Green's function estimation from ambient noise signals with limited illumination in 3D

The goal is to show numerically that the cross correlations of signals emitted by noise sources and recorded by two receivers is related to the Green's function between the sensors even when the noise sources do not surround the region of interest.

1) Geometric set-up.

We consider the solution u of the wave equation in a three-dimensional medium with background propagation speed c_0:

$$\frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} - \Delta_{\boldsymbol{x}} u = n(t, \boldsymbol{x}). \tag{1}$$

The term $n(t, \boldsymbol{x})$ models a random field of noise sources. It is a zero-mean stationary (in time) random process with autocorrelation function

$$\langle n(t_1, \boldsymbol{y}_1) n(t_2, \boldsymbol{y}_2)\rangle = F(t_2 - t_1)\delta(\boldsymbol{y}_1 - \boldsymbol{y}_2)K(\boldsymbol{y}_1). \tag{2}$$

Here $\langle\cdot\rangle$ stands for statistical average with respect to the distribution of the noise sources. K is the probability density function of the distribution $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with a diagonal matrix $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$:

$$K(\boldsymbol{y}) = \prod_{j=1}^{3} \frac{1}{\sqrt{2\pi}\sigma_j} \exp\Big(- \frac{(y_j - \mu_j)^2}{2\sigma_j^2}\Big) \tag{3}$$

Numerically, this can be simulated by sampling N points $(\boldsymbol{y}_s)_{s=1,\ldots,N}$ independently and uniformly according to the prescribed Gaussian distribution and each point emits independent and identically distributed signals $n_s(t)$ with stationary Gaussian statistics, mean zero, and covariance function $\langle n_s(t)n_s(t')\rangle = F(t-t')$:

$$n(t, \boldsymbol{x}) = \frac{1}{\sqrt{N}} \sum_{s=1}^{N} n_s(t)\delta(\boldsymbol{x} - \boldsymbol{y}_s) \tag{4}$$

For the simulations, one may take $\boldsymbol{\mu} = (0, -200, 0)$, $(\sigma_1, \sigma_2, \sigma_3) = (100, 50, 100)$, and $\hat{F}(\omega) = \omega^2 \exp(-\omega^2)$.

There are five receivers. We will consider three situations:
- $\boldsymbol{x}_j = (0, 50(j-1), 0)$, $j = 1, \ldots, 5$.
- $\boldsymbol{x}_j = (0, 5(j-1), 0)$, $j = 1, \ldots, 5$.
- $\boldsymbol{x}_j = (50(j-3), 100, 0)$, $j = 1, \ldots, 5$.

2) Preliminaries.

The homogeneous three-dimensional Green's function $\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y})$ is solution of

$$\Delta_{\boldsymbol{x}} \hat{G}_0 + \frac{\omega^2}{c_0^2}\hat{G}_0 = -\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^2 \tag{5}$$

with the Sommerfeld radiation condition. It is given by

$$\hat{G}_0(\omega, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \exp\Big(i\frac{\omega}{c_0}|\boldsymbol{x}-\boldsymbol{y}|\Big) \tag{6}$$

3) Empirical cross correlation.

The empirical cross correlation of the signals recorded at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ for an integration time T is

$$C_{T,N}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{T-|\tau|}\int_0^{T-|\tau|} u(t, \boldsymbol{x}_1)u(t+\tau, \boldsymbol{x}_2)dt. \tag{7}$$

Its expectation with respect to the distribution of the emitted signals is

$$C_N(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi N}\sum_{s=1}^{N}\int d\omega \hat{F}(\omega)\overline{\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{y}_s)}\hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y}_s)e^{-i\omega\tau}. \tag{8}$$

The expectation of the empirical cross correlation with respect to the distribution of the emitted signals and the source positions is

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{2\pi}\int d\omega \int d\boldsymbol{y} K(\boldsymbol{y})\hat{F}(\omega)\overline{\hat{G}(\omega, \boldsymbol{x}_1, \boldsymbol{y})}\hat{G}(\omega, \boldsymbol{x}_2, \boldsymbol{y})e^{-i\omega\tau}. \tag{9}$$

If $T \gg 1$, then the theory predicts that

$$C_{T,N}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \stackrel{T\to\infty}{\longrightarrow} C_N(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \tag{10}$$

If $N \gg 1$, then the theory predicts that

$$C_N(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \stackrel{N\to\infty}{\longrightarrow} C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \tag{11}$$

If the illumination were isotropic (which is not the case here), then we would have:

$$C^{(1)}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) \approx C_{\text{asy}}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2), \tag{12}$$

$$\text{with } \frac{\partial}{\partial\tau}C_{\text{asy}}(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) = -\big[F * G(\tau, \boldsymbol{x}_1, \boldsymbol{x}_2) - F * G(-\tau, \boldsymbol{x}_1, \boldsymbol{x}_2)\big], \tag{13}$$

up to a multiplicative constant.

4) Questions.

1. Plot $\tau \to C_N(\tau, \boldsymbol{x}_j, \boldsymbol{x}_1)$ for $j = 1, \ldots, 5$ and discuss the results for the three cases (take $N = 10^4$, $c_0 = 1$).

2. Compare $C_{T,N}$ defined by (7), C_N defined by (8), and $C^{(1)}$ defined by (9), for different values of T and N.

3. Study the statistical stability of $C_{T,N}$ with respect to the number N of sources and with respect to the integration time T.

4. The results of the first question are obtained with $c_0 = 1$. Assume that you do not know the velocity c_0 and that you observe the data. Describe and study a method to extract c_0.