LINMA2370 - Project (Part 2)
The Dynamics of Infectious Diseases

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Version 1.2

Introduction

In the first part of the project, you derived a simple "SIR" model. There exist many variants of these models in the literature, corresponding to different kinds of diseases or to different sets of assumptions. You also highlighted the properties of the quantity R_0, known as the *basic reproduction number*. For the record, the existence of such a threshold quantity governing the existence of an endemic state was only made possible thanks to mathematical modelling !

In this second part of the project, we turn our attention towards the analysis of this model and two of its variants taking into account (i) the vaccination of the population and (ii) the seasonality of the disease. For convenience, we will express the models in terms of new variables defined as $s = \frac{S}{N}$, $i = \frac{I}{N}$ and $r = \frac{R}{N}$.

Remark: For this part of the project, consider the *mathematical* existence of the equilibria even if they do not exist *physically*. For instance, an equilibrium outside the positive orthant should be taken into account in your mathematical analysis, although it has no physical meaning.

1 Analysis of an SIR model

Considering that $r(t) = 1 - s(t) - i(t)$, the model is rewritten as follows:

$$\begin{aligned} \dot{s}(t) &= \mu - \beta s(t) i(t) - \mu s(t) \\ \dot{i}(t) &= \beta s(t) i(t) - \gamma i(t) - \mu i(t) \end{aligned} \tag{1}$$

1. The positive orthant is a trivial invariant set of system (1). Find another invariant set $\mathcal{X} \subset \mathbb{R}^2_+$ for this system.
2. Provide the Jacobian matrix associated with system (1).
3. The *basic reproduction number* for system (1) is $R_0 = \frac{\beta}{\gamma+\mu}$. Using the answer to the previous question, characterize (hyperbolicity, type, etc.) the disease-free and the endemic equilibria
   (a) when $R_0 < 1$
   (b) when $R_0 > 1$
4. For the case $R_0 > 1$, use Lasalle's invariance principle to characterize the *asymptotic stability* of the endemic equilibrium.

   Hint. Consider the function $V : \Omega \to \mathbb{R} : V(s,i) = W_1(s - s^* \ln(\frac{s}{s^*})) + W_2(i - i^* \ln(\frac{i}{i^*}))$, where Ω has to be defined appropriately, (s^*, i^*) denotes the endemic equilibrium and W_1, W_2 are real positive constants to be chosen.

2 Analysis of an SIR model with vaccination

Vaccination consists in administrating a weakened (or harmless) version of the pathogen in order to stimulate the adaptive immunity of the human body. It remains one of the most effective way to prevent the spread of infectious diseases. For instance, one of the first vaccines was designed by the physician

Edward Jenner in 1796 in order to fight the smallpox epidemic. In 1979, the disease was finally eradicated.

Let us consider the vaccination of newborns as a strategy to avoid an epidemic. The specific constant rate of vaccination of the newborns is denoted by u, and we consider that vaccinated newborns are directly recovered (since they acquired permanent immunity). The model with vaccination is written as follows:

$$\begin{aligned} \dot{s}(t) &= \mu - u - \beta s(t)i(t) - \mu s(t) \\ \dot{i}(t) &= \beta s(t)i(t) - \gamma i(t) - \mu i(t) \\ \dot{r}(t) &= u + \gamma i(t) - \mu r(t) \end{aligned} \tag{2}$$

Again, we can analyze this system as a planar system by noticing that $r(t) = 1 - s(t) - i(t)$. We denote the fraction of the population being vaccinated as $p = \frac{u}{\mu}$, and we still have that $R_0 = \frac{\beta}{\gamma+\mu}$.

5. What is the fraction of the population $\bar{p}$ that needs to receive the vaccine in order to avoid the *physical* existence of an endemic equilibrium? Express your result as a function of R_0.

6. Analyze the bifurcation occuring at $\bar{p}$, with $R_0 > 1$, when varying u. How can you relate this result to the concept of "herd immunity[1]" ?

7. Provide a bifurcation diagram illustrating your answer to the previous question. Plot the equilibrium value i^* as a function of u. Use different "line styles" and an appropriate legend in order to indicate the type of equilibria that you show.

3 Analysis of an SIR model with seasonality

As stated before, many variants of the SIR model exist in the literature. Here, we will consider the case of "seasonal" diseases for which we can observe a periodic upsurge. Mathematically, this is translated into a time-varying transmision rate $\beta(t)$.

The model that we will consider in this last section is therefore the following:

$$\begin{aligned} \dot{s}(t) &= \mu - \beta(t)s(t)i(t) - \mu s(t) \\ \dot{i}(t) &= \beta(t)s(t)i(t) - \gamma i(t) - \mu i(t) \end{aligned} \tag{3}$$

Note that we do not consider the impact of vaccination here, and we still have that $r(t) = 1 - s(t) - i(t)$. Let us consider the particular case of a periodic transmission rate $\beta(t)$, expressed as

$$\beta(t) \quad = \quad \beta_0 \left[1 + \epsilon \cos\left(\frac{2\pi}{T} t \right) \right]$$

where ϵ denotes the amplitude of the seasonality and T its period.

8. Implement a function that will numerically simulate system (3) as it evolves from the initial condition `x0`, over a time period going from 0 to `Tend`. Provide a plot of the trajectory of the system in the s-i plane, for the following sets of parameters[2]:

μ [years^{-1}]	γ [years^{-1}]	β_0 [years^{-1}]	ϵ	T [years]	`Tend` [years]	`x0`
1/30	73	1200	0.01	1	20	$[0.06, 0.001]$
1/30	73	1200	0.05	1	20	$[0.06, 0.001]$
1/30	73	1200	0.1	1	20	$[0.06, 0.001]$

9. What do you observe for each set of parameters?

10. In the lecture notes, we cover two theorems that strongly restrict the possible dynamics of autonomous planar systems. Do your observations in the previous question follow this restricted behavior of the trajectories of a planar system? Why (not) ?

[1] In French: "immunité collective"

[2] Do not be surprised by these values: they have been chosen on purpose to illustrate some property of the modified system. Also, keep in mind that these specific values may be realistic for particular diseases, or for species other than human.

Practical information

You are asked to keep the **same group** as for the first part of the project, and to write a report (in French or in English) gathering your answers to the above questions. You should submit on Moodle a single zip file containing your report in `pdf` and your codes (Do not include your codes in your report!). The deadline for the submission is **December 4th 2020 at 11:55 pm**. Feel free to ask your questions on the dedicated forum on Moodle and to ask for an appointment with the teaching assistant in charge of the project (Benjamin Chiêm).