Solving the Vasicek model for reversion to the mean of interest rates.

Reminder: Ito Lemma: If

$$dX = a(X,t)dt + b(X,t)dW$$

Then

$$dg(X,t) = \left(ag_x + \frac{1}{2}b^2 g_{xx} + g_t\right) dt + bg_x dW \ .$$

The Vasicek model is

$$dX = \alpha(r - X)dt + sdW$$

Look at $g(X,t) = e^{\alpha t}(X - r)$. From Ito:

$$dg \ = \ (\alpha(r - X)e^{\alpha t} + \alpha e^{\alpha t}(X - r))dt + se^{\alpha t}dW \ = \ se^{\alpha t}dW \ .$$

Integrating, we have

$$\begin{aligned} e^{\alpha t}(X(t) - r) - (X(0) - r) &= s\int_0^t e^{\alpha u}dW(u) \\ &= s\left(e^{\alpha t}W(t) - \alpha\int_0^t W(u)e^{\alpha u}du\right) \ . \end{aligned}$$

To get the last line we have used Ito again, with "g" equal to $e^{\alpha t}W$ (and $X = W$). Rearranging gives

$$X(t) = r + (X(0) - r)e^{-\alpha t} + s\left(W(t) - \alpha\int_0^t W(u)e^{-\alpha(t-u)}du\right) \ .$$

Notes:

1. There is a problem in this model that X can become negative.
2. Everyone not in finance calls the process the "Ohrnstein-Uhlenbeck" process — it has many applications outside finance.