December Maths Puzzle

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(Disclaimer: I have no formal education in number theory.)

The problem can be formulated as

> Find all positive integers a, b with $a^b + 1 \equiv 0 \pmod{b^a + 1}$.

It is first useful to distinguish between three classes of solutions:

1. The 'trivial' solutions:
$$(a, b) = (n, n),\ (4, 2),\ (2n + 1, 1) \quad \forall n \in \mathbb{N}^*.$$
2. The regular solutions: they can be written as
$$(a, b) = (x^m, x^n),$$
where $x, m, n \in \mathbb{N}^*$, $x \geq 2$, and $n > m$.
3. The exotic solutions: solutions that are neither trivial nor regular.

Exotic solutions

I have found the following exotic solutions:

$$\begin{aligned}
a &= 2, \qquad b = 386 = 2^1 + 193^1, \\
a &= 2, \qquad b = 20136 = 2^3 + 3^1 + 839^1, \\
a &= 2, \qquad b = 59140 = 2^2 + 5^1 + 2957^1, \\
a &= 2, \qquad b = 373164544 = 2^9 + 728837^1.
\end{aligned}$$

When $a = 2$ is fixed, the sequence corresponding to solutions for b is on the 'On-line Encyclopaedia of Integer Sequence' (OEIS A247220) since 2014. It is yet unknown if there is a term after 373164544, and I could not solve this problem myself. In addition, I have not found any exotic solutions for $a \neq 2$, but I could not disprove their existence.

Regular solutions

Theorem 1. *Let $x, m, n \in \mathbb{N}^*$, $x \geq 2$, and $n > m$. Then, $(a, b) = (x^m, x^n)$ is a solution if and only if*

$$\frac{mx^{n-m}}{n}$$

is an odd integer.

Proof. Let us assume that $c = mx^{n-m}/n$ is an odd integer. As

$$b^a = (x^n)^{x^m} = (x^{x^m})^n,$$

we obtain

$$\begin{aligned} a^b + 1 = (x^m)^{x^n} + 1 &= (x^{x^m})^{mx^{n-m}} + 1 \\ &= (b^a)^c + 1 \\ &\equiv (-1)^c + 1 = 0 \pmod{b^a + 1}. \end{aligned}$$

Conversely, let us assume that $(a, b) = (x^m, x^n)$ is a solution and that $c = mx^{n-m}/n$ is not odd. Then, c is either even or a fraction. If c is even,

$$a^b + 1 \equiv (-1)^c + 1 = 2 \not\equiv 0 \pmod{b^a + 1}.$$

If c is a non-integer fraction, we can define $i \in \mathbb{N}$ such that $0 < c - i < 1$. Let us also define $y = x^{x^m}$. Then

$$a^b + 1 = y^{cn} + 1 = Q(y^n + 1) + R = Q(b^a + 1) + R,$$

where Q and R are integers defined as

$$\begin{aligned} Q &= y^{n(c-1)} - y^{n(c-2)} + \cdots + (-1)^{i+1} y^{n(c-i)}, \\ R &= (-1)^i y^{n(c-i)} + 1. \end{aligned}$$

Because $0 < n(c - i) < n$, $0 < |R| < y^n + 1 = b^a + 1$ and thus $a^b + 1 \not\equiv 0 \pmod{b^a + 1}$. We have a contraction and c must be odd. $\square$

Conditions on x given m and n

Using Theorem 1, we can generate infinitely many solutions. Let us fix m and n and define the coprime integers m' and n' such that $m'/n' = m/n$. Additionally, let us write the prime factorization of n' as

$$n' = \prod_{p \in P_{n'}} p^{i_p},$$

where $P_{n'}$ is the set of primes appearing in the decomposition of n'.

From Theorem 1, (x^m, x^n) is a solution $\iff$ $m'x^{n-m} = cn'$ and c is odd. Thus, there are solutions if and only if

$$m' \text{ is odd}, \quad \text{and} \quad i_2 \equiv 0 \pmod{n-m}. \tag{1}$$

When the two conditions are satisfied, one can easily check that x is a solutions if and only if

$$x \equiv 0 \pmod{M} \quad \text{and} \quad x/M \text{ is odd}.$$

where

$$M = \prod_{p \in P_{n'}} p^{\alpha}, \qquad \alpha = \lceil i_p/(n-m) \rceil.$$

Thus

$$x = (2k+1)M. \tag{2}$$

is a solution $\forall\ k \in \mathbb{N}$.

In this Jupyter Notebook, I have implemented an algorithm based on Eqs. (1) and (2) to generate the regular solutions for x given m and n, when they exist.