Control Synthesis for Stochastic Switched Systems using the Tamed Euler Method

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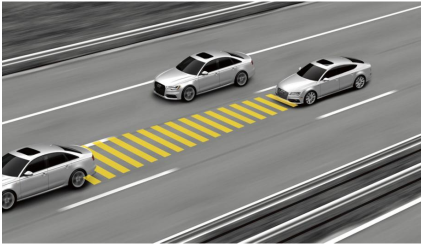
Context: control systems



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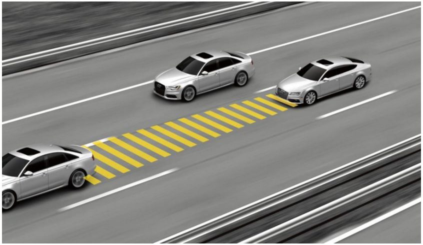

Issues: guaranteed control synthesis for safety/stability/reachability of safety critical systems, nonlinear systems, SDEs

Introduction

Framework

- Framework of the switched control systems: one selects the working modes of the system over time, every mode is described by ordinary/stochastic differential equations (ODEs/SDEs)
- The tools classically used for *correct-by-construction* control synthesis:
  - Tiling or space discretization
  - Polytopes or ellipsoids (symbolic representation of sets)
  - Rigorous simulation

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  - Tiling or space discretization
  - Polytopes or ellipsoids (symbolic representation of sets)
  - Rigorous simulation
- The tools we introduce:
  - A new error bound for the explicit Euler scheme and its equivalent for SDEs
  - Associated set based computations using balls

Outline

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Switched systems

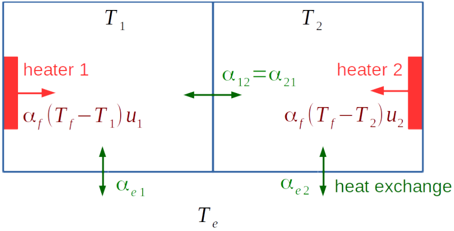
A continuous switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t))$$

- state $x(t) \in \mathbb{R}^n$
- finite set of (switched) modes $U = \{1, \dots, N\}$
- state dependent rule σ which associates a mode $u \in U$ to a state $x(t)$

We focus on sampled switched systems:
given a sampling period $\tau > 0$, switchings will occur at times τ, 2τ, ...

Example: Two-room apartment



$$\begin{pmatrix}\dot{T}_1\\ \dot{T}_2\end{pmatrix} = \begin{pmatrix}-\alpha_{21}-\alpha_{e1}-\alpha_f u_1 & \alpha_{21}\\ \alpha_{12} & -\alpha_{12}-\alpha_{e2}-\alpha_f u_2\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix}+\begin{pmatrix}\alpha_{e1}T_e+\alpha_f T_f u_1\\ \alpha_{e2}T_e+\alpha_f T_f u_2\end{pmatrix}.$$

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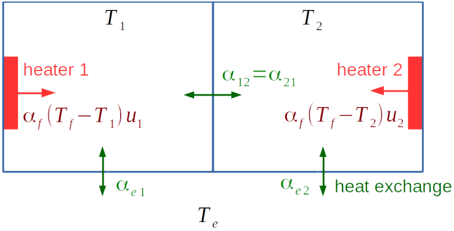

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- Modes: $\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \begin{pmatrix}0\\ 1\end{pmatrix}, \begin{pmatrix}1\\ 0\end{pmatrix}, \begin{pmatrix}1\\ 1\end{pmatrix}$; sampling period τ

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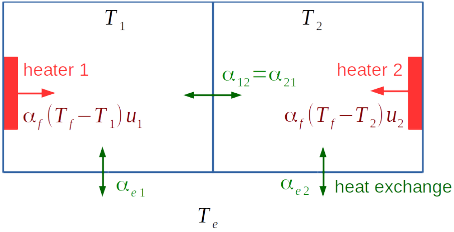

$$T_1(t+\tau) = f_1(T_1(t), T_2(t), u_1)$$
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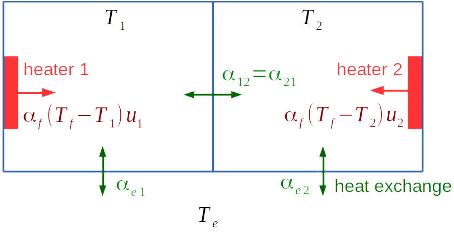

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- A pattern π is a finite sequence of modes, e.g. $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

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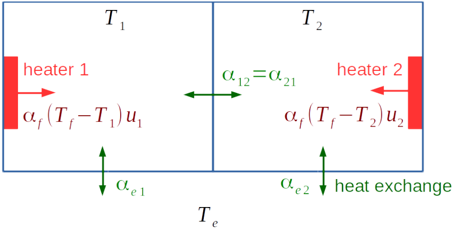

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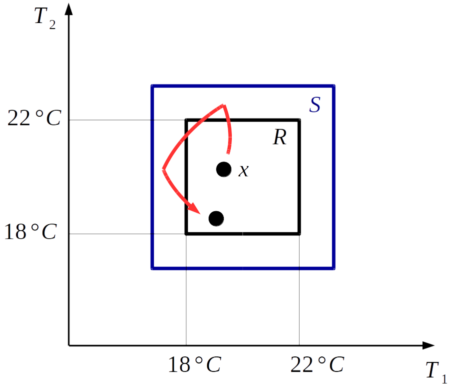
- A state dependent control consists in selecting at each τ a mode (or a pattern) according to the current value of the state.

Control Synthesis Problem

Being given a recurrence set R and a safety set S, we consider the state-dependent control problem of synthesizing σ:
At each sampling time t, find the appropriate switched mode $u \in U$ according to the current value of x, such that

Recurrence: after some steps of time, the state returns into R with safety in S



Numerical integration, reachability analysis

- Classical (non guaranteed) methods: Euler, Runge-Kutta, implicit, explicit schemes...
- Guaranteed reachability analysis: Enclosing solutions, error bounding, additional hypotheses
- State-of-the-art:
  - Monotone, ISS, incrementally stable systems [Girard, Sontag, Zamani, Tabuada...]
  - Reachability analysis using zonotopes [Dang, Girard, Althoff...]
  - Validated simulation, guaranteed integration [Moore, Lohner, Bertz, Makino, Nedialkov, Jackson, Corliss, Chen, Ábrahám, Sankaranarayanan, Taha, Chapoutot,...]
  - Ellipsoid methods [Kurzhanski, Varaiya, Dang...]
  - Sensitivity Analysis [Donzé, Maler...]

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Links can be made with strong monotony, incremental stability (when $\lambda_j < 0$)
Constants computed by constrained optimization.

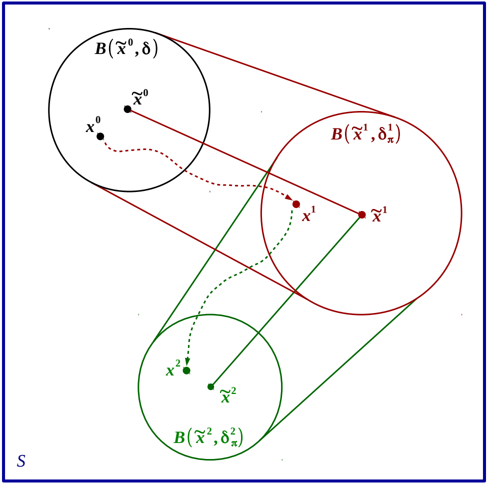
Main result

Theorem

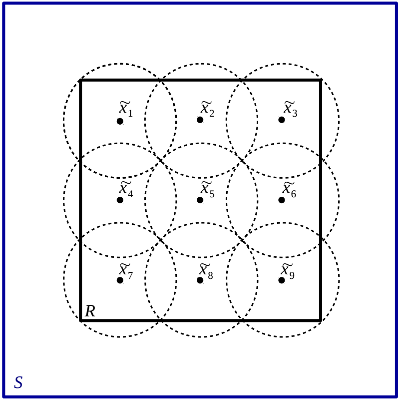
Given a sampled switched system satisfying (H0-H1), consider a point $\tilde{x}^0$ *and a positive real* δ*. We have, for all* $x^0 \in B(\tilde{x}^0, \delta)$, $t \in [0, \tau]$ *and* $j \in U$*:*
$\phi_j(t; x^0) \in B(\tilde{\phi}_j(t; \tilde{x}^0), \delta_j(t))$.
with

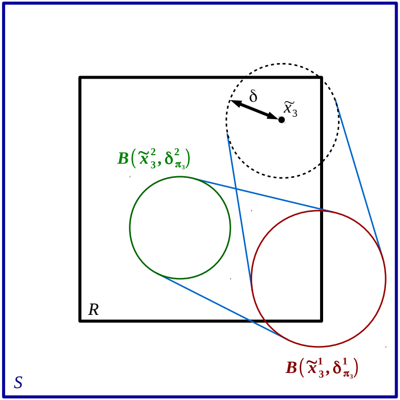
- *if* $\lambda_j < 0$: $\delta_j(t) = \left(\delta^2 e^{\lambda_j t} + \frac{C_j^2}{\lambda_j^2}\left(t^2 + \frac{2t}{\lambda_j} + \frac{2}{\lambda_j^2}\left(1 - e^{\lambda_j t}\right)\right)\right)^{\frac{1}{2}}$
- *if* $\lambda_j = 0$: $\delta_j(t) = \left(\delta^2 e^t + C_j^2(-t^2 - 2t + 2(e^t - 1))\right)^{\frac{1}{2}}$
- *if* $\lambda_j > 0$: $\delta_j(t) = \left(\delta^2 e^{3\lambda_j t} + \frac{C_j^2}{3\lambda_j^2}\left(-t^2 - \frac{2t}{3\lambda_j} + \frac{2}{9\lambda_j^2}\left(e^{3\lambda_j t} - 1\right)\right)\right)^{\frac{1}{2}}$

Application to guaranteed integration



Control synthesis





Building ventilation

[Meyer, Nazarpour, Girard, Witrant, 2014]

Dynamics of a four-room apartment:

$$\frac{dT_i}{dt} = \sum_{j \in \mathcal{N}^*} a_{ij}(T_j - T_i) + \delta_{s_i} b_i (T_{s_i}^4 - T_i^4) + c_i \max\left(0, \frac{V_i - V_i^*}{\bar{V}_i - V_i^*}\right)(T_u - T_i).$$

Control inputs: V_1 and V_4 can take the values 0V or 3.5V, and V_2 and V_3 can take the values 0V or 3V

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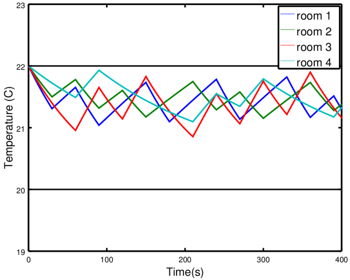

Perturbed Euler scheme

$$\dot{x} = f_j(x, d),$$

where d belongs to a given (compact) set D

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(H_D) (Robustly OSL): for all $j \in U$, there exist constants $\lambda_j \in \mathbb{R}$ and $\gamma_j \in \mathbb{R}_{>0}$ such that $\forall x, x' \in S$ and $\forall y, y' \in D$, the following expression holds

$$\langle f_j(x, y) - f_j(x', y'), x - x' \rangle \leq \lambda_j \|x - x'\|^2 + \gamma_j \|x - x'\| \|y - y'\|.$$

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Links can be made with incremental ISS stability

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(H1) f behaves polynomially: there exist constants $D \in \mathbb{R}_{\geq 0}$, $q \in \mathbb{N}$ such that, for all $x, y \in \mathbb{R}^d$

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(H2) g is Lipschitz: there exists $L_g \in \mathbb{R}_{\geq 0}$ such that, for all $x, y \in \mathbb{R}^d$

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(H3) the SDE satisfies the following one-sided Lipschitz (OSL) condition with constant $\lambda \in \mathbb{R}$:

$$\exists \lambda \in \mathbb{R}\ \forall x, y \in \mathbb{R}^d : \ \langle f(y) - f(x), y - x\rangle \leq \lambda\,\|y - x\|^2$$

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Standard Euler scheme for SDEs: the Euler-maruyama scheme

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More stable version: the tamed Euler scheme

$$\underline{X}_{n+1,z}^N = \underline{X}_{n,z}^N + \frac{\frac{\tau}{N} \cdot f(\underline{X}_{n,z}^N)}{1 + \frac{\tau}{N} \cdot \|f(\underline{X}_{n,z}^N)\|} + g(\underline{X}_{n,z}^N)(W_{\frac{(n+1)\tau}{N}} - W_{\frac{n\tau}{N}})$$

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Time interpolation to get a continuous trajectory:

$$\tilde{X}^N_{t,z} = \tilde{X}^N_{n,z} + \frac{(t - n\tau/N) \cdot f(\tilde{X}^N_{n,z})}{1 + \tau/N \cdot \|f(\tilde{X}^N_{n,z})\|} + g(\tilde{X}^N_{n,z})(W_t - W_{\frac{n\tau}{N}})$$

Main result (2)

Theorem

Given the SDE system satisfying (H1)-(H2)-(H3). Let $\delta_0 \in \mathbb{R}_{\geq 0}$. *Suppose that* z *is a random variable on* $\mathbb{R}^d$ *such that* $\mathbb{E}[\|x_0 - z\|^2] \leq \delta_0^2$. *Then, we have, for all* $\tau \geq 0$*:*

$$\mathbb{E}[\sup_{0 \leq t \leq \tau} \|X_{t,x_0} - \tilde{X}_{t,z}\|^2] \leq \delta_{\tau,\delta_0}^2,$$

with $\delta_{\tau,\delta_0}^2 := \beta(\tau) e^{\gamma\tau}$, *where:*

$$\gamma = 2(\sqrt{\Delta_t} + 2\lambda + L_g^2 + 128 L_g^4)$$

$$\beta(\tau) = 2\delta_0^2 + 2\tau\Delta_t L_g^2(1 + 128 L_g^2)(F_{2,z} d + E_{2,z}\Delta_t) + 4\tau\sqrt{\Delta_t} D(F_{4,z} d + E_{4,z}\Delta_t^2)^{\frac{1}{2}}$$
$$(1 + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\underline{X}_{t,z}\|^{2q} + 4\mathbb{E} \sup_{0 \leq t \leq \tau} \|\tilde{X}_{t,z}\|^{2q})^{\frac{1}{2}}.$$

with $\Delta_t = \tau / N$.

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$$\sup_{0 \leq t \leq \tau} \mathbb{E}\|\underline{X}_{t,z} - \tilde{X}_{t,z}\|^{r} \leq (\Delta_t)^{\frac{r}{2}} (E_{r,z} (\Delta_t)^{\frac{r}{2}} + F_{r,z} d).$$

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Main tools of the proof

- [Hutzenthaler et al., 2012]

$$\sup_{N\in\mathbb{N}} \sup_{n\in\{0,1,\dots,N\}} \mathbb{E}[\|\underline{X}^N_{n,z}\|^p] < \infty$$

- [Lemma 4.3, Higham et al., 2002] For any even integer $r \geq 2$, there exist two constants $E_{r,z}$ and $F_{r,z}$ such that

$$\sup_{0\leq t\leq\tau} \mathbb{E}\|\underline{X}_{t,z} - \tilde{X}_{t,z}\|^r \leq (\Delta_t)^{\frac{r}{2}}(E_{r,z}(\Delta_t)^{\frac{r}{2}} + F_{r,z}d).$$

  with $\Delta_t = \tau/N$
- [Theorem 4.4, Higham et al., 2002]
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Example

[Zamani, Abate, Girard, 2015]

$$dx_1 = (-0.25x_1 + x_2 + 0.25)dt + 0.05x_1 dW_t^1$$
$$dx_2 = (-2x_1 - 0.25x_2 - 2)dt + 0.05x_2 dW_t^2$$

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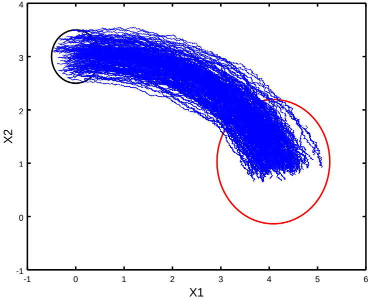
The program gives (for $\tau = 1$, $\Delta_t = \tau/10^4$): $q = 0$, $D = 1.36$, $L_g = 0.05$, $\lambda = 0.25$; and for $z = (-4, -3.8)$: $E_{2,z} = 893.3$, $E_{4,z} = 2.14 \cdot 10^5$, $F_{2,z} = 0.002$, $F_{4,z} = 4.9 \cdot 10^{-6}$.



Control synthesis

We now consider a finite number of SDEs. Each SDE is referred to as a *mode* j, and the set of modes is referred to as $U = \{1, \dots, M\}$. We will denote by X^j_{t,x_0} the solution at time t of the system:

$$\begin{aligned} \mathrm{d}x(t) &= f_j(x(t)) + g_j(x(t))\mathrm{d}W^j_t, \\ x(0) &= x_0. \end{aligned} \tag{1}$$

Control synthesis

Exhibit a finite set of points $z_1, \dots, z_p$ of S, and a positive real $\delta_0 > 0$ such that:

1. all the balls $B(z_i, \delta_0)$, $i = 1, \dots, p$, cover R, and are included into S (i.e. $R \subseteq \bigcup_{i=1}^{p} B(z_i, \delta_0) \subseteq S$);
2. for each $i = 1, \dots, p$, there is a pattern π of length k such that:
   - $B_{i,\pi,t} \subseteq S$ for $t = \tau, 2\tau, \dots, (k-1)\tau$, and
   - $B_{i,\pi,t} \subseteq R$ for $t = k\tau$.

   where $B_{i,\pi,t} := B(\mathbb{E}\tilde{X}^{\pi}_{t,z_i}, \delta^{\pi}_{t,\delta_0})$.

Application

[Zamani, Abate, Girard, 2015]

$dx_1 = (-0.25x_1 + ux_2 + (-1)^u 0.25)dt + 0.01x_1 dW_t^1$
$dx_2 = ((u-3)x_1 - 0.25x_2 + (-1)^u(3-u))dt + 0.01x_2 dW_t^2$
where $u = 1, 2$.

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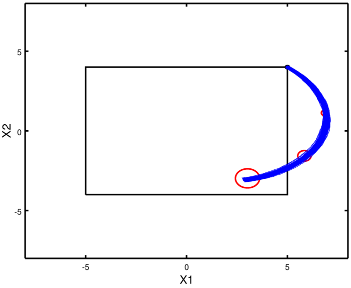
For $\tau = 0.5$, $\Delta_t = 10^{-4}$, one finds (for all modes $u = 1, 2$):
$q = 0$, $D = 1.36$, $L_g = 0.01$, $\lambda = 0.25$; for $z = (-4, -3.8)$: $E_{2,z} = 893.31$, $E_{4,z} = 2.14 \cdot 10^5$, $F_{2,z} = 0.002$, $F_{4,z} = 4.9 \cdot 10^{-6}$; and for $z = (0, 3)$: $E_{2,z} = 543.22$, $E_{4,z} = 7.94 \cdot 10^4$, $F_{2,z} = 0.0442$, $F_{4,z} = 0.00178$.

Pattern: $2 \cdot 2 \cdot 2$



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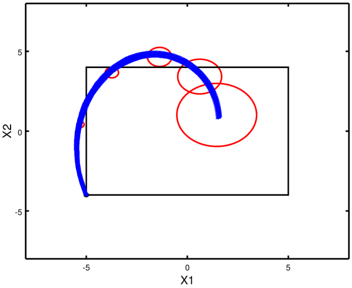

Other application: the slit problem

[Morzfeld, 2015]

$$dX = udt + dW, \qquad X_0 = 1$$

with modes $u \in \{-6, -5, -4, -3, -2, 1, 0, 1, 2, 3, 4, 5, 6\}$.

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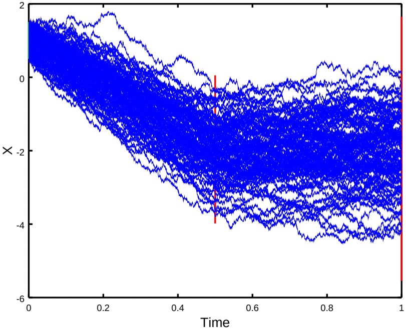

Conclusions and future work

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Future work:

- Experiments with drift function behaving polynomially
- Extend the bound to a probability result
- Find a bound for the variance

Thank you!