Traveling Salesman Problem Formulations

Sub-Tour Elimination Formulation (STE)

$$\min \sum_{e \in E} c_e x_e \tag{1}$$

$$\text{s.t.} \sum_{e \in \delta(\{i\})} x_e = 2 \quad \forall i \in V \tag{2}$$

$$0 \leq x_e \leq 1 \quad \forall e \in E \tag{3}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall S \subset V, 2 \leq |S| \leq |V| - 2 \tag{4}$$

$$x_e \in \{0, 1\} \quad \forall e \in E \tag{5}$$

Miller-Tucker-Zemlin Formulation (MTZ)

$$\min \sum_{i \neq j} c_{ij} x_{ij} \tag{6}$$

$$\text{s.t.} \sum_{i, i \neq j} x_{ij} = 1 \quad \forall j \in V \tag{7}$$

$$\sum_{j, i \neq j} x_{ij} = 1 \quad \forall i \in V \tag{8}$$

$$u_i - u_j + (|V| - 1)x_{ij} \leq |V| - 2 \quad \forall i, j \in V \setminus \{1\}, i \neq j \tag{9}$$

$$1 \leq u_i \leq |V| - 1 \quad \forall i \in V \setminus \{1\} \tag{10}$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j \in V, i \neq j \tag{11}$$

Question 1

We recall the definition of $\bar{x}$ (from the statement) as

$$\bar{x} = \arg \min_{x \in \mathbb{R}^{|E|}} \sum_{e \in E} c_e x_e \text{ s.t.} \sum_{e \in \delta(\{i\})} x_e = 2 \quad \forall i \in V \tag{12}$$

$$0 \leq x_e \leq 1 \quad \forall e \in E \tag{13}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \text{for a subset of all possible } S \subset V \tag{14}$$

In order to answer this question, we will prove that solving the separation problem is equivalent to solve min cut problems. Since the min cut optimization problem is the dual of the max flow problem (max-flow min-cut theorem, section 3.4 of the course notes), considering min cut problems is equivalent to considering max flow problems.

We seek to prove that

$$\begin{gathered} \exists S \subset V \text{ (with } 2 \leq |S| \leq |V| - 2) \text{ such that } \bar{x} \text{ violates } \left(\sum_{e \in \delta(S)} \bar{x}_e \geq 2 \right) \\ \Longleftrightarrow \\ \exists s, t \in V,\ s \neq t \text{ such that min cut}(s, t) < 2 \end{gathered} \tag{15}$$

We will deal with both implications separately in order to prove the equivalence (15).

$\boxed{\Longrightarrow}$ Since it exists some subset $S \subset V$ with $2 \leq |S| \leq |V| - 2$ (thus $S \neq \emptyset$ and $S \neq V$) such that $\sum_{e \in \delta(S)} \bar{x}_e < 2$, then for any $s \in S$ and for any $t \in V \setminus S$ (implying $s \neq t$), the weight of min cut(s, t) will be less than 2 since we can build a minimum cut candidate taking edges in $\delta(S)$. Indeed every other candidate will be either worse or with weight less than 2.

$\boxed{\Longleftarrow}$ Since it exists a node s in some subset S and t not in S (since that $s \neq t$) such that the bipartition $(S, V \setminus S)$ is a cut with min cut$(s, t) < 2$, we know for this S that $\sum_{e \in \delta(S)} \bar{x}_e =$ min cut$(s, t) < 2$ (implying $2 \leq |S| \leq |V| - 2$ since $0 \leq \bar{x}_e \leq 1$ for any $e \in E$ following equation (13)). Thus S from the bipartition $(S, V \setminus S)$ exists and is such that $\left(\sum_{e \in \delta(S)} \bar{x}_e \geq 2 \right)$ is not true.

Moreover, we want to prove that

$$\begin{gathered} \exists s, t \in V,\ s \neq t \text{ such that min cut}(s, t) < 2 \\ \Longleftrightarrow \\ \exists t_2 \in V \setminus \{1\} \text{ such that min cut}(1, t_2) < 2 \end{gathered} \tag{16}$$

And we do the same reasoning by proving each implication.

$\boxed{\Longrightarrow}$ It is equivalent to prove the contrapositive :

$$\forall t_2 \in V \setminus \{1\} : \text{min cut}(1, t_2) \geq 2 \implies \forall s, t \in V,\ s \neq t : \text{min cut}(s, t) \geq 2 \tag{17}$$

Proposition (17) can be proved by contradiction. By hypothesis, we have min cut$(1, t_2) \geq 2$ $\forall t_2 \in V \setminus \{1\}$. We suppose $\exists \tilde{s}, \tilde{t} \in V$, $\tilde{s} \neq \tilde{t}$ such that min cut$(\tilde{s}, \tilde{t}) < 2$. We denote $\tilde{S}^*$ the subset such that the bipartition $(\tilde{S}^*, V \setminus \tilde{S}^*)$ is a cut with min cut$(\tilde{s}, \tilde{t}) < 2$ $(\tilde{s} \in \tilde{S}^*, \tilde{t} \in V \setminus \tilde{S}^*)$. We construct S_{cand} such that $S_{cand} = \tilde{S}^*$ if $1 \in \tilde{S}^*$, otherwise $S_{cand} = V \setminus \tilde{S}^*$ and we take $t_2 \in V \setminus S_{cand}$.

The min cut(s, t) problem can also be defined as finding a subset S with $s \in S$ and $t \in V \setminus S$ with the cut $(S, V \setminus S)$ such that $\sum_{e \in \delta(S)} \bar{x}_e$ is minimal. With this definition, we see that S_{cand} is a candidate (feasible solution) of the min cut$(1, t_2)$ problem. Since it is a minimization problem, we know that the minimum of the objective function is smaller or equal to the objective function evaluated at any feasible solution. Thus, we have that min cut$(1, t_2)$ $\leq$ min cut$(\tilde{s}, \tilde{t}) < 2$ because S_{cand} is a feasible solution of min cut$(1, t_2)$ and S_{cand} is the optimal solution of min cut$(\tilde{s}, \tilde{t})$. That leads to a contradiction since we found some t_2 such that min cut$(1, t_2) < 2$.

$\boxed{\Longleftarrow}$ By construction, we take $s = 1$ and $t = t_2$.

Since (15) and (16) are proved, we know that

$$\begin{gathered} \exists S \subset V \text{ (with } 2 \leq |S| \leq |V| - 2) \text{ such that } \bar{x} \text{ violates } \left(\sum_{e \in \delta(S)} \bar{x}_e \geq 2 \right) \\ \Longleftrightarrow \\ \exists t \in V \setminus \{1\} \text{ such that min cut}(1, t) < 2 \end{gathered} \tag{18}$$

Thus, we propose the following method in order to solve the separation problem.

- Initialization : $\bar{x}$ is the optimal solution for an empty subset of possible $S \subset V$.
- Repeat :
  - Try to find $t \in V \setminus \{1\}$ a cut such that min cut$(1, t) < 2$ with the associated cut $(S, V \setminus S)$
    - if this does not exist, then $\bar{x}$ is feasible for any S since $\boxed{\Longrightarrow}$ in equation (18), separation problem is **solved**.
    - else, $\sum_{e \in \delta(S)} \bar{x}_e < 2$ since $\boxed{\Longleftarrow}$ in equation (18).
  - Add the constraint $\left(\sum_{e \in \delta(S)} \bar{x}_e \geq 2 \right)$.
  - Update $\bar{x}$.

Question 2

Cut and Branch The restricted method of the Branch-and-Cut consists of adding constraints in the root node of the Branch-and-Bound algorithm using the method explained above. Afterwards, we start branching in order to compute the IP solution.

Separation problem As explained above, we used min cut problems to solve the separation problem. If each min cut is greater than 2, the cutting phase is finished. Otherwise, we take the cut $(S, V \setminus S)$ with the minimal min cut (i.e. we select the cut $(S, V \setminus S)$ associated to the target $t \in V \setminus \{1\}$ such that min cut$(1, t)$ is the minimal min cut above all possible targets) and than we add the constraint (14) for this subset S and compute the new optimal LP solution.

In order to solve the min cut problem, for any target $t = 2, \ldots, |E|$, with respect to the current considered LP solution $\bar{x}$, we solve the LP problem

$$\min_{w \in \mathbb{R}^{|E|}, u \in \mathbb{R}^{|V|}} \sum_{e \in E} w_e \bar{x}_e \text{ s.t. } u_i - u_j + w_{e(i,j)} \geq 0 \quad \forall i, j \in V \tag{19}$$

$$u_t - u_1 \geq 1 \tag{20}$$

$$w_e \geq 0 \quad \forall e \in E \tag{21}$$

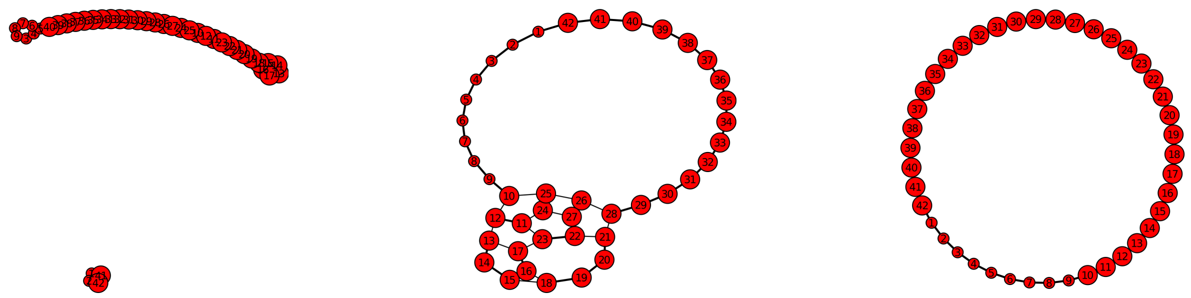
with $e(i, j)$ the edge from node i to node j or inversely. The cut $(S, V \setminus S)$ is given by values of u, which are integers (see section 3.4 in the course notes).

Number of added constraints We only added 7 constraints to make the LP relaxation solution feasible regarding constraint (4) (instead of $2^{|V|} - 2|V| - 2 = 4398046511018$). Thus, we made 328 min cut problems $((7 + 1) \times (|V| - 1))$. Initially, we tried to minimize the number of min cut problems and we succeed to make the LP relaxation solution feasible with 129 min cut problems but we had to add 9 constraints instead of 7.

Analysis of the formulation Table 1 shows relevant information about the Sub-Tour Elimination Formulation. Figure 1 shows the evolution of the solution before the cutting phase, after the cutting phase and after the Branch-and-Bound algorithm.

	Sub-Tour Elimination
LP relaxation objective value	Before cutting phase : 641.00 After cutting phase : 697.00
Number of sub tours	Before cutting phase : 2 After cutting phase : 1
Number of explored nodes in B&B	8
Depth of the B&B tree	3
GAP between LP relaxation and IP objective value	Before cutting phase : 8.30% After cutting phase : 0.29%
IP relaxation objective value	699

Table 1: Solution information of TSP problem with Sub Tour Elimination Formulation



(a) LP relaxation solution (b) Solution after cut (c) IP solution

Figure 1: Evolution of the optimal solution

Question 3

Table 2 shows relevant information about the Sub-Tour Elimination Formulation and Miller-Tucker-Zemlin Formulation.

	Sub-Tour Elimination	Miller-Tucker-Zemlin
LP relaxation objective value	Before cutting phase : 880.00 After cutting phase : 937.00	835.48
Number of sub tours in the LP relaxation solution	Before cutting phase : 4 After cutting phase : 1	4
Number of explored nodes in B&B	0	6880
Depth of the B&B tree	0	42
GAP between LP relaxation and IP objective value	Before cutting phase : 6.08% After cutting phase : 0.00%	10.83%
IP objective value	937	937

Table 2: Solution information of TSP problem with 2 Formulations

As we can observe the Sub-Tour Elimination Formulation seems more efficient than the Miller-Tucker-Zemlin Formulation. Indeed, we only had to add 6 constraints (and thus solve 175 min cut problems) to obtain an LP relaxation feasible solution. In this case, this solution is the IP optimal solution and there is thus no node to explore in the Branch-and-Bound algorithm. With the Miller-Tucker-Zemlin Formulation, there are several thousand of nodes to explore during the Branch-and-Bound. If we look at the Branch-and-bound algorithm, we found an upper bound and lower bound of 937 at the first node with the Sub-Tour Elimination Formulation while the first lower bound of the Miller-Tucker-Zemlin Formulation is 835.48 and the first founded upper bound is 1132 at the 143th explored node.

Also, there are several IP optimal solutions. Indeed there are at least 3 paths with the same cost for the Traveling Salesman Problem with these 26 cities since we observe different solutions (1) after the cutting phase of the STE formulation, (2) after the Branch-and-Bound of the STE formulation and (3) after the Branch-and-Bound of the MTZ formulation. These solutions are shown at the end of the notebook `HW2.ipynb`.