LINMA 2471 – Optimization models and methods II

Homework assignment II – Portfolio optimization

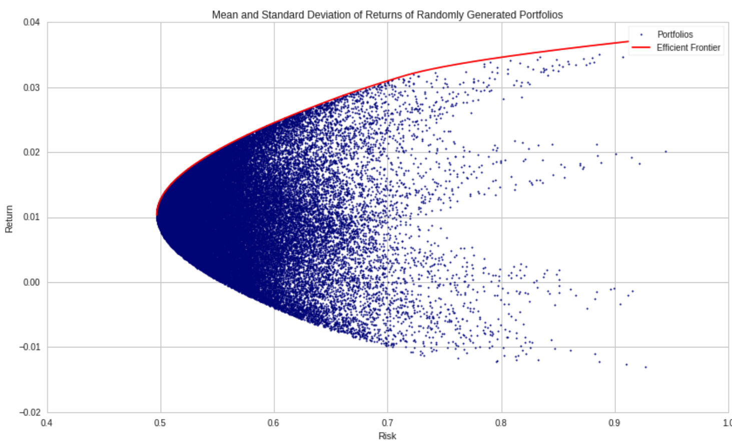
[v1.1]

We consider the classical *Markowitz* model of portfolio optimization. Assume the stock market offers the possibility to buy n different types of assets. Given a total budget $B > 0$ (in euros), we need to pick which amount of each asset to include in a portfolio: letting x_i the budget invested in asset i (in euros) we have the constraints $\sum_{i=1}^{n} x_i = B$ and $x_i \geq 0$ for all i.

The quality of a portfolio can be estimated from two differents points of view:

- Overall return: we want to maximize the expected return of the whole portfolio after a given period of time, say one year. For each asset i, we assume the knowledge of the expected return (i.e. relative price variation of the asset after one year), contained in R_i, and the overall return of the portfolio will be equal to $\sum_{i=1}^{n} R_i x_i = R^T x$.
- Associated risk: a classical way to measure the risk associated to a portfolio is to use the variance of its return. Assuming we know the variance-covariance matrix of the returns Σ, a symmetric $n \times n$ matrix (which we denote by $\Sigma \in \mathbb{S}^n$), that variance is equal to $x^T \Sigma x$ and needs to be minimized.

We have thus a bi-criteria optimization problem: maximize the return of the portfolio while minimizing the associated risk. On the following diagram those two criteria have been plotted for a large number of (random) portfolios (each point is a portfolio, x-axis is risk and y-axis is return):



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A. Preliminaries

A1. Suppose the variance-covariance matrix is estimated using averages over past data, which means it is given by the formula $\Sigma = \frac{1}{N} \sum_{i=1}^{N} (D_i - \bar{D})(D_i - \bar{D})^T$, where D_i is the vector of returns for all assets at time i, and $\bar{D} = \frac{1}{N}\sum_{i=1}^{N} D_i$ is the average of theses vectors (i.e. vectors $D_i - \bar{D}$ contain the deviations of the returns from their averages).

What useful property does the matrix Σ necessarily possess ?

(in addition, for the rest of the assignment, you can assume for simplicity that Σ is nonsingular)

A2. Consider the problem of maximizing the return without any consideration for the risk. Is the problem convex ? Explain how to compute its optimal solutions. Where can you find them on the above diagram ?

B. Risk-variance trade-off

Suppose that, given a target return T, we want to find a portfolio achieving exactly that return with the associated risk as small as possible.

B1. Is the problem convex ? How can you find it solution (depending on T) on the diagram ? Is the solution for a given T necessarily unique ?

B2. Give a theoretical explanation for the parabolic border that is visible on the diagram, and compute its equation (minimum risk in terms of T) involving the data parameters R and Σ.

Hint : relax the nonnegativity constraint on x_i (what effect does this have ?), and solve the optimality conditions for the problem.

C. Markowitz model

C1. The so-called Markowitz model, for which Harry Markowitz received the Nobel prize in economics in 1990, considers the maximization of a linear combination of the risk and overall return. Given a positive parameter μ, it consists in minimizing the objective $x^T\Sigma x - \mu R^T x$. Is the problem convex ? How can you find its solutions (depending on μ) on the diagram ?

C2. Derive the Lagrangian dual problem of the above minimization problem. Simplify the dual function as much as possible, and comment the resulting formulation.

D. Sharpe ratio

Finally, we consider the following different way of scalarizing the bi-criteria portfolio optimization problem. Assume that a risk-free return $r > 0$ is available independently (i.e. on a standard savings account), so that what really matters is the (hopefully positive) difference between the overall return and this risk-free return. One wishes now to maximize this difference divided by the standard deviation of the portfolio's return, where the standard deviation is the square root of the variance ; this leads to an objective equal to $\dfrac{R^T x - r}{\sqrt{x^T\Sigma x}}$.

D1. Is the problem convex ?

D2. Find a convex reformulation of the problem. *Hint* : use homogenization.

Optional: how can you find an optimal solution (depending on r) on the diagram ?

Changelog. 2020-10-23: [v1] initial publication. 2020-10-23: [v1.1] a few typos corrected, second part of D1 moved to "optional". 2020-11-03: [v1.2] reworded to indicate that $\mathbb{S}^n$ is the set of symmetric $n \times n$ matrices, and corrected typo in definition of Σ in A1 (sum until index reaches N, not n).