1 Problem considered

For some function $u : \Omega \to \mathbb{R}$, consider the following diffusion problem:

$$\begin{aligned}\frac{\partial u(t,x,y)}{\partial t} &= \alpha \left(\frac{\partial^2 u(t,x,y)}{\partial x^2} + \frac{\partial^2 u(t,x,y)}{\partial y^2} \right), \\ u(0,x,y) &= u_0(x,y).\end{aligned} \tag{1}$$

Equation (1) is a special case of the diffusion problem considered in the statement, with $D(x,y) = \alpha$ and $f(x,y) = 0$ for all x, y. We consider the following values for the problem:

- $\Omega = [0,300] \times [0,50] \times [0,50]$,
- $\alpha = 2$,
- $u_0 : [0,50] \times [0,50] \to \mathbb{R}$ defined as: for all $y \in [0,50]$ $u_0(0,y) = 100$ and for all $(x,y) \in (0,50] \times [0,50]$, $u_0(x,y) = 0$.

Note that a physical way to interpret Equation (1) is to consider the diffusion of *heat* on a 2D plate of dimensions 50[m] $\times$ 50[m] during 300[s], from an intial state. $u(t,x,y)$ is thus the temperature in [K] at time t[s] at the point at x[m] of the origin of the plate in one direction and y[m] of the origin in the other direction[1]. The intial state is when only the upper part of the plate has temperature 100[K].

Consider that, with the same notations as in the statement, we choose somes values of Δt, Δx, Δy, then the explicit scheme is numerically stable if [2, Equations (3.5), (3.6)]:

$$\frac{\alpha \Delta t}{\Delta x^2} + \frac{\alpha \Delta t}{\Delta y^2} \leq \frac{1}{2} \iff \Delta t \leq \frac{1}{2\alpha} \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2}. \tag{2}$$

2 Results

Following condition (2), we choose to numerically solve (1) (still with the same notations as in the statement):

- $\Delta x = \Delta y = 1$, thus $n_x = n_y = 50$,
- according to Equation 2, $\Delta t = 1/4\alpha = 1/8$, thus $n_t = 2400$.

Results are illustrated in Figure 1.

3 Convergence rate analysis

3.1 Method

We denote by $\tilde{u}_k^{i,j}$ the solution of the implemented finite scheme at point $(t_k, x_i, y_j) := (kL_t/n_t, iL_x/n_x, jL_y/n_y) \in \Omega$. We define the following sequence:

$$e_k^{i,j} = \left| \tilde{u}_k^{i,j} - u(t_k, x_i, y_j) \right|. \tag{3}$$

We are looking for the rates of convergence of the scheme i.e., $r_t, r_x, r_y \in \mathbb{R}$ such that, for some constants C_t, C_x and C_y:

$$\|\tilde{u}(t,x,y) - u(t,x,y)\| \leq C_t \Delta_t^{r_t} + C_x \Delta_x^{r_x} + C_y \Delta_y^{r_y}, \tag{4}$$

where $\tilde{u}$ is a defined by part approximation function from the sequence $\left(\tilde{u}_k^{i,j}\right)$.

In order to approximate the left hand side of (4), we will analyze the following errors, respectively approximations of norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$[2]:

$$E_1(n_t, n_x, n_y) := \frac{\sum_{k=0}^{n_t} \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} e_k^{i,j}}{(n_t+1) + (n_x+1) + (n_y+1)}, \tag{5}$$

[1] [m] stands for *meters*, [s] for *seconds* and [K] for *Kelvin*.

[2] For information, E_2 is chosen in the same fashion as in [1] and E_∞ in the same fashion as in [2].

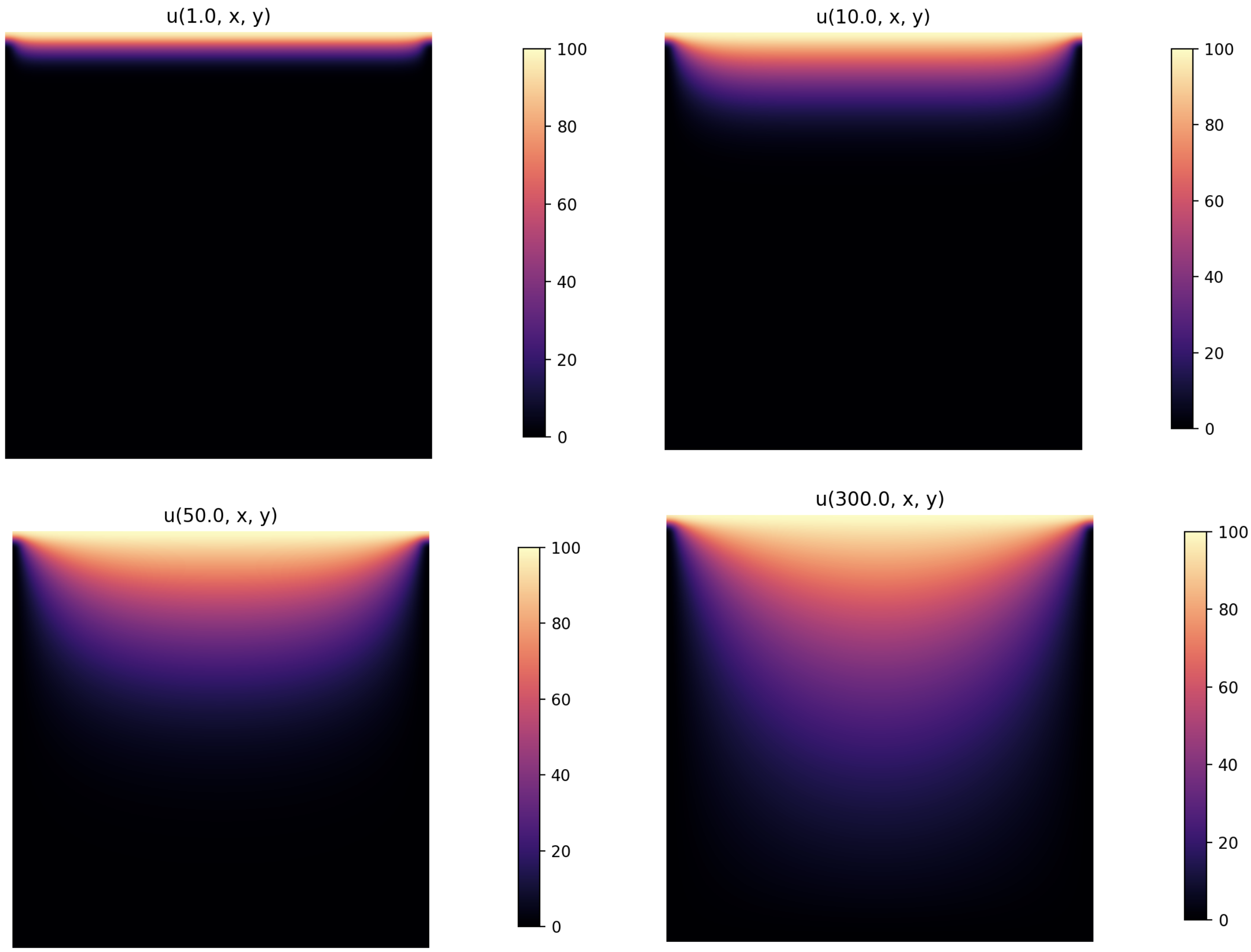

Figure 1: Solution of $u(t, x, y)$ as described in Section 1 for $(x, y) \in [0, 50] \times [0, 50]$, and for $t \in \{1, 10, 50, 300\}$.

$$E_2(n_t, n_x, n_y) := \left(\Delta t \Delta x \Delta y \sum_{k=0}^{n_t} \sum_{i=0}^{n_x} \sum_{j=0}^{n_y} \left(e_k^{i,j} \right)^2 \right)^{1/2}, \tag{6}$$

$$E_\infty(n_t, n_x, n_y) := \max_{\substack{k \in \{0, \ldots, n_t\} \\ i \in \{0, \ldots, n_x\} \\ j \in \{0, \ldots, n_y\}}} e_k^{i,j}. \tag{7}$$

Suppose we want to find r_x. It holds that, for some error $E = E_1, E_2, E_\infty$, for n_x large enough:

$$\log_2 \left(\frac{E(n_t, n_x, n_y)}{E(n_t, 2n_x, n_y)} \right) \approx \log_2 \left(\frac{C_x \Delta x^{r_x}}{C_x \left(\frac{\Delta x}{2} \right)^{r_x}} \right) = \log_2 \left(2^{r_x} \right) = r_x. \tag{8}$$

Thus, for different values of n_x and for all the considered errors, we can compute the quantity written above and consider it as an approximation of r_x. We can apply the same method to find r_t and r_y.

The Euler explicit scheme (applied for t) has a rate of convergence of 1, and the central difference (applied for x and y) has a rate of convergence of 2. We thus expect to find $r_t \approx 1$, $r_x \approx 2$ and $r_y \approx 2$ (see [1, Section 3.6.6]).

3.2 Convergence rate w.r.t. n_t

Since we do not know the analytical solution of PDE (1), we choose large enough values of n_t, n_x and n_y compared to those used for the experiment. We consider in our computations that $\tilde{u}_k^{i,j}$ with these values of (n_t, n_x, n_y) is the analytical solution. For this experiment, we choose to take as reference $(n_t, n_x, n_y) = (76800, 50, 50)$. We will now compute the quantity expressed in (8) with $(n_x, n_y) = (50, 50)$, and $n_t \in \{2400, 4800, 9600, 19200\}$[3]. Results of experiments to approximate r_t are depicted in Table 1.

n_t	2400	4800	9600	19200
$E_1(n_t, 50, 50)$	5.470×10^{-3}	2.651×10^{-3}	1.238×10^{-3}	5.308×10^{-4}
$E_2(n_t, 50, 50)$	6.621×10^{1}	3.146×10^{1}	1.457×10^{1}	6.225×10^{0}
$E_\infty(n_t, 50, 50)$	1.947×10^{1}	1.054×10^{1}	5.221×10^{0}	2.307×10^{0}
$r_t \approx \log_2\left(\frac{E_1(n_t,n_x,n_y)}{E_1(2n_t,n_x,n_y)}\right)$	1.045	1.098	1.222	
$r_t \approx \log_2\left(\frac{E_2(n_t,n_x,n_y)}{E_2(2n_t,n_x,n_y)}\right)$	1.074	1.110	1.227	
$r_t \approx \log_2\left(\frac{E_\infty(n_t,n_x,n_y)}{E_\infty(2n_t,n_x,n_y)}\right)$	0.885	1.013	1.178	

Table 1: Experiments to approximate r_t, for $n_t \in \{2400, 4800, 9600, 19200\}$

One can observe that, as expected, the value of $r_t \approx 1$ (c.f. Section 3.1) for all the considered errors.

3.3 Convergence rate w.r.t. n_x

In the same way as for the experiment above, we choose reference values for (n_t, n_x, n_y) i.e., $(19200, 128, 128)$. We will now compute the quantity expressed in (8) with $(n_t, n_y) = (19200, 128)$, and $n_x \in \{8, 16, 32, 64\}$. Results of experiments to approximate r_x are depicted in Table 2.

n_x	8	16	32	64
$E_1(19200, n_x, 128)$	2.293×10^{-1}	7.797×10^{-1}	2.312×10^{-2}	5.344×10^{-3}
$E_2(19200, n_x, 128)$	4.851×10^{2}	2.235×10^{2}	9.860×10^{1}	3.381×10^{1}
$E_\infty(19200, n_x, 128)$	6.578×10^{0}	6.532×10^{0}	6.347×10^{0}	5.521×10^{0}
$r_x \approx \log_2\left(\dfrac{E_1(n_t,n_x,n_y)}{E_1(n_t,2n_x,n_y)}\right)$	1.556	1.754	2.113	
$r_x \approx \log_2\left(\dfrac{E_2(n_t,n_x,n_y)}{E_2(n_t,2n_x,n_y)}\right)$	1.118	1.181	1.544	
$r_x \approx \log_2\left(\dfrac{E_\infty(n_t,n_x,n_y)}{E_\infty(n_t,2n_x,n_y)}\right)$	0.010	0.041	0.201	

Table 2: Experiments to approximate r_x, for $n_x \in \{8, 16, 32, 64\}$

One can see that, by considering E_1, r_x is approximately its expected value i.e., 2. When E_2 is considered, one can observe that the approximation of r_x tends to grow and goes above 1.5. However, when E_∞ is considered, the maximal value attained by the approximation of r_x is no larger than 0.3.

We interpret these results in the following way. First, a coarse mesh allows for a worse approximation of r_x. Second, a p-norm $\|\cdot\|_p$ is more and more sensitive to inaccuracy for larger and larger p. This allows us to conclude that E_2 would need finer meshes to attain $r_x \approx 2$, and it is even more the case for E_∞.

However, for the approximation to be good, since the reference sequence $\left(\tilde{u}_k^{i,j}\right)$ needs to be very accurate in comparaison to those computed for the experience (i.e. with $n_x \in \{8, 16, 32, 64\}$), we would need to re-do the experiments with larger values of n_x. For computational reasons (in space and time), we cannot find stable values of (n_t, n_x, n_y) for the reference and the computed solutions such that the approximation is good for both E_2 and E_∞. For this reason we choose to restrict ourselves to E_1 for this experiment.

3.4 Convergence rate w.r.t. n_y

In order to approximate r_y, we do exactly the same as for r_x (same reference) but by computing values of $\tilde{u}_k^{i,j}$ for $(n_t, n_x) = (19200, 128)$ and $n_y \in \{8, 16, 32, 64\}$. Results of experiments to approximate r_y are depicted in Table 3.

[3]Note that, all along the report, values of (n_t, n_x, n_y) are chosen such that condition (2) holds i.e., such that the numerical resolution is stable.

n_y	8	16	32	64
$E_1(19200, 128, n_y)$	1.745×10^{-1}	6.138×10^{-2}	1.877×10^{-2}	4.432×10^{-3}
$E_2(19200, 128, n_y)$	4.110×10^{2}	1.990×10^{2}	9.154×10^{1}	3.192×10^{1}
$E_\infty(19200, 128, n_y)$	3.023×10^{0}	2.956×10^{0}	2.540×10^{0}	1.803×10^{0}
$r_y \approx \log_2\left(\frac{E_1(n_t,n_x,n_y)}{E_1(n_t,n_x,2n_y)}\right)$	1.507	1.709	2.082	
$r_y \approx \log_2\left(\frac{E_2(n_t,n_x,n_y)}{E_2(n_t,n_x,2n_y)}\right)$	1.046	1.121	1.520	
$r_y \approx \log_2\left(\frac{E_\infty(n_t,n_x,n_y)}{E_\infty(n_t,n_x,2n_y)}\right)$	0.032	0.219	0.495	

Table 3: Experiments to approximate r_y, for $n_y \in \{8, 16, 32, 64\}$

By symmetry, we obtain very similar results as for r_x.

References

[1] Hans Langtangen and Svein Linge. *Finite Difference Computing with PDEs*, volume 16. 01 2017.

[2] K. W. Morton and D. F. Mayers. *Numerical Solution of Partial Differential Equations: An Introduction*. Cambridge University Press, 2 edition, 2005.