Kerr and Schwarzschild spacetimes derived differently

Adrien Sasportes

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4 Deriving Schwarzschild differently

In this section we will be interested in trying to find Schwarzschild differently using the general Kerr-Shild form :

$$g_{ij} = \eta_{ij} + 2H(r)k_i k_j$$

The Schwarzschild solution is a really important solutionin general relativity. It was discovered a little more than a month after Einstein had formulated General Relativity and it describes the gravitationnal field outside a static spherical mass. This solution can be used as an approximation for slowly rotating spherical astronomical object, like the Earth.

4.1 Classical way of finding the Schwarzschild solution

The usual way of finding the Schwarzschild metric usually start by writing the most general symmetric metric in vacuum :

$$ds^2 = -e^{A(r)}dt^2 + e^{B(r)}dr^2 + r^2 d\Omega^2$$

Then we compute the Einstein tensor, and we solve the Einstein vacuum field equation $G_{ab} = 0$ by taking $G_{00} = 0$ and $G_{11} = 0$. Then we obtain the following equations :

$$e^B + rB' - 1 = 0$$
$$-e^B + rA' + 1 = 0$$

From which we get :

$$e^A = e^{-B} = 1 - \frac{C}{r}$$

And thus we have the Schwarzschild solution :

$$ds^2 = -(1 - \frac{C}{r})dt^2 + (1 - \frac{C}{r})^{-1}dr^2 + r^2 d\Omega^2$$

This method of finding the Schwarzschild solution is work perfectly way but still it is always interesting to find an alternative way to find it.

4.2 Elements of the Kerr-Shild for the Schwarzschild case

We now want to focus on a metric of the following form $g_{ij} = \eta_{ij} + 2H(r)k_i k_j$. First in the Schwarzschild case we consider a spherical object so we consider some r such that $r^2 = x^2 + y^2 + z^2$. η_{ij} is known as the Minkowski metric defines as $\eta_{ij} = diag(+1, +1, +1, -1)$ As for the H we guess that it would only depend on r. We define $k_i = (-1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r})$, a null geodesic. We notice that $k_i k^i = 0 <=> r^2 = x^2 + y^2 + z^2$. Looking at the x, y and z components of k, k looks like a radial vector, but we have a -1 in the t direction.

4.3 Einstein vacuum field equation

From the formula we derived for G_{ij} for metric of this form we can fairly easily obtain the Einstein tensor. Looking at $G_{i0}, i = 1, 2, 3$ we get :

$$G_{i0} = -\frac{8H(r)\left(r^2 H'(r) + rH(r)\right)}{\left(r^2\right)^{3/2}} \tag{1}$$

Our goal is to find $H(r)$ that solves the Einstein vacuum field equation $G_{ij} = 0$ which in our case is $G_{i0} = 0, i = 1, 2, 3$ or :

$$-\frac{8H(r)\left(r^2 H'(r) + rH(r)\right)}{\left(r^2\right)^{3/2}} = 0 \tag{2}$$

Which in general is equivalent to :

$$rH'(r) + H(r) = 0 \tag{3}$$

This equation is quite nice and can easily be solved as follow :

$$
\begin{aligned}
rH'(r) + H(r) &= 0 \\
r\frac{d}{dr}H(r) &= -H(r) \\
\frac{1}{H(r)}\frac{d}{dr}H(r) &= -\frac{1}{r} \\
\frac{1}{H(r)}dH(r) &= -\frac{1}{r}dr \\
&\text{, integrating,} \\
log(H(r)) &= -\log r + C \\
&\text{, with C constant} \\
H(r) &= \frac{C}{r}
\end{aligned}
$$

Giving us :

$$g_{ij} = \eta_{ij} + \frac{2C}{r}k_i k_j \tag{4}$$

This is the Schwarzschild solution. As it is not so obvious in this form, in the next sub-section, we will put in back in the "classical form".

4.4 Back to the classical form of the Schwarzschild solution

In order to be able to find the classical Schwarzschild form we should first write the line element of the metric we found explcitly :

$$
\begin{aligned}
ds^2 &= g_{ij}dX^i dX^j \\
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2C}{r}(-dt + \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz)^2 \\
ds^2 &= (\frac{2C}{r} - 1)dt^2 + (\frac{2C}{r} + \frac{x^2}{r^2})dx^2 + (\frac{2C}{r} + \frac{y^2}{r^2})dy^2 + (\frac{2C}{r} + \frac{z^2}{r^2})dz^2 \\
&\quad + \frac{4C}{r}(dxdy + dxdz + dydz - dtdx - dtdy - dtdz)
\end{aligned}
$$

5 Deriving Kerr differently

Now we want to use the same method we used for the Schwarzschild solution to find the Kerr solution. The Kerr metric is another exact solution of the Einstein field equations found in 1963 by Roy P. Kerr. It is a generalisation of

the Schwarzschild solution, it describes an axially-symmetric rotating object. This metric is often used to describe rotating black-holes.
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5.1 Classical way of finding the Kerr solution

6 Conclusion