Assignment 2

STAT 3150–Statistical Computing

Due on 05/10/2020

This assignment covers the modules on **Generating Random Variabltes** and **Monte Carlo Integration**.

Solutions must be submitted electronically via UM Learn **no later than 11:59PM CDT on Monday October 5$^{\mathbf{th}}$**. Please provide both the `Rmd` file and a PDF version of the output of compiling it.

You are allowed to discuss the problems among yourselves, but your submission must reflect your original work. Note that your `R` code will be analysed for suspicious similarities.

```
library(smoothmest)
```

```
## Loading required package: MASS
```

```
library(scatterplot3d)
```

Problem 1

The standard Laplace distribution has density

$$f(x) = \frac{1}{2}\exp(-|x|), \quad x \in \mathbb{R}.$$

Use the inverse transform method to generate a random sample of size $n = 1000$ from this distribution. Use one of the methods from the lecture to compare the generated sample to the target distribution. First integrate the PDF to get the CDF Since we have an absolute value sign we know the if x<0 then f(x)=1/2exp(x) and if x>=0 then f(x)=1/2exp(-x). We need a piece wise function to properly integrate this.

If x < 0 we integrate 1/2exp(x) from -inf to x this gives us -1/2[exp(x)]-inf to x which is -1/2(exp(x)-0) =-1/2exp(x)

To integrate f(x) from -inf to x when x>0 we must integrate the function first from -inf to x and then from 0 to x. Knowing that f(x)=1/2exp(x) from -inf to 0 and f(x)= 1/2exp(-x) from 0 to inf the integral is as follows 1/2 the integral of exp(x) with respect to x from -inf to 0 + 1/2 the integral of exp(-x) from 0 to x. this gives us 1/2[exp(x)]-inf to 0 + -1/2[exp(-x)]0 to x =1/2(1-0) + (-1/2)(exp(-x)-1) =1/2-1/2exp(-x)-(-1/2) =-1/2exp(-x)+1

So the final CDF is F(x)={(1/2exp(x)) when x>=0, (-1/2exp(x)+1) when x<0}

x=1/2exp(y)=>y=ln(2x) x=-1/2exp(-y)+1 => x-1/(-1/2)=exp(-y) => 2(-x+1)=exp(-y) => ln(2(1-x))=-y => y=-ln(2(1-x))

Now we must invert each piece of this wise function

When x>=0, F^-1(x)=ln(2x) and when x<0 F^-1(x)=-ln(2(1-x)) Now we can apply this to our sample

```
n<-1000
U <- runif(n)
X <- c(1:1000)
for (i in 1:1000) {
  if(U[i]>=0){
    X[i]<-log(2*U[i])
  }
  else{
    X[i]<- (-1)*log(2*(1-U[i]))
  }
}
```

Problem 2

We will generate random variates from a standard normal $N(0,1)$ using the double exponential distribution; its density is given by

$$g(x \mid \lambda) = \frac{1}{2\lambda} \exp\left(-\frac{1}{\lambda}|x|\right), \quad \lambda > 0.$$

a. Let $f(x)$ be the density of the normal distribution. Compute the ratio $f(x)/g(x \mid \lambda)$. Using calculus (or any other analytic method), find a uniform upper bound C for the ratio. (*Hint*: The upper bound C will be a function of λ, but not of x.) b. Find the value $\hat{\lambda}$ that minimises the upper bound C. c. Implement the Accept-Reject algorithm for sampling from $N(0,1)$ using proposals from $g(x \mid \hat{\lambda})$. To generate samples from a double exponential distribution, you can use the function `rdoublex` from the `smoothmest` package.

a. Since we are using N(0,1) f(x)=$\frac{e^{-1/2x^2}}{\sqrt{2\pi}}$ So for our ratio we get

$$\frac{e^{(-1/2)x^2)}2\lambda}{e^{-\frac{|x|}{\lambda}}\sqrt{2\pi}}$$

Now to find the upper bound of this function we must first find the first derivative. Top calculate the derrivative (with respect to lambda) first we factor out the 2/sqrt(2pi). $\frac{d}{dx}(\frac{e^{(-1/2)x^2)}2\lambda}{e^{-\frac{|x|}{\lambda}}\sqrt{2\pi}}) = \frac{2}{\sqrt{2\pi}}\frac{d}{dx}(\frac{e^{(-1/2)x^2)}\lambda}{e^{-\frac{|x|}{\lambda}}}) = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}\frac{d}{dx}(\lambda e^{\frac{-x^2}{2}-\frac{-|x|}{\lambda}}) = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{-x^2}{2}-\frac{-|x|}{\lambda}}\frac{d}{dx}(\frac{-x^2}{2}-\frac{-|x|}{\lambda}) = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{-x^2}{2}-\frac{-|x|}{\lambda}}(-\frac{2x}{2}+\frac{x}{|x|\lambda})$ =

$$\frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{-x^2}{2}-\frac{-|x|}{\lambda}}(\frac{x}{|x|\lambda}-x)$$

Since we know that $\lambda > 0$ that means the only way for this function to equal 0 is when $\frac{x}{|x|y} - x = 0$ $\frac{x}{|x|\lambda} - x = 0 => \frac{x}{|x|\lambda} - \frac{x|x|\lambda}{|x|\lambda} = 0 => \frac{x(1-|x|\lambda)}{|x|\lambda} => 1 - |x|\lambda = 0 => |x| = \frac{1}{\lambda}$ Say $x = \frac{1}{\lambda} + 1 = \frac{1+\lambda}{\lambda}$ then we have $\frac{x(1-\frac{1+\lambda}{\lambda}\lambda)}{\frac{1+\lambda}{\lambda}\lambda} = \frac{x\lambda}{1+\lambda}$ Which is positive when x>0 and negative when x<0.

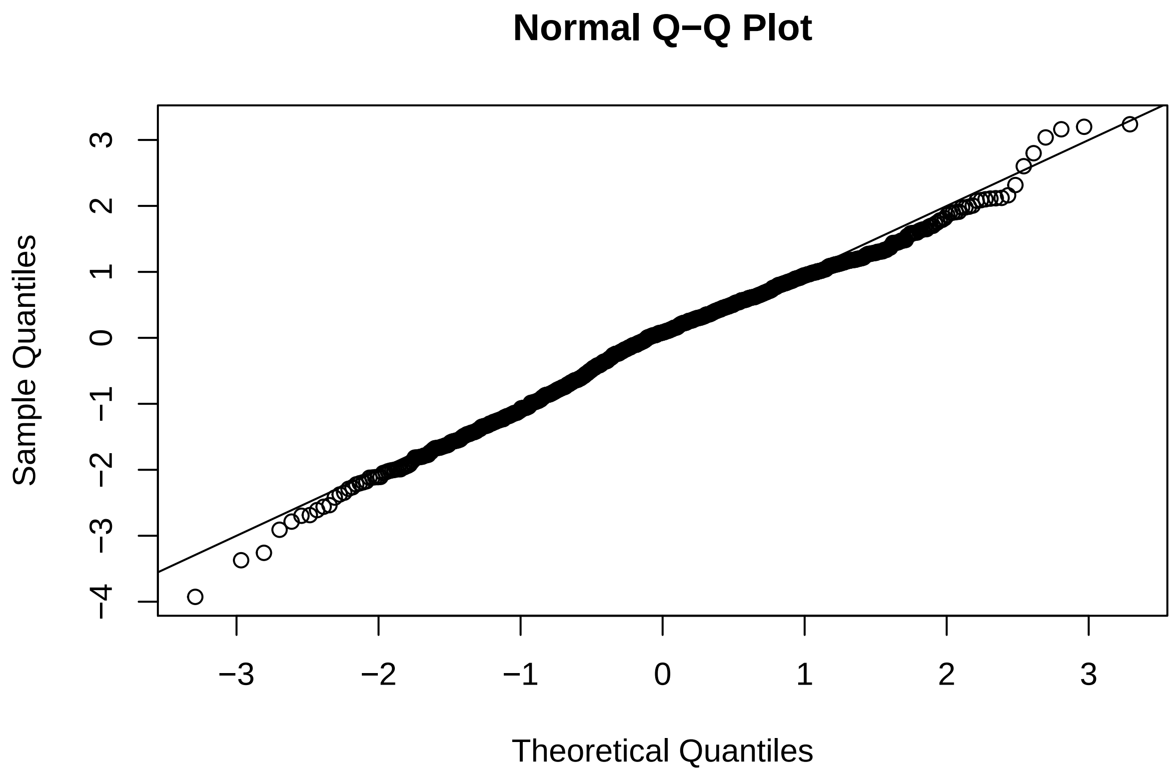
So plugging this into the original function we get an absolute max at $|x| = \frac{1}{\lambda}$ when x<0 $\frac{e^{\frac{-1}{2}x^2}2\lambda}{e^{-\frac{|x|}{\lambda}}\sqrt{2\pi}} = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}\frac{e^{\frac{-1}{2}x^2}}{e^{-\frac{|x|}{\lambda}}}$ $= \frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{-1}{2}x^2+\frac{|x|}{\lambda}}$ plugging in |x|=1/lambda we get $\frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{(-1)}{2\lambda^2}+\frac{1}{\lambda^2}} = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{(-1)}{2\lambda^2}+\frac{1}{\lambda^2}} = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{(-1)}{2\lambda^2}+\frac{2}{2\lambda^2}} = \frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{1}{2\lambda^2}}$ and this is our uniform upper bound for the ratio.

b. The first derivative of $\frac{\sqrt{2}\lambda}{\sqrt{\pi}}e^{\frac{1}{2\lambda^2}}$ is $\frac{\sqrt{2}}{\sqrt{\pi}}\frac{d}{d\lambda}\lambda e^{\frac{1}{2\lambda^2}} = \frac{\sqrt{2}}{\sqrt{\pi}}\lambda e^{\frac{1}{2\lambda^2}}(\frac{d}{d\lambda}\frac{1}{2\lambda^2}) + 1e^{\frac{1}{2\lambda^2}} = \frac{\sqrt{2}}{\sqrt{\pi}}\lambda e^{\frac{1}{2\lambda^2}}\frac{-2}{2\lambda^3} + e^{\frac{1}{2\lambda^2}} = \frac{\sqrt{2}}{\sqrt{\pi}}(e^{\frac{1}{2\lambda^2}}\frac{-1}{\lambda^2} + e^{\frac{1}{2\lambda^2}}) = \frac{\sqrt{2}}{\sqrt{\pi}}(\frac{-e^{\frac{1}{2\lambda^2}}}{\lambda^2} + e^{\frac{1}{2\lambda^2}}) = \frac{\sqrt{2}}{\sqrt{\pi}}(e^{\frac{1}{2\lambda^2}} - \frac{e^{\frac{1}{2\lambda^2}}}{\lambda^2}) = \frac{\sqrt{2}}{\sqrt{\pi}}e^{\frac{1}{2\lambda^2}}(1 - \frac{1}{\lambda^2}) = \frac{\sqrt{2}}{\sqrt{\pi}}e^{\frac{1}{2\lambda^2}}(\frac{\lambda^2}{\lambda^2} - \frac{1}{\lambda^2})$ $= \frac{\sqrt{2}}{\sqrt{\pi}}e^{\frac{1}{2\lambda^2}}(\frac{\lambda^2-1}{\lambda^2}) = \frac{\sqrt{2}(\lambda-1)}{\sqrt{\pi}\lambda^2}e^{\frac{1}{2\lambda^2}}$ this function is 0 at $\lambda = 1, -1$. at $\lambda = -10$ the function is negative because $\lambda - 1 = -10 - 1 = -11$ is negative so therefore $\lambda = -1$ can't be a minimum. At $\lambda = 1/2$ $-1 + 1/2 = -1/2$ which is negative. At $\lambda = 2$ we get $2 - 1 = 1$ which is positive. So at $\lambda = 1$ the first derivative goes from negative to positive which means there is a minimum at that point. We also must ignore any negative λ values because of the domain of the original function. So with that domain the first derivative has one root and it switches from negative to positive at that point and never switches sign again, that means the minimum possible value of the original function is at $\lambda = 1$.

c. $g(x|\lambda = 1) = \frac{1}{2*(1)}e^{-\frac{1}{(1)}|x|} = \frac{1}{2}e^{-|x|}$

```r
# Set parameters----
C <- sqrt(2)*exp(1/2)/(sqrt(pi)) # Constant
n <- 1000 # Number of variates
k <- 0 # counter for accepted
j <- 0 # iterations
norm_vars <- numeric(n) # Allocate memory

# A while loop runs until condition no longer holds
while (k < n) {
  u <- rdoublex(1,lambda=1)
  j <- j + 1
  x <- rnorm(1) # random variate from g
  if (u < sqrt(2)*x*exp((-(x^2)/2)+abs(x))/(sqrt(pi))*(-1+(1/(abs(x))))/(sqrt(pi))/C) {
    k <- k + 1
    norm_vars[k] <- x
    }
}
qqnorm(norm_vars)
abline(a = 0, b = 1)
```



Problem 3

The goal of this problem is to sample *uniformly* points on a sphere, i.e. for every subset of the sphere, the probability a point fall in that subset is proportional to the area of the subset (and not its location on the sphere).

Every point on Earth[1] can be uniquely described by a latitude (between -$\pi/2$ and $\pi/2$) and a longitude (between -π and π). To convert from $(Lat, Long)$ to cartesian coordinates, we can use the following formulas:

- $x = \cos(Lat)\cos(Long)$
- $y = \cos(Lat)\sin(Long)$
- $z = \sin(Lat)$

a. Generate from a uniform $U(-\pi/2, \pi/2)$ and $U(-\pi, \pi)$ a sample from $n = 1000$ points on a sphere. Are these points uniformly distributed? (*Hint*: You don't need to give a probability calculation as justification. You can visualize the sampled points using the function `scatterplot3d::scatterplot3d`.)
b. Generate the longitude using a uniform $U(-\pi, \pi)$, but now generate a latitude according to the following cumulative distribution function:

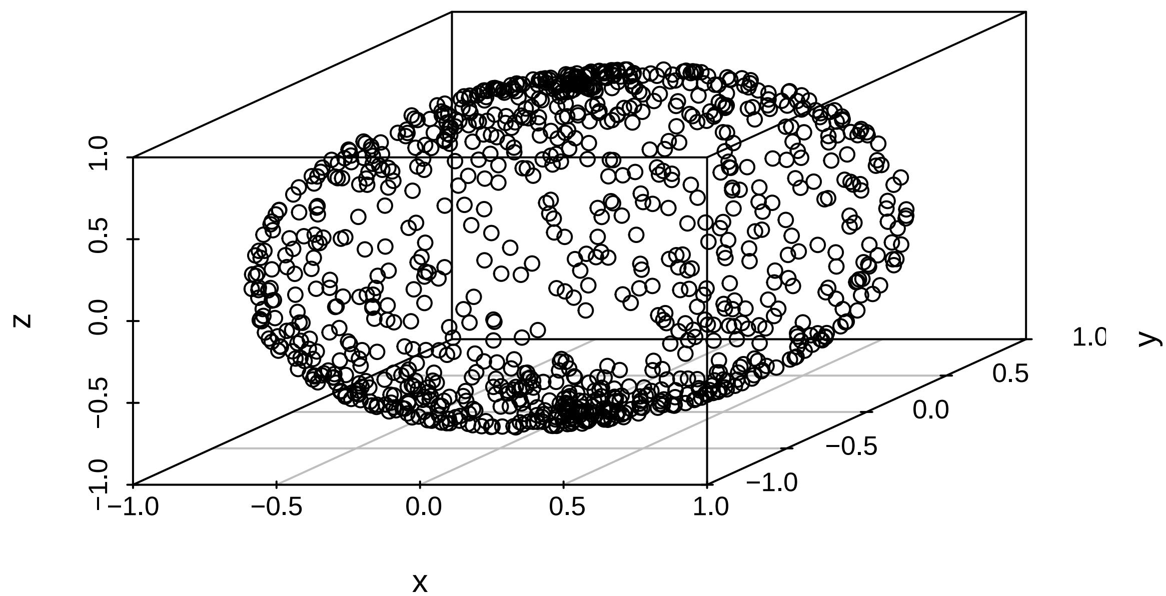
$$F(Lat) = \frac{1}{2}(1 + sin(Lat)), \quad Lat \in (-\pi/2, \pi/2).$$

[1]Except the poles

Are these points uniformly distributed?

a)

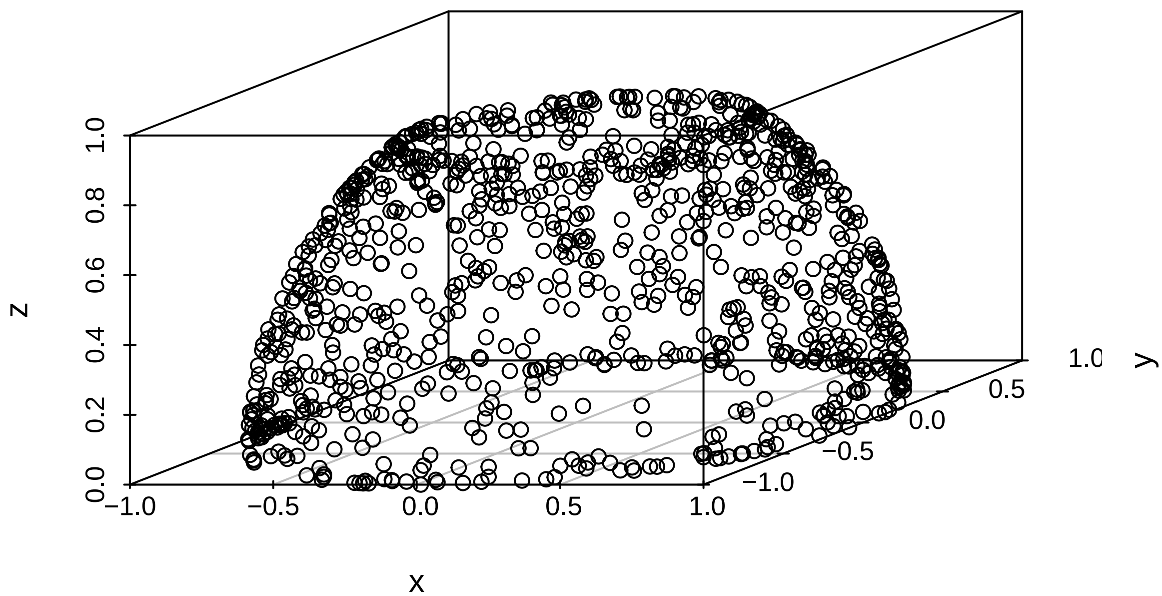
```
n<-1000
coordinates <- data.frame("x"=numeric(),"y"=numeric(),"z"=numeric())
unifVals <- data.frame(lat=1:n,lon=1:n)
for(i in 1:1000){
  lat<-runif(1,-pi/2,pi/2)
  lon<-runif(1,-pi,pi)
  coordinates[i,1]<-cos(lat)*cos(lon)
  coordinates[i,2]<-cos(lat)*sin(lon)
  coordinates[i,3]<-sin(lat)
}
scatterplot3d(coordinates)
```



We can see that the points are very concentrated at the top and bottom of this little egg (i.e. around x=.5,y=.5,z=1 and x=.5,y=.5,z=-1). Due to this concentration it is most likely not uniformly distributed.

b)

```r
n<-1000
coordinates <- data.frame("x"=numeric(),"y"=numeric(),"z"=numeric())
unifVals <- data.frame(lat=1:n,lon=1:n)
for(i in 1:1000){
  lat<-runif(1,-pi/2,pi/2)
  lon<-runif(1,-pi,pi)
  lat<-.5*(1+sin(lat))
  coordinates[i,1]<-cos(lat)*cos(lon)
  coordinates[i,2]<-cos(lat)*sin(lon)
  coordinates[i,3]<-sin(lat)
}
scatterplot3d(coordinates)
```



We tend to see concentrations around the edge of the shape implying this is not uniform

Problem 4

Consider the following integral:

$$\theta = \int_0^{0.5} e^{-x} dx.$$

a. By sampling from a uniform $U(0, 0.5)$, use Monte Carlo integration to estimate θ.
b. By sampling from an exponential distribution $Exp(1)$, find another estimate of θ.
c. Which approach has the smallest standard error? The actual value of this integral is $1 - e^{-.5}$ which equals

```
1-exp(-.5)
```

```
## [1] 0.3934693
```

a. from the law of large numbers we know that $\frac{1}{n}\sum_{i=1}^{n} e^{-X_i} = 2\int_0^2 e^{-x}$ where X~U(0,.5) So to estimate this integral we will need to divide the mean by 2.

```
n<-5000
unifVars<- runif(n,max=.5)
theta<-mean(exp(-unifVars))/2
sigma<-sd(exp(-unifVars))/2
theta
```

```
## [1] 0.3924721
```

```
sigma
```

```
## [1] 0.05651058
```

```
SE<-sigma/sqrt(n)
SE
```

```
## [1] 0.0007991803
```

b.

```
n<-5000
expVars<- rexp(n)
theta<-mean(1/(1+expVars))/2
sigma<-sd(1/(1+expVars))/2
theta
```

```
## [1] 0.2980921
```

```
sigma
```

```
## [1] 0.1089135
```

```r
SE<-sigma/sqrt(n)
SE
```

```
## [1] 0.00154027
```

c. the uniform approach had a smaller standard error. # Problem 5

Consider the following integral:

$$\int_0^1 \frac{e^{-x}}{1+x^2}dx.$$

a. Show that the function $f(x) = \frac{e^{-x}}{1+x^2}$ is monotone.
b. Use the method of antithetic variables to find an estimate of the integral.
c. Compute an approximate 95% confidence interval for the estimate.
d. The first derivative of this function is $\frac{(\frac{d}{dx}e^{-x})(1+x^2)-e^{-x}\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(-e^{-x})(1+x^2)-e^{-x}2x}{(x^2+1)^2} = \frac{-e^{-x}(x^2+2x+1)}{(x^2+1)}$ $= \frac{-e^{-x}(x+1)^2}{(x^2+1)}$ This function is 0 at x=-1 only. When x=0 we get$1*1/1 = 1$ which is positive. Therefore the function is always non decreasing on the interval [0,inf) which means it is also on the definite interval [0,1] which means it is monotone positive on that interval
e.

```r
n<-5000
unifVars <- runif(n)
theta <- mean(exp(-c(unifVars,1-unifVars))/(1+c(unifVars,1-unifVars)^2))
sigma <- sd(exp(-c(unifVars,1-unifVars))/(1+c(unifVars,1-unifVars)^2))
SE<-sigma/sqrt(2*n)
c(theta,sigma,SE)
```

```
## [1] 0.524371940 0.243630558 0.002436306
```

c.

```r
c(theta-1.96*SE,theta+1.96*SE)
```

```
## [1] 0.5195968 0.5291471
```