E}\left[\int \left[V^l(x'(s'),s')\right]\mu(\theta')(\mathrm{d}g')|\theta\right], \tag{9}$$

$$V^b(x,s) = U(c,n,\boldsymbol{\mu}) + \beta\mathbb{E}\left[\int \left[V^b(x'(s'),s')\right]\mu(\theta')(\mathrm{d}g')|\theta\right] \tag{10}$$

The policy functions for consumption and labor of the Fund contract must solve the first-order conditions of the SPFE. In particular, $c(x,s)$ and $n(x,s)$ satisfy

$$u'(c(x,s)) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}, \tag{11}$$

$$\frac{h'(1-n(x,s))}{u'(c(x,s))} = \theta f'(n(x,s)). \tag{12}$$

These conditions are standard. The borrower's consumption is determined by its endogenous relative Pareto weight and, given that preferences are separable, the labor supply is not distorted. Turning to the optimal distribution, note that, given $\{\nu_b, \nu_l, \boldsymbol{\varrho}'\}$ and $\{c, n\}$, $\tilde{\boldsymbol{\mu}}'$ maximizes $FV(x,s)$ if it maximizes

$$\mathbb{E}\Big[\int \Big[\frac{1+\nu_l}{1+r}\tilde{FV}(x'(s'),s') - x(1+\nu_b)v_{\mu'(\theta')}(g') - x\varrho(s')v_{\tilde{\mu}'(\theta')}(g')\Big]\tilde{\mu}'(\theta')(\mathrm{d}g')|\theta\Big]$$

where $\tilde{FV}(x,s) = x\left[V^b(x,s) - V_0^b(x,\theta)\right] + \left[V^l(x,s) - V_0^l(x,\theta)\right]$ with $V_0^b(x,\theta) = V^b(x,\{\theta,-\overline{g}\})$ and $V_0^l(x,\theta) = V^l(x,\{\theta,-\overline{g}\})$. Given this, the first-order condition can be expressed as $\Pi_{\mu(\theta')}(s') = 0$, where

$$\begin{aligned}\Pi_{\mu(\theta')}(s') \equiv\ & x\left((1+\nu_b) + \varrho(s')\right)\left[\beta\left(V^b(x',s') - V_0^b(x',\theta')\right) - v_{\mu'(\theta')}(g')\right] \\ &+ \frac{1+\nu_l}{1+r}\left[V^l(x',s') - V_0^l(x',\theta')\right] - x\int\left((1+\nu_b) + \varrho(\theta',z)\right)w_{\mu(\theta')}(z,g')\mu(\theta')(\mathrm{d}z),\end{aligned}$$

The second derivative of v, $w_\mu(z,g')$, represents the change in the marginal cost of generating expenditure z associated with a slight increase in the probability of expenditure g'.[3] We can further simplify the first-order condition as

$$\begin{aligned}v_{\mu'(\theta')}(g') =\ & \beta\left[V^b(s') - V_0^b(x',\theta')\right] \\ &+ \frac{1+\nu_l}{1+\nu_b}\frac{1}{1+r}\frac{1}{x}\left[V^l(x',s') - V_0^l(x',\theta')\right] - Q_{\mu(\theta')}(g'),\end{aligned} \tag{13}$$

where $Q_{\mu(\theta')}(g') = \int(1+\tilde{\varrho}(\theta',z))w_{\mu(\theta')}(z,g')\mu(\theta')(\mathrm{d}z)$ and $\tilde{\varrho}(\theta',z) = \frac{1+\varrho(\theta',z)}{1+\nu_b}$. Since the first line of (13) is the ICC condition (4), the optimal distribution is such that

$$\frac{1+\nu_l}{1+\nu_b}\frac{1}{1+r}\left[V^l(x',s') - V_0^l(x',\theta')\right] = xQ_{\mu(\theta')}(g') \tag{14}$$

The following proposition characterize the optimal distribution. Before this, define $\delta_{g'(\theta')}$ as the Dirac distribution generating expenditure $g'(\theta')$ with probability 1.

Proposition 1 (Optimal Distribution) *If $w_{\mu(\theta')}(g')$ is constant for every g' and $\mu(\theta')$, then $\Pi_{\mu(\theta')}$ is strictly quasiconcave for every $\mu(\theta')$. The optimal distribution has only one expenditure. The Fund's problem is thus given by*

$$\begin{aligned}FV(x,s) = \mathrm{SP}\min_{\{\nu_b,\nu_l,\boldsymbol{\varrho}\}}\max_{\{c,n,\boldsymbol{g'}\}}\Bigg\{ & x\Big[(1+\nu_b)U(c,n,\boldsymbol{\delta}) - \nu_b V^D(s)\Big] \\ &+ \Big[(1+\nu_l)[\theta f(n) - c - g] - \nu_l Z(s)\Big] \\ &+ \mathbb{E}_{\theta'|\theta}\left[\frac{1+\nu_l}{1+r}FV(x'(s'),s') - x\varrho_{\delta_{g'(\theta')}}(g'(\theta'))\left(v_{\delta_{g'(\theta')}}(g'(\theta')) + V_0^b(x',\theta')\right)\right]\Bigg\}.\end{aligned}$$

where $\boldsymbol{g'} \equiv \{g'(\theta_1),\ldots,g'(\theta_N)\}$, $\boldsymbol{\delta} \equiv \{\delta_{g'(\theta_1)},\ldots,\delta_{g'(\theta_N)}\}$ and $\boldsymbol{\varrho} \equiv \{\varrho_{\delta_{g'(\theta_1)}}(g'(\theta_1)),\ldots,\varrho_{\delta_{g'(\theta_N)}}(g'(\theta_N))\}$,

The proposition is made of two parts. First, if the second derivative of the expenditure effort is constant, then $\Pi_{\mu(\theta')}$ is strictly quasiconcave. The reason is that both $m(s)$ and $V^l(x,s)$ are concave in s. Second, when $\Pi_{\mu(\theta')}$ is strictly quasiconcave, the problem of choosing an optimal distribution is the same as the one of choosing g' directly. In other words, the optimal distribution choice collapses to the direct choice of a θ-contingent sequence of of expenditures.

[3] We assume that this second derivative exists (and it does in our quantitative analysis); see, the discussion of Assumption 3 in Georgiadis et al. (2024).

4.3 Moral Hazard and Limited Enforcement

In the economies we study, with the need of risk-sharing, avoiding default or undesired permanent transfers, moral hazard problems arise when these problems could be alleviated with appropriate expenditure distribution, but such distribution is not contractible. The following lemma provides a characterization of the interaction between limited enforcement and moral hazard constraints.

Lemma 2 *When $g' = \overline{g}$, then $\varrho_{\mu(\theta')}(s') = 0$. Otherwise, $\varrho_{\mu(\theta')}(s') > 0$.*

The lemma states that when the highest expenditure realizes, the multiplier attached to the ICC is zero. Otherwise, it is strictly positive. The rationale behind this result is the following. The borrower incurs zero cost when $\overline{g}$ realizes. However, for any $g' < \overline{g}$, the cost is positive and the borrower needs to be compensated accordingly. Hence, when $g' = \overline{g}$, the borrower only gets the basis $V_0^b(x', \theta')$ meaning that $\varrho_{\mu(\theta')}(s') = 0$. For any other realization, $\varrho_{\mu(\theta')}(s') > 0$ to compensate the borrower for incurring more costs.

A direct corollary is that the law of motion of the relative Pareto weight is a left bounded positive submartingale. To see this, take the expectations of the law of motion

$$\mathbb{E}_t x_{t+1}(s^{t+1}) \equiv \mathbb{E}_t\left[\bar{x}_{t+1}(s^t) + \hat{x}_{t+1}(s^{t+1})\right] = \mathbb{E}_t\left[\frac{1+\nu_{b,t}(s^t)}{1+\nu_{l,t}(s^t)}x(s^t) + \frac{\varrho_{\mu_{t+1}}(s^{t+1}|s^t)}{1+\nu_{l,t}(s^t)}x(s^t)\right]\eta,$$

where $\hat{x}_{t+1}(s^{t+1})$ accounts for the dynamic effect of the moral hazard constraint. There are two forces working against the effect of the ICC multiplier: impatience $\eta < 1$ and the lender's participation constraint $\nu_{l,t}(s^t) \geq 0$. Hence, with neither the lender's participation constraint and nor impatience, we get that $\mathbb{E}_t x_{t+1}(s^{t+1}) \geq x_t(s^t)$. This means that without either one of these two elements, the relative Pareto weight would go towards infinity. This is the reverse of the *immiseration* result of Atkeson and Lucas (1992). We call it *bliss*.

Corollary 1 (Bliss) *When $\eta = 1$ and $\nu_{l,t}(s^t) = 0$ in all states, $\mathbb{E}_t x_{t+1}(s^{t+1}) \geq x_t(s^t)$.*

This result is important as it uncovers a different provision of incentives as the canonical moral hazard problem. In the next section we analyze such problem and contrast it with the outcome of Lemma 2.

5 The Fund under Canonical Moral Hazard

We now relax the assumption of flexible moral hazard. In particular, the borrower cannot anymore manipulate in an arbitrary way the relative likelihood of any collection of expenditures. It is limited to choose among a restricted set of measures defined on the σ-algebra of Borel sets. In addition, the cost function does not depend on the chosen distribution anymore.

5.1 The Constraints

The borrower's choice of distribution is restricted to a family of distributions in $\mathcal{M}$. In particular, define Q as a mixture distribution $Q = \zeta(e)Q_L + (1 - \zeta(e))Q_H$ for $Q_L, Q_H \in \mathcal{M}$ with weighting function $\zeta(e)$. The borrower can manipulate the weights by choosing the (non-contractible) effort $e \in [0, 1]$ to change Q. We assume that Q_H first-order stochastically dominates Q_L and $\zeta'(e) < 0$ and and $\zeta''(e) < 0$.

To recover the formulation of the canonical moral hazard problem, we consider a different effort cost than what we had so far. The cost of effort $\hat{v} : [0, 1] \to \mathbb{R}$ is a mapping from $[0, 1]$ instead of $\mathcal{M}$. We then have that $\hat{U}(c, n, e) = u(c) + h(1 - n) - \hat{v}(e)$. Additionally, we assume independence between θ and g. This implies that a single shock variable, g, depends on effort. We denote the joint distribution of θ and g by $\Upsilon(s'|s, e)$.

Notice that the shape of the cost function $\hat{v}(e)$ prevents us to use the apparatus developed previously. In the next chapter we analyze the case of restricted distribution with the previously adopted cost function $v(\mu)$.

Given the re-definition of the utility function – which is now defined over e instead of μ – the participation constraint of the lender is given by (3), while the participation constraint of the borrower reads

$$\mathbb{E}_t\left[\sum_{j=t}^{\infty} \beta^{j-t} U(c(s^j), n(s^j), e(s^j))\right] \geq V^D(s_t). \tag{15}$$

The main change relates to the *incentive compatibility constraint* (ICC). Instead of choosing an entire distribution, the borrower picks an effort level. Define $\hat{V}^b(s^t) = \mathbb{E}_t[\sum_{j=0}^{\infty} \beta^j \hat{U}(c(s^{t+j}), n(s^{t+j}), e(s^{t+j}))]$ as the value of the borrower at time t. The optimal choice of effort is given by

$$e(s^t) = \underset{\tilde{e}}{\operatorname{argmax}}\ \hat{U}(c(s^t), n(s^t), \tilde{e}) + \beta \sum_{s^{t+1}|s^t} \Upsilon(s^{t+1}|s_t, \tilde{e}) \hat{V}^b(s^{t+1}).$$

The ICC is therefore

$$\hat{v}_e(e(s^t)) = \beta \sum_{g^{t+1}|g^t} \frac{\partial \Upsilon(g^{t+1}|g_t, e(s^t))}{\partial e(s^t)} \hat{V}^b(s^{t+1}). \tag{16}$$

The difference with respect to the case of flexible moral hazard is that government expenditures are assumed to follow a fixed Markovian distribution given e.

For the first-order approach of Rogerson (1985) to be valid, the cumulative distribution function of g should be differentiable, convex and satisfy the monotone likelihood-ratio condition. This mirrors our Assumption 1. Moreover, the ICC in (16) relies on the informativeness of the realization of g. This is because the information content of a specific expenditure realization can be directly measured by the relative likelihood given that we know Υ. In the case of flexible moral hazard, the probability distribution is a choice variable which is not contractible. This precludes any informativeness of the expenditure realization.

Assumption 4 (*Differentiability, Monotonicity and Convexity*) *For every s, if $e \geq \tilde{e} > 0$ the ratio $\frac{\Upsilon(s'|s,\tilde{e})}{\Upsilon(s'|s,e)}$ is nonincreasing in g', and, for every (e, s), $F_j(e, s) = \sum_{i=1}^{j} \Upsilon(\{\theta_j, g'\}|s, e)$ is differentiable in e, with $\partial_e F_j(e, s) \leq 0$ and $\partial_e^2 F_j(e, s) \geq 0$,.*

These conditions simply generalize the assumptions of Rogerson (1985) so that we can apply his first-order condition approach in a simple static Pareto-optimization problem to our dynamic contracting problem with limited enforcement and moral hazard frictions.

5.2 The Long Term Contract

With *two-sided limited enforcement* and *moral hazard*, an optimal Fund contract is a solution to the following **Fund problem**:

$$\max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}_0 \left[\alpha_{b,0} \sum_{t=0}^{\infty} \beta^t \hat{U}(c(s^t), n(s^t), e(s^t)) + \alpha_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \left[\theta_t f(n(s^t)) - c(s^t) - g(s^t) \right] \right]$$
$$\text{s.t.} \quad (3), \ (15), \ \text{and } (16), \ \forall s^t, t \geq 0.$$

In terms of structure, the Fund problem is very similar to what we had under flexible moral hazard. The main change is that the government exercises effort $e(s^t)$ instead of directly choosing a distribution μ_{t+1}. As before, to ensure the uniform boundedness of the Lagrange multipliers, we posit an interiority assumption.

Assumption 5 (*Interiority*) *There is an $\epsilon > 0$, such that, for all $s_0 \in S$ there is a program $\{\tilde{c}(s^t), \tilde{n}(s^t), \tilde{e}(s^t)\}_{t=0}^{\infty}$ satisfying constraints (15) and (3) when, on the right-hand side, $V^D(s_t)$ and $Z(s_t)$ are replaced by $V^D(s_t) + \epsilon$ and $Z(s_t) + \epsilon$, respectively, and similarly, when in (16) $\partial_e \hat{v}(e(s^t))$ is replaced by $\partial_e \hat{v}(e(s^t)) + \epsilon$ and $=$ is replaced by $\leq$.*

The interiority of effort can be guaranteed if full risk sharing is not the only feasible allocation and appropriate Inada conditions are imposed on the cost $\hat{v}(e)$ and benefit $\Upsilon(g'|g, e)$ of effort. The following assumption parallels Assumption 2 for the economy with canonical moral hazard.

Assumption 6 (Monotonicity of $V^D(s)$ with respect to the s partial-order) *If $\theta \geq \theta'$ and $g \leq g'$ then $V^D(s) \geq V^D(s')$, with inequality if, in addition, $s \neq s'$.*

Following the previous section, we can formulate the Fund problem in recursive form. We find that the *Saddle-Point Functional Equation* (**SPFE**) — i.e. the saddle-point version of Bellman's equation — is given by

$$\hat{FV}(x,s) = \text{SP} \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} \Big\{ x\Big[(1+\nu_b)\hat{U}(c,n,e) - \nu_b V^D(s) - \varrho \hat{v}_e(e)\Big] \tag{17}$$

$$+ \Big[(1+\nu_l)(\theta(s)f(n) - c - g(s)) - \nu_l Z\Big] + \frac{1+\nu_l}{1+r}\mathbb{E}\big[\hat{FV}(x',s')\big|s,e\big]\Big\}$$

$$\text{s.t. } x'(s') \equiv \bar{x}'(s) + \hat{x}'(s') = \left[\frac{1+\nu^b}{1+\nu_l} + \frac{\varphi(s'|s)}{1+\nu_l}\right]\eta x \text{ and } \varphi(s'|s) = \varrho\frac{\partial_e \Upsilon(s'|s,e)}{\Upsilon(s'|s,e)}. \tag{18}$$

The Fund's value functions can be decomposed as in the case with flexible moral hazard. Similarly, the policy functions for consumption and labor of the Fund contract are the solution to (11) and (12). This is because of additive separability in the utility function. The formulation of the moral hazard does not directly affect the optimal choice of consumption and labor.

Notice that the multiplier $\varphi(s'|s)$ is defined as $\varrho\frac{\partial_e \Upsilon(s'|s,e)}{\Upsilon(s'|s,e)}$. It does not explicitly depend on e since, as multiplier, the action is taken as given. Moreover, it can be positive or negative depending on the sign of $\partial_e \Upsilon(s'|s,e)$. This reflects the main difference with the flexible moral hazard approach we discussed previously. As the choice of distribution is restricted, the relative likelihood is informative about the realization of g'.

The effort policy $e(x,s)$ is determined by the first order condition of the SPFE with respect to e, which can be conveniently expressed as:

$$\hat{v}'(e(x,s)) = \beta \sum_{s'|s} \partial_e \Upsilon(s'|s,e)\hat{V}^b(x'(s'),s')$$

$$+ \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}\frac{1}{1+r}\sum_{s'|s}\partial_e \Upsilon(s'|s,e)\hat{V}^b(x'(s'),s')$$

$$- \frac{\varrho(x,s)}{1+\nu_b(x,s)}\left[v''(e(x,s)) + \beta\sum_{s'|s}\partial_e^2 \Upsilon(s'|s,e)\hat{V}^b(x'(s'),s')\right]. \tag{19}$$

Equation (19) balances the marginal cost of effort with the benefits. The first line is the life-time utility benefit of effort to the borrower; the second line is the marginal benefit of effort to the lender, in terms of the borrower's marginal utility, given by (11); the third line accounts for the marginal relaxation/tightening effect of the moral hazard constraint (4) when there is a change in effort. With contractible effort, the Fund problem would not have the *incentive compatibility constraint* (4) and the effort decision would be given by the first two lines, with the second one accounting for the social value of effort. In contrast, with non-contractible effort, as we assume, constraint (4) is present and the first line is equal to zero, namely:

$$\hat{v}'(e(x,s)) = \beta \sum_{s'|s} \partial_e \Upsilon(s'|s,e)\hat{V}^b(x'(s'),s'). \tag{20}$$

In this case, (19) reduces to

$$\frac{1}{1+r}\sum_{s'|s}\partial_e\Upsilon(s'|s,e)\hat{V}^b(x'(s'),s') \\ = \vartheta(x,s)\left[v''(e(x,s))-\beta\sum_{s'|s}\partial_e^2\Upsilon(s'|s,e)\hat{V}^b(x'(s'),s')\right], \tag{21}$$

where $\vartheta(x,s)\equiv\frac{x\varrho(x,s)}{1+\nu_l(x,s)}$ can be interpreted as the marginal value of relaxing the ICE constraint in terms of the lender's valuation; that is, (21) accounts for the external effect of effort on the lender's value through its effect on the incentive compatibility constraint. Note that, although incentive compatibility implies that only the borrower's returns affect the effort decision directly, the benefits represented in (21) will affect incentives as they affect $\varrho(x,s)$ and hence the whole future path of allocations through (18).

5.3 Limited Enforcement and Moral Hazard

In the economies we study, with the need of risk-sharing, avoiding default or non sustainable debt levels, moral hazard problems arise when these problems could be alleviated with effort, but such effort is not contractible. Therefore it is reasonable to model limited enforcement contracts that satisfy the following property:

Definition 2 *The limited enforcement constraints (15) and (3) satisfy the 'no-free-lunch condition' if, given (x,s), whenever $\nu_b(x'(x,s),s')>0$, then $\partial_e\Upsilon(s'\mid s,e)>0$ and whenever $\nu_l(x'(x,s),s')>0$, then $\partial_e\Upsilon(s'\mid s,e)<0$, respectively.*

Conversely, if $\partial_e\Upsilon(s'\mid s,e)=0$ (or the inequality signs were reversed) exercising more effort would not have any effect on the limited enforcement constraints (or a perverse effect) and, on those grounds, moral hazard would not be an issue. The following lemma provides a characterization of the interaction between limited enforcement and moral hazard constraints. Before we state it, note that the law of motion of x which can be written as:

$$x_{t+1}(s^{t+1})\equiv\left[\bar{x}_{t+1}(s^t)+\hat{x}_{t+1}(s^{t+1})\right]=\left[\frac{1+\nu_{b,t}(s^t)}{1+\nu_{l,t}(s^t)}x(s^t)+\frac{\varphi_{t+1}(s^{t+1}|s^t)}{1+\nu_{l,t}(s^t)}x(s^t)\right]\eta,$$

where $\hat{x}_{t+1}(s^{t+1})$ accounts for the dynamic effect of the moral hazard constraint.

Lemma 3

i) In recursive contracts, limited enforcement constraints have an effect on the expected law of motion of the Pareto weights, when they are binding; in contrast, moral hazard constraints do not have an effect on $\{\mathbb{E}_t x_{t+1}\}$, even if they bind; i.e. $\mathbb{E}_t x_{t+1}=\bar{x}_{t+1}(s^t)$.

ii) If limited enforcement constraints satisfy the 'no-free-lunch condition', moral hazard constraint make no-default constraints (15) *more likely to bind and sustainability constraints* (*3*) *less likely to bind and, in both cases,* $\mathbb{E}_t \frac{1}{u'(c_{t+1})}$ *increases.*

To see *i)*, note that, given the above decompostion of the law-of-motion of x, $\sum_{s^{t+1}|s^t} \Upsilon(s^{t+1}|s^t, e(s^t))\hat{x}_{t+1}(s^{t+1}) = 0$, since independently of effort $\sum_{s^{t+1}|s^t} \Upsilon(s^{t+1}|s^t, e(s^t)) = 1$, hence $\sum_{s^{t+1}|s^t} \partial_e \Upsilon(s^{t+1}|s^t, e(s^t)) = 0$. Therefore $\mathbb{E}_t \hat{x}_{t+1} = 0$ and $\mathbb{E}_t x_{t+1} = \bar{x}_{t+1}(s^t)$. Alternatively, the expected law of motion of x can also be expressed as

$$\mathbb{E}_t x_{t+1} = \mathbb{E}_t \left[\frac{1}{u'(c_{t+1})} \frac{1 + \nu_{l,t+1}(s^t)}{1 + \nu_{b,t+1}(s^t)} \right] = \frac{1}{u'(c_t)} \eta,$$

where the last equality is the *inverse Euler equation* of the recursive contract (Ábrahám et al. (2022), Lemma 4).

To see *ii)*, note that, since the limited enforcement multipliers are either zero or at most one of the two is positive, we can have the following decomposition

$$\mathbb{E}_t \frac{1}{u'(c_{t+1})} = \mathbb{E}_t \left[x_{t+1} \frac{1 + \nu_{b,t+1}(s^{t+1})}{1 + \nu_{l,t+1}(s^{t+1})} \right] = \mathbb{E}_t x_{t+1} + \mathbb{E}_t x_{t+1} \nu_{b,t+1}(s^t) - \mathbb{E}_t x_{t+1} \frac{\nu_{l,t+1}(s^t)}{1 + \nu_{l,t+1}(s^t)},$$

where $\mathbb{E}_t x_{t+1} = \eta x_t$ and, without incentive constraints, the last two terms simply denote the change in the relative Pareto weight when either the no-default or the sustainability constraints binds. However, if limited enforcement constraints satisfy the 'no-free-lunch condition', the no-default constraint is more likely to bind, while the sustainabiliy constraint is less likely to bind and, as a result, in both cases expected consumption increases.

6 Flexible vs. Canonical Moral Hazard

In this section, we contrast the two moral hazard formulations. In particular, we show that the expression for the cost of effort is key for the provision of incentives. Even if the choice of distributions is restricted, the Fund may function as in the case of flexible moral hazard. For this, the cost of effort has to incorporate the properties of the probability distribution.

6.1 Back-loaded moral hazard

In the canonical moral hazard problem, the provision of incentives is generated by a system of rewards and punishment associated with the moral-hazard constraint (16). Such system is not unique. We analyze contracts that consist of an infinite sequence of subprograms, whereby within each subprogram rewards and punishment are *back-loaded* to the end.

We consider a contract with subprograms. The length of such subprograms is directly determined by the binding limited enforcement constraints. The reason is that whenever a subprogram violates one of the limited enforcement constraints, one of the contracting parties would find it optimal to terminate the contract. Hence, the binding limited enforcement constraints endogenously determine the subprogram's length.

The Fund contract can be expressed as the solution to a sequence of recursive (SPFE) problems. As the program resets when one of the limited enforcement constraint binds, the start of a subprogram is such that

$$
\begin{aligned}
\hat{FV}(x,s) = \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} \Big\{ & x\big[(1+\nu_b)(u(c)+h(1-n)-\hat{v}(e)) - \nu_b V^0(s) - \varrho v'(e)\big] \\
& + [(1+\nu_l)(\theta f(n)-c-g) - \nu_l Z] \\
& + \frac{1+\nu_l}{1+r}\mathbb{E}\left[\mathbb{I}_{\{(x',s')\}}\hat{FV}(x',s') + (1-\mathbb{I}_{\{(x',s')\}})\overline{FV}(x',s';\bar{x}') \mid s,e\right] \Big\} \\
\text{s.t.} \quad & x'(s') = \eta x \frac{1+\nu_b+\psi(s' \mid s,e)}{1+\nu_l} \\
& \bar{x}' = \eta x \frac{1+\nu_b}{1+\nu_l}
\end{aligned}
$$

where $\mathbb{I}_{\{(x',s')\}}$ is an indicator function where $\mathbb{I}_{\{(x',s')\}} = 1$ if one of the limited enforcement constraints is binding – i.e. $\nu_b(x',s') + \nu_l(x',s') > 0$. Then within the subprogram

$$
\begin{aligned}
\overline{FV}(x,s;\bar{x}) = \min_{\{\varrho\}} \max_{\{c,n,e\}} \Big\{ & \bar{x}\,[u(c)+h(1-n)] - x\big[\hat{v}(e) + \varrho v'(e)\big] + (\theta f(n) - c - g) \\
& + \frac{1}{1+r}\mathbb{E}\left[\mathbb{I}_{\{(x',s')\}}\hat{FV}(x',s') + (1-\mathbb{I}_{\{(x',s')\}})\overline{FV}(x',s';\bar{x}') \mid s,e\right] \Big\} \\
\text{s.t.} \quad & x' = \eta x\left[1+\psi(s' \mid s,e)\right] \\
& \bar{x}' = \eta\bar{x},
\end{aligned}
$$

As it can be seen, within the subprogram x' is the *latent aggregated multiplier* and the consumption and labor policies are given by the same first-order condition, (11) and (12), resulting in $c(\overline{x},s)$ and $n(\overline{x},s)$, while the optimal effort $e(\overline{x},s)$ requires a reformulation of (21). For this, it is useful to recall that, as in the benchmark Fund contracts, $\hat{FV}(x,s) = x\hat{V}^b(x,s) + V^l(x,s)$. However, $\overline{FV}(x,\overline{x},s)$ depends on the within the subprogram relative Pareto weight $\overline{x}$ and the *latent relative Pareto weight* x; therefore, we first decompose

$$
\begin{aligned}
\overline{V}^b_1(\overline{x},s) &= u(c(\overline{x},s)) + h(1-n(\overline{x},s)) + \beta\mathbb{E}\overline{V}^b_1(x',s'), \\
\overline{V}^b_2(x,s) &= -\hat{v}(e(x,s)) + \beta\mathbb{E}\overline{V}^b_2(x',s'),
\end{aligned}
$$

then the value of the borrower is simply $\overline{V}^b(x,s,\overline{x}) = \overline{V}^b_1(\overline{x},s) + \overline{V}^b_2(x,s)$ implying that

$$
\overline{FV}^b(x,s,\overline{x}) = \overline{x}\overline{V}^b_1(\overline{x},s) + x\overline{V}^b_2(x,s) + \overline{V}^l(x,s,\overline{x}).
$$

Note that, except for the distinction of $\overline{x}$, $\overline{FV}(x,s;\overline{x})$ is the same as $\hat{FV}(x,s)$ since we can always incorporate in the minimization $\{\nu_b,\nu_l\}$ which will satisfy $\nu_b=\nu_l=0$, by construction, within the subprogram.

In Appendix A, we provide some more details about the behavior of consumption and labor in a subprogram. We further consider additional stochastic and deterministic criteria determining the subprogram length.

6.2 Restricted flexible moral hazard

In Section 4, the cost of effort is given by $v(\mu)$ which is a mapping from $\mathcal{M}$ to the real. In Section 5, the cost of effort becomes $\hat{v}(e)$ which is a mapping from $[0,1]$ to the real. This cost function does not incorporate anything about the distribution implied by the choice e.

Now if we were to re-formulate the cost function in the canonical moral hazard case to get closer to the flexible moral hazard, we would get

$$v(Q_e)=K\left[\int(\overline{g}-g)Q_e(\mathrm{d}g)\right],$$

where $Q_{\tilde{e}}=\zeta(\tilde{e})Q_L+(1-\zeta(\tilde{e}))Q_H$ is the distribution obtained for a specific effort $e=\tilde{e}$.

Clearly, this cost function integrates the properties of the chosen distribution. This enables the Fund to provides incentive as described in Section 4 – and in particular Proposition 1. The reason behind this is that the borrower incurs a cost to provide effort directly related to the chosen distribution. Moreover, as Q is a mixture distribution, the family of distribution is convex enabling local flexibility in the choice of the distribution.

7 Quantitative Analysis

In this section, we present the parametrization of the model and the results of the quantitative analysis. In future versions, we will properly calibrate the model to match specific moments.

7.1 Parametrization

We parametrize the model following Ábrahám et al. (2022). The utility of the borrower is additively separable in consumption, leisure and effort. In particular, we assume that $u(c)=\log(c)$ and $h(1-n)=\gamma\frac{(1-n)^{1-\sigma}-1}{1-\sigma}$. For the flexible moral hazard, we consider $v(\mu)=\omega[\int(\overline{g}-g)\mu(\mathrm{d}g)]^2$ and for the canonical moral hazard $v(e)=\omega e^2$ so that the third derivative is zero in both cases.

The preference parameters are set to $\sigma=0.6887$, $\gamma=1.4$, $\omega=0.1$ and the discount factor to $\beta=0.945$. The risk free interest rate is set to $r=2.48\%$, the average short-term real interest rate of Germany. The fact that the borrower is less patient than the lender implies that the borrower would like to front-load consumption.

Table 1: Parameter Values

α	β	σ	γ	r	λ	ψ	δ	ω	Z
0.566	0.945	0.6887	1.4	0.0248	0.1	0.8099	0	0.1	0

The participation constraint of the lender is set to $Z = 0$, implying no expected permanent transfers between the borrower and the lender at any time or state. In other words, the Fund is not built on an assumption of solidarity which would require permanent transfers.

We assume that *Fund-exit is irreversible*, with the interpretation that the fund can commit to exclusion of the borrower. If a country defaults, it is also subject to an asymmetric default penalty as in Arellano (2008)

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \geq \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

where $\psi = 0.8099$. In addition, we adopt a probability to re-enter the market of 0.1. To simplify computation, we set $\delta = 0$ in the incomplete market economy with default.

Regarding the technology, we assume that $f(n) = n^\alpha$ with the labor share of the borrower set to $\alpha = 0.566$ to match the average labor share across the Euro Area 'stressed' countries. Table 1 summarizes the parameter values.

Regarding the calibration of the productivity shocks, we consider that $\theta \in \{0.81, 1.01, 1.12\}$ and the transition matrix is given by

$$\pi = \begin{bmatrix} 0.980 & 0.015 & 0.005 \\ 0.005 & 0.975 & 0.020 \\ 0.015 & 0.025 & 0.960 \end{bmatrix} \tag{22}$$

Regarding the expenditure shock g, we allow for 3 realizations $g \in \{0.0785, 0.0415, 0.0185\}$. For the flexible moral hazard, this is all we need to define as the borrower is free to pick any distribution over the vector of g. However, for the canonical moral hazard, the borrower is restricted in the choice of distribution. Hence, we additionally need to determine Q.

Recall that $Q = \zeta(e)Q_L + (1-\zeta(e))Q_H$ for $Q_L, Q_H \in \mathcal{M}$. Effort affects the probability distribution over next period's realisation of the government expenditure g'. We consider that Q_L first-order stochastically dominates Q_H and $\zeta'(e) < 0$ and and $\zeta''(e) < 0$. For this, we set $\zeta(e) = (e-1)^2$ and

$$Q_L = \begin{bmatrix} 0.93 & 0.0466 & 0.0234 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}, \; Q_H = \begin{bmatrix} 1 & 0 & 0 \\ 0.06 & 0.94 & 0 \\ 0.03 & 0.04 & 0.93 \end{bmatrix}$$

This functional form implies simple expressions for $\frac{\partial Q(g'|g,e)}{\partial e}$ and $\frac{\partial^2 Q(g'|g,e)}{\partial e^2}$ as follows:

$$\frac{\partial Q(g'|g,e)}{\partial e} = -\zeta'(e)\big[Q_L(g'|g) - Q_H(g'|g)\big] = 2(1-e)\big[Q_L(g'|g) - Q_H(g'|g)\big]$$
$$\frac{\partial^2 Q(g'|g,e)}{\partial e^2} = -\zeta''(e)\big[Q_L(g'|g) - Q_H(g'|g)\big] = -2\big[Q_L(g'|g) - Q_H(g'|g)\big].$$

Note that with this functional forms Assumption 4 is satisfied.

7.2 Policy functions

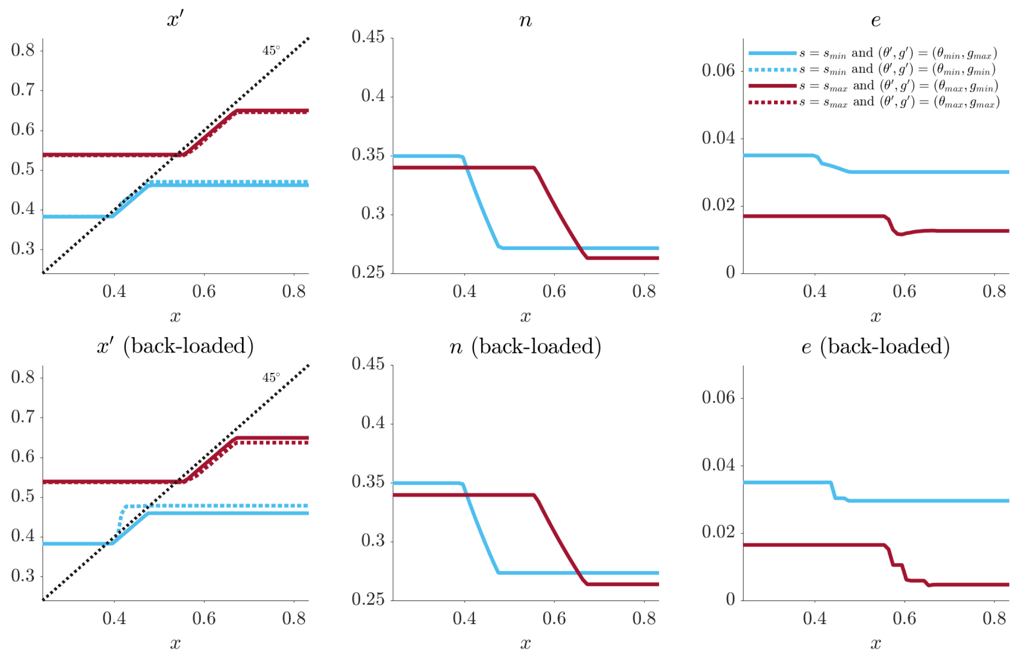

Figure 1: Main Policy Functions – Canonical and Back-loaded MH

In this subsection, we expose the main policies associated to the Fund problem. Especially, we present the policies for the relative Pareto weight, consumption, labor and government expenditures. We first focus on the flexible moral hazard before going to the canonical moral hazard.

Figure 1 depicts the main policies related to the canonical moral hazard as a function of the relative Pareto weight x. In particular, we distinguish between the standard provision of incentives and the back-loaded one. The red lines (either solid or dotted) relate to the highest value of s and the blue lines to the lowest value of s. For each color, the solid line corresponds to a transition to the same g, while the dotted corresponds to a transition to the opposite one.

In each specification, the horizontal line on the left hand side is determined by the borrower's binding participation constraint, while the horizontal line on the right hand side is determined by

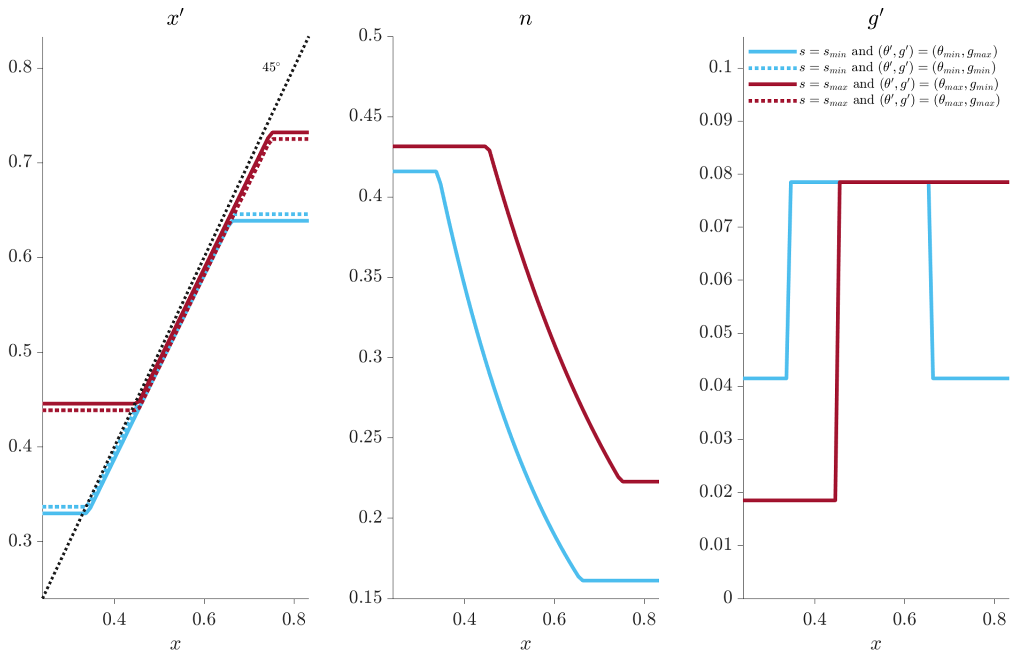

Figure 2: Main Policy Functions – Flexible MH

the Fund's binding participation constraint. The line rejoining both horizontal lines is determined by the first best allocation and has a slope of $\eta < 1$.

As we assume log-utility, $c = x'/\eta$. We therefore do not depict the policy for consumption as it closely follows the relative Pareto weight. In opposition, labor follows the inverse direction. As the relative Pareto weight increases, consumption increases but labor reduces if the constraints are not binding, however as it can be seen in Figure 1 when the borrowing constraint is binding at low values of x and θ, there may be a reversal: n is higher in our simulation, signaling that even with the Fund, there may be some austerity at these low values. These results follow from the first-order conditions in (11) and in (12).[4]

We find significant differences in the law of motion of the relative Pareto weights between the two specifications. In particular, in the back-loaded case, we note a larger wedge between the solid and the dotted line of each color. This means that the relative Pareto weight reacts more strongly to changes in g. Regarding consumption and labor, we observe little changes. This is because the determination of such variables does not directly depend on the value of the multiplier attached to the ICC, ϱ.

Figure 2 depicts the main policies related to the flexible moral hazard as a function of the relative Pareto weight x. The figure uses the same color system as the previous one.

[4]See Lemma 1 in Ábrahám et al. (2022); in contrast with them, our simulations have the indicated reversal effect on n with respect to θ when the borrower's constraint binds at low values of s, while we do not have -as they do – a reversal effect of e with respect to g when the lender's constraint binds at high values of s.

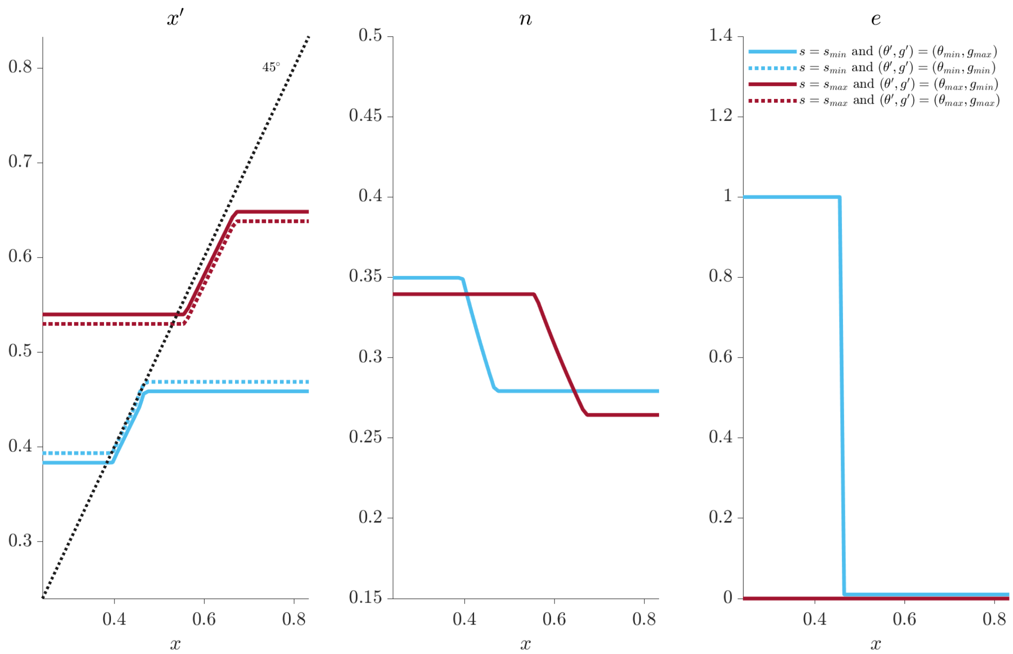

Figure 3: Main Policy Functions – Restricted Flexible MH

In terms of relative Pareto weight, we observe two elements. First, when the borrower picks a $g' > g$, then the relative Pareto weight decreases. In opposition, when the borrower picks $g' < g$, the opposite holds true. This is because the Fund has to compensates the borrower when it chooses a lower g' because it is more costly to implement.

As consumption follows the same dynamic as the relative Pareto weight, we do not depict it. Moreover, as in the canonical moral hazard, labor reduces as the relative Pareto weight increases. This is because the Fund with flexible and canonical moral hazard, lead to the same first-order conditions with respect to c and n.

Finally, looking at the policy function for g', we observe that the level of expenditures is not necessarily monotonic in x. This is due to the lender's participation constraint as shown in Ábrahám et al. (2022). This non-monotonicity disappears once we reomve the lender's participation constraint.

Figure 3 depicts the main policies related to the restricted flexible moral hazard as a function of the relative Pareto weight x. The main difference with the flexible moral hazard is that the borrower is restricted to choose the distribution of g' among a specific family of distribution Q.

We observe a similar dynamic of the relative Pareto weight as in Figure 2. Moreover, we directly see that the borrower provides more effort than what is depicted in Figure 1. This is because the cost structure internalizes the cost of choosing a specific distribution.

7.3 Steady state analysis

We analyze the Fund's allocation in steady state. We compute welfare gains with respect to the Fund with canonical MH and distinguish two cases. First, the Fund contracts which have outside options given by (1). Second, we re-compute the different Fund contracts with the same outside option.

Figure 4 depicts the ergodic set of the relative Pareto weights for the different Fund contracts. The definition of the ergodic set is given in Definition A.1. The dark grey region corresponds to the ergodic set, while the light grey region is the basin of attraction. The dark grey region is therefore the steady state in which we will contact simulation exercises.

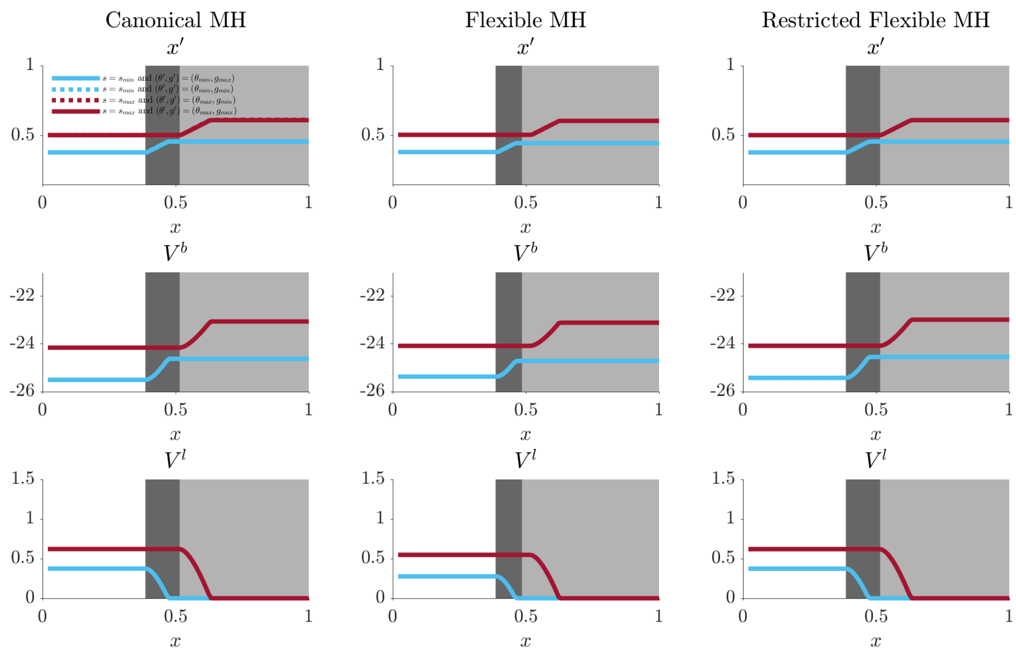

Figure 4: Ergodic Set – Canonical and Back-loaded MH

We observe only little differences in the determination of the ergodic set of the different specifications. The ergodic set of the Fund under flexible MH has a slightly narrower width, though.

Figure 5 depicts the Pareto frontiers for the different Fund contracts with outside options given by (1) (i.e. IMD outside option). We see that the Fund contract under flexible MH is Pareto dominated by all the other contracts. The most efficient contract is the restricted flexible MH.

Figure 6 depicts the Pareto frontiers for the different Fund contracts with the same outside options. The IMD outside option given by (1) differ with the MH regime. That is why we re-compute all the contracts with the same outside option. We see that the Fund contract under flexible MH Pareto dominates all the other contracts when the outside option is the same. This is

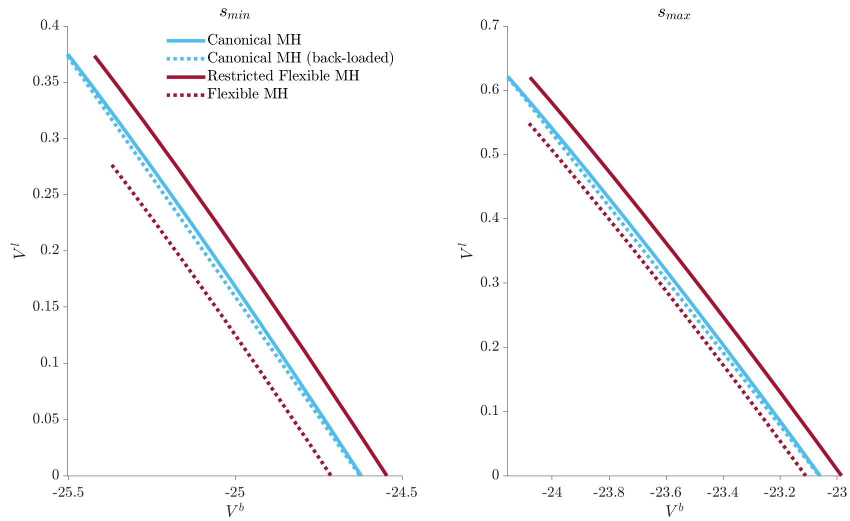

Figure 5: Pareto Frontiers – IMD outside option

the opposite of what Figure 5 shows. It comes from the fact that the flexible MH leads to a higher value of the outside option, which is more costly in terms of resource the Fund must provide.

We complement this with a welfare analysis. Welfare is calculated as a percent of consumption-equivalent changes. Results are consistent with Figures 5 and 6. Under the IMD outside option, the welfare losses under flexible MH come from the Fund side. The borrower is actually better off given the larger outside option.

Table 2: Welfare Gains

Outside option: IMD					
Borrower			Fund		
Canonical (back-loaded)	Flexible	Restricted flexible	Canonical (back-loaded)	Flexible	Restricted flexible
0.04	0.62	0.56	-0.01	-1.69	-0.09
Outside option: same for all					
Borrower			Fund		
Canonical (back-loaded)	Flexible	Restricted flexible	Canonical (back-loaded)	Flexible	Restricted flexible
0.03	1.21	0.23	0.00	2.67	1.43

8 Conclusion

From the perspective of economic theory, since the pioneer work of Prescott and Townsend (1984) it is understood that under appropriate convexity assumptions moral hazard (and adverse selection) problems can be incorporated in the problem of efficiently assigning resources subject to technological and feasibility constraints, by introducing Incentive Compatibility (IC) constraints

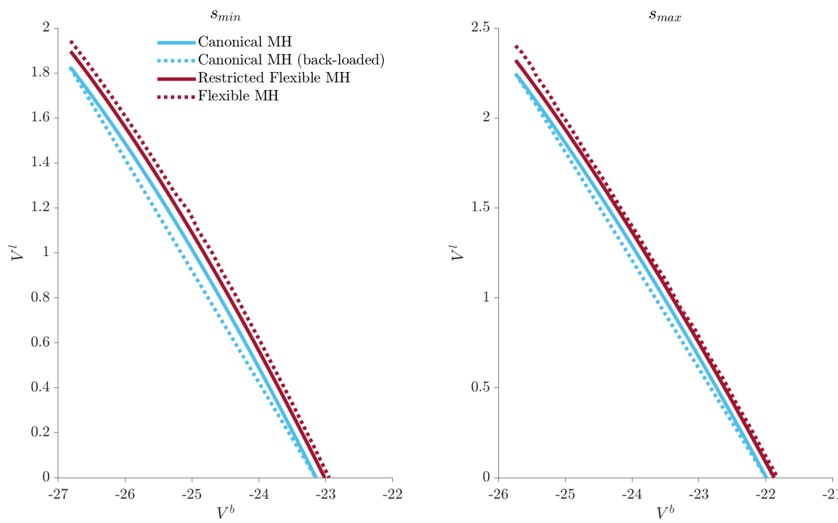

Figure 6: Pareto Frontiers – Same outside option

in parallel to other constraints. Furthermore, it is also understood that under these, and other standard assumptions, the corresponding competitive equilibrium exists and the First and Second Welfare Theorems are satisfied for constrained-efficiency allocations. Extensive follow up work has extended their results to dynamic economies – e.g. with debt or other financial assets, etc. However, not much work has been done in studying different forms of implementation; in particular, in dynamic economies with debt and risk-sharing contracts. In particular, given its novel nature, to our knowledge, no work has been done extending the *flexible moral hazard* approach to dynamic contracts – hence, to recursive contracts with limited enforcement constraints. Here, we have pursued this enquire and have shown how different forms to implement IC constraints result in different constraint-efficient allocations and, correspondingly, in a different (not obvious) split of the surplus between the risk-averse borrower and the risk-neutral lender, while satisfying limited enforcement (LE) constraints. The *back-loaded* design helps to close the gap between the theory and the current practices of official lenders who have a long-term relation with their borrowing countries (as it is the case with the ESM and euro area countries), while our development of the *flexible moral hazard* approach opens a new venue for the design of new official lending programs.

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Appendix

A Back-Loaded Moral Hazard

We first now give the characteristics of a subprogram in terms of length, consumption and labor.

The long-run property of the Fund contract is related to the definition of an ergodic set of relative Pareto weights, x. The term ergodic refers to the fact that the contract will move around the same set of relative Pareto weights over time and over histories.

Definition A.1 (Steady State) *A* Steady State Equilibrium *is defined by an ergodic set* $X^{ss} \equiv [x^{lb}, x^{ub}]$ *where the lower bound is* $x^{lb} = \min_{s \in S}\{x : V^b(x,s) = V^D(s)\}$ *and the upper bound is* $x^{ub} = \max_{s \in S}\{x : V^b(x,s) = V^D(s)\}$*, satisfying* $x^{lb} \leq x^{ub}$*.*

The lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the borrower accepts in the contract, which keeps it away from immiseration. The upper bound represents the highest relative Pareto weight that makes the borrower's constraint bind; therefore it is the highest weight that the lender may need to accept.[5]

Using Definition A.1, we can determine subprogram lengths in steady state. For this define $x_l(s) = \min\{x : \nu_l(x,s) > 0\}$ as the relative Pareto weight for which the limited enforcement constraint of the lender binds in s.

Lemma A.1 (Subprogram Length) *The steady state is such that the subprogram resets every time* $s' > s$*. In addition, it resets every time* $s' < s$ *if* $x_l(s') \in X^{ss}$*. Finally, for* $\eta < \tilde{\eta}$*, the subprogram resets more often with* η *than with* $\tilde{\eta}$*.*

The lemma is made of three parts. First, the subprogram resets as productivity improves. This is due to the binding constraint of the borrower coupled with the fact that $\eta < 1$. Second, the subprogram may not necessarily reset as productivity declines. This depends on whether the lender's constraint binds in steady state. More precisely, the subprogram resets as production declines only if this productivity state is associated to a binding constraint of the lender. Finally, the programs' length reduces as the borrower becomes more impatient.

Lemma A.2 (Subprogram Consumption) *Within a subprogram, consumption,* $c(\overline{x}, s)$*, decays at rate* η*. When a subprogram resets, consumption jumps up if* $x > \overline{x}$ *and jumps down if* $x < \overline{x}$*.*

[5] It should be noted that if the sovereign and the Fund are equally patient (i.e. $\eta = 1$), then the upper bound would be determined by $\min_{s \in S}\{x : V^l(x,s) = Z(s)\}$.

The lemma states that within a subprogram, there is perfect risk sharing adjusted for the relative impatience of the borrower η. Conversely, when a subprogram resets, consumption is adjusted up if the latent multiplier is larger than $\overline{x}$. In other words, consumption increases if the history of past expenditures is associated with a large marginal cost of effort. The opposite is true when $x < \overline{x}$.

Lemma A.3 (Subprogram Labor) *Within a subprogram, labor, $n(\overline{x}, s)$, increases at rate η whenver θ remains fixed. Otherwise, labor increases in θ. When a subprogram resets, labor jumps down if $x > \overline{x}$ and jumps up if $x < \overline{x}$.*

Unlike consumption, labor is affected by changes in θ within the subprogram as we can see in (12). Other than that labor follows the inverse pattern of consumption.

We can add more structure on subprograms. As we have seen before, the length of a subprogram is determined by the binding enforcement constraints. As these constraints bind according to the realization of s, the length of the subprogram is stochastic. However, as long as none of the enforcement constraints are binding, we can add additional (exogenous) criteria determining the length of a program. We consider a stochastic and a deterministic criteria.

Besides limited enforcement, consider that with exogenous probability χ a subprogram resets, while with probability $1-\chi$ the subprogram continues. In this case, the recursive SPFE reads,

$$\begin{aligned}
\hat{FV}(x,s) = \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} \Bigg\{ & x\big[(1+\nu_b)(u(c)+h(1-n)-\hat{v}(e)) - \nu_b V^0(s) - \varrho v'(e)\big] \\
& + [(1+\nu_l)(\theta f(n) - c - g) - \nu_l Z] \\
& + \frac{1+\nu_l}{1+r}\mathbb{E}\left[\mathbb{I}_{\{(x',s');\chi\}}\hat{FV}(x',s') + (1-\mathbb{I}_{\{(x',s');\chi\}})\overline{FV}(x',s';\bar{x}') \mid s,e\right] \Bigg\} \\
\text{s.t.} \quad & x'(s') = \eta x \frac{1+\nu_b+\psi(s' \mid s,e)}{1+\nu_l} \\
& \bar{x}' = \eta x \frac{1+\nu_b}{1+\nu_l}
\end{aligned}$$

where $\mathbb{I}_{\{(x',s');\chi\}} = \mathbb{I}_{\{(x',s')\}}(1-\chi)+\chi$ is such that the subprogram resets whenever one of the limited enforcement constraints is binding or resets with probability χ when no such constraints bind. Then within the subprogram

$$\begin{aligned}
\overline{FV}(x,s;\bar{x}) = \min_{\{\varrho\}} \max_{\{c,n,e\}} \Bigg\{ & \bar{x}\,[u(c)+h(1-n)] - x\big[\hat{v}(e)+\varrho v'(e)\big] + (\theta f(n) - c - g) \\
& + \frac{1}{1+r}\mathbb{E}\left[\mathbb{I}_{\{(x',s');\chi\}}\hat{FV}(x',s') + (1-\mathbb{I}_{\{(x',s');\chi\}})\overline{FV}(x',s';\bar{x}') \mid s,e\right] \Bigg\} \\
\text{s.t.} \quad & x' = \eta x\left[1+\psi(s' \mid s,e)\right] \\
& \bar{x}' = \eta\bar{x},
\end{aligned}$$

Except for the definition of the indicator function $\mathbb{I}_{\{(x',s');\chi\}}$, the problem is the same as the one considered in the previous section.

Instead of the stochastic component χ, we can add a deterministic component besides the limited enforcement constraint. In particular, suppose that the subprogram resets after a deterministic number of periods m if none of the limited enforcement constraints have bound until then. Starting with $\hat{FV}(x,s)$ for the start of the subprogram,

$$\begin{aligned}
\hat{FV}(x,s) = \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} \Bigg\{ & x\big[(1+\nu_b)(u(c)+h(1-n)-\hat{v}(e)) - \nu_b V^0(s) - \varrho v'(e)\big] \\
& + [(1+\nu_l)(\theta f(n) - c - g) - \nu_l Z] \\
& + \frac{1+\nu_l}{1+r}\mathbb{E}\left[\mathbb{I}_{\{(x',s')\}}\hat{FV}(x',s') + (1-\mathbb{I}_{\{(x',s')\}})\overline{FV}_1(x',s';\bar{x}') \mid s,e\right] \Bigg\} \\
\text{s.t.} \quad & x'(s') = \eta x \frac{1+\nu_b+\psi(s' \mid s,e)}{1+\nu_l} \\
& \bar{x}' = \eta x \frac{1+\nu_b}{1+\nu_l}
\end{aligned}$$

then, for $k = 1, ..., m-1$,

$$\begin{aligned}
\overline{FV}_k(x,s;\bar{x}) = \min_{\{\varrho\}} \max_{\{c,n,e\}} \Bigg\{ & \bar{x}\left[u(c)+h(1-n)\right] - x\big[\hat{v}(e)+\varrho v'(e)\big] + (\theta f(n)-c-g) \\
& + \frac{1}{1+r}\mathbb{E}\left[\mathbb{I}_{\{(x',s');\chi\}}\hat{FV}(x',s') + (1-\mathbb{I}_{\{(x',s');\chi\}})\overline{FV}_{k+1}(x',s';\bar{x}') \mid s,e\right] \Bigg\} \\
\text{s.t.} \quad & x' = \eta x\left[1+\psi(s' \mid s,e)\right] \\
& \bar{x}' = \eta\bar{x},
\end{aligned}$$

and finally for m,

$$\begin{aligned}
\overline{FV}_k(x,s;\bar{x}) = \min_{\{\varrho\}} \max_{\{c,n,e\}} \Bigg\{ & \bar{x}\left[u(c)+h(1-n)\right] - x\big[\hat{v}(e)+\varrho v'(e)\big] + (\theta f(n)-c-g) + \frac{1}{1+r}\mathbb{E}\left[\hat{FV}(x',s') \mid s,e\right. \\
\text{s.t.} \quad & x' = \eta x\left[1+\psi(s' \mid s,e)\right]
\end{aligned}$$

Obviously, the subprogram resets in m only if none of the limited enforcement constraint binds in all the m periods of the subprogram. Given this, the characterization is the same as before except that in period m,

$$\begin{aligned}
v'(e(x,s)) =& \beta\sum_{s'|s}\partial_e\Upsilon(s'|s,e)\hat{V}^b(x',s') + \frac{1+\nu_l}{1+\nu_b}\frac{1}{x}\frac{1}{1+r}\sum_{s'|s}\partial_e\Upsilon(s'|s,e)\hat{V}^b(x',s') \\
&- \varrho(x,s)\left[v''(e(x,s)) + \beta\sum_{s'|s}\partial_e^2\Upsilon(s'|s,e)\hat{V}^b(x',s')\right].
\end{aligned}$$

For any period $t \neq m$, the first-order condition is as the one derived previously.

B Proofs

B.1 Proof of Lemma 1

This is straightforward as larger θ involves greater consumption. The rest follows the standard argument about the monotonicity of V^D.

B.2 Proof of Proposition 1

As shown by Ábrahám et al. (2019), (17) is concave under Assumptions 1-3 and the fact the the third derivative of v is zero. We can therefore apply Corollary 3 in Georgiadis et al. (2024) stating that the distribution has at most one g' in its support. As a result, we can reformulate (17) as a problem of choosing g' directly instead of μ'.

B.3 Proof of Lemma 2

From (14), when $g' = \bar{g}$, $V^l(x', s') = V_0^l(x', \theta')$ meaning that the FOC is satisfied only if $\varrho_{\mu(\theta')}(s') = 0$. In opposition, when $g < \bar{g}$, $V^l(x', s') > V_0^l(x', \theta')$ meaning that the FOC is satisfied only if $\varrho_{\mu(\theta')}(s') > 0$.

B.4 Proof of Corollary 1

When $\eta = 1$ and $\nu_{l,t}(s^t) = 0$ in all states, the law of motion of the relative Pareto weight simplifies to the following expression

$$\mathbb{E}_t x_{t+1}(s^{t+1}) \equiv \mathbb{E}_t \left[\bar{x}_{t+1}(s^t) + \hat{x}_{t+1}(s^{t+1})\right] = \mathbb{E}_t \left[(1 + \nu_{b,t}(s^t))x(s^t) + \varrho_{\mu_{t+1}}(s^{t+1}|s^t)x(s^t)\right].$$

Since $(\nu_{b,t}(s^t), \varrho_{\mu_{t+1}}(s^{t+1}|s^t)) \geq 0$, we get that $\mathbb{E}_t x_{t+1}(s^{t+1}) \geq x_t(s^t)$.

B.5 Proof of Lemma 3

To see *i)*, note that, given the above decompostion of the law-of-motion of x, $\sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t, e(s^t))\hat{x}_{t+1}(s^{t+1}) = 0$, since independently of effort $\sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t, e(s^t)) = 1$, hence $\sum_{s^{t+1}|s^t} \partial_e \pi(s^{t+1}|s^t, e(s^t)) = 0$. Therefore $\mathbb{E}_t \hat{x}_{t+1} = 0$ and $\mathbb{E}_t x_{t+1} = \bar{x}_{t+1}(s^t)$. Alternatively, the expected law of motion of x can also be expressed as

$$\mathbb{E}_t x_{t+1} = \mathbb{E}_t \left[\frac{1}{u'(c_{t+1})} \frac{1 + \nu_{l,t+1}(s^t)}{1 + \nu_{b,t+1}(s^t)}\right] = \frac{1}{u'(c_t)}\eta,$$

where the last equality is the *inverse Euler equation* of the recursive contract (Ábrahám et al. (2022), Lemma 4).

To see *ii)*, note that, since the limited enforcement multipliers are either zero or at most one of the two is positive, we can have the following decomposition

$$\mathbb{E}_t \frac{1}{u'(c_{t+1})} = \mathbb{E}_t \left[x_{t+1} \frac{1 + \nu_{b,t+1}(s^{t+1})}{1 + \nu_{l,t+1}(s^{t+1})}\right] = \mathbb{E}_t x_{t+1} + \mathbb{E}_t x_{t+1}\nu_{b,t+1}(s^t) - \mathbb{E}_t x_{t+1} \frac{\nu_{l,t+1}(s^t)}{1 + \nu_{l,t+1}(s^t)},$$

where $\mathbb{E}_t x_{t+1} = \eta x_t$ and, without incentive constraints, the last two terms simply denote the change in the relative Pareto weight when either the no-default or the sustainability constraints binds. However, if limited enforcement constraints satisfy the 'no-free-lunch condition', the no-default constraint is more likely to bind, while the sustainabiliy constraint is less likely to bind and, as a result, in both cases expected consumption increases.