# Moral Hazard with Risk-Sharing and Safe Debt

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**Abstract**

We study an economy where a Financial Stability Fund provides long-term credit and insurance contracts to sovereign countries making their debt liabilities safe, without ever incurring in expected losses. Making sovereign debt safe can exacerbate moral hazard problems. We explore different ways to introduce incentive compatibility (IC) constraints to address these problems. We analyze the tradeoff between the effectiveness of the IC constraints in enhancing effort and their disruption of perfect-risk sharing, as well as the interaction between the IC constraints and the limited enforcement (LE) constraints, which make debt safe and sustainable. Our benchmark is a Fund contract where the IC rewards and punishments take place every period, as in Ábrahám et al. (2022). We first consider less disruptive contracts where perfect risk-sharing is preserved for spans of time, while the IC incentives are back-loaded to the end of these spans. Under reasonable assumptions a constrained-efficient solution to a Fund contract exists and it is unique. However, different specifications of the IC constraints in the Fund contract result in different constrained-efficient solutions; in particular the borrower prefers a contract with back-loaded incentives. Second, our framework allows us to formally analyze 'conditionality': unobservable (non-contractible) effort not only has an effect on the final distribution of the observable outcome (e.g. output or government liabilities), but also can help to fulfil an observable (and contractible) condition informative of the improved risk profile of the country. With this we open the theoretical-quantitative enquire of what is the value of 'conditionality'. These alternative formulations of moral hazard constraints can help to bridge the theoretical modeling of Fund contracts with existing official lending practices.

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# 1 Introduction

In models of debt, or risk-sharing, moral hazard is usually a concern: with effort the risk-averse borrower can improve its risk-profile; however, effort is not contractible (e.g. non observable) and, therefore, lenders and insurers can only infer the risk-profile with the *ex-post* observable outcomes. When debt is a risky asset, or risk-sharing is imperfect, the market discipline of prices is a mechanism to curb moral hazard problems, but an imperfect one. Because, the closer it is to the efficient outcome of safe debt, or perfect risk-sharing, the weaker the price signal is, aggravating the moral hazard problem. Mechanism design, with *incentive compatibility* (IC) constraints, can overcome this problem. More precisely, can find a constrained-efficient debt-and-insurance contract where debt safety and sustainability are preserved and there is an optimal balance between effort and imperfect risk-sharing. However, there is not a unique way to introduce IC constraints and, correspondingly, there is no unique constrained-efficient outcome. While IC constraints are not new in the debt and risk-sharing theoretical literature, as we refer below, the issue of how IC constraints can be introduced remains mostly open. This is the first focus of this paper.

On the other hand, it is common practice – or even a rule – among Official Lenders (IMF, ESM, etc.) to make debt contracts *conditional* to observed outcomes; for example, debtors must comply with observable conditions in order to receive disbursements of funds. In other words, as insurance companies change the terms of their contracts based on observable facts (e.g. a security alarm), Official Lenders address the moral hazard problem with observable conditions (e.g. a structural reform). This raises several questions that, to our knowledge, have not been formally studied: is *conditionality* a substitute for, or a better instrument than, IC constraints? What is the value of *conditionality* vs IC constraints? To open up this enquire is the second focus of this paper.

We consider economies where there are impatient risk-averse representative-agent countries and a more-patient risk-neutral Financial Stability Fund with long-term state-contingent contracts. The Fund has country-specific contracts that guarantee that the sovereign debt is safe and sustainable (i.e. there are no *ex-post* expected losses for the lender). If there is no moral hazard risk-sharing is perfect, except when *limited enforcement* (LE) constraints are binding. We build on Ábrahám et al. (2022), where defaultable sovereign debt is transformed into a safe Fund contract, which accounts for moral hazard. They assume that the Fund has an exclusivity contract (i.e. absorbs all sovereign debt of the country which, as it is common in debt models, corresponds to the country's current account). Furthermore, the IC constraints are always binding – i.e. risk-sharing is always imperfect – and there are no conditionality conditions. Although, since the Fund contract is based on a rigorous risk-assessment of the country, it can always have a menu of contracts adapted to different risk-profiles; that is, the country may exercise effort *ex-ante* to be eligible for a better contract (as insurance companies do).

We exploit the fact that IC constraints are disruptions to perfect risk-sharing. Therefore, if the Fund has exclusivity in providing risk-sharing (as it is in Liu et al. (2020)), the IC constraints are part of the Fund contract, and the Fund it is only required to be able to absorb a *minimal amount of debt*. We consider Fund contracts as an infinite sequence of subprogrammes, with different types of contract corresponding to different ending conditions; that is, they can end: *i*) every period (a reinterpretation of the current infinite horizon benchmark); end *ii*) after a fixed number $m > 1$ of periods;[1] *iii*) randomly; *iv*) when a LE constraint binds; *v*) when a specific state-contingency is realized, or *vi*) different combinations of (*ii*) - (*v*). All these types have in common that within a subprogramme: there is perfect risk-sharing and the risk-improving effort through the subprogramme determines the initial conditions of the follow-up subprogramme; i.e IC constraints are *back-loaded*. With an impatient risk-averse country there may be a trade-off between the benefit of perfect risk-shairing within the subprogramme and less effort since there is back-loading of IC constraints. Our simulations of the calibrated model shows that, with a proper design, the contract with back-loading is not characterized by a significant losses of effort and, in fact, can be welfare improving for the borrower and, therefore worse for the lender, although its limited enforcement constraints are satisfied – implying no expected losses beyond the contract's terms. This is our first contribution to the existing literature.

The second is that we provide a rigorous way to introduce *conditionality*. There can be a verifiable and contractible condition – which depends on the accumulated non-contractible effort – being a terminal state-contingency, as in ($v$). We show how this form of *conditionality* can improve the Fund contract design.[2] This is our second contribution to the existing literature.

Given the focus of our analysis, we abstain from discussing the 'decentralized' of our contract designs. As it has already been said, they can be implemented in the economies studied in Liu et al. (2020) and Callegari et al. (2023), where the Fund only absorbs a *minimal* share of the sovereign debt, while maintaining the exclusivity of risk-sharing. In these economies, most sovereign debt, most of the time, is held by private lenders and the same decentralization works for our contracts with moral-hazard, which reinforces the role of the Fund in these economies.

## 2 Literature Review

This paper combines elements of the literature on optimal contracts with elements of the literature about sovereign debt and defaults.[3]

[1] This is similar to Official Lenders's contracts, which determine the length of the contract (disbursments and repayments) and often have *ex-post* follow-ups. However, differs from them in that continuations are part of the design of the Fund contract and, in fact, IC constraints define them.

[2] In this version, it is not exactly a variant of ($v$) since there is no perfect risk-sharing before conditionality is satisfied.

[3] It is also related to the literature on *conditionality* in Official Lending programmes but, as we acknowledge in the conclusions, we plan to further pursue the analysis of *conditionality* and, therefore, we postpone the corresponding literature review to when it will become relevant.

The paper derives the optimal contract between a lender and a borrower therefore relates to the seminal contributions of Kehoe and Levine (1993, 2001) and Thomas and Worrall (1994) who considered the case of limited enforcement. The difference with our approach is that we consider two-sided limited enforcement, while the literature has focused on one-sided limited enforcement analyzing the borrower's perspective.

We develop an optimal contract combining limited enforcement and moral hazard constraints. Our analysis is close to Atkeson (1991) who – similar to Thomas and Worrall (1994) – studies lending contracts in international contexts. However, Atkeson (1991) models moral hazard with respect to consuming or investing the borrowed funds, while we focus on risk-reduction policies. Quadrini (2004) also combine moral hazard and limited enforcement to study when and how contracts are renegotiation-proof. Similarly, Dovis (2019) shows that the combination of moral hazard and limited enforcement can generate a region of *ex post* inefficiency. This is not the focus of our analysis as our contract is both *ex ante* and *ex post* efficient. In addition, Müller et al. (2019) study dynamic sovereign lending contracts with moral hazard, with respect to reform policy efforts, and limited enforcement. They provide an interesting characterization and decentralization of the constrained-efficient allocation in a model that, in relation to ours, is more stylised (normal times are an absorbing state) and their debt contracts rely heavily on complex *ex-post* default procedures.

We contribute to the literature on moral hazard in dynamic macroeconomic models. Our work relates to the pioneer work Prescott and Townsend (1984) who show that, in a static economy, with moral hazard and adverse selection problems a constrained efficient allocation can be the allocation of competitive equilibrium if the space of contracts is restricted to satisfy the corresponding incentive compatibility constraints. Since then the feasible set of contracts may not be convex, they amplify and convexify allowing for lotteries over incentive compatible contracts.[4] Atkeson and Lucas (1992) show that models with moral hazard feature immiseration. This comes from the incentive-compatible mechanism which rewards high types with larger future utility and low types with lower future utility. This mechanism is present in our framework to some extent. In particular, limited enforcement acts as a stopper for immiseration. Besides this, combining the moral hazard with limited enforcement constraints, we show moral hazard generates only part of the risk sharing. More precisely, we define different ways of introducing incentive compatibility depending on the desired risk sharing.[5]

We solve the optimal contract by means of the Lagrangian approach of Marcet and Marimon (2019). This method has been widely used to account for limited enforcements (e.g. Kehoe and Perri (2002) and Ferrari et al. (2021)) and its combination with moral hazard (e.g. Ábrahám et al. (2019) and Simpson-Bell (2020)). Similar to Kapička (2013), we rely on the first order approach of

[4]This convexification technique has been extensively used in game theory and in macroeconomic environments without moral hazard. See for instance Rogerson (1988) and more recently Iskhakov et al. (2017).

[5]Given this, we recast the result of Tsyrennikov (2013) who argues that limited enforcement generates little dynamic compared to moral hazard.

Rogerson (1985). An alternative would be to use the approach of Fernandes and Phelan (2000).[6] However, it is not properly suited to quantitative analyses.

Our work more closely contributes to the recent literature on the design of an optimal stability Fund. Roch and Uhlig (2018), Liu et al. (2020) and Callegari et al. (2023) focus on the lender's side of the contract and therefore disregard moral hazard issues. In opposition, Dovis and Kirpalani (2023) account for moral hazard and show that the provision of effort is back-loaded. Similarly, Ábrahám et al. (2022) expressively accounts for moral hazard with both a theoretical and quantitative focus. As we have already mentioned, their moral hazard IC constraints are always binding, while we show that back-loaded IC constraints may improve the welfare of the borrowing country. Also they *decentralize* the IC constraints with 'Pigou taxes' on all sovereign debt and risk-sharing liabilities. In their economy, the Fund holds all the debt, therefore, by construction there are no taxes to private lenders. However, the natural extension to their decentralization – say the economies of Liu et al. (2020) with moral hazard – would be a 'Pigou taxes' also on privately held sovereign debt. As we have mentioned, there is no need to tax privately held debt with our Fund contacts.

## 3 The Economy

We consider an infinite-horizon small open economy where the 'benevolent government' acts as a representative agent with preferences for current leisure, $\ell = 1 - n \in [0, 1]$, consumption, $c \geq 0$, and effort, $e \in [0, 1]$, valued by $U(c, n, e) \equiv u(c) + h(1 - n) - v(e)$. We make standard assumptions on preferences: $u, h, v$ are differentiable; $u'(x) > 0, h'(x) > 0$ for $x \geq 0$, $v'(x) > 0$ for $x > 0$, and $v'(0) = 0$; $u''(x) < 0, h''(x) < 0$, and $v''(x) > 0$; and $v'''(x) \geq 0$. The government discounts the future at the rate $\beta$, satisfying $\beta \leq 1/(1+r)$, where $r$ is the risk-free world interest rate. In general, we will assume the inequality to be strict.

The country has access to a decreasing-returns labour technology $y = \theta f(n)$, where $f'(n) > 0$, $f''(n) < 0$, and $\theta$ is a productivity shock, assumed to be Markovian, $\theta \in \{\theta_1, ..., \theta_N\}$, $\theta_i < \theta_{i+1}$. The distribution of $\theta$ depends on the effort $e$ the country exercises to improve its productivity, with costly higher effort resulting in a distribution of expenditures that first order stochastically dominates the distribution with lower effort. We assume that effort is not contractible. We also denote the state of the economy by $s(i) \equiv \theta_i$, a TFP shock with a distribution which, in part, depends on the effort the country does to improve its risk profile. As it is standard in models with private effort, we assume full support: $\pi(s'|s, e) > 0$ for all $s'$, $s$ and $e > 0$. This implies that (i) for interior effort, our model generates an ergodic set of $S$ that includes all possible combinations of shocks with positive probability and (ii) our incentive problem is well-defined as there are no states of the world where infinite punishments can be used.

[6] See Halac and Yared (2014) for an application of the method of Fernandes and Phelan (2000).

Since the country is an open economy, its trade balance $\theta f(n) - c$, does not need to be zero period by period. Most of our our analysis focuses on a set-up where the country can manage its private and public debt liabilities with the help of a Financial Stability Fund (Fund), which acts as a benevolent risk-neutral principal/planner who has access to the international capital markets at the risk-free rate.

The country can default on its liabilities. We assume that the country's outside option corresponds to the autarky value of the standard incomplete market model with default which is

$$V^o(\theta) = \max_{n,e} \left\{U(\theta^d f(n), n, e)\right\} + \beta \mathbb{E}\left[(1-\lambda)V^o(\theta') + \lambda J(\theta', 0)\middle|\theta, e\right], \tag{1}$$

where $\theta^d \leq \theta$ contains the penalty for defaulting and $\lambda$ is the probability to re-access the private bond market. Furthermore, $V^o$ corresponds to the value under financial autarky and $J$ to the value of reintegrating the private bond market without the Fund. More precisely, $J(\theta, b) = \max\{V^i(\theta, b), V^o(\theta)\}$, with

$$\begin{aligned} V^i(\theta, b) = & \max_{\{c,n,e,b'\}} U(c, n, e) + \beta \mathbb{E}\left[J(\theta', b')\middle|\theta, e\right] \\ & \text{s.t. } c + q(b')(b' - \delta b) \leq \theta f(n) + (1 - \delta + \delta\kappa)b. \end{aligned}$$

In the private bond market, the country can borrow long-term defaultable bonds, $b'$, at a unit price of $q_p(\theta, b')$. A fraction $1-\delta$ of each bond matures today and the remaining fraction $\delta$ is rolled-over and pays a coupon $\kappa$. Private lenders are competitive and the price of one unit of private bond is given by $q_p(\theta, b') = \frac{1}{1+1} \sum_{\theta'|\theta} \pi(\theta'|\theta)(1 - D)[1 - \delta + \delta\kappa + \delta q_p(\theta', b'')]$ where $D$ is the default policy taking value one in case of default and zero otherwise.

### 3.1 The Economy with a ***Financial Stability Fund*** (Fund)

The Financial Stability Fund (Fund) is modeled as a long-term contract between a fund (also called lender) and an individual partner (also called country or borrower) who is the government of the small open economy. The Fund contract chooses a state-contingent sequence of consumption, leisure and effort that maximises the life life-time utility of the borrower given some initial level of the borrower's debt. The optimal contract is self-enforcing through the presence of two limited-enforcement constraints. First, we assume that if the country ever defaults on the Fund contract, it will not be able to sign a new contract with the Fund and will enter the markets for defaultable long-term debt as a defaulter. The Fund contract, however, makes sure that the country never finds it optimal to renege the contract. Second, the contract also prevents the Fund from ever incurring undesired expected losses, i.e. undesired permanent transfers. In addition, the contract also has an incentive compatibility constraint, since effort to achieve a better distribution of government liabilities is non-contractible (i.e. it is private information, or a sovereign right of the country).

Thus, the long term contract must provide sufficient incentives for the country to implement a (constrained) efficient level of effort. In sum, the Fund contract can provide risk-sharing and consumption smoothing with state-contingent transfers. However, these transfers are constrained by limited enforcement and moral hazard frictions. These frictions preclude perfect risk-sharing. Note also that the Fund contract is based on a country-specific risk-assessment, as the allocation depends on all the underlying parameters describing preferences, technology and the shock process.

### 3.1.1 The Long Term Contract

In its extensive form, the *Fund contract* specifies that in state $s^t = (s_0, \ldots, s_t)$, the country consumes $c(s^t)$, uses labour $n(s^t)$ and exercises effort $e(s^t)$, resulting in a transfer to the Fund of $c_l(s^t) = \theta f(n(s^t)) - c(s^t)$, with $c_l(s^t) < 0$ implying that the country is effectively borrowing. With *two-sided limited enforcement* and *moral hazard*, an optimal Fund contract is a solution to the following **Fund problem**:

$$\max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}\left[\mu_{b,0}\sum_{t=0}^{\infty}\beta^t U(c(s^t),n(s^t),e(s^t)) + \mu_{\ell,0}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_l(s^t)\middle| s_0\right] \tag{2}$$

$$\text{s.t.} \quad \mathbb{E}\left[\sum_{j=t}^{\infty}\beta^{j-t}U(c(s^j),n(s^j),e(s^j))\middle| s^t\right] \geq V^o(s_t), \tag{3}$$

$$\partial_e v(e(s^t)) = \beta \sum_{s^{t+1}|s^t}\frac{\partial \pi(s^{t+1}|s_t, e(s^t))}{\partial e(s^t)} V^b(s^{t+1}), \tag{4}$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty}\left(\frac{1}{1+r}\right)^{j-t} c_l(s^j)\middle| s^t\right] \geq Z(s), \tag{5}$$

$$\text{and} \quad c_l(s^t) = \theta(s^t)f(n(s^t)) - c(s^t) - g(s^t), \quad \forall s^t, t \geq 0, \tag{6}$$

$$V^b(s^t) = \mathbb{E}\left[\sum_{j=0}^{\infty}\beta^j U(c(s^{t+j}),n(s^{t+j}),e(s^{t+j}))\middle| s^t\right]$$

Note that $(\mu_{b,0},\ \mu_{\ell,0})$ are the initial Pareto weights, which are key for our interpretation of the Fund contract as a lending contract. In particular, we show in Section 3.3, that the initial relative Pareto weight determines uniquely the level of debt that the Fund takes over when the country joins. Note also that the notation is implicit about the fact that expectations are conditional on the implemented effort sequence, as it affects the distribution of the shocks.

Constraints (3) and (5) are the *limited enforcement constraints* for the borrower and the lender, respectively, in state $s^t$. The outside value for the borrower is denoted by $V^o(s_t)$ and is given by (1). For now, we just need to assume that $V^o(s)$ is bounded below, increasing in $s$, and that there is always room for risk-sharing, in spite of the limited enforcement constraints.[7] In principle, a diverse set of default scenarios can satisfy these requirements.

[7] In particular, $V^o(s(1)) < V^o(s(N))$.

The finite outside option of the lender $Z(s) \leq 0$ measures the extent of *ex-post* redistribution the Fund is willing to tolerate.[8] That is, if $Z(s) < 0$ the Fund is allowed to make a permanent loss in terms of lifetime expected net present value – i.e. the Fund can find better investment opportunities in the international financial market and if it does not renege it is because it has committed to sustaining $Z(s) < 0$. Clearly, the level of $Z(s)$ has an important impact on the amount of risk sharing in our environment and it can thus be interpreted as the extent of solidarity the Fund is willing to accept in state $s$, as in Tirole (2015). In our benchmark calibration, we assume that $Z = 0$ implying that the lender does not accept any permanent level of *ex-ante* (at the time of signing the contract) or *ex-post* (at any later period) redistribution. At the same time, the period by period transfers $c_l$ can be positive or negative, hence the Fund can still generate risk sharing. In fact, we will show that, even with $Z = 0$, the Fund can be superior to other financial mechanisms, since it can still provide significant risk-sharing gains and a higher debt capacity to the government.

Constraint (4) is the *moral hazard* (i.e. incentive compatibility) constraint with respect to the borrower's effort, which is not contractible and $V^b(s^{t+1})$ represents the value of the Fund contract for the borrower in state $s^{t+1}$. The interpretation of this constraint is standard: the marginal cost of increasing effort has to be equal to the marginal benefit. The latter is measured as the change in life-time utility due to the change in the distribution of future shocks as a result of the increasing effort. Note that (4) uses implicitly the *first-order condition approach*, that is, we replace the agent's full optimization problem with respect to effort by its necessary first-order conditions. Following Rogerson (1985), we now introduce assumptions to guarantee that this condition is also sufficient. We denote the cumulative distribution function of $s'$ with:

$$F_j(e, s) = \sum_{i=1}^{j} \pi(s' = s(i)|s, e).$$

**Assumption 1 (*Differentiability, Monotonicity and Convexity*)** *For every $s$, if $e \geq \tilde{e} > 0$ the ratio $\frac{\pi(s'=s(i)|s,\tilde{e})}{\pi(s'=s(i)|s,e)}$ is nonincreasing in $i$, and, for every $j$ and $(e, s)$, $F_j(e, s)$ is differentiable in $e$, with $\partial_e F_j(e, s) \leq 0, \partial_e^2 F_j(e, s) \geq 0$, and $\partial_e^3 F_j(e, s) = 0$.*

Except for the last assumption, (i.e. $\partial_e^3 F_j(e, s) = 0$), these conditions simply generalize the assumptions of Rogerson (1985) to our dynamic contracting problem.[9] The strength of the proof of the sufficiency of the *first-order condition approach* in the simple static Pareto-optimization problem in Rogerson (1985) is that it generalizes to our dynamic contractual problem with limited enforcement constraints. The last assumption guarantees that, if we replace the equality in (4) with a weak inequality, $\leq$, the corresponding set of feasible efforts, $e$, is convex and the Lagrangean

[8] We can introduce state-dependence of this constraint without any conceptual difficulty.

[9] More precisely, the monotonicity condition is Rogerson's monotone likelihood-ratio condition (MLR) (which, actually implies $\partial_e F_j(e, s) \leq 0$, his Lemma 1) and $\partial_e^2 F_j(e, s) \geq 0$ is Rogerson's convexity of the distribution condition (CDF).

of the contract problem is concave in $e$.[10] Finally, and in addition to the assumptions we have already made on preferences, technologies and the stochastic process we also assume:

**Assumption 2 (*Interiority*)** *There is an $\epsilon > 0$, such that, for all $s_0 \in S$ there is a program $\{\tilde{c}(s^t), \tilde{n}(s^t), \tilde{e}(s^t)\}_{t=0}^{\infty}$ satisfying constraints (3) and (5) when, on the right-hand side, $V^o(s_t)$ and $Z(s_t)$ are replaced by $V^o(s_t) + \epsilon$ and $Z(s_t) + \epsilon$, respectively, and similarly, when in (4) $\partial_e v(e(s^t))$ is replaced by $\partial_e v(e(s^t)) + \epsilon$ and $=$ is replaced by $\leq$.*

This *strict interiority* assumption guarantees the uniform boundedness of the Lagrange multipliers. For constraints (3) and (5), this assumption requires that, in spite of the *enforcement constraints*, there are strictly positive rents to be shared since otherwise there may not be a constrained-efficient risk-sharing contract. The last part of this assumption is satisfied if an interior effort $0 < e < 1$ is feasible for all states.[11]

### 3.1.2 Recursive Formulation

It is known from Marcet and Marimon (2019) and Mele (2011) that we can rewrite the general fund contract problem as a saddle-point Lagrangian problem:[12]

$$\begin{aligned}
\text{SP} \min_{\{\gamma_b(s^t),\gamma_l(s^t),\xi(s^t)\}} \max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}\Bigg[ \sum_{t=0}^{\infty} \beta^t \Big( & \mu_{b,t}(s^t) U(c(s^t), n(s^t), e(s^t)) - \xi(s^t)\partial_e v(e(s^t)) \\
& + \gamma_b(s^t)\big[U(c(s^t), n(s^t), e(s^t)) - V^o(s_t)\big]\Big) \\
+ \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \Big( \mu_{l,t+1}(s^t) & [\theta(s^t) f(n(s^t)) - c(s^t) - g(s^t)] - \gamma_l(s^t) Z \Big) \Bigg| s_0 \Bigg] \\
\text{s.t. } \mu_{b,t+1}(s^{t+1}) = \mu_{b,t}(s^t) & + \gamma_b(s^t) + \xi(s^t) \frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}, \\
\mu_{l,t+1}(s^t) = \mu_{l,t}(s^{t-1}) & + \gamma_l(s^t), \text{ with } \mu_{b,0}(s^0) \equiv \mu_{b,0}, \mu_{l,0}(s^{-1}) \equiv \mu_{l,0} \text{ given},
\end{aligned}$$

where $\beta^t \pi(s^t|s_0, e(s^t))\gamma_b(s^t)$, $\left(\frac{1}{1+r}\right)^t \pi(s^t|s_0, e(s^t))\gamma_l(s^t)$ and $\beta^t \pi(s^t|s_0, e(s^t))\xi(s^t)$ are the Lagrange multipliers of the limited enforcement constraints (3), (5), and incentive compatibility constraint (4), respectively, in state $s^t$. The above formulation of the problem defines two new co-state variables $\mu_b(s^t)$ and $\mu_l(s^t)$, which represent the temporary Pareto weights of the borrower and the

[10] As we show in Section 7, our functional form satisfies Assumption 1.

[11] Note that these assumption can easily be satisfied since there are potential positive gains from risk-sharing (and borrowing an lending) in a contract between a risk-averse agent and a risk-neutral agent as long as there is a sufficiently high penalty for default. In other words, for $\beta$ high enough, the risk sharing gains are strictly positive if $Z = 0$. The interiority of effort can be guaranteed if full risk sharing is not the only feasible allocation and appropriate Inada conditions are imposed on the cost $v(e)$ and benefit $\pi(s'|s, e)$ of effort.

[12] Following Marcet and Marimon (2019), we only consider saddle-point solutions and their corresponding saddle-point multipliers. That is, given $\Phi(a, \lambda)$, $(a^*, \lambda^*)$ solves SP $\min_\lambda \max_a \Phi(a, \lambda)$ if and only if $\Phi(a, \lambda^*) \leq \Phi(a^*, \lambda^*) \leq \Phi(a^*, \lambda)$, for any feasible action $a$ and Lagrangian multiplier $\lambda$. See Appendix A for more details on the derivation.

lender respectively. These variables are initialized at the original Pareto weights and they become time-variant because of the limited commitment and moral hazard frictions. In particular, a binding participation constraint of the borrower (lender) will imply a higher welfare weight of the borrower (lender) so that he does not leave the contract. In addition, the moral hazard friction (whenever $e > 0$ and $\xi > 0$, i.e., whenever the incentive compatibility constraint is binding) implies that the co-state variable of the borrower will increase or decrease depending on the sign of the likelihood ratio $\frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}$. In particular, a positive likelihood ratio, which occurs with a high realization of productivity, provides a good signal about effort and hence the borrower will be rewarded with a higher temporary Pareto weight.

It turns out that, in the previous problem, only relative Pareto weights, defined as $x_t(s^t) \equiv \mu_{l,t}(s^t)/\mu_{b,t}(s^t)$, matter for the allocations, and this allows us to reduce the dimensionality of the co-state vector and write the problem recursively by using a convenient normalization. Let $\eta \equiv \beta(1+r) \le 1$. We normalize the multipliers as follows:

$$
\begin{aligned}
&\nu_b(s^t) = \frac{\gamma_b(s^t)}{\mu_{b,t}(s^t)},\ \nu_l(s^t) = \frac{\gamma_l(s^t)}{\mu_{l,t}(s^{t-1})}, \varrho(s^t) = \frac{\xi(s^t)}{\mu_{b,t}},\\
&\varphi(s^{t+1}|s^t) \equiv \varrho(s^t)\frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}.
\end{aligned} \tag{7}
$$

With this normalization, $\nu_b$ and $\nu_l$ become the multipliers of the limited enforcement constraints, corresponding to (3) and (5) and $\varrho$ the multiplier of the incentive compatibility constraint corresponding to (4). Moreover, the state vector for the problem (including the new co-state) becomes $(x, s)$.

Note that, on the left-hand side of (7), the multiplier $\varphi_{t+1}(s^{t+1}|s^t)$ does not explicitly depend on $e(s^t)$ since, as multiplier, the action is taken as given. Moreover, it can be positive or negative depending on the sign of $\partial_e \pi(s^{t+1}|s^t, e(s^t))$. The *Saddle-Point Functional Equation* (**SPFE**) — i.e. the saddle-point version of Bellman's equation — is given by (see Marcet and Marimon, 2019):[13]

$$
\begin{aligned}
FV(x,s) = \mathrm{SP} \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} \Big\{ & x\Big[(1+\nu_b)U(c,n,e) - \nu_b V^o(s) - \varrho v'(e)\Big] \\
& + \Big[(1+\nu_l)(\theta(s)f(n) - c - g(s)) - \nu_l Z\Big] + \frac{1+\nu_l}{1+r}\mathbb{E}\big[FV(x',s')\big|s,e\big]\Big\}
\end{aligned} \tag{8}
$$

$$
\text{s.t. } x'(s') \equiv \bar{x}'(s) + \hat{x}'(s') = \left[\frac{1+\nu^b}{1+\nu_l} + \frac{\varphi(s'|s)}{1+\nu_l}\right]\eta x \text{ and } \varphi(s'|s) = \varrho\frac{\partial_e \pi(s'|s,e)}{\pi(s'|s,e)}. \tag{9}
$$

Later on, we will provide an existence result for this recursive saddle point problem, but we first provide a decomposition of the Fund's value functions and a preliminary charaterization of the Fund allocation, two results that will be useful for proving existence. To simplify notation, let the policy for the relative Pareto weight be given by $x'_{xs}(s') \equiv x'(x,s,s')$. The Fund's value functions

[13]See the appendix for more details on the derivation of the SPFE with moral hazard.

can be decomposed as follows:

$$FV(x,s) = xV^b(x,s) + V^l(x,s), \text{ with} \tag{10}$$

$$V^b(x,s) = U(c(x,s), n(x,s), e(x,s)) + \beta \mathbb{E}\big[V^b(x'_{xs}(s'), s')\big|s, e(x,s)\big], \text{ and} \tag{11}$$

$$V^l(x,s) = c_l(x,s) + \frac{1}{1+r}\mathbb{E}\big[V^l(x'_{xs}(s'), s')\big|s, e(x,s)\big], \text{ where} \tag{12}$$

$$c_l(x,s) = \theta(s)f(n(x,s)) - g(s) - c(x,s). \tag{13}$$

The policy functions for consumption of the Fund contract must solve the first-order conditions of the SPFE. In particular, $c(x,s)$ and $n(x,s)$ satisfy:

$$u'(c(x,s)) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x} \tag{14}$$

$$\frac{h'(1-n(x,s))}{u'(c(x,s))} = \theta f'(n(x,s)). \tag{15}$$

These conditions are standard, the borrower's consumption is determined by its endogenous relative Pareto weight and, given that preferences are separable, the labor supply is undistorted. The effort policy $e(x,s)$ is determined by the first order condition of the SPFE with respect to $e$, which can be conveniently expressed as:

$$\begin{aligned} v'(e(x,s)) = \beta \sum_{s'|s} \partial_e \pi(x'_{xs}(s'), s') V^b(x'_{xs}(s'), s') \\ + \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}\frac{1}{1+r}\sum_{s'|s}\partial_e \pi(x'_{xs}(s'), s') V^l(x'_{xs}(s'), s') \\ - \frac{\varrho(x,s)}{1+\nu_b(x,s)}\left[v''(e(x,s)) + \beta\sum_{s'|s}\partial_e^2 \pi(x'_{xs}(s'), s') V^b(x'_{xs}(s'), s')\right]. \end{aligned} \tag{16}$$

Equation (16) balances the marginal cost of effort with the benefits. The first line is the life-time utility benefit of effort to the borrower; the second line is the marginal benefit of effort to the lender, in terms of the borrower's marginal utility, given by (14); the third line accounts for the marginal relaxation/tightening effect of the moral hazard constraint (4) when there is a change in effort. With contractible effort, the Fund problem would not have the *incentive compatibility constraint* (4) and the effort decision would be given by the first two lines, with the second one accounting for the social value of effort. In contrast, with non-contractible effort, as we assume, constraint (4) is present and the first line is equal to zero, namely:

$$v'(e(x,s)) = \beta \sum_{s'|s} \partial_e \pi(s'|s, e(x,s)) V^b(x'_{xs}(s'), s'). \tag{17}$$

In this case, (16) reduces to

$$\frac{1}{1+r}\sum_{s'|s}\partial_e\pi(x'_{xs}(s'),s')V^l(x'_{xs}(s'),s')$$
$$=\vartheta(x,s)\left[v''(e(x,s))-\beta\sum_{s'|s}\partial_e^2\pi(x'_{xs}(s'),s')V^b(x'_{xs}(s'),s')\right], \quad (18)$$

where $\vartheta(x,s)\equiv\frac{x\varrho(x,s)}{1+\nu_l(x,s)}$ can be interpreted as the marginal value of relaxing the ICE constraint in terms of the lender's valuation; that is, (18) accounts for the external effect of effort on the lender's value through its effect on the incentive compatibility constraint. Note that, although incentive compatibility implies that only the borrower's returns affect the effort decision directly, the benefits represented in (18) will affect incentives as they affect $\varrho(x,s)$ and hence the whole future path of allocations through (9).

Given the policy function $e(x,s)$, we denote by $\{s\}_{e(x,s)}$ the resulting Markov process of $\{\theta,g\}$ shocks. Furthermore, a recursive constrained-efficient Fund allocation also satisfies the following endogenous limited enforcement (constraint qualification) constraints:

$$\nu_b(x,s)\left[V^b(x,s)-V^o(s)\right]=0 \text{ with } \nu_b(x,s)=0 \text{ if } V^b(x,s)>V^o(s), \quad (19)$$
$$\nu_l(x,s)\left[V^l(x,s)-Z\right]=0 \text{ with } \nu_l(x,s)=0 \text{ if } V^l(x,s)>Z. \quad (20)$$

**Definition 1 (Recursive Constrained Efficient Fund Contract)** *Given an initial relative Pareto weight $x(s_0)$ and outside options $\{V^o(s),Z(s)\}$ for the borrower and lender, the policies for the allocations $\{c(x,s),c_l(x,s),n(x,s),e(x,s)\}$, multipliers $\{\nu_l(x,s),\nu_b(x,s),\varrho(x,s)\}$, value functions $\{V^b(x,s),V^l(x,s)\}$, relative Pareto weight $x'_{xs}(s')$, and the laws of motion for $\{\theta\}_{e(x,s)}$ are a recursive constrained efficient Fund contract if they satisfy conditions (9)–(15) and (17)–(20) for all $(x,s)$.*

Without an endogenous effort process and, therefore, without moral hazard constraints, the following propositions would be a direct application of Marcet and Marimon (2019, Theorem 2 & 3) and no further assumptions would be needed. We generalize their result to contracts with moral hazard.

**Proposition 1** *Given our assumptions, for any $s_0$, $x(s_0)$, and outside options $\{V^o(s),Z\}$, there is a unique recursive constrained-efficient Fund contract.*

**Proof**: See Appendix A

Note that *uniqueness* refers to the Fund contract as solution to the Fund problem (2)-(6). As we will see in Section 5, different specifications of the Fund problem can result in different 'unique' Fund contracts.

Our assumptions ensure the necessary convexity and interiority conditions for the *Saddle-Point Functional Equation* (SPFE) to be well defined; i.e. non-empty for every $(x, s)$.[14] One can then show that other results of Marcet and Marimon (2019) and Mele (2011) — such as the *Contraction Mapping* property of the SPFE — also extend to the Fund contract. The following corollaries sharpen our existence result.

**Corollary 1** *The value function $FV(x, s)$ and its components given by (10) are bounded and uniquely determined and $FV(x, s)$ is increasing, concave and differentiable in $(x)$ — and strictly increasing and strictly concave, whenever neither limited enforcement constraint is binding.*

To see how the above properties of $FV$ — in particular, differentiability — are derived from the properties of $V^b$ and $V^l$, it is convenient to denote by $\underline{x}(s)$ the maximum value of $x$ for which the borrower's limited enforcement constraint is binding — i.e. $\underline{x}(s) \equiv \max\{x : V^b(x, s) = V^o(s)\}$ — and by $\overline{x}(s)$ the minimum value of $x$ for which the lender's limited enforcement constraint is binding — i.e. $\overline{x}(s) \equiv \min\{x : V^l(x, s) = Z\}$. Note that neither $V^b$ or $V^l$ are differentiable at $\underline{x}(s)$ or $\overline{x}(s)$. Nevertheless,

$$\begin{aligned} \partial_x FV(x, s) &= V^b(x, s) + x\partial_x V^b(x, s) + \partial_x V^l(x, s) \\ &= V^b(x, s), \end{aligned} \tag{21}$$

As it is shown in Lemma A.1, the identity (21) follows from the Euler's Theorem on degree-monotonicity functions. It implies that $x\partial_x V^b(x, s) = -\partial_x V^l(x, s)$ which, in our context, characterizes *efficient risk-sharing*, otherwise the contract values would not be in the (constrained) Pareto frontier. Furthermore, (21) shows that $FV(x, s)$ is differentiable in $x$ at $\underline{x}(s)$, even if $V^b(x, s)$ is not.[15]

The following corollary below establishes that the first-order conditions are not only necessary but also sufficient.

**Corollary 2** *Given an initial relative Pareto weight $x(s_0)$ and outside options $\{V^o(s), Z(s)\}$ for the borrower and the lender, the policies for the allocations $\{c(x, s), c_l(x, s), n(x, s), e(x, s)\}$, multipliers $\{v_l(x, s), v_b(x, s), \xi(x, s)\}$, value functions $\{V^b(x, s), V^l(x, s)\}$, relative Pareto weight $x'_{xs}(s')$, and the laws of motion for $\{\theta, g\}_{e(x,s)}$ are a recursive constrained efficient Fund contract if they satisfy conditions (9)–(15) and (17)–(20) for all $(x, s)$.*

[14]In contrast with Mele (2011, Proposition 1), who only considers a MH constraint and implicitly assumes the sufficiency of the *first-order condition* (4), as well as the convexity of the corresponding feasible set, Proposition 1 accounts for these elements with the *Monotonicity and Convexity* assumption and, furthermore, integrates MH and LE constraints. In this sense, Proposition 1 is the first result on existence of a *Recursive Contract* as as solution to a SPFE with limited enforcement and moral hazard constraints.

[15]Note that at $\underline{x}(s)$ the left-hand side and the right-hand side $x$-partial derivative of $FV(x, s)$ are equal to $V^o(s)$ and, hence, $FV(x, s)$ is differentiable in $x$ at $\underline{x}(s)$, and similarly for $\overline{x}(s)$. In fact, this differentiability result also follows from the uniqueness of the saddle-point Bellman equation (see Marimon and Werner, 2021).

Above, we have established that the constrained optimal Fund allocation is unique and so are the associated value functions. Given our assumptions on preferences and technologies, uniqueness of the multipliers and policy functions follows from the uniqueness of the value functions.

## 4 On the interaction between Moral Hazard and Limited Enforcement constraints

In the economies we study, with the need of risk-sharing, avoiding default or non sustainable debt levels, moral hazard problems arise when these problems could be alleviated with effort, but such effort is not contractible. Therefore it is reasonable to model limited enforcement contracts that satisfy the following property:

**Definition 2** *The limited enforcement constraints (3) and (5) satisfy the 'no-free-lunch condition' if, given $(x, s)$, whenever $\nu_b(x'(x, s), s') > 0$, then $\partial_e \pi(s' \mid s, e) > 0$ and whenever $\nu_l(x'(x, s), s') > 0$, then $\partial_e \pi(s' \mid s, e) < 0$, respectively.*

Conversely, if $\partial_e \pi(s' \mid s, e) = 0$ (or the inequality signs were reversed) exercising more effort would not have any effect on the limited enforcement constraints (or a perverse effect) and, on those grounds, moral hazard would not be an issue. The following lemma provides a characterization of the interaction between limited enforcement and moral hazard constraints. Before we state it, note that the law of motion of $x$ which can be written as:

$$x_{t+1}(s^{t+1}) \equiv \left[\bar{x}_{t+1}(s^t) + \hat{x}_{t+1}(s^{t+1})\right] = \left[\frac{1 + \nu_{b,t}(s^t)}{1 + \nu_{l,t}(s^t)} x(s^t) + \frac{\varphi_{t+1}(s^{t+1}|s^t)}{1 + \nu_{l,t}(s^t)} x(s^t)\right] \eta,$$

where $\hat{x}_{t+1}(s^{t+1})$ accounts for the dynamic effect of the moral hazard constraint.

**Lemma 1**

*i) In recursive contracts, limited enforcement constraints have an effect on the expected law of motion of the Pareto weights, when they are binding; in contrast, moral hazard constraints do not have an effect on $\{\mathbb{E}_t x_{t+1}\}$, even if they bind; i.e. $\mathbb{E}_t x_{t+1} = \bar{x}_{t+1}(s^t)$.*

*ii) If limited enforcement constraints satisfy the 'no-free-lunch condition', moral hazard constraint make no-default constraints (3) more likely to bind and sustainability constraints (5) less likely to bind and, in both cases, $\mathbb{E}_t \frac{1}{u'(c_{t+1})}$ increases.*

To see *i)*, note that, given the above decompostion of the law-of-motion of $x$, $\sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t, e(s^t))\hat{x}_{t+1}(s^{t+1}) = 0$, since independently of effort $\sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t, e(s^t)) = 1$, hence $\sum_{s^{t+1}|s^t} \partial_e \pi(s^{t+1}|s^t, e(s^t)) = 0$.

Therefore $\mathbb{E}_t\hat{x}_{t+1} = 0$ and $\mathbb{E}_t x_{t+1} = \bar{x}_{t+1}(s^t)$. Alternatively, the expected law of motion of $x$ can also be expressed as

$$\mathbb{E}_t x_{t+1} = \mathbb{E}_t\left[\frac{1}{u'(c_{t+1})}\frac{1+\nu_{l,t+1}(s^t)}{1+\nu_{b,t+1}(s^t)}\right] = \frac{1}{u'(c_t)}\eta,$$

where the last equality is the *inverse Euler equation* of the recursive contract (Ábrahám et al. (2022), Lemma 4).

To see *ii)*, note that, since the limited enforcement multipliers are either zero or at most one of the two is positive, we can have the following decomposition

$$\mathbb{E}_t\frac{1}{u'(c_{t+1})} = \mathbb{E}_t\left[x_{t+1}\frac{1+\nu_{b,t+1}(s^{t+1})}{1+\nu_{l,t+1}(s^{t+1})}\right] = \mathbb{E}_t x_{t+1} + \mathbb{E}_t x_{t+1}\nu_{b,t+1}(s^t) - \mathbb{E}_t x_{t+1}\frac{\nu_{l,t+1}(s^t)}{1+\nu_{l,t+1}(s^t)},$$

where $\mathbb{E}_t x_{t+1} = \eta x_t$ and, without incentive constraints, the last two terms simply denote the change in the relative Pareto weight when either the no-default or the sustainability constraints binds. However, if limited enforcement constraints satisfy the 'no-free-lunch condition', the no-default constraint is more likely to bind, while the sustainabiliy constraint is less likely to bind and, as a result, in both cases expected consumption increases.

## 5 Contracts with back-loaded moral hazard incentive constraints

In this section, we analyze contracts that consist of an infinite sequence of subprograms, whereby within each subprogram rewards and punishment associated with the moral-hazard constraint (4) are *back-loaded* to the end and take the form of better or worse terms for the follow-up subprogram. *Back-loading* rewards is a common feature of optimal managerial contracts with risk-neutral managers. In contrast, we consider a contract with an impatient risk-averse country who can benefit from perfect risk-sharing within the subprogram, although impatience may weaken the effect of back-loaded incentives.

We explore different types of contracts with a common structure of subprograms. In particular, subprograms of finite length – say, of $m + 1$ periods – and subprograms which random endings. To help the exposition we first consider them without limited enforcement constraint. Next we introduce limited enforcement constraints, in which case the subprograms terminate if one of these constraints, (3) or (5), binds. In all the contracts, within the subprogram the moral hazard constraint (4) is not active as relative Pareto weight determining the allocation $(c, n)$, but it is accounted for as a *latent relative Pareto weight.* This accumulated multiplier – a version of the law of motion (9) – determines effort $e$ and at the final state is used as the starting relative Pareto weight of the following subprogramme.

### 5.1 Contracts with subprograms of finite length without LE constraints

We first consider subprograms of finite length where the LE constraints do not bind. The Fund contract can be expressed as the solution to a sequence of recursive (SPFE) problems. Starting with $FV(x,s)$ for the start of the subprogram, with $(x,s)=(x_0,s_0)$ at the beginning of the contract (i.e. the first subprogram).

$$\begin{aligned}
FV(x,s) = &\min_{\{\varrho\}} \max_{\{c,n,e\}} \left\{ x\big[u(c,n) - v(e) - \varrho v'(e)\big] + (\theta(s)f(n) - c) + \frac{1}{(1+r)}\mathbb{E}\left[F\bar{V}_1(x', s'; \bar{x}) \mid s, e\right] \right\} \\
&\text{s.t.} \quad x' = \eta x \left[1 + \varphi(s'|s,e)\right] \\
&\qquad\;\; \bar{x}' = \eta x,
\end{aligned}$$

then, for $k = 1, ..., m-1$,

$$\begin{aligned}
F\bar{V}_k(x,s,\bar{x}) = &\min_{\{\varrho\}} \max_{\{c,n,e\}} \Big\{ \bar{x}u(c,n) - x\big[v(e) + \varrho v'(e)\big] + (\theta(s)f(n) - c) \\
&+ \frac{1}{(1+r)}\mathbb{E}\left[F\bar{V}_{k+1}(x', s', \bar{x}') \mid s, e\right] \Big\} \\
&\text{s.t.} \quad x' = \eta x \left[1 + \varphi(s'|s,e)\right] \\
&\qquad\;\; \bar{x}' = \eta \bar{x}
\end{aligned}$$

and finally for $m$,

$$\begin{aligned}
F\bar{V}_m(x,s,\bar{x}) = &\min_{\{\varrho\}} \max_{\{c,n,e\}} \Big\{ \bar{x}u(c,n) - x\big[v(e) + \varrho v'(e)\big] + (\theta(s)f(n) - c) \\
&+ \frac{1}{(1+r)}\mathbb{E}\left[FV(x', s') \mid s, e\right] \Big\} \\
&\text{s.t.} \quad x' = \eta x \left[1 + \varphi(s'|s,e)\right].
\end{aligned}$$

As it can be seen, within the subprogram $x'$ is the *latent aggregated multiplier* and the allocation policies are given by the same first-order conditions, (14) and (15), resulting in $c(\bar{x},s)$ and $n(\bar{x},s)$, while the effort policy $e(x,s)$ requires a reformulation of (16). For this, it is useful to recall that, as in the benchmark Fund contracts, $FV(x,s) = xV^b(x,s) + V^l(x,s)$. However, $F\bar{V}(x,s,\bar{x})$ depends on the within the subprogram relative Pareto weight $\bar{x}$ and the *latent relative Pareto weight* $x$; therefore, we first decompose

$$\begin{aligned}
\bar{V}_1^b(\bar{x},s) &= u(c(\bar{x},s), n(\bar{x},s)) + \beta\mathbb{E}\bar{V}_1^b(\bar{x}',s'), \text{ and} \\
\bar{V}_2^b(x,s) &= -v(e(x,s)) + \beta\mathbb{E}\bar{V}_2^b(x',s'), \text{ then} \\
\bar{V}^b(x,s,\bar{x}) &= \bar{V}_1^b(\bar{x},s) + \bar{V}_2^b(x,s) \text{ and} \\
F\bar{V}^b(x,s,\bar{x}) &= \bar{x}\bar{V}_1^b(\bar{x},s) + x\bar{V}_2^b(x,s) + \bar{V}^l(x,s,\bar{x})
\end{aligned}$$

With this notation, the effort policy $e(x,s)$ solves the following equations, for $k = 1, ..., m-1$:

$$
\begin{aligned}
v'(e(x,s)) =& \beta \sum_{s'|s} \partial_e \pi(s' \mid s) \bar{V}^b_{k+1}(x', s', \bar{x}') + \frac{1}{x}\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s' \mid s) \bar{V}^l_{k+1}(x', s', \bar{x}') \\
& - \varrho(x,s) \left[ v''(e(x,s)) + \beta \sum_{s'|s} \partial^2_e \pi(s' \mid s) \bar{V}^b_{k+1,2}(x', s') \right],
\end{aligned}
$$

and for $m$:

$$
\begin{aligned}
v'(e(x,s)) =& \beta \sum_{s'|s} \partial_e \pi(s' \mid s) V^b(x', s') + \frac{1}{x}\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s' \mid s) V^l(x', s') \\
& - \varrho(x,s) \left[ v''(e(x,s)) + \beta \sum_{s'|s} \partial^2_e \pi(s' \mid s) V^b(x', s') \right].
\end{aligned}
$$

## 5.2 Contract with subprograms of stochastic length

We now consider a contract with subprograms of stochastic length, instead of a deterministic number of periods $m$ as in the previous subsection. With exogenous probability $\chi$ a subprogram continues, while with probability $1-\chi$ the contract terminates. As in the previous subsection we start considering the case that LE constraints do not bind within the subprogram, there is perfect risk-sharing within the period, there are *latent aggregated multipliers*, which define the starting Pareto weights when the follow-up subprogram, after the subprogram randomly ends. The recursive SPFE reads,

$$
\begin{aligned}
FV(x,s) =& \min_{\{\varrho\}} \max_{\{c,n,e\}} \Big\{ x\big[u(c,n) - v(e) - \varrho v'(e)\big] + (\theta(s) f(n) - c) \\
& + \frac{1}{(1+r)} \mathbb{E}\left[\chi F\bar{V}(x', s'; \bar{x}') + (1-\chi) FV(x', s') \mid s, e\right] \Big\} \\
\text{s.t.} \quad & x'(s') = \eta x \left[1 + \varphi(s' \mid s, e)\right] \\
& \bar{x}' = \eta x
\end{aligned}
$$

$$
\begin{aligned}
F\bar{V}(x,s;\bar{x}) =& \min_{\{\varrho\}} \max_{\{c,e\}} \Big\{ \bar{x} u(c,n) - x[v(e) + \varrho v'(e)] + (\theta(s) f(n) - c) + \\
& \frac{1}{(1+r)} \mathbb{E}\left[\chi F\bar{V}(x', s'; \bar{x}') + (1-\chi) FV(x', s') \mid s, e\right] \Big\} \\
\text{s. to} \quad & x' = \eta x \left[1 + \varphi(s' \mid s, e)\right] \\
& \bar{x}' = \eta \bar{x}.
\end{aligned}
$$

The effort policy $e(x,s)$ is determined by the first order condition of the SPFE with respect to $e$ which, as it can be seen, is the same for $FV(x,s)$ and $F\bar{V}(x,s;\bar{x})$, and can be conveniently expressed as:

$$\begin{aligned}
v'(e(x,s)) =& \beta \sum_{s'|s} \partial_e \pi(s' \mid s) \left[\chi \bar{V}^b(x',s',\bar{x}') + (1-\chi)V^b(x',s')\right] \\
&+ \frac{1}{x}\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s' \mid s) \left[\chi \bar{V}^l(x',s',\bar{x}') + (1-\chi)V^l(x',s')\right] \\
&- \varrho(x,s) \left[ v''(e(x,s)) + \beta \sum_{s'|s} \partial_e^2 \pi(s' \mid s) \left[\chi \bar{V}_2^b(x',s') + (1-\chi)V^b(x',s')\right] \right].
\end{aligned}$$

We now account for limited enforcement constraints. The SPFE (8), now takes the form:

$$\begin{aligned}
FV(x,s) = \min_{\{\nu_b,\nu_l,\varrho\}} \max_{\{c,n,e\}} & \Big\{ x\big[(1+\nu_b)(u(c,n)-v(e)) - \nu_b V^0(s) - \varrho v'(e)\big] + \big[(1+\nu_l)(\theta(s)f(n)-c) - \nu_l Z\big] \\
&+ \frac{1+\nu_l}{1+r} \mathbb{E}\left[\chi \mathbb{I}_{\{(x',s')\}} F\bar{V}(x',s';\bar{x}') + (1-\chi \mathbb{I}_{\{(x',s')\}}) FV(x',s') \mid s,e\right] \Big\} \\
\text{s.t.} \quad & x'(s') = \eta x \frac{1+\nu_b+\psi(s' \mid s,e)}{1+\nu_l} \\
& \bar{x}' = \eta x \frac{1+\nu_b}{1+\nu_l}
\end{aligned}$$

$$\begin{aligned}
F\bar{V}(x,s;\bar{x}) = \min_{\{\varrho\}} \max_{\{c,e\}} & \Big\{ \bar{x}u(c,n) - x\big[v(e) + \varrho v'(e)\big] + (\theta(s)f(n)-c) \\
& \frac{1}{1+r} \mathbb{E}\left[\chi \mathbb{I}_{\{(x',s')\}} F\bar{V}(x',s';\bar{x}') + (1-\chi \mathbb{I}_{\{(x',s')\}}) FV(x',s') \mid s,e\right] \Big\} \\
\text{s.t.} \quad & x' = \eta x \left[1+\psi(s' \mid s,e)\right] \\
& \bar{x}' = \eta \bar{x},
\end{aligned}$$

where $\mathbb{I}_{\{(x',s')\}}$ is an indicator function with $\mathbb{I}_{\{(x',s')\}} = 1$ if one of the limited enforcement constraints is binding – i.e. $\nu_b(x',s')\nu_l(x',s') > 0$. Note that, except for the distinction of $\bar{x}$, $F\bar{V}(x,s;\bar{x})$ is the same than $FV(x,s)$ since we can always incorporate in the minimization $\{\nu_b,\nu_l\}$ which will satisfy $\nu_b = \nu_l = 0$, by construction, within the subprogram. This allow us to express a unique first-order condition for the effort policy, as in the case without limited enforcement constraints ending the subprograms.

$$
\begin{aligned}
v'(e(x,s)) &= \beta \sum_{s'|s} \partial_e \pi(s' \mid s) \left[\chi \mathbb{I}_{\{(x',s')\}} \bar{V}^b(x',s',\bar{x}') + (1-\chi \mathbb{I}_{\{(x',s')\}}) V^b(x',s')\right] \\
&+ \frac{1+\nu_l}{1+\nu_b} \frac{1}{x} \frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s' \mid s) \left[\chi \mathbb{I}_{\{(x',s')\}} \bar{V}^l(x',s';\bar{x}') + (1-\chi \mathbb{I}_{\{(x',s'\bar{x}')\}}) V^l(x',s')\right] \\
&- \frac{\varrho(x,s)}{1+\nu_b} \left[ v''(e(x,s)) + \beta \sum_{s'|s} \partial_e^2 \pi(s' \mid s) \left[\chi \mathbb{I}_{\{(x',s')\}} \bar{V}_2^b(x',s') + (1-\chi \mathbb{I}_{\{(x',s')\}}) V^b(x',s')\right]\right]. \qquad (22)
\end{aligned}
$$

Note that (22) is a generalized version of the benchmark (16), which is now the particular case with $\chi = 0$, and it can be decomposed into the *incentive compatibility constraint*:

$$
v'(e(x,s)) = \beta \sum_{s'|s} \partial_e \pi(s' \mid s) \left[\chi \mathbb{I}_{\{(x',s')\}} \bar{V}^b(x',s',\bar{x}') + (1-\chi \mathbb{I}_{\{(x',s')\}}) V^b(x',s')\right], \qquad (23)
$$

which determines $e(x,s)$, and

$$
\begin{aligned}
&\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s' \mid s) \left[\chi \mathbb{I}_{\{(x',s')\}} \bar{V}^l(x',s';\bar{x}') + (1-\chi \mathbb{I}_{\{(x',s')\}}) V^l(x',s')\right] \\
&= \vartheta(x,s) \left[ v''(e(x,s)) + \beta \sum_{s'|s} \partial_e^2 \pi(s' \mid s) \left[\chi \mathbb{I}_{\{(x',s')\}} \bar{V}_2^b(x',s') + (1-\chi \mathbb{I}_{\{(x',s')\}}) V^b(x',s')\right]\right], \qquad (24)
\end{aligned}
$$

where $\vartheta(x,s) \equiv \frac{x\varrho(x,s)}{1+\nu_l}$ is determined by (24).

## 6 Contracts with conditionality

In this section, we introduce conditionality. Unobservable (non-contractible) effort not only affects the final distribution of the observable outcome (e.g. government liabilities) but also can help to fulfill an observable (and contractible) condition informative of the improvement of the risk profile of the country. We then analyze how conditionality can improve the performance of the benchmark and long-span contracts.

Let $\zeta$ be cumulative effort, which we assume follows an autoregressive process

$$
\zeta' = \phi \zeta + e, \qquad \phi \in [0,1],
$$

and introduce a new state $\psi = \{H, L\}$ which indicates the risk profile of a country. By accumulating effort, the borrowing government can increase the country's probability $\chi(\zeta)$ to jump from a high risk profile, $H$, to a low risk profile, $L$. We assume that once the country reaches the low risk status, it remains there forever, i.e. we assume that the low risk state is absorbing. The risk profile affects the conditional distribution of output, so that we express transition probabilities as $\pi(s' \mid s, e(s), \psi)$.

### 6.1 Derive the Borrower's ICC

Conditional on being in a high-risk state, the value of the borrower is

$$V^b(s,\zeta;H) = \max_{\zeta'} \ u(c,n) - v(\zeta' - \phi\zeta) + \beta\chi(\zeta')\mathbb{E}V^b(s';L) + \beta(1-\chi(\zeta'))\mathbb{E}V^b(s',\zeta';H),$$

Conditional on being in a low-risk state, the value of the borrower is

$$V^b(s;L) = \max_{e} \ u(c,n) - v(e) + \beta\mathbb{E}V^b(s';L),$$

Note that once $\psi = L$ realizes, $\zeta$ becomes irrelevant for the value of the borrower, since it is only current effort that determines the transition probability.

Taking FOCs conditional on being in a low-risk state, the condition is:

$$v'(e(x,s)) = \beta \sum_{s'|s} \partial_e \pi(s'|s, e(x,s)) V^b(s',L). \tag{25}$$

Taking FOCs conditional on being in a high-risk state:

$$\begin{aligned}
& v'(\zeta' - \phi\zeta) - \beta\mathbb{E}\left[\chi'(\zeta')\left(V^b(s';L) - V^b(s',\zeta',H)\right)\right] \\
& - \beta\mathbb{E}\left[\frac{\partial_{\zeta'}\pi(s' \mid s, \zeta'-\phi\zeta, H)}{\pi(s' \mid s, \zeta'-\phi\zeta, H)}\left(\chi(\zeta')V^b(s';L) + (1-\chi(\zeta'))V^b(s',\zeta',H)\right)\right] \\
& = \beta\left(1-\chi(\zeta')\right)\phi\mathbb{E}\Bigg[v'(\zeta''-\phi\zeta') \\
& - \beta\mathbb{E}\left[\frac{\partial_{\zeta'}\pi(s'' \mid s', \zeta''-\phi\zeta', H)}{\pi(s'' \mid s', \zeta''-\phi'\zeta, H)}\left(\chi(\zeta'')V^b(s'';L) + (1-\chi(\zeta''))V^b(s'',\zeta'',H)\right) \mid s'\right]\Bigg].
\end{aligned} \tag{26}$$

Note that without accumulation the equality is equal to zero, while with accumulation, but without direct effect within the conditional distribution (i.e. no direct moral hazard effect), the terms with partial derivatives ($\partial_{\zeta'}\pi$) on the left and right hand sides are zero.

## 6.2 Fund contract

The Fund contract for the high-risk state is the solution to the following recursive saddle-point problem, starting at $\zeta = \mu_\xi = 0$:

$$
\begin{aligned}
& FV(\mu_b, \mu_l, \mu_\xi, s, \zeta; H) = \\
& \quad \min_{\{\gamma_b, \gamma_l, \xi\}} \max_{\{c, n, \zeta'\}} \Big\{ \big[(\mu_b + \gamma_b)(u(c,n) - v(\zeta' - \phi\zeta)) - \gamma_b V^0(s) - (\xi - \phi\mu_\xi) v'(\zeta' - \phi\zeta)\big] \\
& \quad + [(\mu_l + \gamma_l)(\theta(s) f(n) - c) - \gamma_l Z] \\
& \quad + \frac{1}{1+r} \mathbb{E}\left[\chi(\zeta') FV(\mu'_{b,L}, \mu'_l, \mu'_\xi, s'; L) + (1 - \chi(\zeta')) FV(\mu'_{b,H}, \mu'_l, \mu'_\xi, s', \zeta'; H) \mid s, e; H\right] \Big\} \\
\text{s.t.} \quad & \mu'_l = \mu_l + \gamma_l \\
& \mu'_{b,L} = \left(\mu_b + \gamma_b + \frac{\chi'(\zeta')}{\chi(\zeta')}\xi + \frac{\partial_{\zeta'} \pi(s' \mid s, \zeta' - \phi\zeta; H)}{\pi(s' \mid s, \zeta' - \phi\zeta; H)}(\xi - \beta\phi\mu_\xi)\right)\eta, \\
& \mu'_{b,H} = \left(\mu_b + \gamma_b - \frac{\chi'(\zeta')}{1 - \chi(\zeta')}\xi + \frac{\partial_{\zeta'} \pi(s' \mid s, \zeta' - \phi\zeta; H)}{\pi(s' \mid s, \zeta' - \phi\zeta; H)}(\xi - \beta\phi\mu_\xi)\right)\eta, \\
& \mu'_\xi = \xi
\end{aligned}
$$

which can also be expressed in terms of relative Pareto weights, $x = \frac{\mu_b}{\mu_l}$, and normalized multipliers $\nu_l = \frac{\gamma_l}{\nu_l}$, $\nu_b = \frac{\gamma_b}{\nu_b}$ and $\varrho = \frac{\xi}{\mu_b}$. Then, starting at $\zeta = \nu_\epsilon = 0$, the Fund contract in the high-risk state solves:

$$
\begin{aligned}
& FV(x, \nu_\xi, s, \zeta; H) = \\
& \quad \min_{\{\nu_b, \nu_l, \varrho\}} \max_{\{c, n, \zeta'\}} \Big\{ x\big[(1 + \nu_b)(u(c,n) - v(\zeta' - \phi\zeta)) - \gamma_b V^0(s) - (\varrho - \phi\nu_\xi) v'(\zeta' - \phi\zeta)\big] \\
& \quad + [(1 + \nu_l)(\theta(s) f(n) - c) - \gamma_l Z] \\
& \quad + \frac{1 + \nu_l}{1+r} \mathbb{E}\left[\chi(\zeta') FV(x'_L, \nu'_\xi, s'; L) + (1 - \chi(\zeta')) FV(x', \nu'_\xi, s', \zeta'; H) \mid s; H\right] \Big\} \\
\text{s.t.} \quad & x'_L = \frac{1}{1 + \nu_l}\left(1 + \nu_b + \frac{\chi'(\zeta')}{\chi(\zeta')}\varrho + \frac{\partial_{\zeta'} \pi(s' \mid s, \zeta' - \phi\zeta; H)}{\pi(s' \mid s, \zeta' - \phi\zeta; H)}(\varrho - \phi\nu_\xi)\right)\eta x, \\
& x' = \frac{1}{1 + \nu_l}\left(1 + \nu_b - \frac{\chi'(\zeta')}{1 - \chi(\zeta')}\varrho + \frac{\partial_{\zeta'} \pi(s' \mid s, \zeta' - \phi\zeta; H)}{\pi(s' \mid s, \zeta' - \phi\zeta; H)}(\varrho - \phi\nu_\xi)\right)\eta x, \\
& \nu'_\xi = \varrho
\end{aligned}
$$

Taking the first-order conditions with respect to effort, we obtain

$$
\begin{aligned}
&x\left[(1+\nu_b)v'(\zeta'-\phi\zeta)+(\varrho-\phi\nu_\xi)v''(\zeta'-\phi\zeta)\right] \qquad (27)\\
&=\beta\left(\chi'(\zeta')((1+\nu_b)+\chi'(\zeta')\varrho\right)\mathbb{E}\left[V^b(s',L)-V^b(s',\zeta',H))\mid s\right]x\\
&+\frac{1+\nu_l}{1+r}\chi'(\zeta')\mathbb{E}\left[V^l(s',L)-V^l(s',\zeta',H))\mid s\right]\\
&+\beta\mathbb{E}\left[\frac{\partial_{\zeta'}\pi(s'\mid s,\zeta'-\phi\zeta;H)}{\pi(s'\mid s,\zeta'-\phi\zeta;H)}\left[(1+\nu_b)\left(\chi(\zeta')V^b(s',L)+(1-\chi(\zeta'))V^b(s',\zeta',H)\right)+\chi'(\zeta')\varrho\left(V^b(s',L)-V^b(s',\zeta',H)\right)\right]\mid s\right]x\\
&+\frac{1+\nu_l}{1+r}\mathbb{E}\left[\frac{\partial_{\zeta'}\pi(s'\mid s,\zeta'-\phi\zeta;H)}{\pi(s'\mid s,\zeta'-\phi\zeta;H)}\left(\chi(\zeta')V^l(s',L)+(1-\chi(\zeta'))V^l(s',\zeta',H)\right)\mid s\right]\\
&+\beta\mathbb{E}\ \chi'(\zeta')[\frac{\chi''(\zeta')}{\chi'(\zeta')}\varrho\left(V^b(s',L)-V^b(s',\zeta',H)\right)\mid s\Big]x\\
&+\beta(\varrho-\phi\nu_\xi)\mathbb{E}\left[\frac{\partial^2_{\zeta'}\pi(s'\mid s,\zeta'-\phi\zeta;H)}{\pi(s'\mid s,\zeta'-\phi\zeta;H)}\left(\chi(\zeta')V^b(s',L)+(1-\chi(\zeta'))V^b(s',\zeta',H)\right)\mid s\right]x\\
&+\beta(1-\chi(\zeta'))\phi\Big[\left(1+\nu_b-\chi'(\zeta')\varrho\left[v'(\zeta''-\phi\zeta')+\nu_b'v'(\zeta''-\phi\zeta')+(\varrho'-\phi\nu_\xi')v''(\zeta''-\phi\zeta')\right]\right]x\\
&-\beta(1-\chi(\zeta'))\phi\Big[\left(1+\nu_b-\chi'(\zeta')\right)\varrho\beta\mathbb{E}\left[\frac{\partial_{\zeta'}\pi(s''\mid s,\zeta''-\phi\zeta';H)}{\pi(s''\mid s',\zeta''-\phi\zeta';H)}\left(\chi(\zeta'')V^b(s'',L)+(1-\chi(\zeta''))V^b(s'',\zeta'',H)\right)\mid s'\right]\Big]x\\
&-\frac{(1+\nu_l')(1+\nu_l)}{(1+r)^2}(1-\chi(\zeta'))\phi\mathbb{E}\left[\frac{\partial_{\zeta'}\pi(s''\mid s',\zeta''-\phi\zeta';H)}{\pi(s''\mid s',\zeta''-\phi\zeta';H)}\left(\chi(\zeta'')V^l(s'',L)+(1-\chi(\zeta''))V^l(s'',\zeta'',H)\right)\mid s'\right]
\end{aligned}
$$

This first-order condition is, in fact, composed of three first-order conditions: first in red, the first-order condition of the incentive compatibility constraint faced by the borrower (26), which determines effort, $e$; second in blue, the first-order condition of the planner problem with moral hazard without accumulated effort, once the incentive compatibility constraint is accounted for, which in (26) determines $\varrho$ and in (27) determines $(\varrho-\phi\nu_\xi)$, and third, the first-order condition to fulfill the targeted condition through accumulated effort –i.e. the remaining terms, all multiplied by $\chi'(\zeta')$ – which determines $\varrho$.

# 7 Quantitative Analysis

In this section, we present the parametrization of the model and the results of the quantitative analysis. In future versions, we will properly calibrate the model to match specific moments.

## 7.1 Parametrization

We parametrize the model following Ábrahám et al. (2022). The utility of the borrower is additively separable in consumption, leisure and effort. In particular, we assume that $u(c)=\log(c)$, $h(1-n)=\gamma\frac{(1-n)^{1-\sigma}-1}{1-\sigma}$ and $v(e)=\omega e^2$ so that:

$$
U(c,n,e)=\log(c)+\gamma\frac{(1-n)^{1-\sigma}-1}{1-\sigma}-\omega e^2
$$

The preference parameters are set to $\sigma=0.6887$, $\gamma=1.4$, $\omega=0.1$ and the discount factor to $\beta=0.945$. The risk free interest rate is set to $r=2.48\%$, the average short-term real interest rate of Germany. The fact that the borrower is less patient than the lender implies that the borrower would like to front-load consumption. As it is well known, in the absence of any frictions (limited commitment or moral hazard) consumption of the borrower would converge towards zero in the long run.

The participation constraint of the lender is set to $Z=0$, implying no expected permanent transfers between the borrower and the lender at any time or state. In other words, the Fund is not built on an assumption of solidarity which would require permanent transfers.

Table 1: Parameter Values

| $\alpha$ | $\beta$ | $\sigma$ | $\gamma$ | $r$ | $\lambda$ | $\psi$ | $\omega$ | $Z$ |
|---|---|---|---|---|---|---|---|---|
| 0.566 | 0.945 | 0.6887 | 1.4 | 0.0248 | 0 | 0.8099 | 0.1 | 0 |

We assume that *Fund-exit is irreversible*, with the interpretation that the fund can commit to exclusion of the borrower. If a country defaults, it is also subject to an asymmetric default penalty as in Arellano (2008)

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \geq \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

where $\psi = 0.8099$. In addition, we consider permanent autarky upon default meaning that $\lambda = 0$.

Regarding the technology, we assume that $f(n) = n^\alpha$ with the labor share of the borrower set to $\alpha = 0.566$ to match the average labor share across the Euro Area 'stressed' countries. Table 1 summarizes the parameter values.

Our basic calibration of the productivity shocks assumes that $\theta$ is independent of effort. It implies that the calibration of this process requires setting the levels of this variable and a standard Markov transition matrix describing its law of motion. In particular, we allow three realizations: $\Theta = \{\theta^1, \theta^2, \theta^3\}$, with $0 < \theta^1 < \theta^2 < \theta^3$. More precisely, the transition matrix and productivity shock values of $\theta$ are given below:

$$\pi = \begin{bmatrix} 0.9650 & 0.0233 & 0.0117 \\ 0.0300 & 0.9650 & 0.0050 \\ 0.0150 & 0.0200 & 0.9650 \end{bmatrix} \tag{28}$$
$$\theta \in \{0.9, 1.0, 1.1\}.$$

In our benchmark model, (policy) effort affects the probability distribution over next period's realisation of the productivity $\theta$. In order to parametrize the full model, we provide more structure by assuming that, given current productivity $\theta$, there are two possible distributions of tomorrow's liabilities, $\pi^l(\cdot|\theta)$ and $\pi^h(\cdot|\theta)$ where $\pi^h(\cdot|\theta)$ first-order stochastically dominates $\pi^l(\cdot|\theta)$ for all $\theta$. In particular, there is $\Gamma(e) \in (0,1)$, with $\Gamma'(e) < 0$ and and $\Gamma''(e) < 0$, such that $\pi(\theta'|\theta, e) = \Gamma(e)\pi^l(\theta'|\theta) + (1 - \Gamma(e))\pi^h(\theta'|\theta)$.

To determine these two matrices, we assume that with $\bar{\Gamma} = \mathbb{E}\Gamma(e)$ evaluated at the ergodic distribution of effort in autarky, $\pi^h$ and $\pi^l$ replicate the transition matrix of $\theta$ without moral hazard in (28), subject to the requirement that $\pi^h$ first order stochastically dominates $\pi^l$. There are many combination of matrices satisfying this requirement. We chose among those the matrices which allow effort to have the most effect on the probability distribution of next period productivity. More specifically, the matrices we use are:

$$\pi^h = \begin{bmatrix} 0.93 & 0.0466 & 0.0234 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}, \pi^l = \begin{bmatrix} 1 & 0 & 0 \\ 0.06 & 0.94 & 0 \\ 0.03 & 0.04 & 0.93 \end{bmatrix}$$

Last, for the $\Gamma(e)$ function determining how effort decreases the weight of the *bad distribution*, we assume it to be $\Gamma(e) = (e-1)^2$, which, together with the specification of $v(e) = \omega e^2$, allows us to have a simple closed form solution for effort. I This functional form implies simple expressions for $\frac{\partial \pi(\theta'|\theta,e)}{\partial e}$ and $\frac{\partial^2 \pi(\theta'|\theta,e)}{\partial e^2}$ as follows:

$$\frac{\partial \pi(\theta'|\theta, e)}{\partial e} = -\Gamma'(e)\left[\pi^h(\theta'|\theta) - \pi^l(\theta'|\theta)\right] = 2(1-e)\left[\pi^h(\theta'|\theta) - \pi^l(\theta'|\theta)\right]$$
$$\frac{\partial^2 \pi(\theta'|\theta, e)}{\partial e^2} = -\Gamma''(e)\left[\pi^h(\theta'|\theta) - \pi^l(\theta'|\theta)\right] = -2\left[\pi^h(\theta'|\theta) - \pi^l(\theta'|\theta)\right].$$

Note that with this functional forms **Assumption 1 is satisfied**. In addition, with $\omega$ set to 0.1, the optimal effort policy in equilibrium implies a value of $\bar{\Gamma} = \mathbb{E}\Gamma(e)$ such that $\pi = \bar{\Gamma}\pi^l + (1-\bar{\Gamma})\pi^h$, replicating the transition matrix matrix of $\theta$ for the economy without moral hazard.

For the contract with conditionality, we need to specify two additional processes. First, the transition probability from a high to a low risk state is given by $\chi(e)$ where $e = \zeta' - \phi\zeta$. More precisely, the probability is composed by $\chi^l$ and $\chi^h$ where the latter first-order stochastically dominates the former. There is $Z(e) \in (0,1)$, with $Z'(e) < 0$ and

and $Z''(e) < 0$, such that $\chi(e) = Z(e)\chi^l + (1 - Z(e))\chi^h$. At this stage, we assume that $Z(e) = (e-1)^2$ as for $\Gamma(e)$ for simplicity and set $\chi^l = 0.01$ and $\chi^h = 0.3$.

Second, we set the transition matrices in the high risk state to

$$\pi_H^h = \pi^h,\ \pi_H^l = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0.05 & 0.1 & 0.85 \end{bmatrix}$$

The high distribution is the same in the high and low risk state. However, the low distribution is worse in the high risk state.

## 7.2 Back-Loaded Moral Hazard

We first present the law of motion of the relative Pareto weight for Fund with standard moral hazard (MH) presented in Section 3 and for the Fund with back-loaded MH presented in Section 5. We then expose some simulation exercises and welfare comparisons.

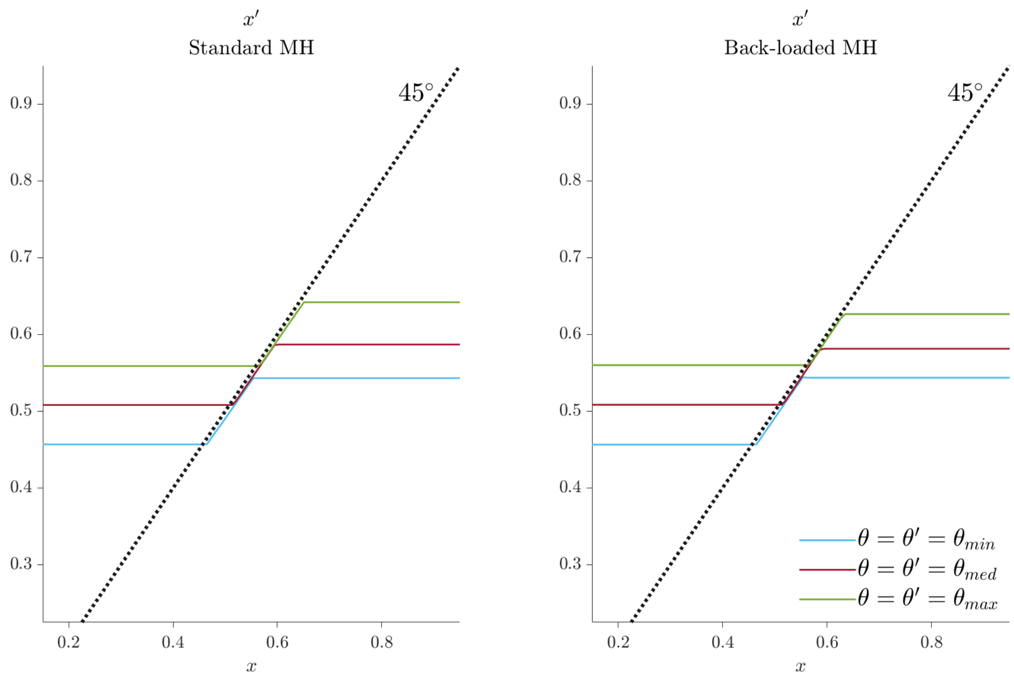


Figure 1: Main Relative Pareto Weights – Different MH specifications

Figure 1 depicts the law of motion of the relative Pareto weight in the standard MH specification and the back-loaded MH specification. In each specification, the horizontal line on the left hand side is determined by the borrower's binding LE constraint, while the horizontal line on the right hand side is determined by the Fund's binding LE constraint. The line rejoining both horizontal lines is determined by the first best allocation and has a slope of $\eta < 1$. We do not observe much differences between the two. This indicates that the LE dominates the MH in the given parametrization.

Looking specifically at the back-loaded MH, Figure 2 presents the law of motion of the main (i.e. $\bar{x}$) and the auxiliary (i.e. $x$) relative Pareto weight. The main difference between the two weights can be found in the intervals in which the LE constraints do not bind. In this region, the main weight decays at rate $\eta$, whereas the auxiliary weight is either above (i.e. in the transition from a low to a high shock) or below (i.e. in the transition from a high to a low shock) the 45° line.

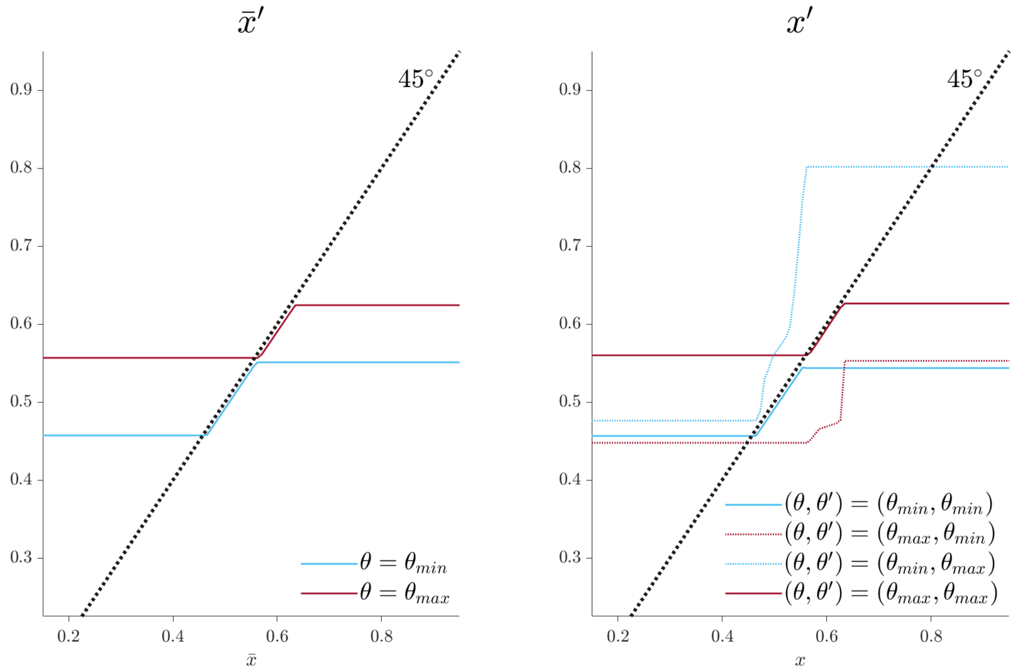


Figure 2: Main and Auxiliary Relative Pareto Weights – Back-Loaded MH

We complement the analysis of the different law of motions with simulations. We simulate the economy within the ergodic set of relative Pareto weights. For this purpose, we generate one history of shocks for 500 periods in steady state starting with the lowest Pareto weight in the ergodic set. To avoid that the initial conditions blur the results, the first 100 periods are discarded.

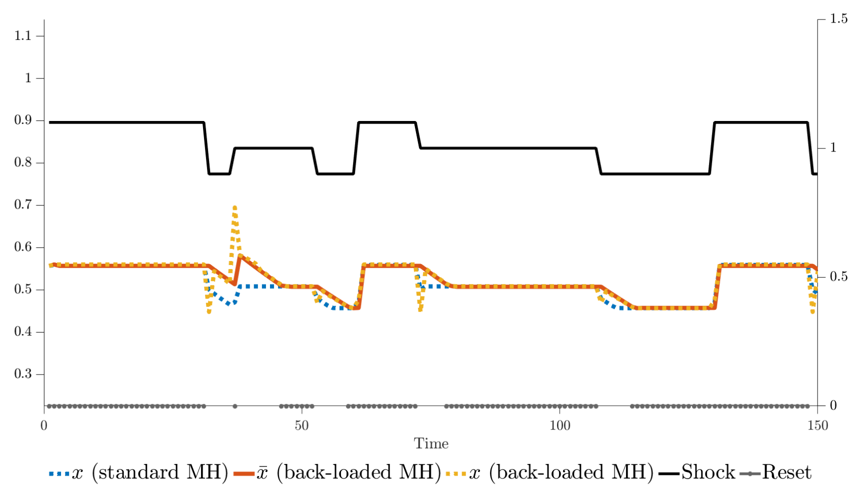


Figure 3: Simulation Path of Relative Pareto Weights

Figure 3 depicts the evolution of the relative Pareto weight in the Fund with standard MH (i.e. the blue line) and back-loaded MH (i.e. the orange and yellow lines). The grey dots on the x-axis correspond to periods in which the Fund resets the program. As one can see, the auxiliary weight in yellow (over) under-shoot when a (high) low productivity shock realizes. This is entirely due to the MH. In addition, given the high persistence of the productivity states, the main relative Pareto weight in orange closely follows the relative Pareto weight in the standard MH in blue. Only in the transition between two states the two weights differ. Most notably, the main weight in the back-loaded MH adjust more slowly than the weight in the standard MH.

Similarly, Figure 4 compares the simulated path of consumption, labor, effort and output in the Fund with standard MH (i.e. the blue line) and back-loaded MH (i.e. the orange line). Consistent with the evolution of the relative Pareto weight, the borrower tends to consume more while supplying less effort and labor in the Fund with back-loaded MH. The wedge between the two specifications of the Fund is not too pronounced. Nonetheless, the borrower is enjoying a greater consumption smoothing in the Fund with back-loaded MH. This comes at the cost of a lower primary surplus (or a greater primary deficit). As we will see next, this generates welfare gains for the borrower and losses for the Fund.

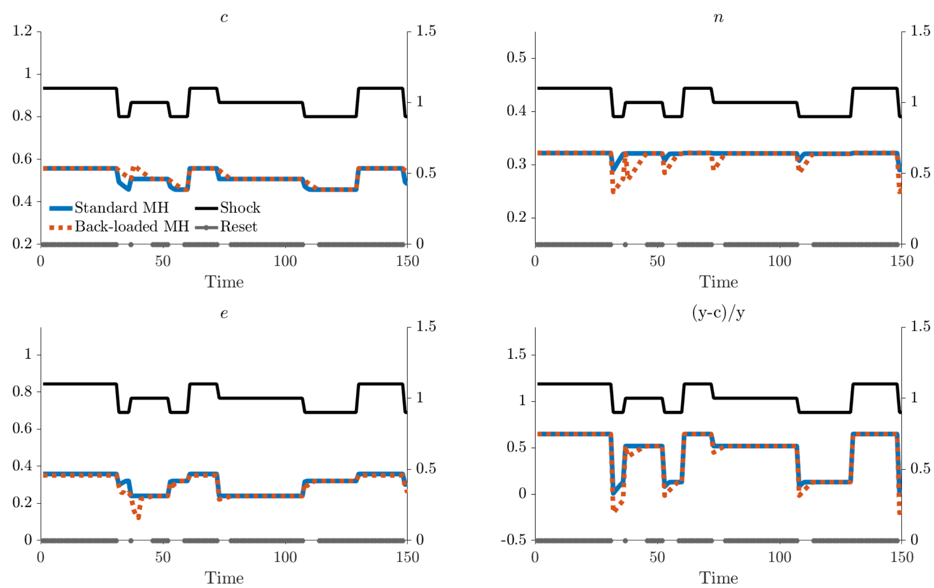


Figure 4: Simulation Path of Consumption, Labor, Effort and Output

Finally, Table 2 presents the welfare gains in consumption equivalent terms relative to the Fund with standard MH. In particular, a positive number indicates welfare gains for the borrower being in the Fund with back-loaded MH relative to the Fund with standard MH. A negative number indicates the opposite.

Table 2: Welfare Gains – Back-Loaded MH

| | Welfare Gains | |
|---|---|---|
| | Borrower (percent) | Fund (percent) |
| $\theta_{min}$ | 0.55 | -0.027 |
| $\theta_{med}$ | 0.88 | -0.025 |
| $\theta_{max}$ | 1.65 | -0.031 |

We note two points on welfare. First, the borrower records welfare gains in the Fund with back-loaded MH relative to the Fund with standard MH. Gains are monotone and are the highest in the highest productivity state.

Second and in opposition to the previous point, the Fund records welfare losses. Those are more pronounced in the high productivity state.

## 7.3 Conditionality

We now discuss the results of the Fund contract with conditionality. For this, we focus on what happens in the high risk state. We compare the Fund contracts with and without conditionality in the high risk state.[16]

Figure 5 depicts the law of motion of the relative Pareto weight for the Fund with and without conditionality in the high risk state. As one can see, the weights related to the borrower's binding constraint are closer from each other in the Fund contract with conditionality. This means that consumption, labor and effort will vary less in this Fund contract in steady state. On the other hand, the lender's limited enforcement constraints are looser with conditionality, reflecting greater confidence that debt is sustainable in the Fund with conditionality.

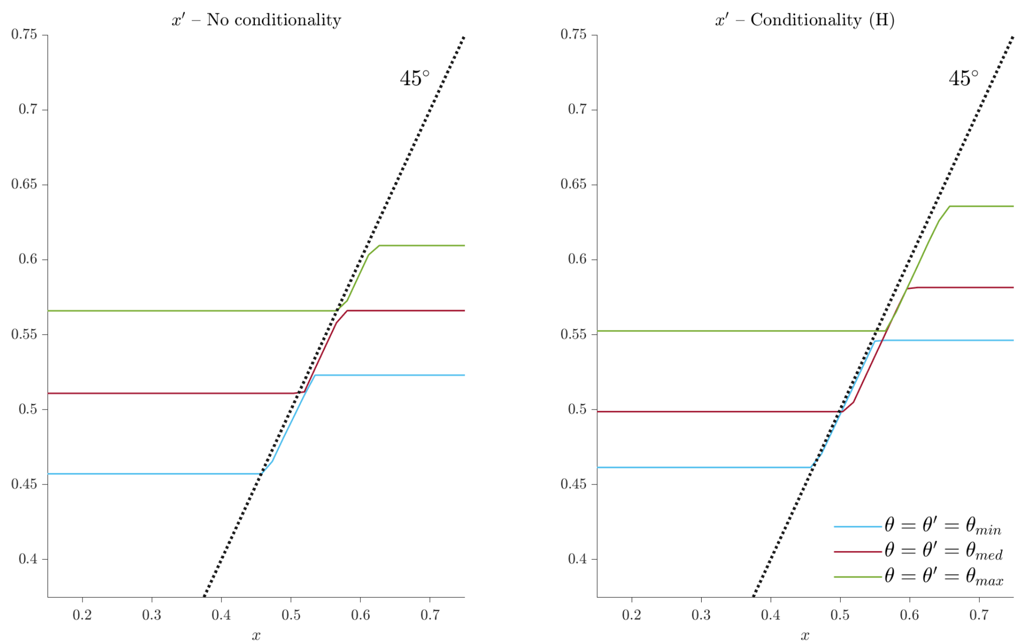


Figure 5: Relative Pareto Weights – Conditionality

Figure 6 compares the simulated path of relative Pareto weight, consumption, labor, effort and output in the Fund without conditionality (i.e. the blue line) and with conditionality (i.e. the orange line) in the high risk state. We use the same procedure as in the previous section to run simulations. The simulated paths follow a similar trend in the two Fund contracts. However, we note that the level of effort (and thus $\zeta'$) is almost always lower in the contract with conditionality. This might appear surprising at first glance. In the Fund with conditionality, effort affects both the probability $\pi$ and $\chi$ at the same time, while it generates the same disutility. Given this "double" benefit, it is optimal for the country to provide less effort than in the case without conditionality.

[16]The comparison is preliminary here as it does not properly reflect the true gains of conditionality. In future versions, we will offer a better comparison.

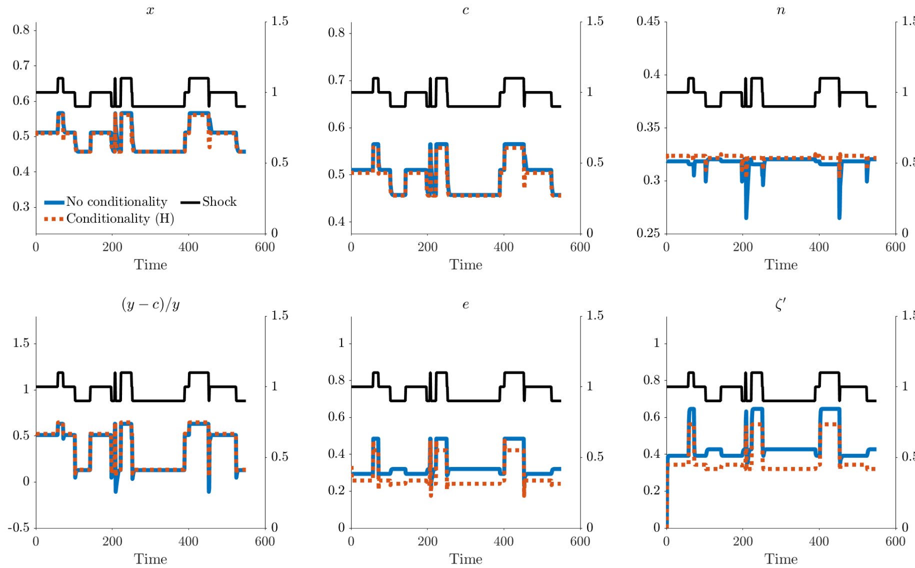


Figure 6: Simulation Path High Risk State (H)

Figure 7 compares the simulated path in the high risk state. Compared to Figure 7, we observe a similar dynamic of consumption and labor. However, effort is somewhat different. The borrower provides even less effort in the contract with conditionality. This is because the country enjoys a better distribution of shocks in the low risk state.

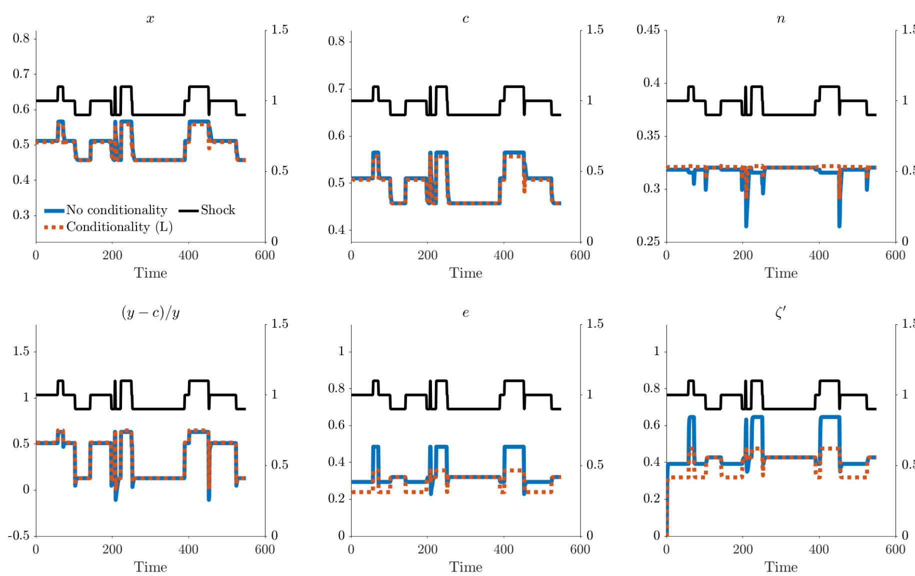


Figure 7: Simulation Path Low Risk State (L)

Table 3 presents the consumption-equivalent welfare gains of the contract with conditionality relative to the contract without conditionality. As one can see there are welfare gains for the borrower for most state in the contract with conditionality. The only exception is for $\theta_{max}$ in the high risk state for which the borrower records a welfare loss. The opposite holds true for the Fund itself.

Table 3: Welfare Gains – Conditionality

| | Welfare Gains | | | |
|---|---|---|---|---|
| | High Risk (H) | | Low Risk (L) | |
| | Borrower (percent) | Fund (percent) | Borrower (percent) | Fund (percent) |
| $\theta_{min}$ | 0.03 | -0.08 | 0.88 | -0.06 |
| $\theta_{med}$ | 0.01 | -0.07 | 0.25 | -0.04 |
| $\theta_{max}$ | -0.11 | 0.01 | 0.65 | -0.05 |

# 8 Conclusion

From the perspective of economic theory, since the pioneer work of Prescott and Townsend (1984) it is understood that under appropriate convexity assumptions moral hazard (and adverse selection) problems can be incorporated in the economists' problem of efficiently assigning resources subject to technological and feasibility constraints, by introducing Incentive Compatibility (IC) constraints in parallel to other constraints. Furthermore, it is also understood that under these, and other standard, assumptions a the corresponding competitive equilibrium exists and the First and Second Welfare Theorems are satisfied for constrained-efficiency allocations. Extensive follow up work has extended their results to dynamic economies – e.g. with debt or other financial assets, etc. However, not much work has been done in studying different forms of implementation; in particular, in dynamic economies with debt and risk-sharing contracts. Here, we have pursued this enquire and have shown how different forms to implement IC constraints result in different constraint-efficient allocations and, correspondingly, in a different (not obvious) split of the surplus between the risk-averse borrower and the risk-neutral lender, while satisfying limited enforcement (LE) constraints.

From the perspective of existing lending and insurance practices, the existence of moral hazard problems is well recognized, but aside from accumulated experience (e.g within the IMF), there is little guidance on how to confront them in practice. The range, from denying lending to imposing relatively *ad-hoc* conditions to be fulfilled for lending to take place, is wide open. Here we have provided a framework to rigorously narrow this range, which should help making lending and risk-sharing – possibly, with conditionality – more efficient.

Nevertheless, we see our contributions – particularly, regarding *conditionality* – as first steps in closing the gap between theory and practice; which is likely to require to further develop both: contractual theory and design, as well as existing Official Lending practices.

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# Appendix

## A Proofs

### A.1 Preliminary lemmas

The first lemma states and proves (21):

**Lemma A.1**

$$\begin{aligned}\partial_x FV(x,s) &= V^b(x,s) + x\partial_x V^b(x,s) + \partial_x V^l(x,s)\\ &= V^b(x,s).\end{aligned}$$

***Proof 1*** *Let $FW(\mu, s)$, where $\mu \equiv (\mu_b, \mu_l)$, be the value function of the recursive formulation of the saddle-point Lagrangian problem. $FW$ is monotone of degree-one in $\mu$ and, therefore, it has the Euler representation:*

$$FW(\mu,s) = \mu_b W^b(\mu,s) + \mu_l W^l(\mu,s) \equiv \mu_b \partial_{\mu_b} FW(\mu,s) + \mu_l \partial_{\mu_l} FW^l(\mu,s),$$

*where $W^b$ and $W^l$ are homogeneous of degree-zero in $\mu$. In particular, provided that it is differentiable in $\mu$,*

$$\begin{aligned}\partial_{\mu_b} FW(\mu,s) &= W^b(\mu,s) + \left[\mu_b \partial_{\mu_b} W^b(\mu,s) + \mu_l \partial_{\mu_b} W^l(\mu,s)\right]\\ &= W^b(\mu,s);\end{aligned}$$

*that is, $\left[\mu_b \partial_{\mu_b} W^b(\mu,s) + \mu_l \partial_{\mu_b} W^l(\mu,s)\right] = 0$. In particular, letting $x = \frac{\mu_b}{\mu_l}$, $FV(x,s) = \frac{1}{\mu_l} FW(\mu,s), V^b(x,s) = W^b(\mu,s)$ and $V^l(x,s) = W^l(\mu,s)$, and if $\mu = (\mu_b, 1)$*

$$\partial_x FV(x,s) = \partial_{\mu_b} FW(\mu,s) = V^b(x,s).$$

*Note we have already used the fact that $\left[x\partial_x V^b(x,s) + \partial_x V^l(x,s)\right] = 0$ in deriving the FOC with respect to e (16), in letting*

$$M(s)\partial_e \mathbb{E}\left[FV(x',s')\big|s,e\right] = \sum_{s'|s} \partial_e^2 \pi(x'_{xs}(s'),s') V^b(x'_{xs}(s'),s'), \tag{A.1}$$

*where $M(s)$ summarizes the components of $FV(x',s')$ which do not depend on $s'$ or $e$. Furthermore, this also shows that in solving the Fund's saddle-point contract problem, the Fund does not act as a Ramsey planner, who in (A.1) would have taken into account that, in $V^b(x'_{xs}(s'),s')$, $x'_{xs}$ depends on $e$ and, therefore, taking the corresponding derivative would be part of the Ramsey's FOC. Instead the Fund simply takes into account the actions of the borrowing country and its external effects.*

***Lemma A.2*** *Given the monotonicity assumption (MLR), the law of motion $x'_{xs}(s'(i))$ is nondecreasing in $i$ and it is constant if $\varrho(x,s) = 0$.*

***Proof 2*** *It follows from Assumption 1 (i.e., the monotone likelihood-ratio condition) — i.e., $\frac{\partial_e \pi(s'(i)|s,e)}{\pi(s'(i)|s,e)}$ is nondecreasing in $i$ for every $e$. Recall (9):*

$$x'_{xs}(s'(i)) = \frac{1+\nu^b + \varphi(s'(i)|s,e)}{1+\nu_l}\eta x \text{ and } \varphi(s'(i)|s,e) = \varrho \frac{\partial \pi(s'(i)|s,e)/\partial e}{\pi(s'(i)|s,e)}.$$

*Note that Lemma 1 implies that $V^b(x'_{xs}(s'(i)), s'(i))$ is nondecreasing in $i$. Let $\omega^h_{xs}(e) \equiv \sum_{s'|s} \pi(s'|s,\, e) V^{hf}(x'_{xs}(s'), s')$, for $h = b, l$, and note that it can also be written as:*

$$\omega^h_{xs}(e) \equiv \sum_{i=1}^{\overline{N}} \Delta V^h_{xs}(s'(i)) \left[\sum_{j=i}^{\overline{N}} \pi(s'(j)|s,\, e)\right],$$

*where*

$$\Delta V^h_{xs}(s'(i)) = \begin{cases} V^{hf}(x'_{xs}(s'(i))) - V^{hf}(x'_{xs}(s'(i-1))), & i > 1, \\ V^{hf}(x'_{xs}(s'(1))), & i = 1, \end{cases} \tag{A.2}$$

*and, similarly, we can define*

$$\Delta\partial_x V^h_{xs}(s'(i)) = \begin{cases} \partial_x V^{hf}(x'_{xs}(s'(i))) - \partial_x V^{hf}(x'_{xs}(s'(i-1))), & i > 1, \\ \partial_x V^{hf}(x'_{xs}(s'(1))), & i = 1, \end{cases} \tag{A.3}$$

*where the terms $\Delta\partial_x V^h_{xs}(s'(i))$ are non-positive if $V^{hf}$ is concave in $x$. Furthermore, let*

$$\overline{\omega}^{h\prime}_{xs}(e) \equiv \sum_{i=1}^{\overline{N}} \Delta V^h_{xs}(s'(i)) \left[ \sum_{j=i}^{\overline{N}} \frac{\partial \pi(s'(j)|s,\, e)}{\partial e} \right]; \tag{A.4}$$

*that is, function $\overline{\omega}^h_{xs}(e)$ is the function $\omega^h_{xs}(e)$ when taking derivatives, with respect to e, only the direct effect of e on the distribution of s is taken into account. We use $\overline{\omega}^h_{xs}(e)$ in the derivations that follow since we are accounting from the fact that the solution to the Fund contract problem satisfies* efficient risk-sharing*; i.e (21), which in this notation is: $x\omega^{b\prime}_{xs}(e) = -\omega^{l\prime}_{xs}(e)$.*

***Lemma A.3*** *Functions $\overline{\omega}^b_{xs}(e)$ and $\overline{\omega}^l_{xs}(e)$ are non decreasing and concave. The saddle-point Lagrangean $\mathcal{L}(x, s)$ (i.e. of the saddle-point Bellman equation) is also concave in e.*

***Proof 3*** *Since the value of the borrower and the lender are non-decreasing in s and by Assumption 1 $\partial_e F_n(e, s) \leq 0$, which implies that all the terms within brackets in (A.4) are non-negative, $\overline{\omega}^{h\prime}_{xs}(e) \geq 0$. Note that, using the latter definition of $\overline{\omega}^h_{xs}(e)$,*

$$\overline{\omega}^{h\prime\prime}_{xs}(e) = \sum_{i=1}^{\overline{N}} \Delta V^h_{xs}(s'(i)) \left[ \sum_{j=i}^{\overline{N}} \frac{\partial^2 \pi(s'(j)|s,\, e)}{\partial e^2} \right]. \tag{A.5}$$

*Similarly, by Assumption 1 $\partial^2_e F_n(e, s) \geq 0$ , which implies that all the terms within brackets in (A.5), are non-positive, $\overline{\omega}^{h\prime\prime}_{xs}(e) \leq 0$.*

*To see that the above conditions guarantee that, given our assumptions, the Lagrangean $\mathcal{L}(x, s)$ is concave, note that*

$$\begin{aligned} \partial_e \mathcal{L}(x, s) = & - x(1+\nu_b)v'(e) - x\varrho v''(e) \\ & + x(1+\nu_b)\beta\overline{\omega}^{b\prime}_{xs}(e) + \frac{1+\nu_l}{1+r}\overline{\omega}^{l\prime}_{xs}(e) \\ & + x\varrho\beta\overline{\omega}^{b\prime\prime}_{xs}(e). \end{aligned}$$

*Note that $\partial_e \mathcal{L}(x, s) = 0$ is (18) expressed in this more synthetic notation. Therefore,*

$$\begin{aligned} \partial^2_e \mathcal{L}(x, s) = & - x(1+\nu_b)v''(e) - x\varrho v'''(e) \\ & + x(1+\nu_b)\beta\overline{\omega}^{b\prime\prime}_{xs}(e) + \frac{1+\nu_l}{1+r}\overline{\omega}^{l\prime\prime}_{xs}(e) \\ & + x\varrho\beta\overline{\omega}^{b\prime\prime\prime}_{xs}(e). \end{aligned}$$

*By assumption the first two terms (RHS first line) are negative and we have just shown that the next two (RHS second line) are also non-positive; by Assumption 1 $\partial^3_e F_n(e, s) \geq 0$ and therefore $\overline{\omega}^{b\prime\prime\prime}_{xs}(e) \leq 0$.*

## A.2 Proof of Proposition 1.

*The proof parallels and extends the proof of Marcet and Marimon (2019) Theorem 3.*

***Proof 4*** ***Step 1:*** Checking that the necessary assumptions are satisfied.

*Our functional forms, and Fund contract problem, satisfy — Marcet and Marimon (2019) assumptions: A2 on functions (continuous, in $(c, n, e)$, and measurable, in $s$); A3 on non-empty feasible sets; given our Assumption 1, A5 on convex technologies, and on concavity: A6 (for the lender) and A6b (strict concavity, for the borrower). They also satisfy the uniform boundedness assumption A4 since, $n$, consumption $c$ and effort $e$ are bounded, therefore the rewards ($U(c, n, e)$ and $c_l$) are bounded as well and the finiteness of $V^o(s)$ and $Z$ implies that the constraint functions are uniformly bounded as well. Finally, our* interiority *assumptions is a version of A7b.*

***Step 2:*** The 'Relaxed Fund Contract problem'and the existence of solutions to this problem.

*We will show first that a solution exists to a 'Relaxed Fund Contract problem ', that is the same than as* Fund Contract problem *except that constraint (4) is replaced by a weak inequality version:*

$$\beta \sum_{s^{t+1}|s^t} \frac{\partial \pi(s^{t+1}|s_t,\, e(s^t))}{\partial e(s^t)} V^b(s^{t+1}|s^t) - v'(e(s^t)) \geq 0, \tag{A.6}$$

*which can also be written as*

$$\beta \overline{\omega}^b_{xs}{}'(e) - v'(e) \geq 0.$$

*In particular, given our assumptions,*

$$\beta \overline{\omega}^b_{xs}{}'''(e) - v'''(e) \leq 0,$$

*therefore (A.6) defines a convex set of feasible efforts.*

*Then, we will show that any solution to the 'Relaxed Fund Contract problem' is also a solution of the original problem. Let $A(s) = \{(c, n, e) \in \mathbb{R}^3_+ : n \leq 1, e \leq 1\}$. This set is obviously compact and convex. Note that the pay-off of the fund $c_l = \theta f(n) - g - c$ is concave given our concavity assumption on $f$.*

*We first decompose the saddle-point recursive contract problem into the choice of actions, $a = (c, n, e)$, and* *multipliers, $\gamma = (\nu_b, \nu_l, \varrho)$, given $FV(x,s)$, as follows:*

$$
\begin{aligned}
SP^a(\gamma) = \Bigg\{ & a \in A(s): \text{ for all } \widetilde{a} \in A(s), \\
& x\Big[U(a) + \beta \sum_{s'|s} \pi(s'|s,e) V^b(x'(s'),s')\Big] \\
& + \Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,e) V^l(x'(s'),s')\Big] \\
& + x\nu_b\Big[U(a) + \beta \sum_{s'|s} \pi(s'|s,e) V^b(x'(s'),s') - V^o(s)\Big] \\
& + \nu_l\Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,e) V^l(x'(s'),s') - Z\Big] \\
& + x\varrho\Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} V^b(x'(s'),s') - v'(e)\Big] \\
& \geq x\Big[U(\widetilde{a}) + \beta \sum_{s'|s} \pi(s'|s,\widetilde{e}) V^b(\widetilde{x}'(s'),s')\Big] \\
& + \Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,\widetilde{e}) V^l(\widetilde{x}'(s'),s')\Big] \\
& + x\nu_b\Big[U(\widetilde{a}) + \beta \sum_{s'|s} \pi(s'|s,\widetilde{e}) V^b(\widetilde{x}'(s'),s') - V^o(s)\Big] \\
& + \nu_l\Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,\widetilde{e}) V^l(\widetilde{x}'(s'),s') - Z\Big] \\
& + x\varrho\Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, \widetilde{e})}{\partial e} V^b(\widetilde{x}'(s'),s') - v'(\widetilde{e})\Big] \Bigg\},
\end{aligned}
$$

*where $\widetilde{x}'(s') = \frac{1+\nu_b}{1+\nu_l} + \varrho \frac{\partial \pi(s'|s,\widetilde{e})/\partial e}{(1+\nu_l)\pi(s'|s,\widetilde{e})}$. Note that our origional problem is homogenous of degree one in $(\mu_{b,0}, \mu_{l,0})$ and that allows us to reformulate the problem using $x$ as a co-state variable. This guarantees together with our* interiority *assumption (a version of A7b used in Lemma 6A in Marcet and Marimon (2019) ), there exists a positive constant $C$ such that for if $\gamma$ is Lagrange multiplier vector $\|\gamma\| \leq C\|x\|$, but the lender's participation constraint $Z$ sets an upper bound on $\|x\|$ for any feasible contract. Therefore, there exists a $\overline{C}$ such that $\|\gamma\| \leq \overline{C}$, and the set of feasible Lagrange multipliers, $\Gamma = \{\gamma \in \mathbb{R}^3_+ : \|\gamma\| \leq \overline{C}\}$, is also compact and convex. The minimization problem can be written* *as:*

$$
\begin{aligned}
SP^\gamma(a) = \Bigg\{ & \gamma \in \Gamma: \text{ for all } \widehat{\gamma} \in \Gamma, \\
& x\Big[U(a) + \beta \sum_{s'|s} \pi(s'|s,e) V^b(x'(s'),s')\Big] \\
& + \Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,e) V^l(x'(s'),s')\Big] \\
& + x\nu_b \Big[U(a) + \beta \sum_{s'|s} \pi(s'|s,e) V^b(x'(s'),s') - V^o(s)\Big] \\
& + \nu_l \Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,e) V^l(x'(s'),s') - Z\Big] \\
& + x\varrho \Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} V^b(x'(s'),s') - v'(e)\Big] \\
& \le x\Big[U(a) + \beta \sum_{s'|s} \pi(s'|s,e) V^b(\widehat{x}'(s'),s')\Big] \\
& + \Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,e) V^l(\widehat{x}'(s'),s')\Big] \\
& + x\widehat{\nu}_b \Big[U(a) + \beta \sum_{s'|s} \pi(s'|s,e) V^b(\widehat{x}'(s'),s') - V^o(s)\Big] \\
& + \widehat{\nu}_l \Big[c_l + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s,e) V^l(\widehat{x}'(s'),s') - Z\Big] \\
& + x\widehat{\varrho} \Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} V^b(\widehat{x}'(s'),s') - v'(e)\Big] \Bigg\},
\end{aligned}
$$

*where* $\widehat{x}'(s') = \frac{1+\widehat{\nu}_b}{1+\widehat{\nu}_l} + \widehat{\varrho}\frac{\partial\pi(s'|s,e)/\partial e}{(1+\widehat{\nu}_l)\pi(s'|s,e)}$.

*Now, if we define the correspondence* $SP : A(x,s) \times \Gamma arrow A(x,s) \times \Gamma$ *by* $SP(a,\gamma) = (SP^a(\gamma), SP^\gamma(a))$ *one can show — given Lemma A.3) — that it is non-empty, convex-valued and upper hemicontinuous, as in Lemma 7A in Marcet and Marimon (2019) , which at this point applies Kakutani's fixed point theorem to prove the existence of solutions to the saddle-point contracting problem (Theorem 3).*

*In our case, this means that, with the additional (A.6), there is a contract satisfying equations (9)–(10), (13)–(15), (16), (19)–(20) and the following constraint qualification condition:*

$$
\varrho[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} V^b(x'(s'),s') - v'(e)] = 0, \tag{A.7}
$$

*with* $\varrho(x,s) = 0$ *if the term in brackets in* (A.7) *is non-zero. Now we show that* $\varrho(x,s) \neq 0$. *Suppose,* $\varrho(x,s) = 0$, *then (16) reduces to*

$$
\begin{aligned}
& \beta \sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^b(x'_{xs}(s'),s') - v'(e(x,s)) \\
& + \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)} \frac{1}{x} \frac{1}{1+r} \sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'),s') = 0.
\end{aligned}
$$

*Note that is nondecreasing in i for any non-negative* $\varrho(x,s)$. *Then we can rewrite*

$$
\sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'),s')
$$

*as*

$$
\sum_{i=1}^{N} \Delta V^h_{xs}(s'(i)) \Bigg[\sum_{j=i}^{\overline{N}} \frac{\partial \pi(s'|s,e(x,s))}{\partial e}\Bigg],
$$

*where $\Delta V^h_{xs}(s'(i))$ is defined by (A.2). Note that the first term is equal to zero in this summation. Monotone likelihood ratio implies that all other terms are non-negative as the terms in the bracket are strictly positive and $\Delta V^h_{xs}(s'(i)) \geq 0$ for all $i > 1$. The fact that we assume that some some risk sharing occurs in this economy (the lender's participation constrain is slack in at least one state realization) implies that some of the tems in the summation will be strictly posiive. Given that $\sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'), s')$ is positive, the first line must be negative, but that would violate (A.6), hence we reached a contradiction and $\varrho(x,s) > 0$ must be true. In sum, the contract satisfies all the conditions of Definition 2.*

***Step 3:*** The relaxed Fund Contract problem and the Fund Contract problem have the same solution. *This is the consequence of $\varrho(x,s) > 0$. Given this (A.6) implies that the incentive compatibility constraint is satisfied as equality in the relaxed Fund problem, hence the solution is equivalent to the original problem when tis constraint was introduced as equality.*

***Step 4:*** Uniqueness. *$FV$ is monotone in $x$, further it is constant either limited enforcement constraints are binding and concave when both are slack. The same contraction mapping argument used in Theorem 3 of Marcet and Marimon (2019) shows that $FV$ is unique. The strict concavity/convexity assumptions on $u, f$ and $v$ imply that the Recursive Contract allocation is unique and $FV$ strictly concave in $x$ whenever neither participation constraint is binding and, uniquely defined when either is binding. Therefore, the saddle-point solution is unique.*

Corollary 1 *It follows from the proof of Proposition 1. The* contraction mapping *guarantees that there is unique $FV(x,s)$ value function that is strictly concave in $x$ when neither participation constraint is binding and constant if either participation constraint is binding. By Step 3, the decomposition into $V^b(x,s)$ and $V^l(x,s)$ are also also unique and, given that the underlying strict monotonicity and concavity assumptions, both value functions are strictly monotone and $V^b(x,s)$ is strictly concave.*