Compositional Analysis of Boolean Networks Using Local Fixed-Point Iterations

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Outline

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Synchronous Boolean Networks

A Boolean network (BN) is a discrete-time dynamical system subject to the rules

$$x(t+1) = f(x(t)) \tag{1}$$

where

- x is a vector of n Boolean variables and
- f is a vector of n Boolean functions on these variables.

Let $S = \{0,1\}^n$ be the set of all instances of x. We have: $f : S \to S$.

Decomposition

We suppose x decomposed into two vectors x_1 and x_2 of n_1 and n_2 components resp. ($n = n_1 + n_2$).

Let $S_1 = \{0,1\}^{n_1}$ and $S_2 = \{0,1\}^{n_2}$ be the sets of all instances of x_1 and x_2 resp. We have: $S = S_1 \times S_2$.

The BN can be written:

$$x_1(t+1) = f_1(x_1(t), x_2(t)) \quad (2)$$

$$x_2(t+1) = f_2(x_1(t), x_2(t)) \quad (3)$$

with $f_1 : S_1 \times S_2 \to S_1$ and $f_2 : S_1 \times S_2 \to S_2$.

NB: Way of partioning x, e.g., by exploiting the oriented graph associated to f, out of the scope of this work (see, e.g., [Akutsu]).

Example

Consider $x = (A, B, C, E, F, G, H, I)$,
$x_1 = (A, F, G, H, I)$, $x_2 = (B, C, E)$
and the functions f_1 and f_2 defined by the systems:

$$\boxed{\begin{array}{l} A(t+1) = 1 \wedge H(t) \\ F(t+1) = E(t) \wedge (E(t) \vee G(t)) \\ G(t+1) = 1 \wedge (B(t) \vee E(t)) \\ H(t+1) = F(t) \wedge (F(t) \vee G(t)) \\ I(t+1) = H(t) \wedge (H(t) \vee I(t)) \end{array}}$$

$$\boxed{\begin{array}{l} B(t+1) = A(t) \wedge (A(t) \vee C(t)) \\ C(t+1) = I(t) \\ E(t+1) = 1 \wedge C(t) \wedge (C(t) \vee F(t)) \end{array}}$$

Attractors

- Major interest of BNs: Finding its cycles.
  In industrial case studies, such as railway interlocking, one for example wants to show no cycle of length >1.

- Since f deterministic and S finite, every derivation ends to a cycle.
  The set of elements composing this cycle is called an attractor.
  An attractor of length 1 is called a stationary state

- Complexity of finding all the attractors of a BN is NP-hard [zhang2007].

Let F^* denote the union of all the attractors of the BN defined by f.

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Lifting f to the powerset

- The "lifted" version $f : 2^S \to 2^S$ is defined, for all $X \subseteq S$, by:

$$f(X) = \{y \mid y = f(x) \text{ for } x \text{ in } X\}$$

- A fixed-point of f is a set $X \subseteq S$ s.t. $f(X) = X$. By Knaster-Tarski th., there is a greatest fixed-point of f given by:

$$gfp(f) = \bigcap_{k \geq 0} f^k(S)$$

- <u>Proposition:</u>
  1. The set F^* of attractors is given by: $F^* = gfp(f)$
  2. If $X \supseteq F^*$ then $F^* = \bigcap_{k \geq 0} f^k(X)$

Constructing an overapproximation of F^*

- Abstraction ("separation"): $\alpha : 2^S \to 2^{S_1} \times 2^{S_2}$ is defined, for all set $X \subseteq S$, by:

$$\alpha(X) = (\pi_1(X), \pi_2(X))$$

  where π_1 and π_2 are the 1st and 2nd projection of S to S_1 and S_2

- Concretization ("glueing"): $\gamma : 2^{S_1} \times 2^{S_2} \to 2^S$ is defined, for all sets $X_1 \subseteq S_1$, $X_2 \subseteq S_2$ by:

$$\gamma(X_1, X_2) = X_1 \times X_2$$

Proposition: α and γ satisfy the properties of a Galois connection:

- $\alpha(\gamma(X_1, X_2)) \subseteq (X_1, X_2)$, for all $X_1 \subseteq S_1$, $X_2 \subseteq S_2$
- $\gamma(\alpha(X)) \supseteq X$, for all $X \subseteq S$

Constructing an overapproximation of F^* (cont'd)

- Abstract function $\tilde{f}$
  Let $\tilde{f} : 2^{S_1} \times 2^{S_2} \to 2^{S_1} \times 2^{S_2}$ defined by $\tilde{f} = \alpha f \gamma$, i.e., for all $X_1 \subseteq S_1$ and $X_2 \subseteq S_2$:

$$\tilde{f}(X_1, X_2) = (f_1(X_1, X_2), f_2(X_1, X_2))$$

- By Cousot's theorem, we have:

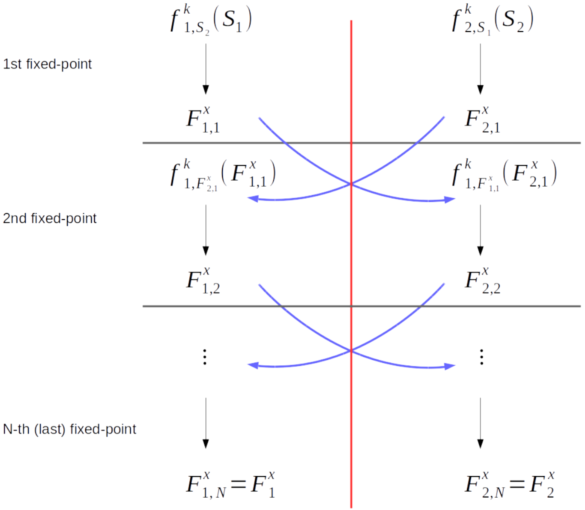
$\gamma(gfp(\tilde{f})) \supseteq gfp(f)$, hence: $\gamma(\bigcap_{k \geq 0} \tilde{f}^k((S_1, S_2))) \supseteq F^*$

Let us denote $\bigcap_{k \geq 0} \tilde{f}^k((S_1, S_2))$ by (F_1^*, F_2^*). We have:

Proposition:

1. $F_1^* \times F_2^* \supseteq F^*$
2. $F^* = \bigcap_{k \geq 0} f^k(F_1^* \times F_2^*)$

Construction of F_1^*, F_2^* by "iterative reduction"



Extensions

1. Decomposition of the BN into more than 2 sub-systems
2. Basins of attraction
3. Controlled Boolean networks ($x(t+1) = f(x(t), u(t))$)
4. Use of the ℓ-th power $g = f^\ell$ of f in order to refine the over-approximation, i.e., use of $\tilde{g} = \tilde{f}^\ell$ and computation of $G_1^* \times G_2^* \subseteq F_1^* \times F_2^*$

Example of (A,B,C,E,F,G,H,I) system

S has 256 elements. Let $\ell = 2$

$G_{1,0} = S_1$	$G_{2,0} = S_2$
$G_{1,1} = \{00000, 00001, 00100, \ldots, 11110\}$	$G_{2,1} = \{000, 001, 011, 101\}$
$G_{1,2} = \{10000, 11000, 11010, \ldots, 10100\}$	$G_{2,2} = G_{2,1}$
$G_{1,3} = G_{1,2}$	$G_{2,3} = G_{2,1}$

Hence $G_1^* \times G_2^*$ has $9 \times 4 = 36$ elements.

$F^* = f^{10}(G_1^* \times G_2^*)$ has 8 elements (unique cycle).

Comparison with other compositional methods for BNs

Our method

1. Partitioning the set of variables
2. Iterative reduction: repeated computation of local fixed-points (F_1^* and F_2^*)
3. Global f-reduction of $F_1^* \times F_2^*$

Closer related method (e.g. [Guo-Yang-Wu-Le-Sun 2014])

1. Decomposing the set of variables with common shared variables
2. Finding local attractors for each subsystem
3. Recombining local attractors in order to maintain compatibility for shared variables

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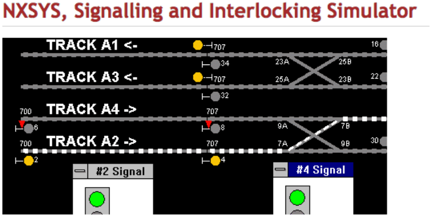
Application to railways interlocking (NXSYS)



- 50 Boolean variables
- Decomposition into 4 subsystems
- Computation of $F_1^* \times F_2^* \times F_3^* \times F_4^*$ takes 2 hours
- Computation of F^* by subsequent f-applications takes 12 hours.
- $|F^*| = 24M$

NB: Other tools of the state-of-art seem slower/unable to finding the attractors of such a BN with high node connectivity (indegree > 7)

Contribution and future work

Contribution

- First application of "iterative reduction" compositionality method to compute attractors of BNs
- Successful application to a railways example with 50 variables

Future work

- parallel implementation
- symbolic representation (BDD, SAT)
- application to a French railways station model (200 variables)