Amortized Time Complexity of Union-Find in Isabelle/HOL

Colloquium to the Bachelor's Thesis

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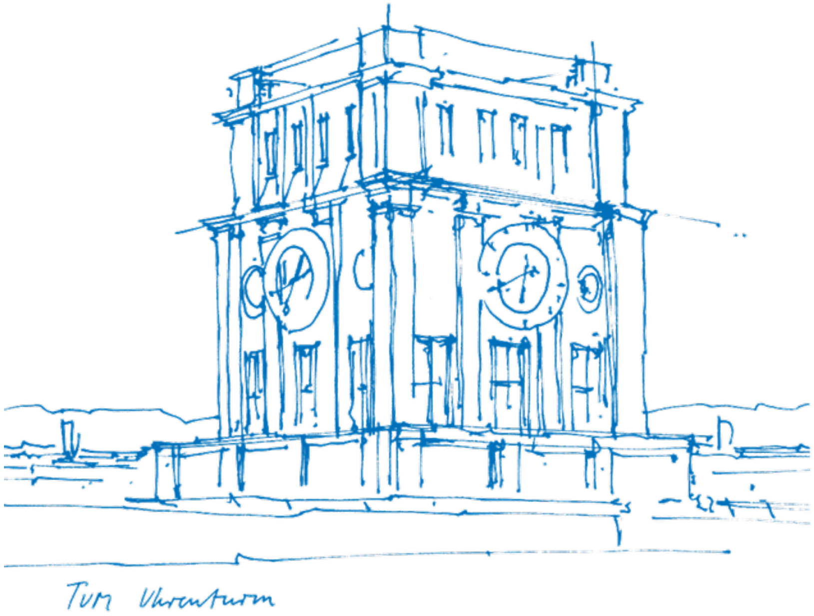

Overview

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Union-Find

- Models a partial equivalence relation over a finite domain
- Implemented by disjoint set forests
- The graph structure is represented by an array
- Supports **Union** and **Find** operations

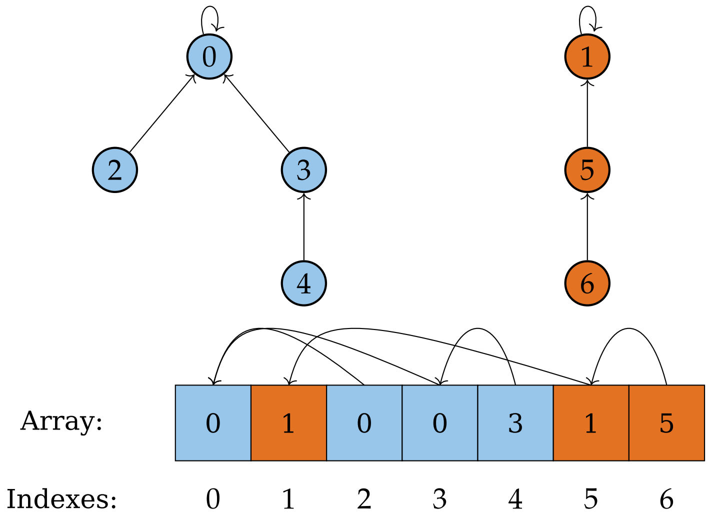


Figure: Two equivalence classes

Path Compression and Union by Rank

- As with many tree-based data structures, trees should be kept flat
- This is done in two ways:
  1. **Path Compression**
  2. **Union by Rank**

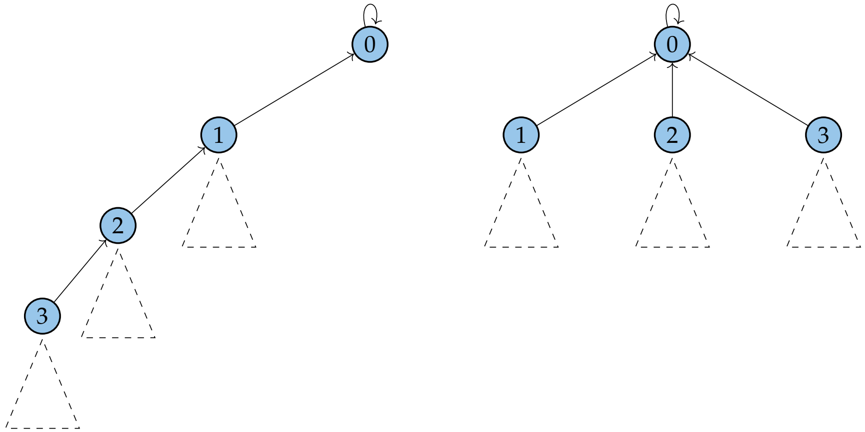


Figure: Illustration of Path Compression

Design Choices

- Done in Imperative/HOL, which can then be exported to several languages
- The arrays used and therefore the domain are static
- We define the operations in Imperative/HOL:
  - **partial_function** (heap) uf_rep_of :: nat array ⇒ nat ⇒ nat Heap
  - **partial_function** (heap) uf_compress :: nat ⇒ nat ⇒ nat array ⇒ unit Heap
  - **definition** uf_rep_of_c :: nat array ⇒ nat ⇒ nat Heap
  - **definition** uf_cmp :: uf ⇒ nat ⇒ nat ⇒ bool Heap

```
definition uf_union :: uf ⇒ nat ⇒ nat ⇒ uf Heap where
uf_union u i j ≡ do {
      let (r,p) = u;
      ci ← uf_rep_of_c p i;        ci ← uf_rep_of p i;
      cj ← uf_rep_of_c p j;        cj ← uf_rep_of p j;
                  .
                  .
                  .
```

  -

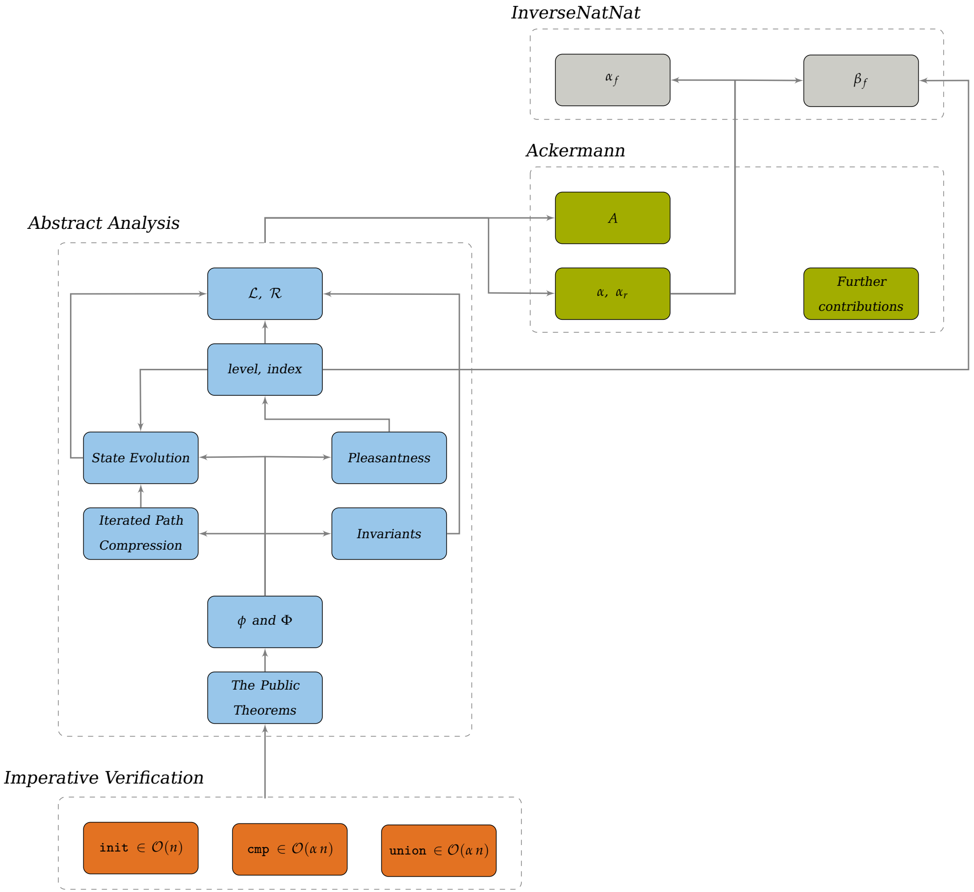
History of the Proof

- Several, slightly different results about the amortized time complexity of the operations of Union-Find
- The first result involving α by Tarjan in 1975. The proof was simplified until the version in CLRS
- In 1989 Fredman and Saks prove (in some sense) the optimality of the result
- The latest result by Alstrup et al. in 2014 tightens the bound
- In 2017, Charguéraud and Pottier formalize the result in Coq
- Now, we present this formalization in Isabelle/HOL
- In all these proofs, but especially since Alstrup et al., most work is needed for the abstract analysis

Timeline of the Project

- **May** — Background work: Familiarisation with Sepreftime and the paper by Charguéraud and Pottier.
- **May 15th** — Official thesis registration.
- **June** — Cleanup of the previous work by Lammich and Haslbeck.
- **June 5th** — Armaël Guéneau visits TUM.
- **June** — **Bottom-up**: Ackermann and InverseNatNat theories and important definitions.
- **July** — First abstract proofs: rank, level, index.
- **July 16th** — **Top-down**: Sanity check of definitions done, skipping the rest and proof of the Hoare-Triples.
- **August** — Exam phase.
- **September** — Parallel writing of the thesis, proof gap-filling and correction of definitions.
- **September 8th** — Complete sound theory.
- **Semptember 13th** — Submission of the thesis.

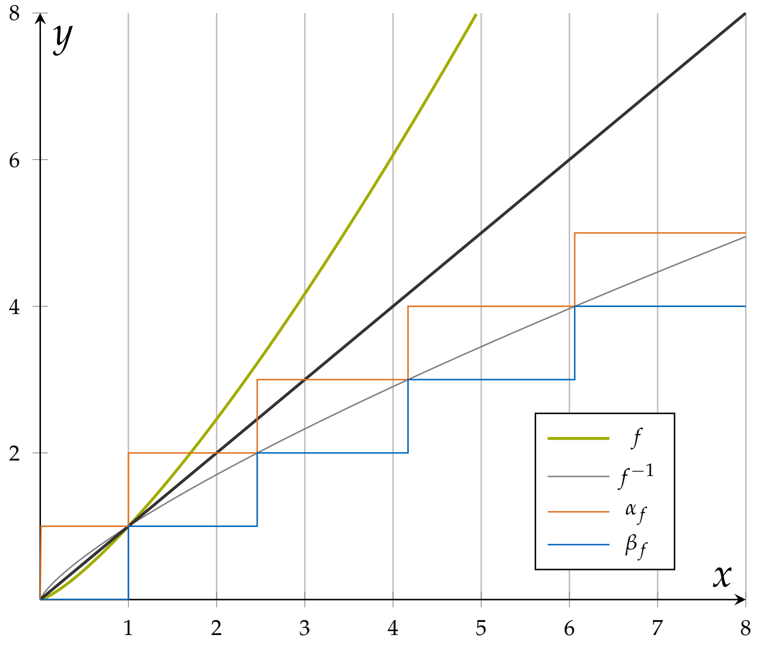
Overview



InverseNatNat
α_f
β_f
Ackermann
A
$\alpha,\ \alpha_r$
Further contributions
Abstract Analysis
$\mathcal{L},\ \mathcal{R}$
level, index
State Evolution
Pleasantness
Iterated Path Compression
Invariants
ϕ and Φ
The Public Theorems
Imperative Verification
init $\in \mathcal{O}(n)$
cmp $\in \mathcal{O}(\alpha\, n)$
union $\in \mathcal{O}(\alpha\, n)$

Inverses of Functions $f : \mathbb{N} \to \mathbb{N}$

- Defined in InverseNatNat.thy
- Upper inverse α_f and Lower inverse β_f
- Lemmas to change the proof obligations between inverses and the original function
- Used for Inverse Ackermann Function, but also the “index” of a node



The Ackermann Function and its Inverse

- There is no single Ackermann Function "A" or inverse Ackermann Function "α"
- All definitions share the property of growing faster than any primitive recursive function
- α grows **very** slowly
- We use the definition of A and α by Tarjan:

Definition
Ackermann Function

$$A\,0\,x = x + 1$$
$$A\,(k+1)\,x = (A\,k)^{(x+1)}\,x$$

Definition
Inverse Ackermann Function

$$\alpha\,n = \min\{k \mid A\,k\,1 \geq n\}$$
$$\alpha_r\,n = 1 + \min\{k \mid A\,k\,r \geq (n+1)\}$$

- In Isabelle/HOL, the definitions are expressed slightly differently and make use of InverseNatNat.thy

The Ackermann Function and its Inverse

Lemma

observable_universe_α

Assume: $n \leq 10^{80}$

$$\alpha\, n \leq 4 \tag{1}$$

- 10^{80} is an estimate of the number of atoms in the universe

Lemma

α_n_0_α_logn

Assume: $16 \leq n$

$$\alpha\, n \leq 1 + (\alpha\, (\log n)) \tag{2}$$

- α and $\alpha(\log n)$ are asymptotically equivalent
- True for every primitive recursive strictly monotonic function

Separation Logic with Time Credits

- Logic to reason about mutable resources in a heap. Enables Hoare-Triple definition
- It also allows for “pure” assertions, predicates independent of the heap content
- Time Credits can also appear in assertions and are required to execute atomic operations
- The components of the logic are:
  - $\uparrow(P)$
  - true and false
  - $p \mapsto_a xs$
  - $P_1 * P_2$
  - $\exists_A x.\, P$
- The Framework by Haslbeck implementing this for Isabelle/HOL, as well as this thesis as a usage example will be published in the AFP.

Amortized Analysis with Time Credits

- We define a potential function Φ that measures the “entropy” of the data structure
- The idea is to “hide” Φ credits in the assertion defining the data structure
- If we are able to prove a Hoare-Triple of the form:

$$\langle \text{invar}(\mathcal{D}) * \$(\Phi(\mathcal{D})) * \underbrace{\$(f(\mathcal{D}))}_{\text{Advertised Cost}} \rangle \quad \texttt{op}(\mathcal{D}) \quad \langle \text{invar}(\mathcal{D}') * \$(\Phi(\mathcal{D}')) \rangle$$

  for any Φ and $f \in \mathcal{O}(g)$, we can conclude that the operation `op` has an amortized cost in $\mathcal{O}(g)$
- Remaining questions:
  - What Φ do you choose? How does Φ evolve?
  - How do you prove the functional correctness?

    $\Longrightarrow$ Mathematical analysis of the data structure

The Potential Function Φ

Definition

Potential for a single node

$$\phi\,\mathcal{L}\,\mathcal{R}\,i := \begin{cases} \alpha_r\,(\mathcal{R}_r\,i)\cdot(1+(\mathcal{R}_r\,i)) & \text{if}\,\mathcal{L}!i = i \\ (\alpha_r\,(\mathcal{R}_r\,i) - \text{level}_{\mathcal{L},\mathcal{R}}\,i)\cdot\mathcal{R}_r\,i - \text{index}_{\mathcal{L},\mathcal{R}}\,i + 1 & \text{if}\,\alpha_r\,(\mathcal{R}_r\,i) = \alpha_r\,(\mathcal{R}_r\,(\mathcal{L}!i)) \\ 0 & \text{otherwise} \end{cases}$$

Definition

Potential of the data structure

$$\Phi\,\mathcal{L}\,\mathcal{R} := \sum_{i=0}^{|\mathcal{L}|-1} \phi\,\mathcal{L}\,\mathcal{R}\,i \tag{3}$$

- $\{0,\ldots,|\mathcal{L}|-1\}$ is in this case the domain of our equivalence relation

Results of the Thesis

The $\mathrm{is\_uf}$ Assertion takes a relation and two arrays as arguments and ensures:

- The well-formedness of the disjoint set forest and the ranks
- That the relation modeled by the array is the one given
- That the necessary potential is stored as Time Credits

Definition

$$
\begin{aligned}
\mathrm{is\_uf}\,\mathcal{X}\,(s,p) :=\; & \exists_A \mathcal{L}\,\mathcal{R}.\, p \mapsto_a \mathcal{L} * s \mapsto_a \mathcal{R} * \\
& \uparrow (\mathrm{ufa\_}\alpha_{\mathcal{L}} = \mathcal{X} \wedge \mathrm{invar\_rank}\,\mathcal{L}\,\mathcal{R}) * \\
& \$(4 \cdot \Phi\,\mathcal{L}\,\mathcal{R})
\end{aligned} \tag{4}
$$

Results of the Thesis

Lemma

$$\text{uf\_cmp\_time} \in \mathcal{O}(\alpha_r\, n)$$

Theorem

$$\langle \text{is\_uf}\, \mathcal{X}\, u * \$(\text{uf\_cmp\_time}\, |\text{Dom}\, \mathcal{X}|) \rangle$$
$$\texttt{uf\_cmp}\, u\, i\, j$$
$$\langle \text{is\_uf}\, \mathcal{X}\, u * \uparrow (r \leftrightarrow (i, j) \in \mathcal{X}) \rangle_t$$

Lemma

$$\text{uf\_union\_time} \in \mathcal{O}(\alpha_r\, n)$$

Theorem

Assumes: $i, j \in \text{Dom}\, \mathcal{R}$

$$\langle \text{is\_uf}\, \mathcal{X}\, u * \$(\text{uf\_union\_time}\, |\text{Dom}\, \mathcal{X}|) \rangle$$
$$\texttt{uf\_union}\, u\, i\, j$$
$$\langle \text{is\_uf}\, (\text{per\_union}\, \mathcal{X}\, i\, j) \rangle_t$$

- `uf_union` corresponds to the **Union** and `uf_cmp` to the **Find** operation
- These theorems follow from abstract results about disjoint set forests
- The whole proof is about 5KLoc long in Isabelle/HOL

Conclusions

- Formalization of the state-of-the-art result about Union-Find
- Comprehensive theory about Ackermann, including quantitative and asymptotic results
- The proofs of the imperative programs are still too long:
  - The automation does not deal well with arithmetic
  - It also instantiates existentials too aggressively
  - Repetitive proofs generated by branching $\Longrightarrow$ optimization possible
- Formal verification of non-trivial results about runtime, not only correctness are possible
- Some overhead, but this helps to reveal hidden assumptions

Thank You for listening!

Any questions?

Important Definitions

Definition
invar_rank

$$
\begin{aligned}
\text{invar\_rank}\,\mathcal{L}\,\mathcal{R} :=& \text{ufa\_invar}\,\mathcal{L} \land & (5)\\
& |\mathcal{L}| = |\mathcal{R}| \land & (6)\\
& (\forall (i, j) \in \text{ufa\_}\beta\text{\_start}_{\mathcal{L}}.\, \mathcal{R}!i < \mathcal{R}!j) & (7)\\
& (\forall i < |\mathcal{L}|.\, \mathcal{L}!i = i \longrightarrow 2^{\mathcal{R}!i} \le |\text{descendants}_{\mathcal{L}}\, i|) & (8)
\end{aligned}
$$

Definition
ufa_invar

$$\text{ufa\_invar}\,\mathcal{L} := \forall i < |\mathcal{L}|.\, i \in \text{Dom}\,\text{rep\_of}_{\mathcal{L}} \land \mathcal{L}!i < |\mathcal{L}| \tag{9}$$

Definition
ufa_β_start

$$\text{ufa\_}\beta\text{\_start}_{\mathcal{L}} := \{(x, y) \mid x < |\mathcal{L}| \land y < |\mathcal{L}| \land x \neq y \land \mathcal{L}!x = y\} \tag{10}$$