1 Introduction

The first claim of this work is to approximate the optimal path for a baseball player to run through all bases on the field. We will do this with two different models, using discretized physics or continuous physics with a certain assumption on the acceleration. We will then implement these optimization problems with `AMPL` in order to solve them.

The second goal is to analyse our results, especially we want to higlight the limits of such an optimal solution in comparaison to the real optimal path.

Finally, we want to analyse our models. In order to do that, we will look at the convexity of both optimization problems and we will observe their accuracy and their sensitivity with respect to different parameters.

2 Models and optimization problems

2.1 Discretized physics

In order to lighten the equations, we define $k' := \left\lfloor \dfrac{k}{K} \right\rfloor \in \{0,..,3\}$ (with $\lfloor \cdot \rfloor$ the integer division, k and K are defined below).

We consider $p_k, v_k, a_k \in \mathbb{R}^2$ respectively position, speed and acceleration at time $t_k = \left(\sum_{j=0}^{k'-1} K h_j \right) + (k - k'K) h_{k'}$ with :

- $k \in \{0, ..., 4K\} \subset \mathbb{N}$, where $K \in \mathbb{N}$ is the number of subsegments for each segment between two bases,
- $h_i \in \mathbb{R}$, where $i \in \{0,..,3\}$ is the time interval to run through a subsegment of segment i.

We want to solve the problem for an initial speed v_{initial} and an initial acceleration a_{initial}.

Since K is a parameter, the variables of our optimization problem are positions p_k, speeds v_k, accelerations a_k (two components for each k) and time intervals h_i. Therefore, we have $6(4K+1)+4$ scalar variables.

We know that the objective function is the total time to go through all bases and go back to the initial point since we want to minimize it.

We also define b_j for $j \in \{0,..,3\}$ respectively position of the initial base and the three other bases on the field.

The optimization problem becomes :

$$
\begin{aligned}
\min_{\substack{p_k, v_k, a_k, h_i \\ k \in \{0,...,4K\} \\ i \in \{0,..,3\}}} \quad & K \sum_{i=0}^{3} h_i \\
\text{s.t.} \quad & ||a_k||_2 \leq 10 \quad (\forall k \in \{0, ..., 4K\}) \\
& v_0 = v_{\text{initial}} \\
& a_0 = a_{\text{initial}} \\
& p_0 = b_0 \\
& p_K = b_1 \\
& p_{2K} = b_2 \\
& p_{3K} = b_3 \\
& p_{4K} = b_0 \\
& v_{k+1} = v_k + h_i a_k \quad (\text{for } i = k' \quad \forall k \in \{0, ..., 4K-1\}) \\
& p_{k+1} = p_k + h_i v_k \quad (\text{for } i = k' \quad \forall k \in \{0, ..., 4K-1\})
\end{aligned} \tag{1}
$$

2.2 Continuous physics

The continuous physics problem is almost the same as the discretized physics, except that the acceleration is partly continuous on a certain interval, and speed and position are continuous functions. Let us compute them.

First, we will look at the acceleration function $a(t)$, continous by parts. Hence, for all $t \in [t_k, t_{k+1}[$:

$$a(t) = a_k \tag{2}$$

Then, we can compute the speed function $v(t)$ since we know the equation bonding speed and acceleration. Therefore, for all $t \in [t_k, t_{k+1}[$:

$$\begin{aligned} v(t) &= v(t_k) + \int_{t_k}^{t} a(t)\mathrm{d}t \\ &= v(t_k) + a(t_k)(t - t_k) \end{aligned} \tag{3}$$

Since we know that the function is continous on the entire domain $\left[0, K\sum_{i=0}^{3} h_i\right]$, then we have that $v(t_{k+1}) = \lim\limits_{t \to t_{k+1}} v(t)$. Hence :

$$\begin{aligned} v_{k+1} &= v(t_{k+1}) \\ &= v(t_k) + a(t_k)(t_{k+1} - t_k) \\ &= v_k + a_k h_{k'} \end{aligned} \tag{4}$$

After that, we can compute the position function $f(t)$ since we also know the equation bonding position and speed. Therefore, for all $t \in [t_k, t_{k+1}[$:

$$\begin{aligned} p(t) &= p(t_k) + \int_{t_k}^{t} v(t)\mathrm{d}t \\ &= p(t_k) + \int_{t_k}^{t} [v(t_k) + a(t_k)(t - t_k)]\mathrm{d}t \\ &= p(t_k) + v(t_k)(t - t_k) + \frac{a(t_k)}{2}(t^2 - t_k^2) - a(t_k)(t - t_k)t_k \\ &= p(t_k) + v(t_k)(t - t_k) + \frac{a(t_k)}{2}(t - t_k)^2 \end{aligned} \tag{5}$$

We also know that this function is continuous on $\left[0, K\sum_{i=0}^{3} h_i\right]$, then we also have $p(t_{k+1}) = \lim\limits_{t \to t_{k+1}} p(t)$:

$$\begin{aligned} p_{k+1} &= p(t_{k+1}) \\ &= p(t_k) + v(t_k)(t_{k+1} - t_k) + \frac{a(t_k)}{2}(t_{k+1} - t_k)^2 \\ &= p_k + v_k h_{k'} + \frac{a_k}{2} h_{k'}^2 \end{aligned} \tag{6}$$

Therefore, the optimization problem using continuous physics with the assumption made above on the acceleration is almost the same as with discretized physics (equation (1)), except for the last (set of) constraints. They become :

$$p_{k+1} = p_k + h_i v_k + \frac{h_i^2}{2} a_k \quad (\text{for } i = k' \quad \forall k \in \{0, ..., 4K - 1\}) \tag{7}$$

3 Results and comments

Note that the following results were obtained with $v_{\text{initial}} := 0$ and $a_{\text{initial}} := 0$.

3.1 Convex optimization problem

In order to know more about our models and to choose the solver we want in `AMPL`, we will look at the convexity of the problem : are our optimization problems (with discretized and continuous physics) convex ?

Recall that a minimization optimization problem is convex if and only if (with $x \in \mathbb{R}^n$ the vector of the n variables) :

- The objective function $c(x)$ is convex
- The feasible region is convex :

- The functions in inequality constraints (e.g. constraints of shape $g_i(x) \leq 0$ for the i-th constraint) are convex
- The functions in equality constraints (e.g. constraints of shape $f_i(x) = 0$ for the i-th constraint) are linear

Hence, if some function $f_i(x)$ in equality constraints is not linear, then our optimization problem is not convex.

In our problems, the objective function $c(x) = K\sum_{i=0}^{3} h_i$ (with $x \in \mathbb{R}^{6(4K+1)+4}$ the vector containing all variables) is convex. Initial acceleration and speed constraints are linear, as well as positions at the bases. Finally, in the set of inequality constraints, the functions are convex since, for all $k \in \{0, ..., 4K\}$:

$$||a_k||_2 \leq 10 \iff ||a_k||_2^2 \leq 100 \iff a_{k,x}^2 + a_{k,y}^2 - 100 \leq 0 \tag{8}$$

With $a_{k,\beta}$ the β-component of the acceleration a_k at a certain time. We see that these functions are the sum of two parabolas and one negative scalar, wich are convex.

However, in the first problem (equation (1) - discretized physics), we have :

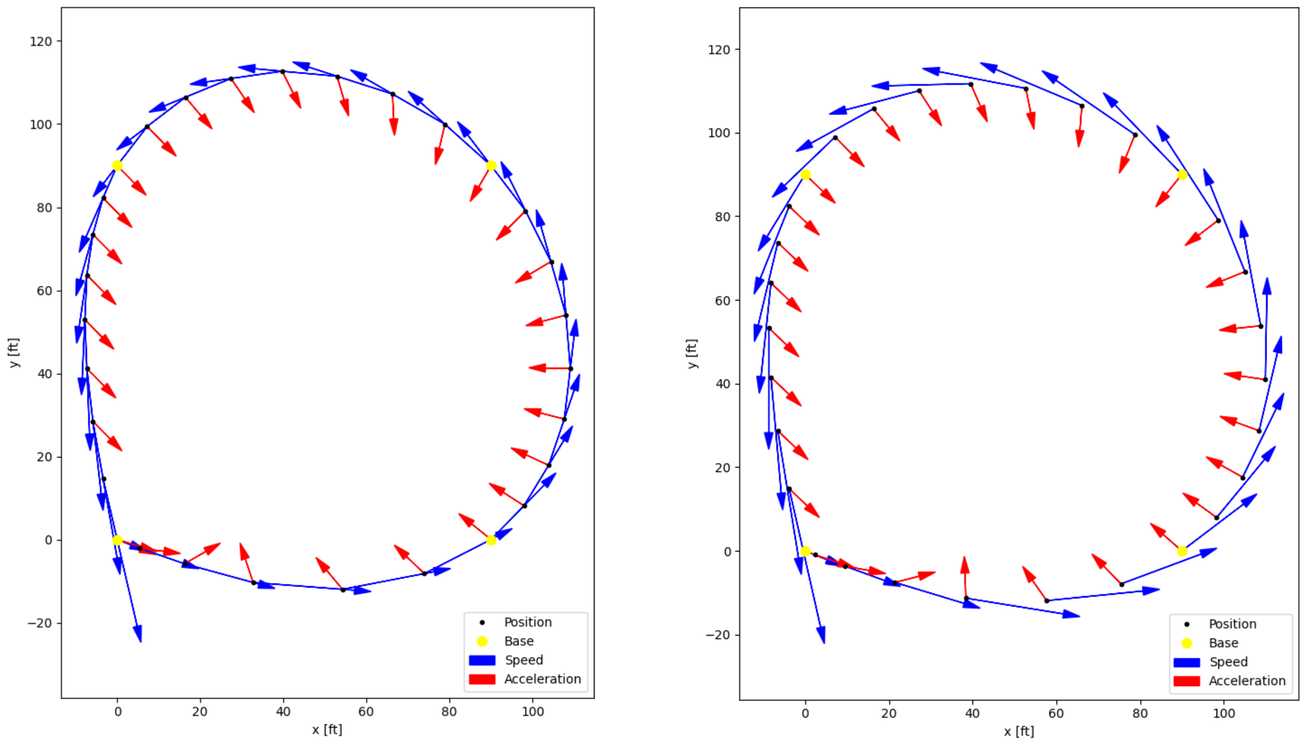
$$\begin{aligned} f_j(x) &= v_{k+1} - v_k - h_i a_k \quad (\text{for } i = k', j = 9 + k \quad \forall k \in \{0, ..., 4K-1\}) \\ f_j(x) &= p_{k+1} - p_k - h_i v_k \quad (\text{for } i = k', j = 9 + 4K + k \quad \forall k \in \{0, ..., 4K-1\}) \end{aligned} \tag{9}$$

And, in the second problem (continuous physics), we have :

$$f_j(x) = p_{k+1} - p_k - h_i v_k - \frac{h_i^2}{2} a_k \quad (\text{for } i = k', j = 9 + 4K + k \quad \forall k \in \{0, ..., 4K-1\}) \tag{10}$$

In both situations, functions f_j for $j \in \{9, ..., 8+8K)\}$ are not linear. Therefore, our optimization problems are not convex : we must choose a solver in `AMPL` that deals with non-linear and non-convex constraints. Moreover, we know that we do not have the guarantee that the solution is a global minimum.

3.2 Feasibility and exactitude of the optimal value provided by the model



(a) Discretized physics (b) Continuous physics

Figure 1: Spatial representation of position, speed and acceleration (discretization with 32 points)

The second model is closer to reality because constraints between consecutive discretized points take physical laws of movement into account. Indeed, in our first model, on one hand the speed is piecewise constant for the computation of the position but on the other hand, the speed on a segment is a linear function of the acceleration. In the second model, the computation of the position takes the speed variation into account. We can see this difference in Figure 1 : for the first model, the direction of the speed vector of one position passes through the next position while for the second model, the direction will still change between two consecutive positions.

Moreover, our models do not exactly reproduce the reality because we observe there is no feasable solution when we take one or two discretized points per segment for the first model or one discretized point per segment for the second model. Indeed, with zero initial speed and acceleration, the position at time t_2 for the first model (and time t_1 for the second model) is still the initial base so the position could not be the first base, for any value of h' :

$$\begin{aligned} &\text{(discretized physics)} \quad p_2 = p_1 + h'v_1 = (p_0 + h'v_0) + h'(v_0 + h'a_0) = p_0 \\ &\text{(continuous physics)} \quad p_1 = p_0 + h'v_0 + \frac{(h')^2}{2}a_0 = p_0 \end{aligned} \tag{11}$$

If the number of discretized points is higher, the solver will most of the time converges to a solution but the position at t_1, thus after h_0 seconds, will still be the initial base. In reality, a baseball player will almost immediately leave the initial base.

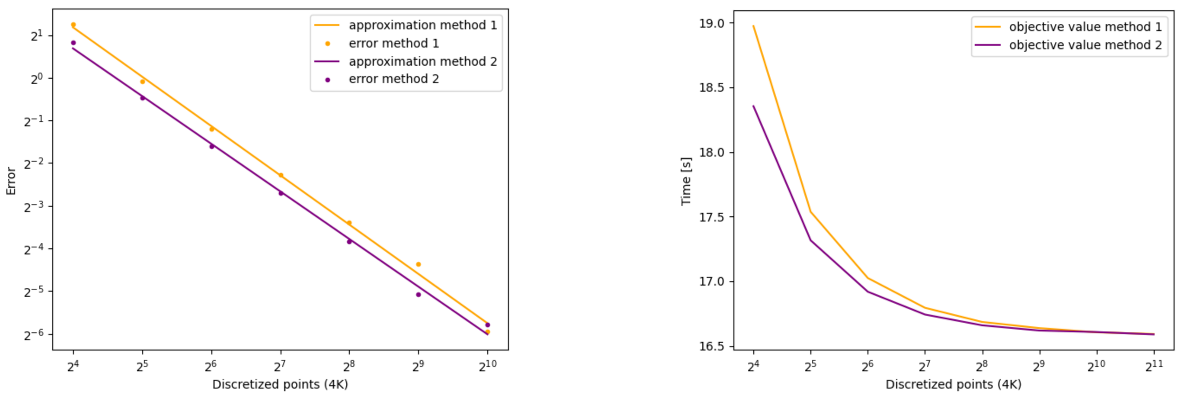
On top of that, these simplified models reflect a "perfect" physical world, solution are not exactly feasable for the problem because there is, for example :

- no friction forces by the wind, the ground, etc.
- no real human constraints on acceleration and speed (e.g. we could have a speed of 50 [ft/s] for a model where only acceleration is bounded while the record of Usain Bolt is 40 [ft/s])

Therefore, our models do not compute the exact optimal value for this problem. From a physical point of view, our models provide a lower bound on the optimal value because they do not take some physical phenomena (which limit the progress of the runner) into account. From a mathematical point of view, if the models corresponded exactly to the reality, they will provide an upper bound on the optimal value. Indeed, our problem is not convex thus there could be other local minima more optimal than the solution computed. In addition, more there are discretized points, more the model tends to approach the real continuous problem. Thus for this problem, with approximately the same trajectory, the objective value will decrease with the number of discretized points and it is always possible to provide a better solution.

3.3 Accuracy

It is interesting to analyse the evolution of the discretization. As explained in section 3.2, with high number of discretized points, our discretized models tends to approach a continuous model.



(a) Error (and approximations of it)

(b) Evolution of optimal solution

Figure 2: Accuracy of the models with respect to the number of sub-segments

Figure 2 shows for different numbers of discretized points (a) the difference between the objective value and the best objective value computed by our models (16.58 seconds), which is the best approximation of the error we can provide, (b) the objective value.

As we can see, solutions provided by the second model are better that those provided by the first model for the same number of discretized points. The addition of the term depending on the acceleration in constraints for the position allows to better adapt the trajectory of the runner and allows to start moving at time t_1 and not at t_2 as for the first model (c.f. section 3.2).

We computed a linear regression (in logarithmic scales) of the error and it approximately seems to be inversely proportional to the number of discretized points in Figure 2a.

3.4 Sensitivity with respect to the parameters

3.4.1 Maximal acceleration

Even if we do not know the real maximal acceleration, we can suppose that the values that we choose here (in Table 1) are not possible for a human. Here, we deviate from the physical feasibility in order to observe some interesting behaviors of the chosen `AMPL` solver.

Maximal acceleration [ft/s^2]	40	45	50
Objective value [s]	8.50	7.91	9.25

Table 1: Optimal solution found by the solver for different maximal accelerations

We observe on Table 1 that the optimal total time (the solution found by the solver) increases when we go from $a_{\max} = 45$ [ft/s^2] ($a_{\max}$ is the maximal acceleration allowed) to $a_{\max} = 50$ [ft/s^2] while we allow the runner to reach a larger acceleration. The conclusion is that there might be a problem with the solver.

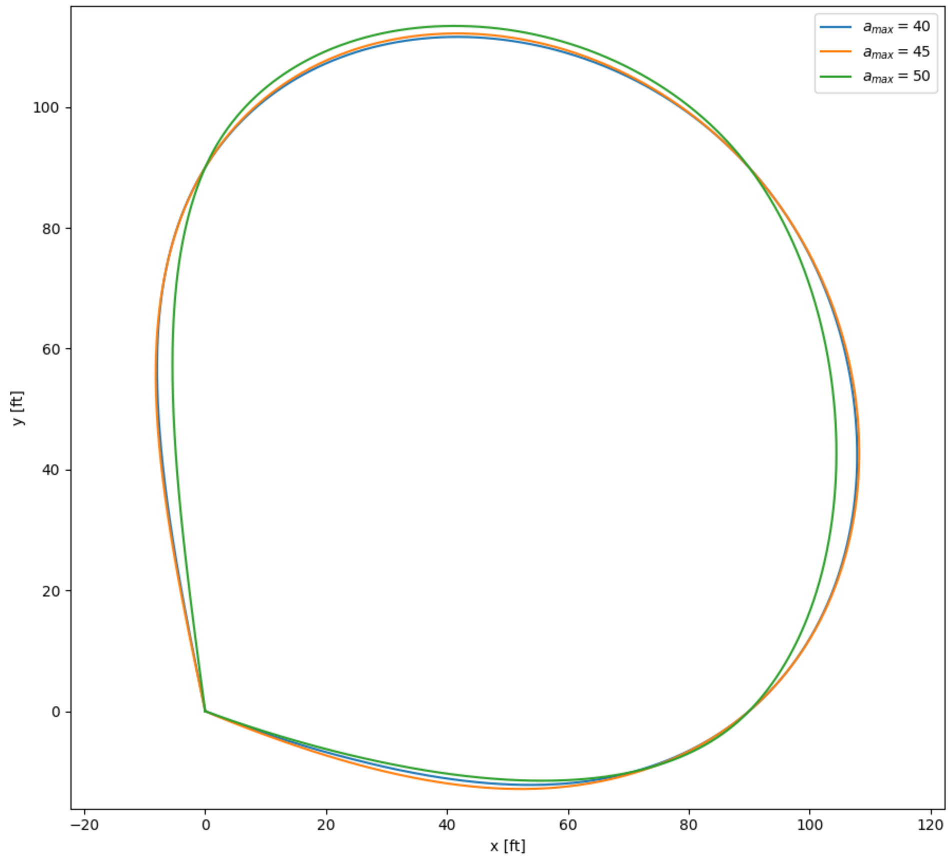


Figure 3: Optimal trajectory found by the solver for different maximal accelerations (using method 2)

We can take as a hypothesis that, since our optimization problems are not convex (c.f. section 3.1), then maybe the optimal solutions found by the solver are not the global minimum (we do not have any guarantee that the solution is global). Then, we could explain the observation made above by the fact that the solver "switches" from one local minimum (that may be global) to another (see Figure 3). This one cannot be global since the previous solution was better and the increasing of maximal acceleration does not prevent to reproduce the same optimal trajectory.

Figure 3 corroborates the hypothesis made above since we see that the optimal paths for $a_{\max} = 40$ [ft/s^2] and $a_{\max} = 45$ [ft/s^2] are significantly different from the optimal path for $a_{\max} = 50$ [ft/s^2].

3.4.2 Number of discretized points

As explained in sections 3.2 and 3.3, theoretically, more there are discretized points, more the optimal time will be high.

K (number of discretized points per segment)	2^3	2^6	2^9	2^{12}
Objective value [s]	17.32	16.66	16.59	21.33

Table 2: Optimal solution found by the solver for different number of sub-segments

We observe on Table 2 that the optimal total time (the solution found by the solver) drastically increases when we go from 2^{11} to 2^{14} discretized points (2^9 to 2^{12} for the value of parameter K). Given that the optimal solution of the model with 2^{11} discretized points can be reproduce with the model with 2^{14} discretized points, there difference needs to come from the solver.

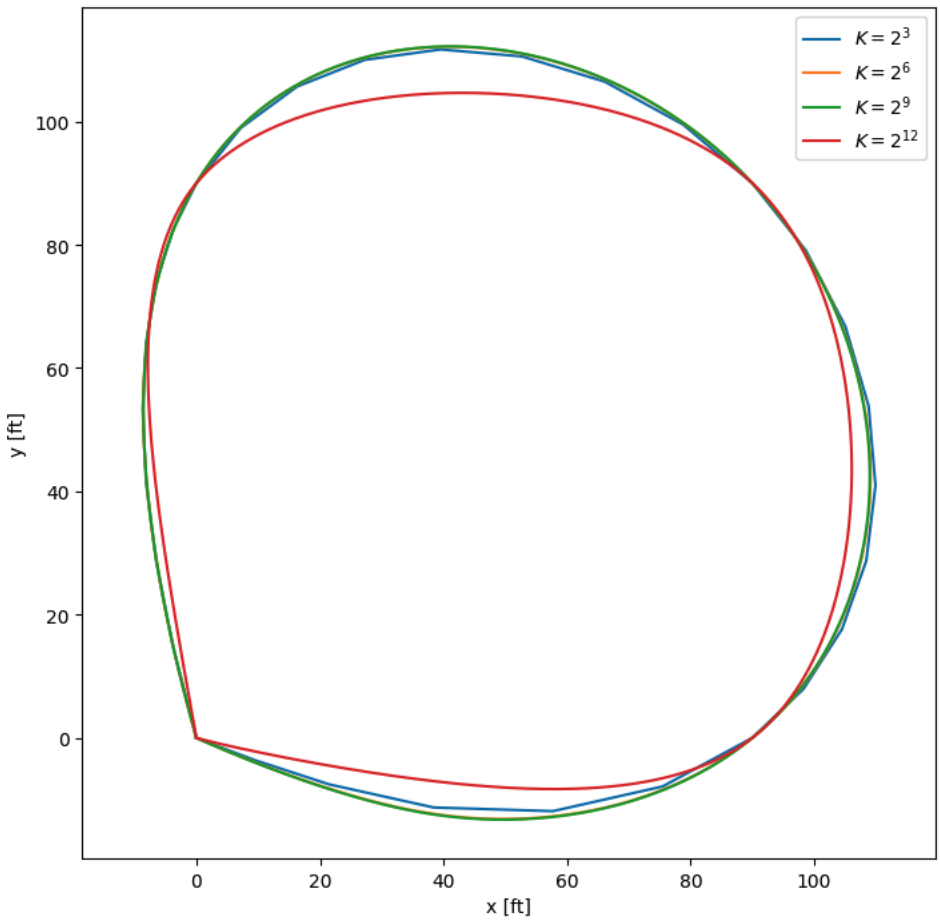


Figure 4: Optimal trajectory found by the solver for different number of sub-segments (using method 2)

We can take the same hypothesis as for the different maximal accelerations : since our problem is not convex (c.f. section 3.1), the solver could converge to different local minima depending on the parameters. It could explain the increasing of the optimal solution for a discretization with more points. As in section 3.4.1, we can observe this on Figure 4.

3.4.3 Initial acceleration and speed

Two other parameters are the initial acceleration and speed. We consider here zero initial acceleration and speed because it seems coherent to start at zero when the baseball player starts running. The player does not have initial speed after hitting the ball and he does not have initial acceleration (initial force) in his legs.

We could also analyse the optimal trajectory and optimal objective value with nonzero initial parameters but it is obvious that having initial acceleration or/and speed in the direction of optimal trajectory with initial zero parameters will take less time to achieve the bases tour. Conversely, if acceleration or/and speed are not in the direction of optimal trajectory with initial zero parameters, it will take more time and the optimal trajectory will be adapted.

We made the hypothesis that the solver could converge to different local minima due to the fact that our problem is not convex (c.f. section 3.1). Thus, as for the two previous parameters discussed, with more benefic parameters the solver could also find an optimal trajectory which is less optimal than a previous one.

4 Conclusion

The best time computed by our models is 16.58 seconds to go through the four bases (with the second model and 2^{11} discretized points) which is less that the 16.7 seconds provided by the mathematical article with the same conditions. However, as explained in this report, this value is provided by our models which are the discretization of a continuous problem and are simplified without some physical phenomena.