Transformaciones en 3D

Fundamentos Matemáticos
Máster en Programación de Videojuegos
Profesor: José María Benito

Vectores

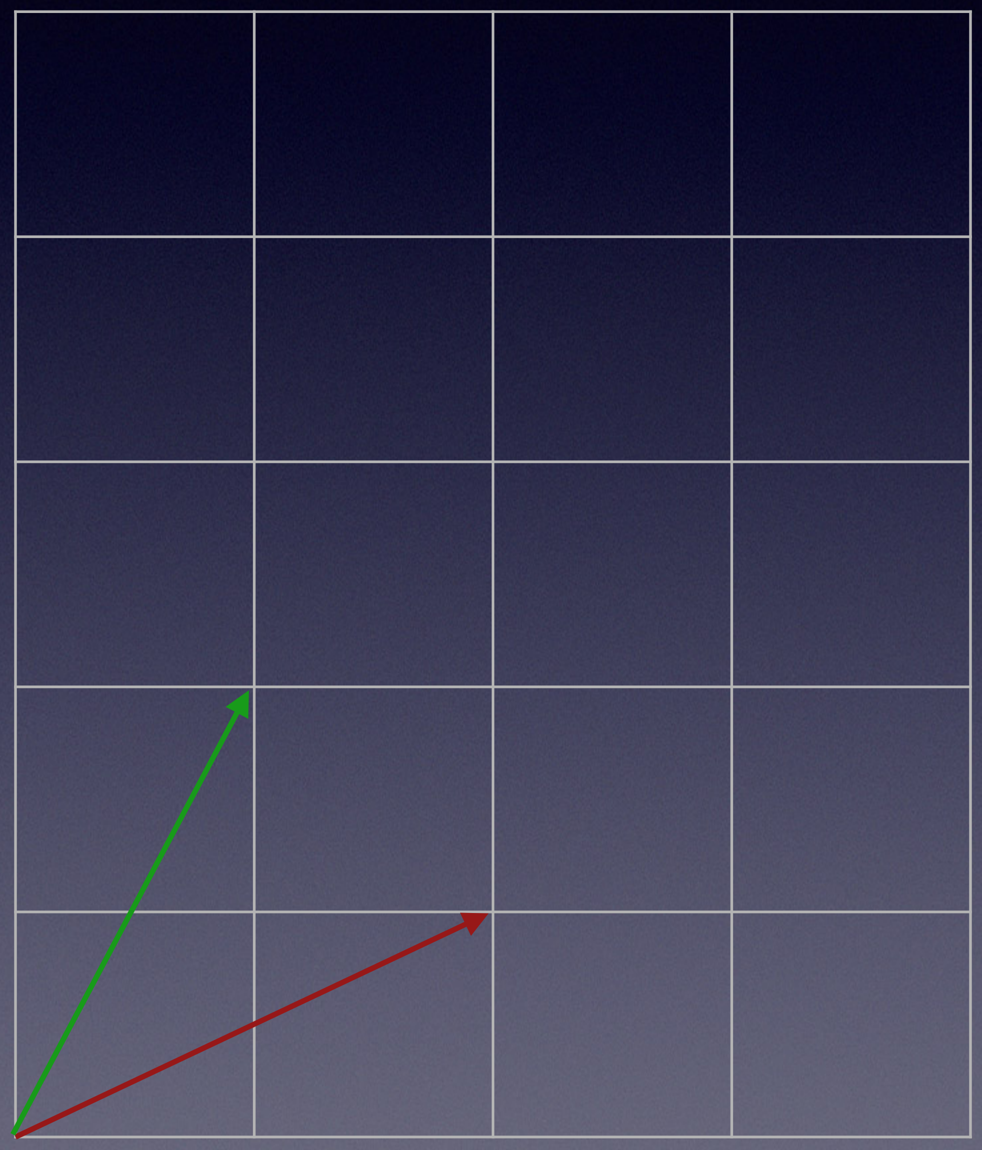
- Surgen de la necesidad de ubicar puntos en el espacio:
- Se pueden extrapolar a 3D (y más)
- Sirven para hacer juegos, p.ej.: "Hundir la Flota"
- v = {x,y,z}

Sumar y Restar

En 3D:

$a = \{x_a, y_a, z_a\}$
$b = \{x_b, y_b, z_b\}$

$s = \{x_a + x_b, y_a + y_b, z_a + z_b\}$
$r = \{x_a - x_b, y_a - y_b, z_a - z_b\}$



Sumar y Restar

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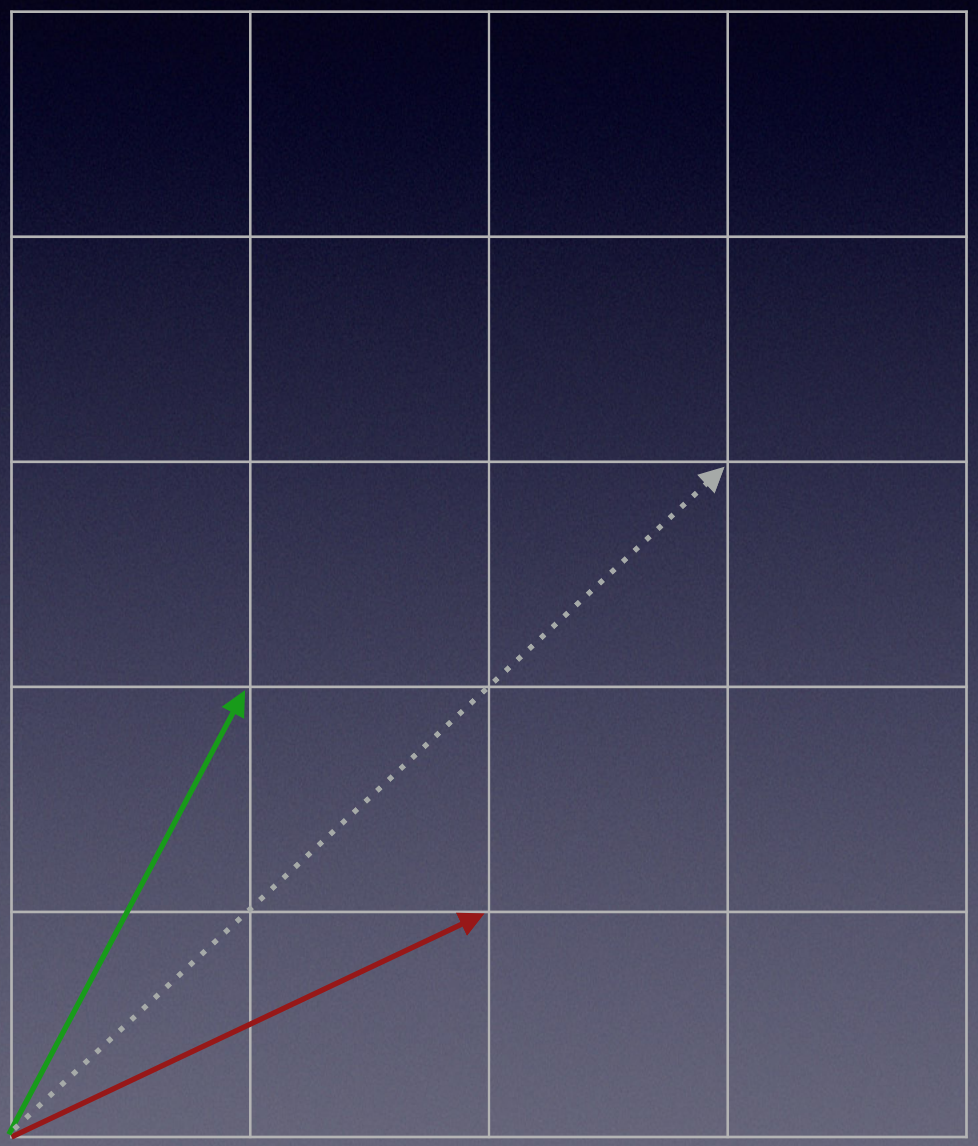

Sumar y Restar

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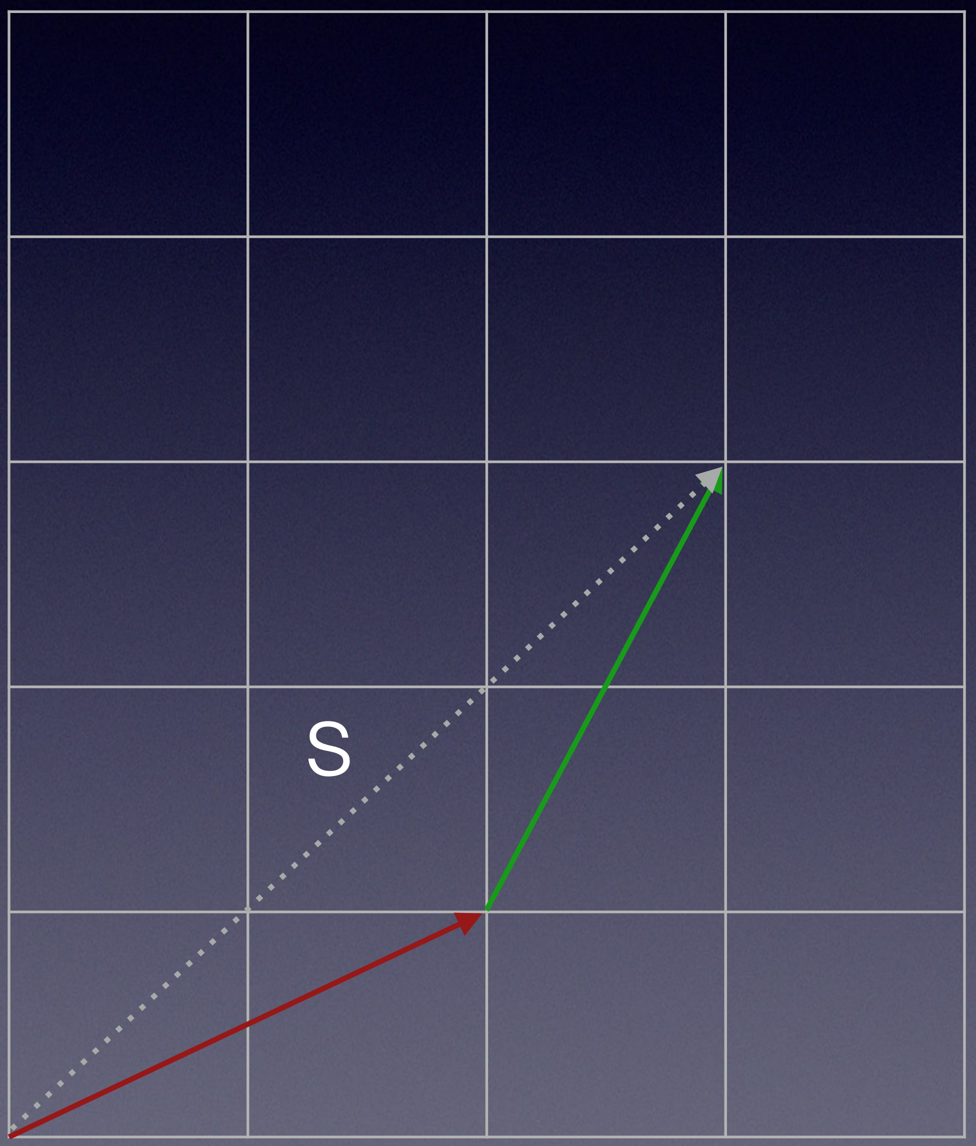

Sumar y Restar

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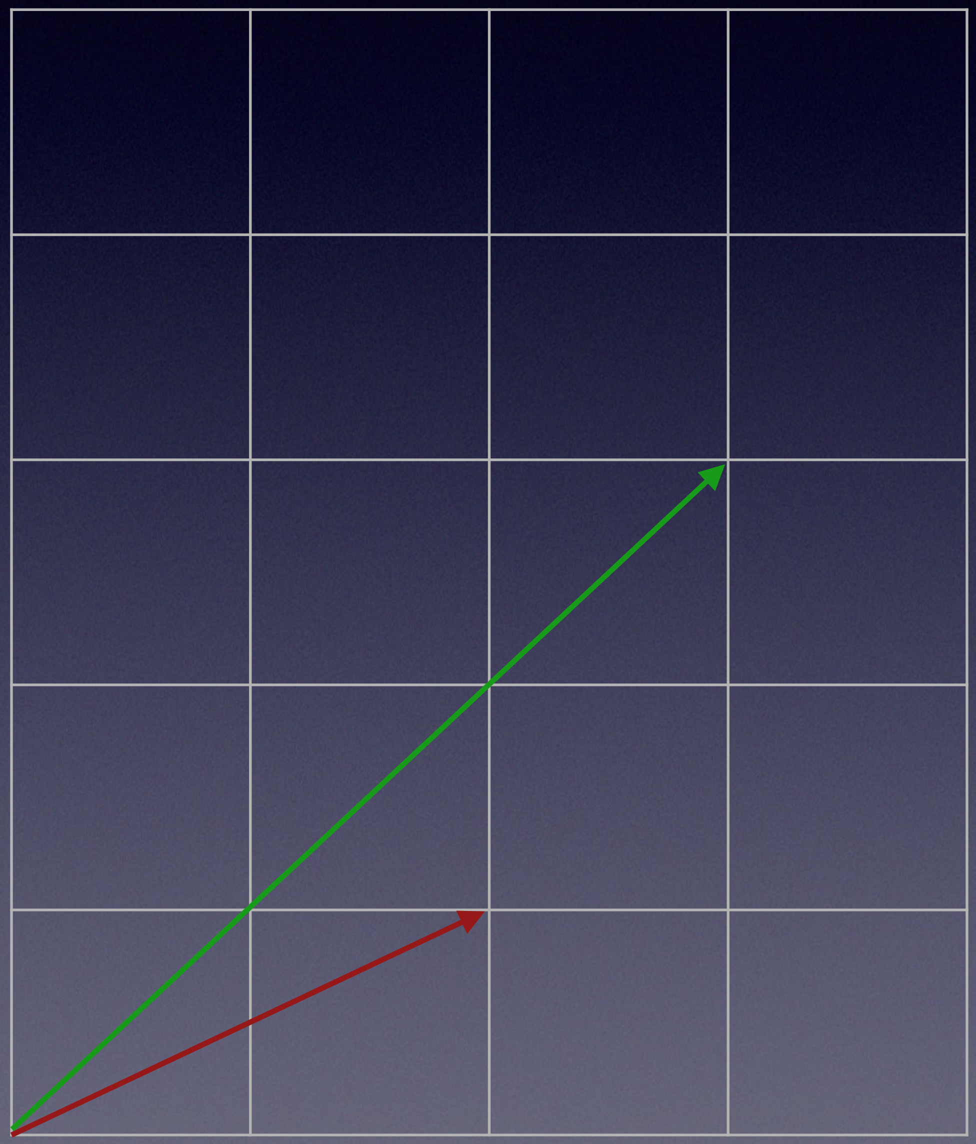

Sumar y Restar

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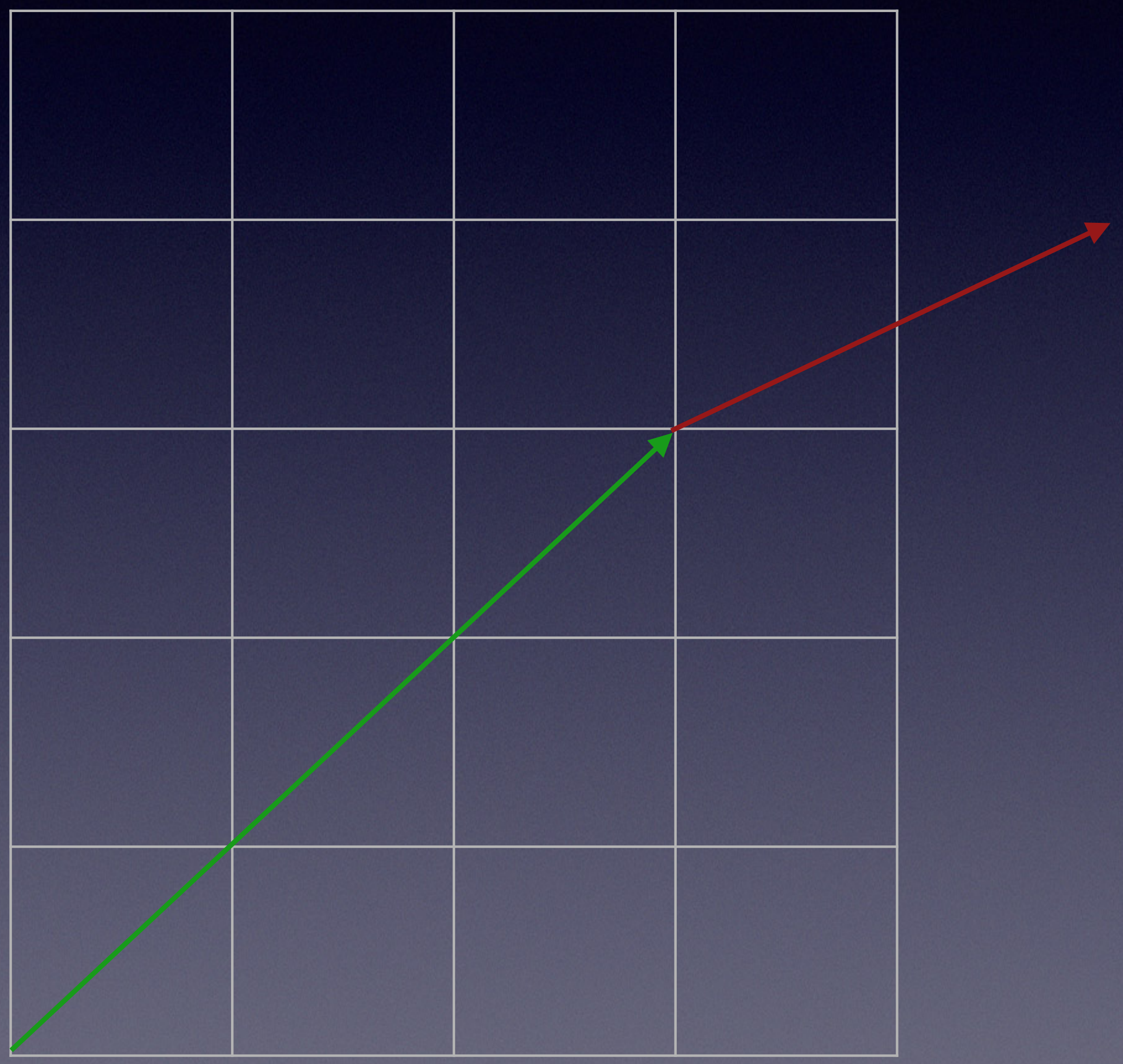

Sumar y Restar

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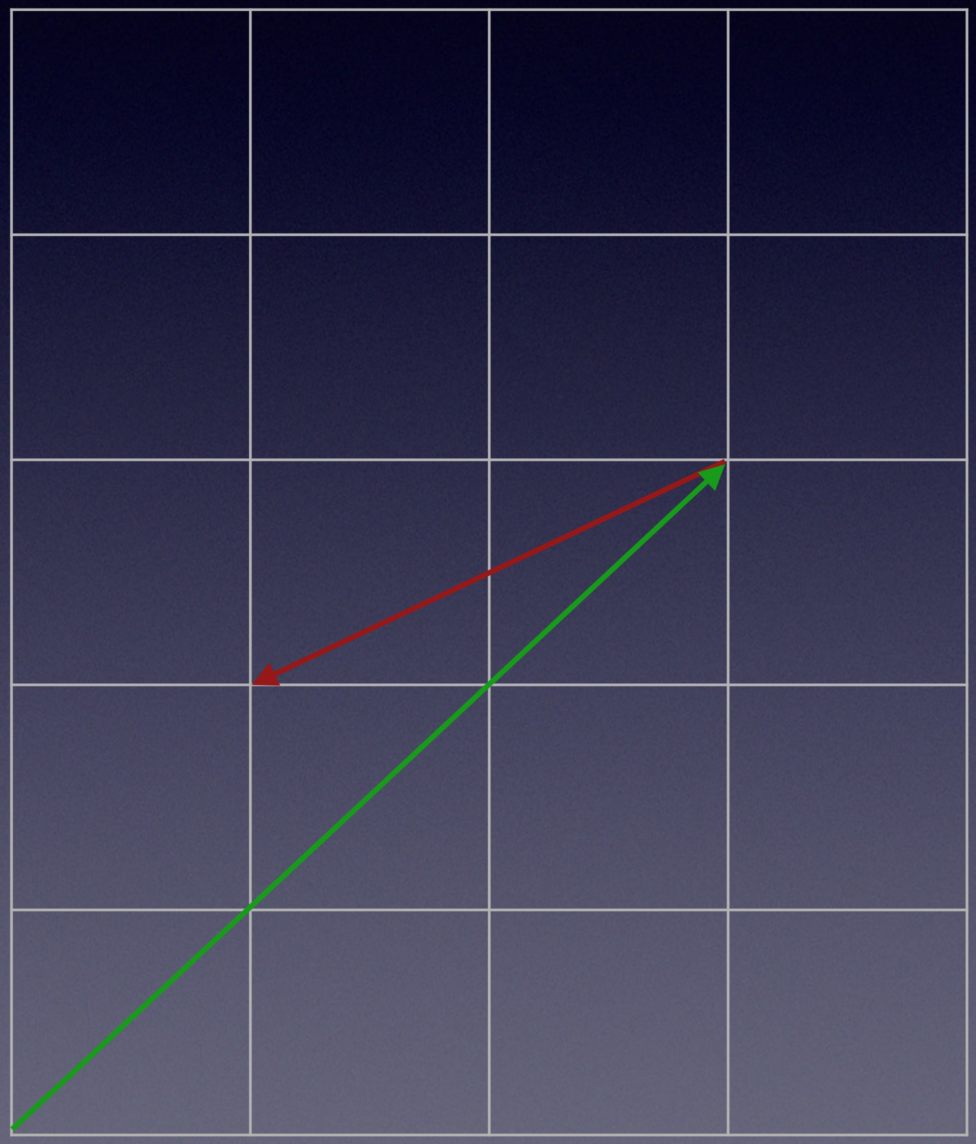

Sumar y Restar

En 3D:

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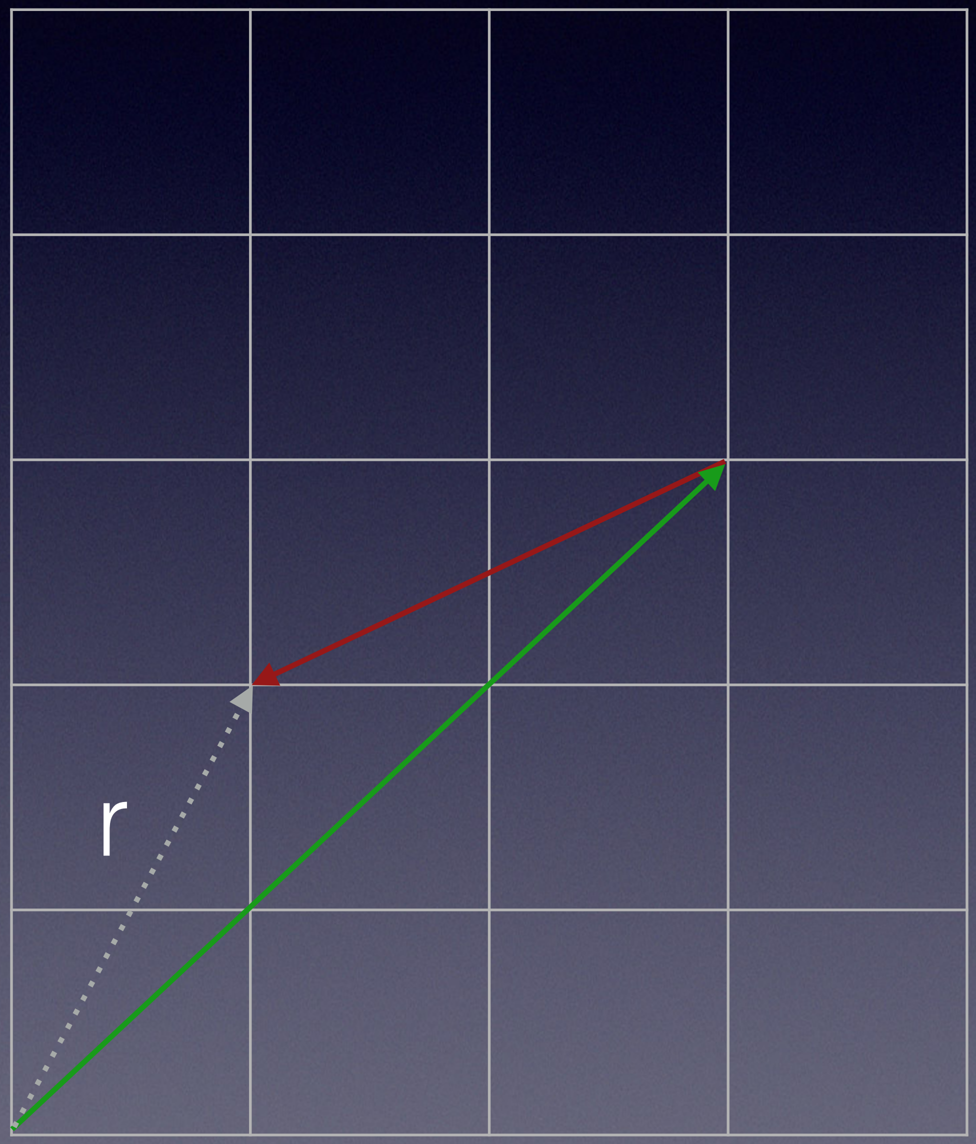

Sumar y Restar

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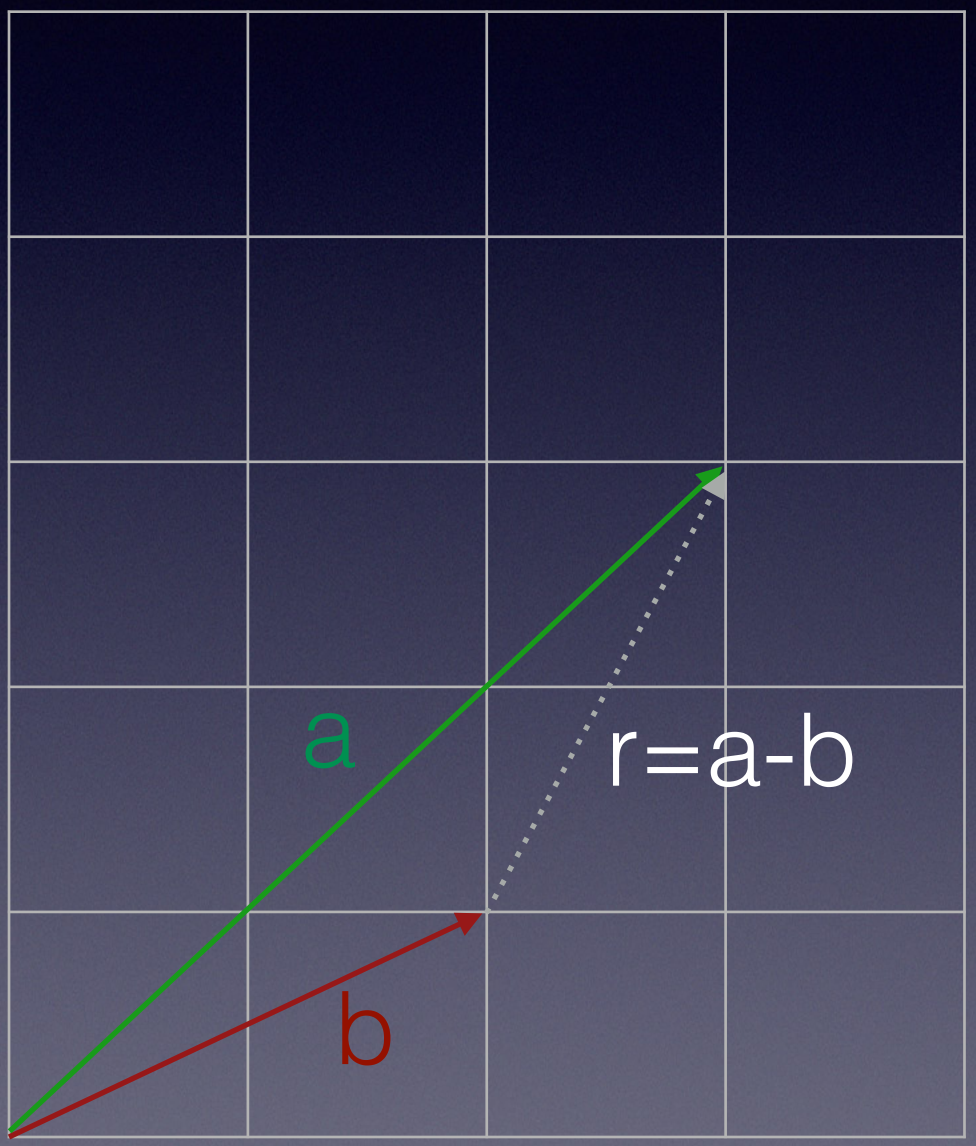
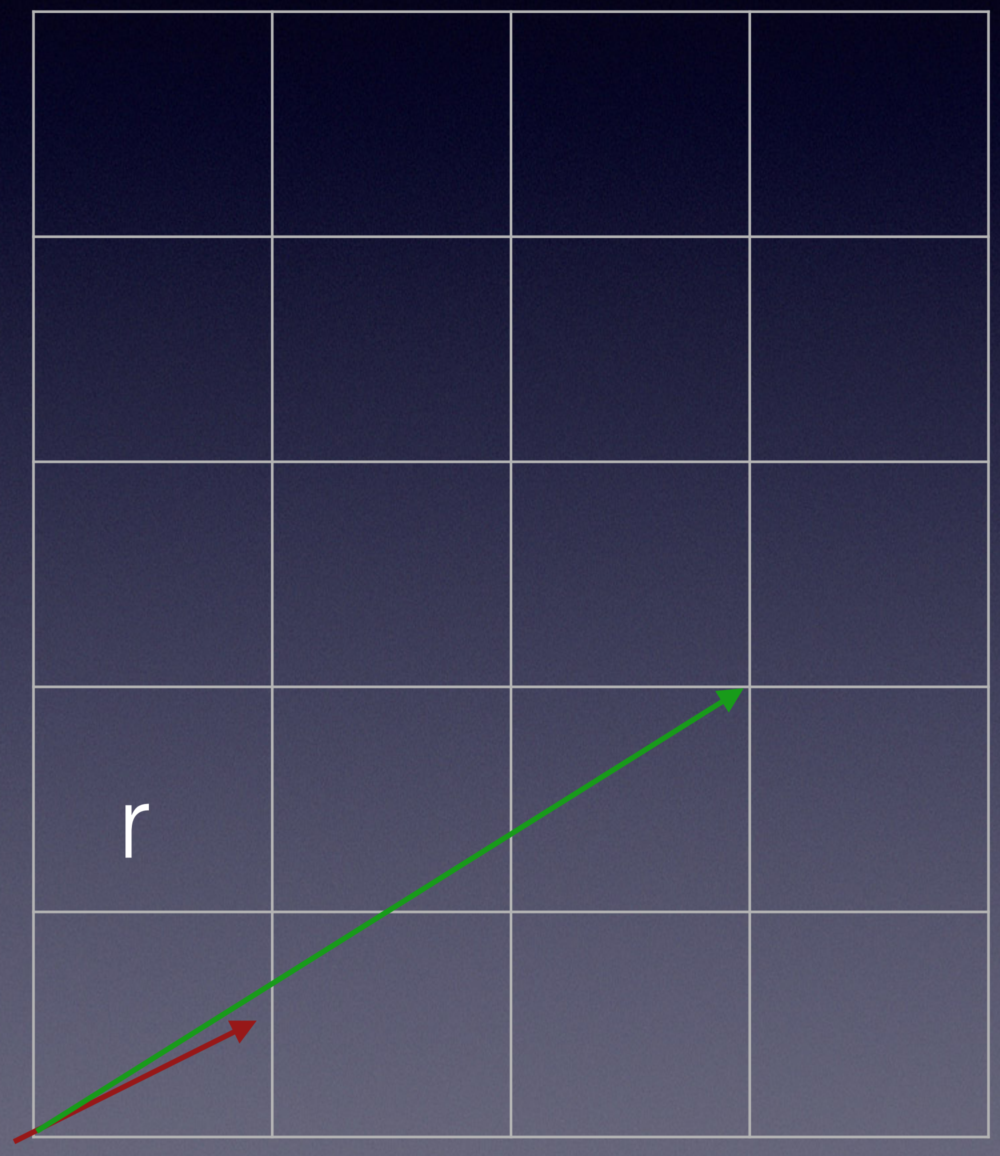


¿Multiplicar?

En 3D:

$a = \{3,2,0\}$
$b = \{1,0.5,1\}$

$p = \{x_a * x_b, y_a * y_b, z_a * z_b\}$

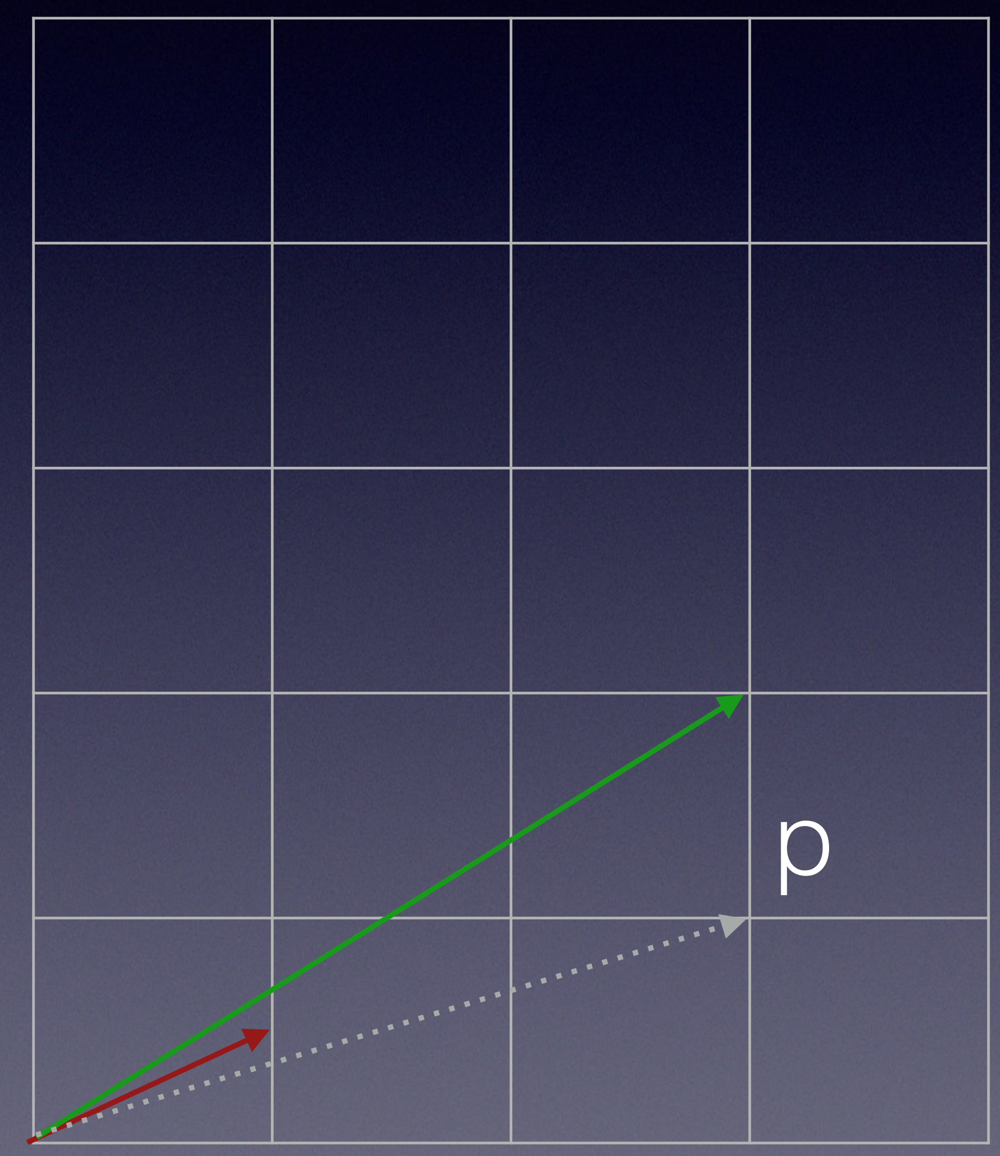


Multiplicar = Escalar

En 3D:

a = {3,2,0}
b = {1,0.5,1}

p = {3,1,0}

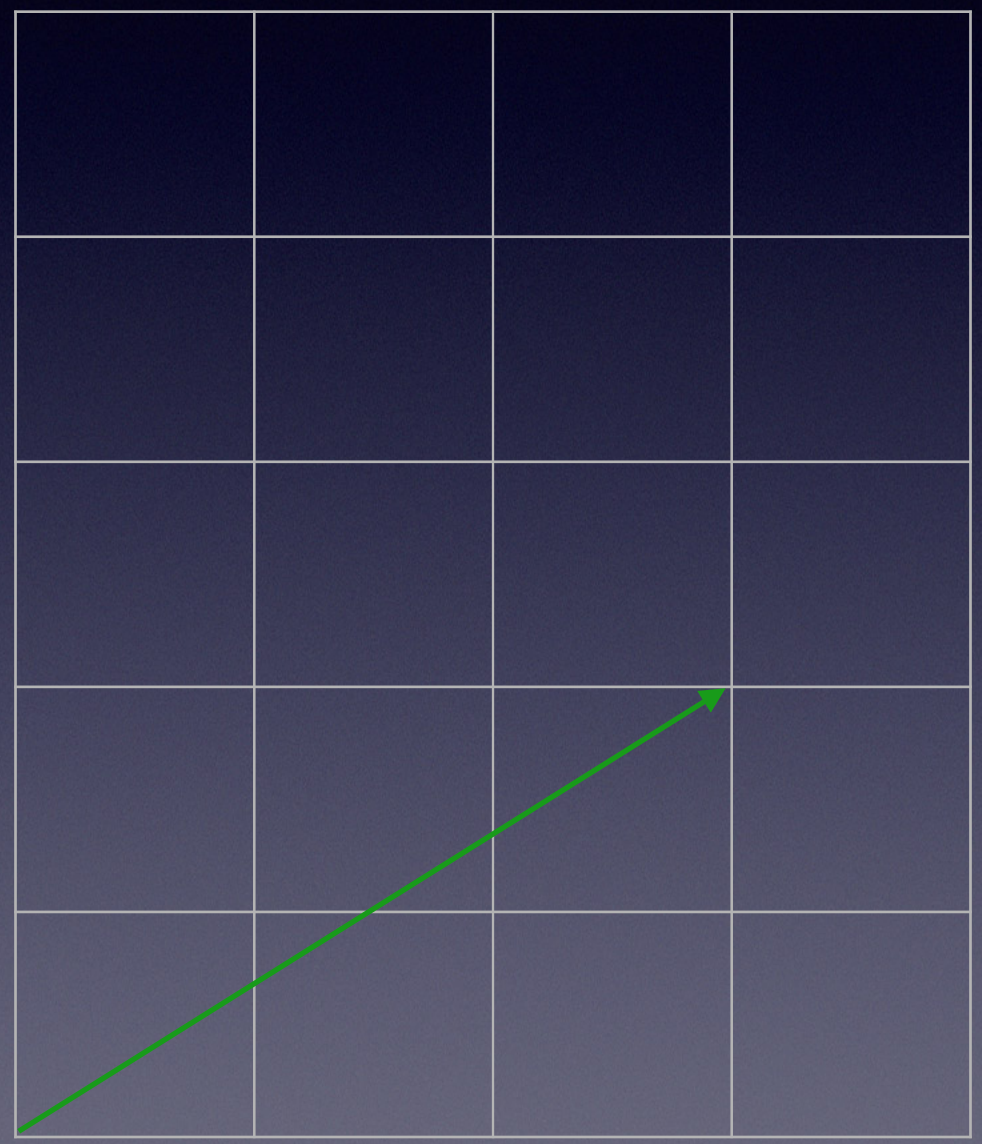


Magnitud

En 3D:

$a = \{x_a, y_a, z_a\}$

$|a| = \sqrt{(x_a^2 + y_a^2 + z_a^2)}$



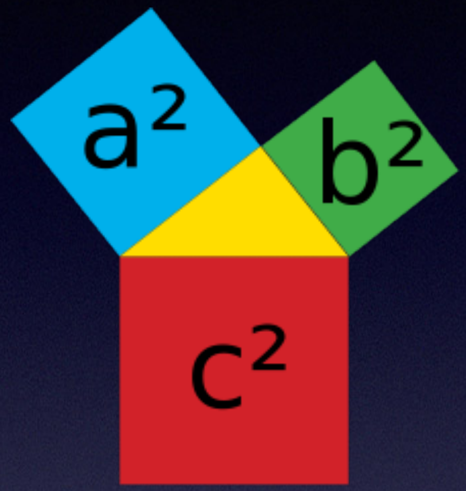
Magnitud

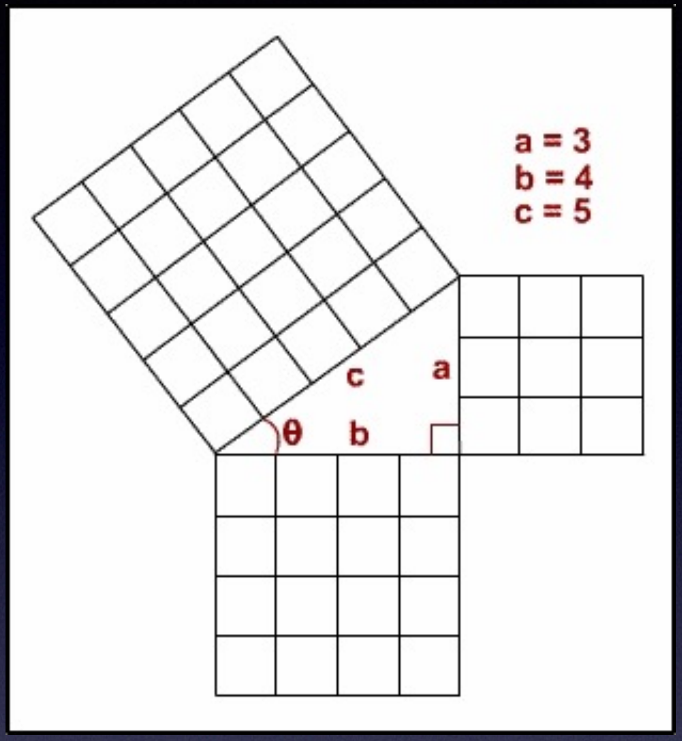
En 3D:

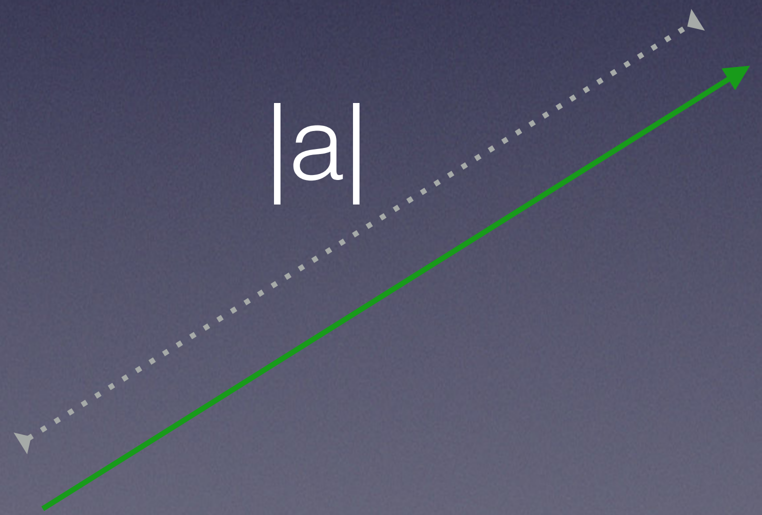
$a = \{x_a, y_a, z_a\}$

$|a| = \sqrt{(x_a^2 + y_a^2 + z_a^2)}$

Pitágoras: $c^2 = a^2 + b^2$





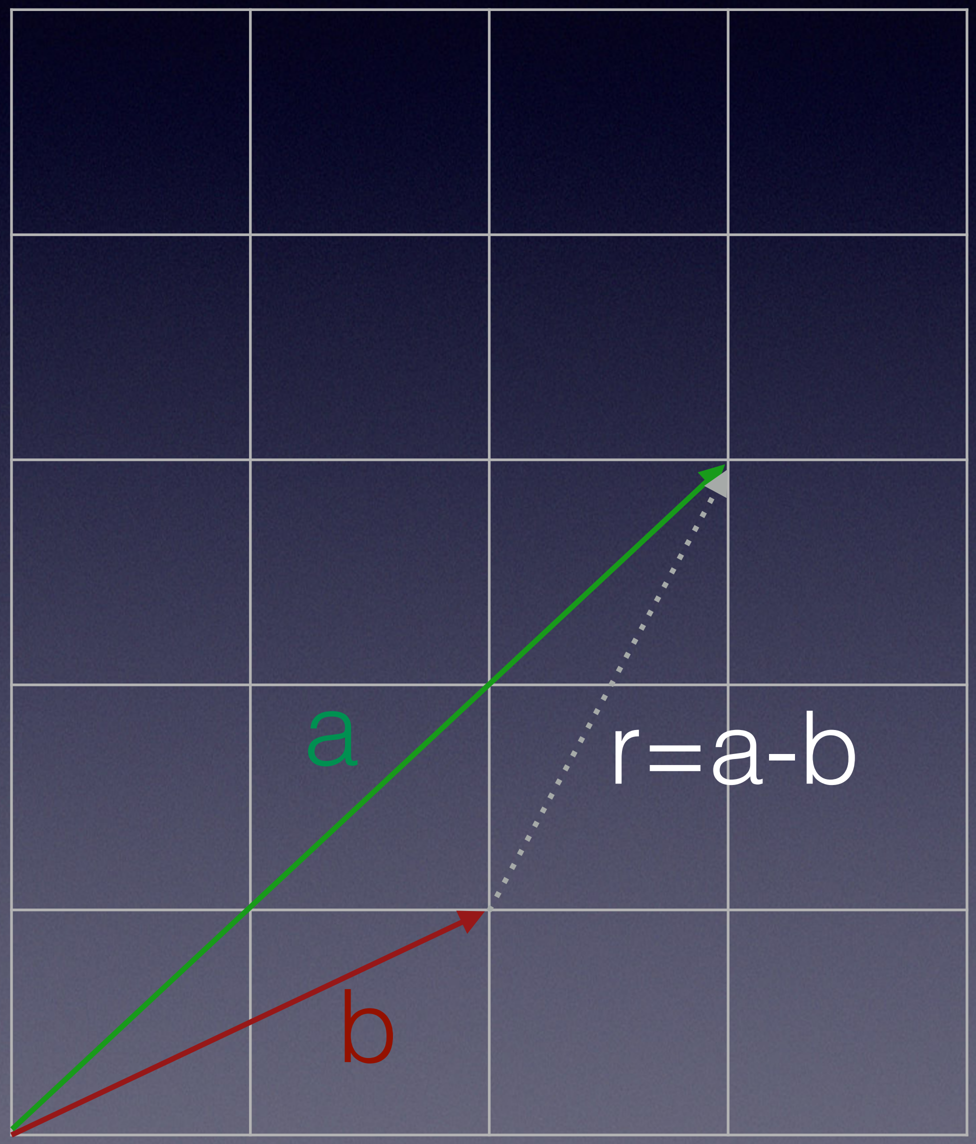


Magnitud y Distancia

En 3D:

$r = \{x_r, y_r, z_r\}$

$|r| = \sqrt{(x_r^2 + y_r^2 + z_r^2)}$

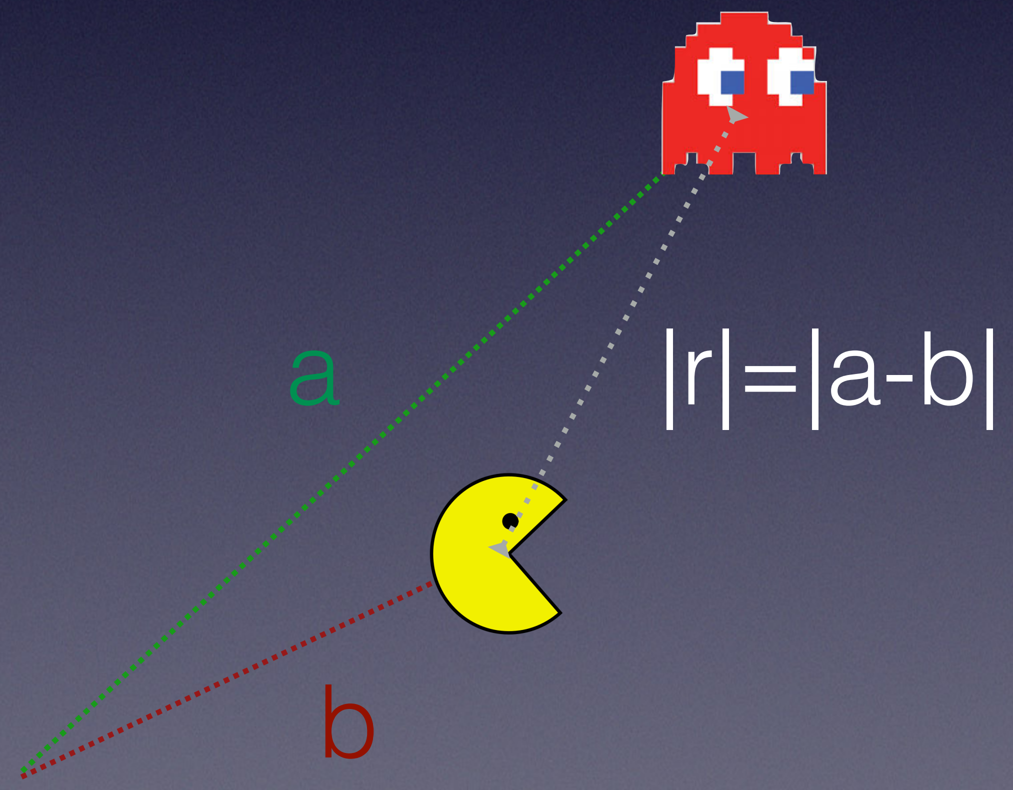


Distancia

En 3D:

$r = \{x_r, y_r, z_r\}$

$|r| = \sqrt{(x_r^2 + y_r^2 + z_r^2)}$



Magnitud=Norma

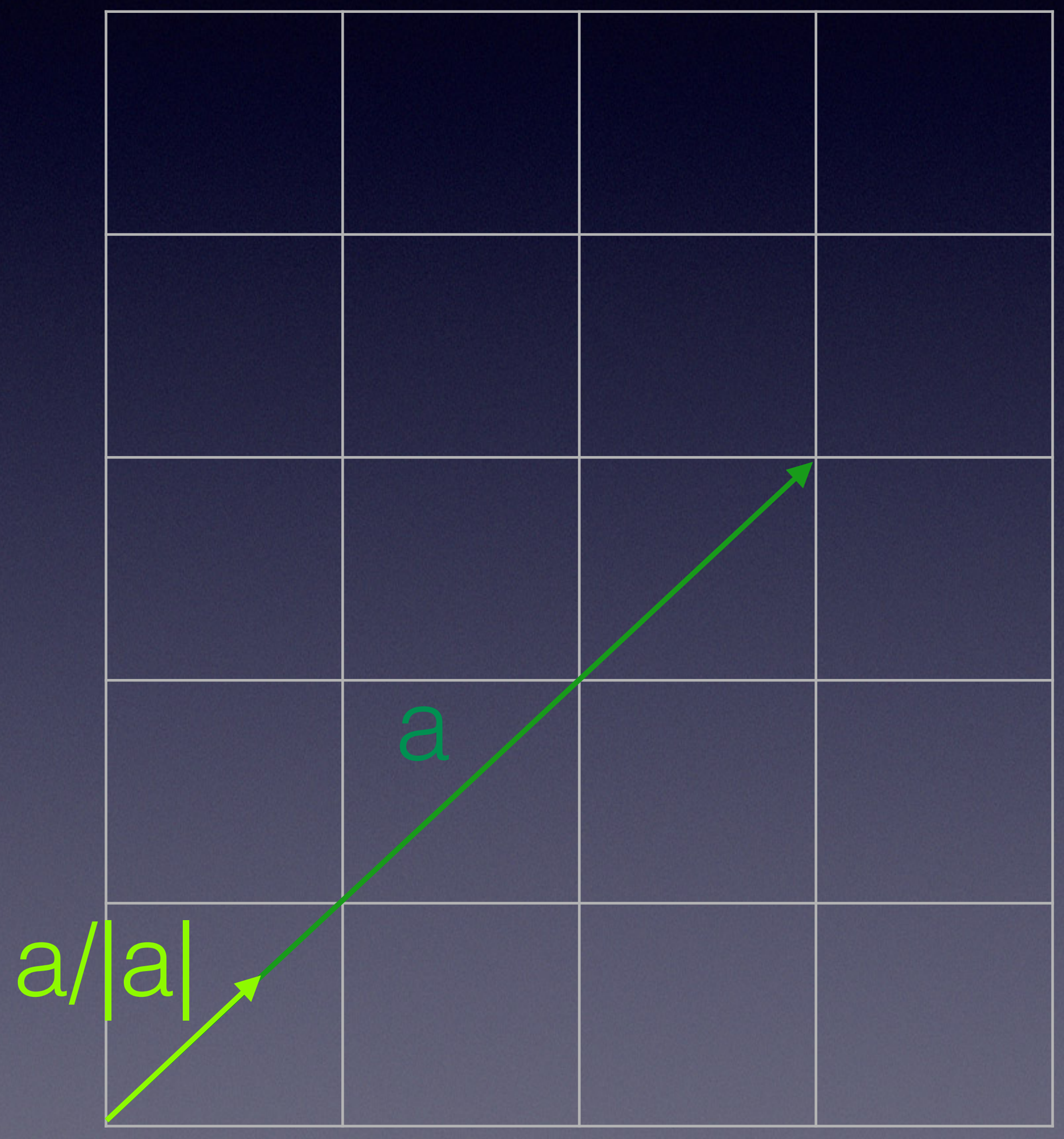
En 3D:

$a = \{x_a, y_a, z_a\}$

$|a| = \sqrt{(x_a^2 + y_a^2 + z_a^2)}$

normalizar = $a/|a|$

¿Magnitud $a/|a|$?



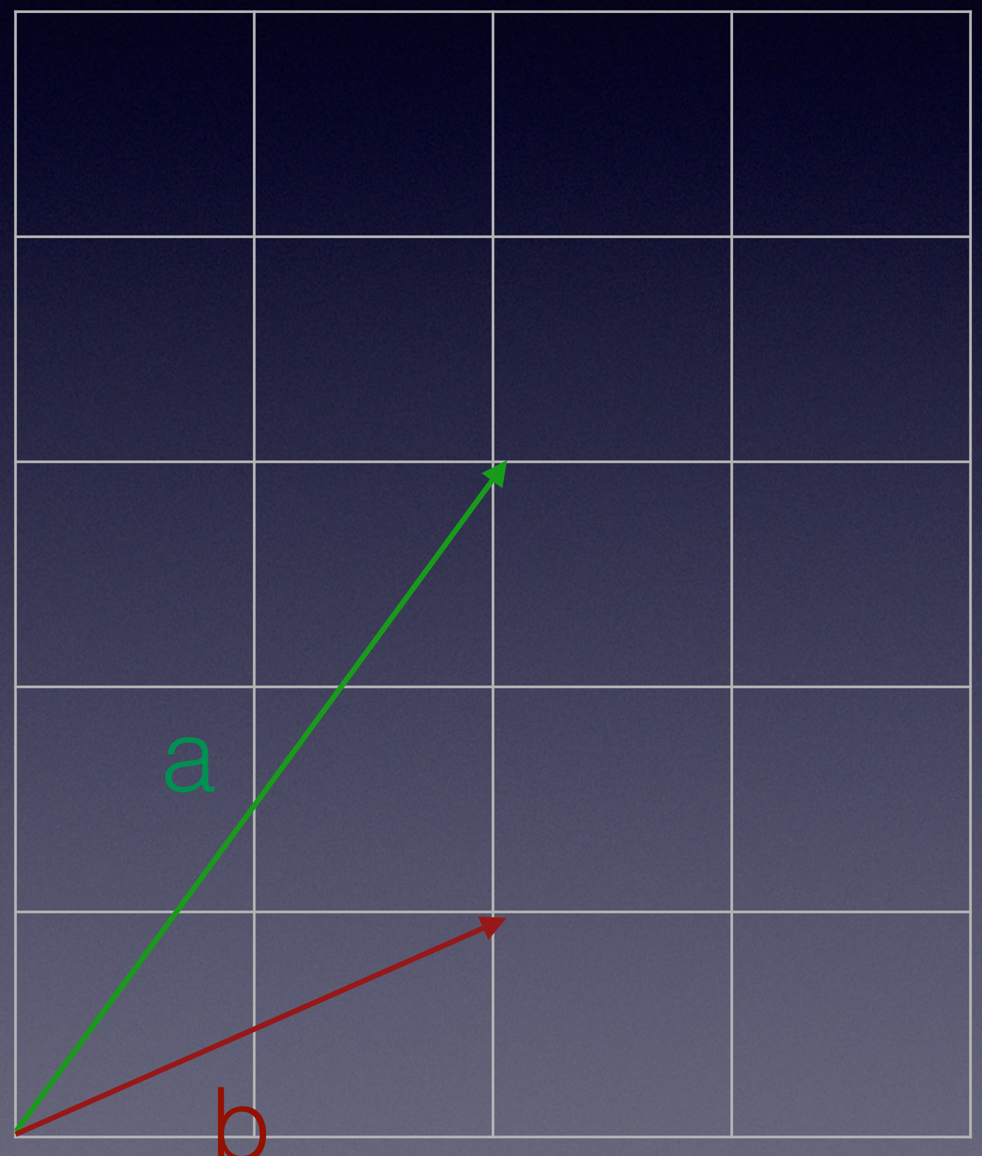
Producto Escalar

En 3D:

$a = \{x_a, y_a, z_a\}$
$b = \{x_b, y_b, z_b\}$

Dot product a·b:
Paso 1 $p = \{x_a{*}x_b, y_a{*}y_b, z_a{*}z_b\}$
Paso 2 $dot = x_p + y_p + z_p$

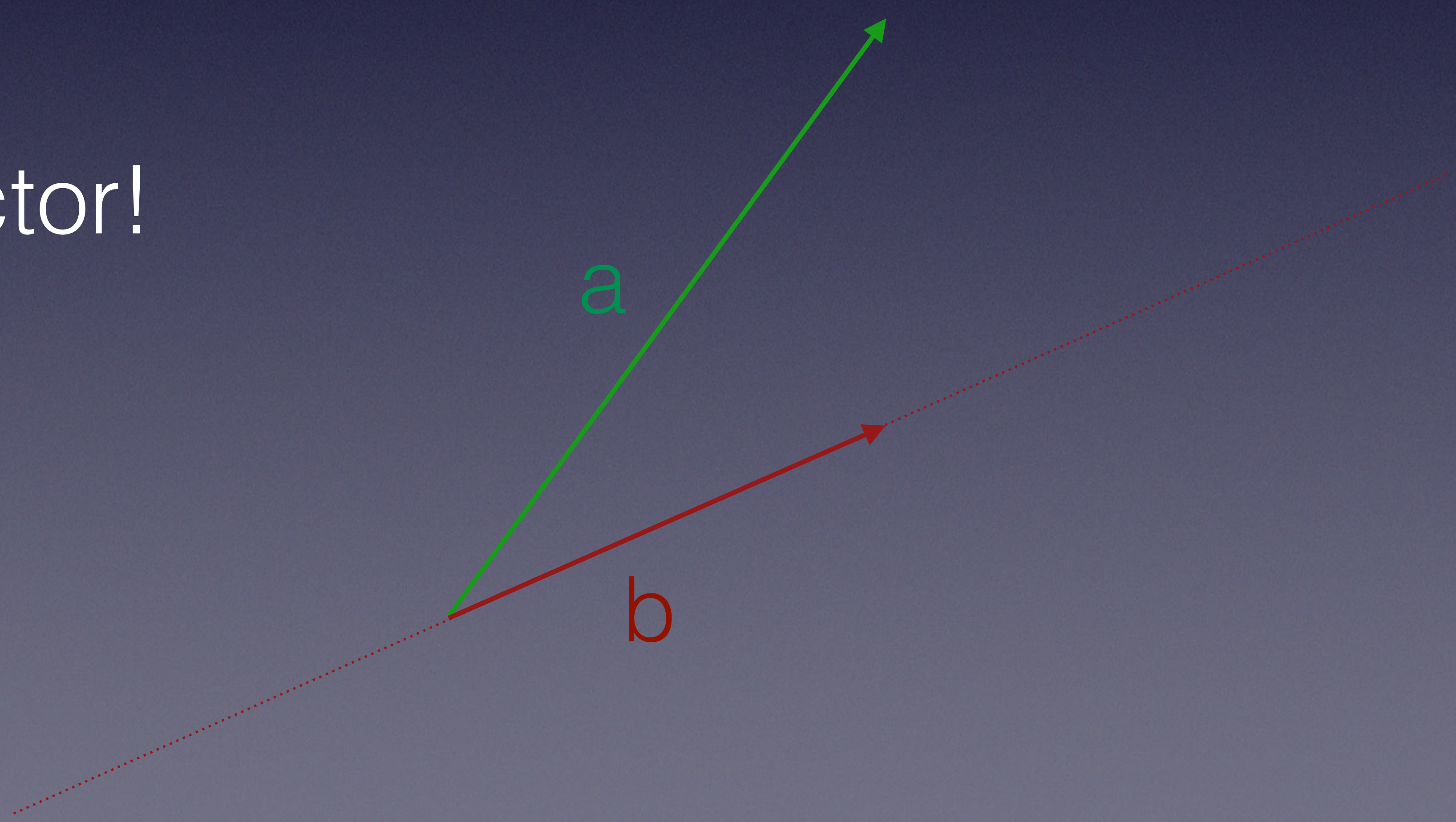
¡a·b no es un vector!



Producto Escalar

a·b

¡a·b no es un vector!



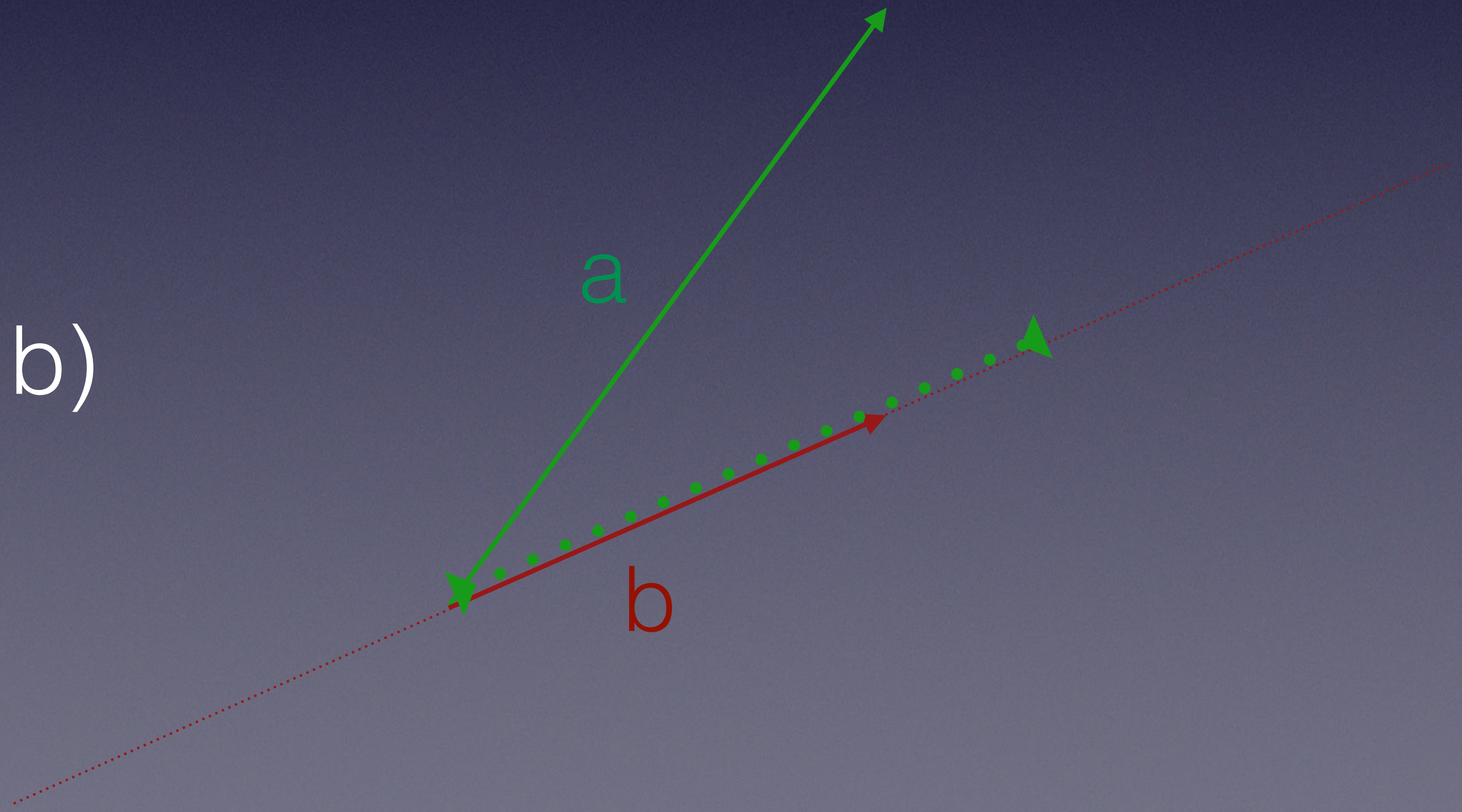
Producto Escalar

a·b

sii |b|=1

a·b = |proj(a,b)|

a·b = |a||b|cos(a,b)



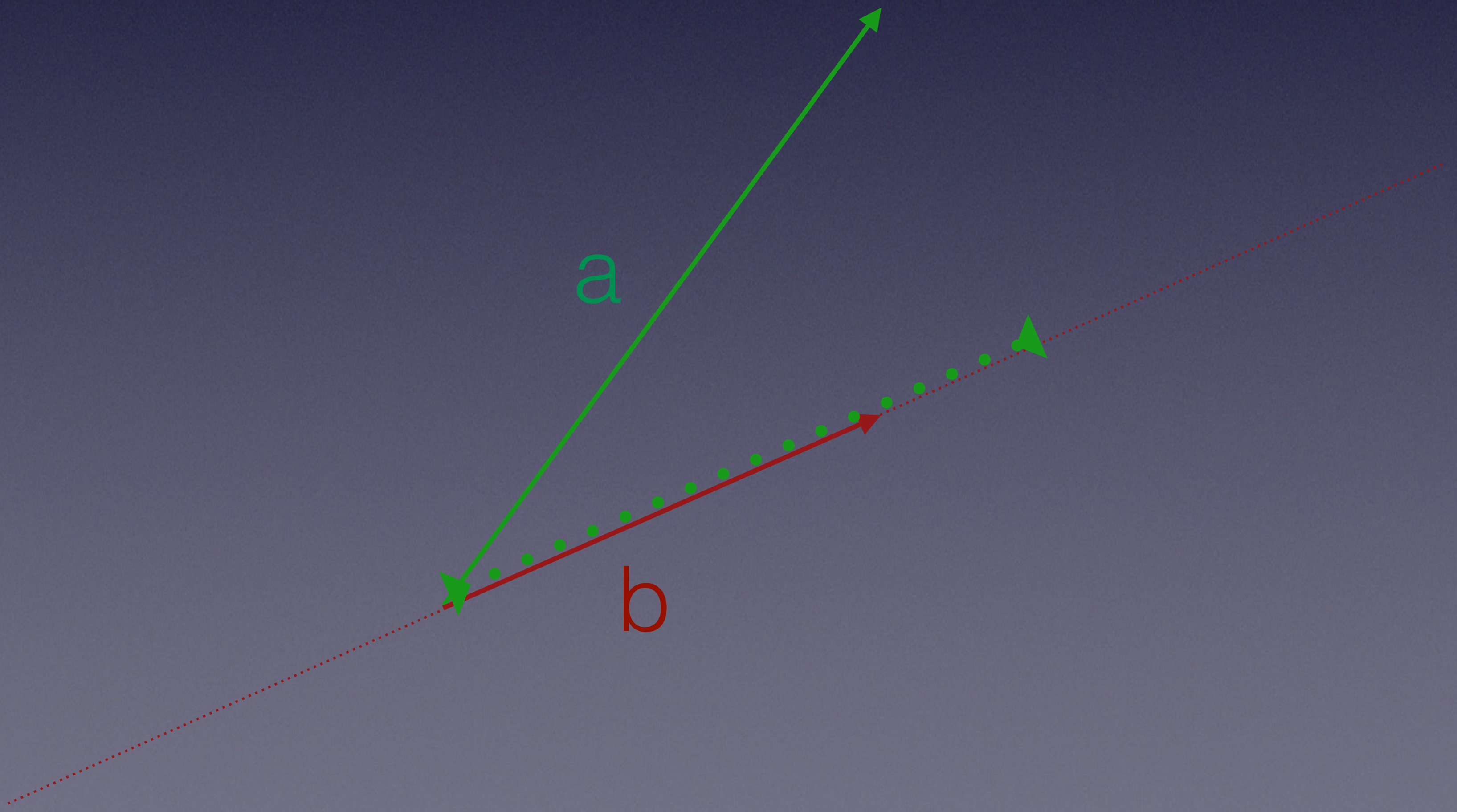
Producto Escalar

$a \cdot b$

sii $|b|=1$

$a \cdot b = |proj(a,b)|$

¿y si $a \perp b$?

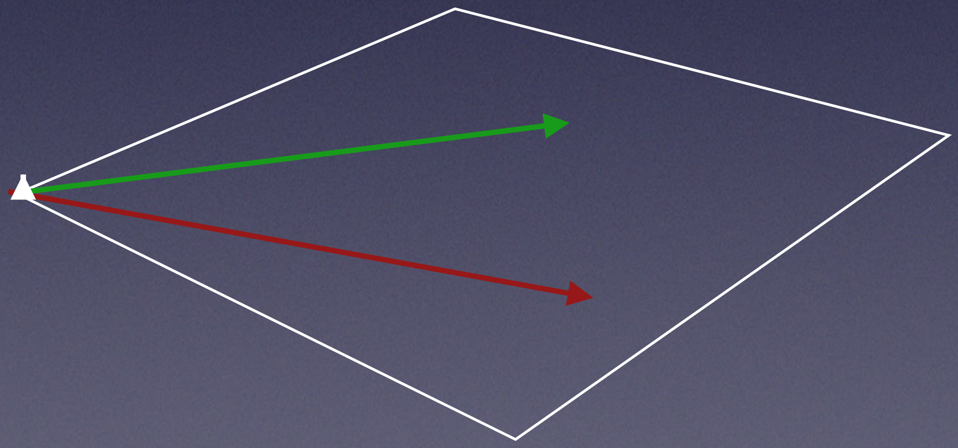


Producto Vectorial

En 3D:
$E = \{e_x, e_y, e_z\}$
$v = \{x_v, y_v, z_v\}$
$w = \{x_w, y_w, z_w\}$

$e_x \; e_y \; e_z$
$x_v \; y_v \; z_v$
x_w, y_w, z_w

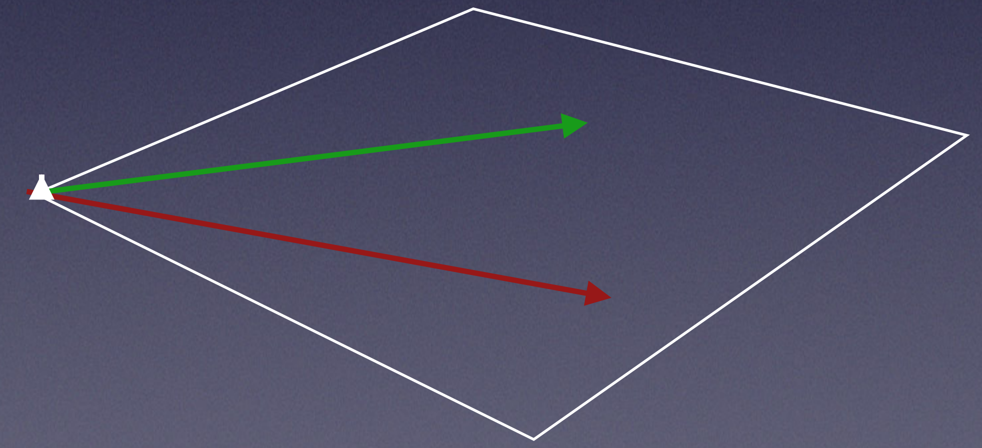
Cross product:
Paso 1 aprendérselo
Paso 2 olvidarlo y buscarlo en internet



Producto Vectorial

$$\text{Matrix}(E,v,w)= \begin{matrix} e_x & e_y & e_z \\ x_v & y_v & z_v \\ x_w, & y_w, & z_w \end{matrix}$$

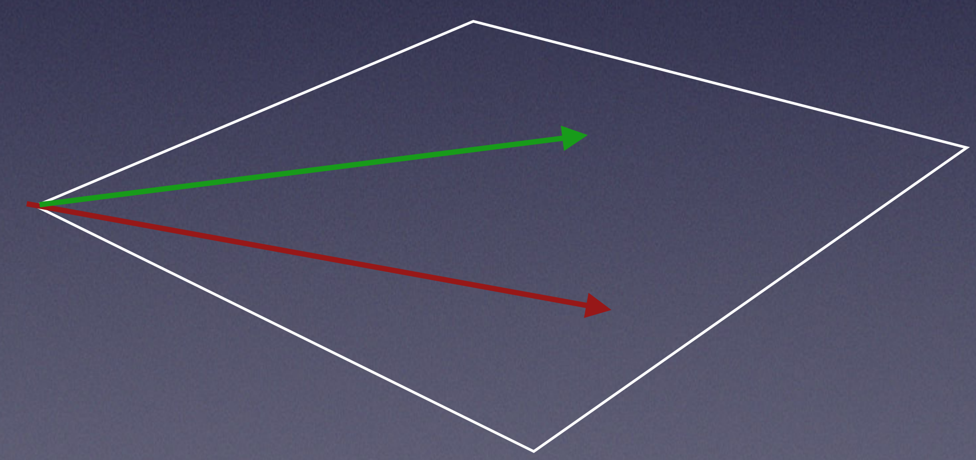
Cross product:
Paso 1 Crear matriz
Paso 2 Sacar determinante
v x w = {??,??,??}



Producto Vectorial

Determinante

$$\begin{array}{ccccc} e_x & e_y & e_z & e_x & e_y \\ x_v & y_v & z_v & x_v & y_v \\ x_w, & y_w, & z_w & x_w, & y_w \end{array}$$

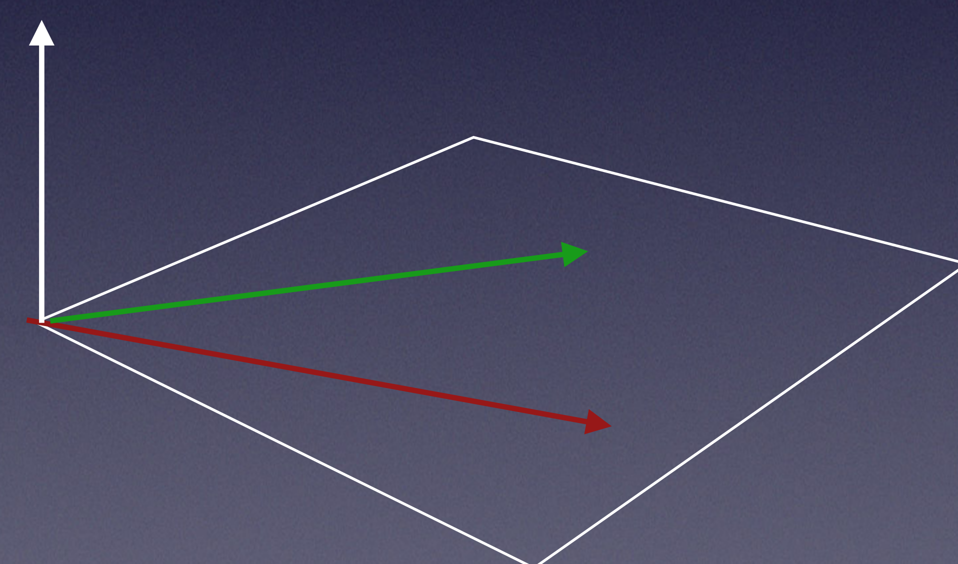


Cross product:
Paso 1 sumar las q bajan
Paso 2 restar las q suben
v x w = {??,??,??}

Vector Cross Prod

Determinante

$$\begin{array}{ccccc} e_x & e_y & e_z & e_x & e_y \\ x_v & y_v & z_v & x_v & y_v \\ x_w, & y_w, & z_w & x_w, & y_w \end{array}$$



Cross product:
Paso 1 sumar las q bajan
Paso 2 restar las q suben

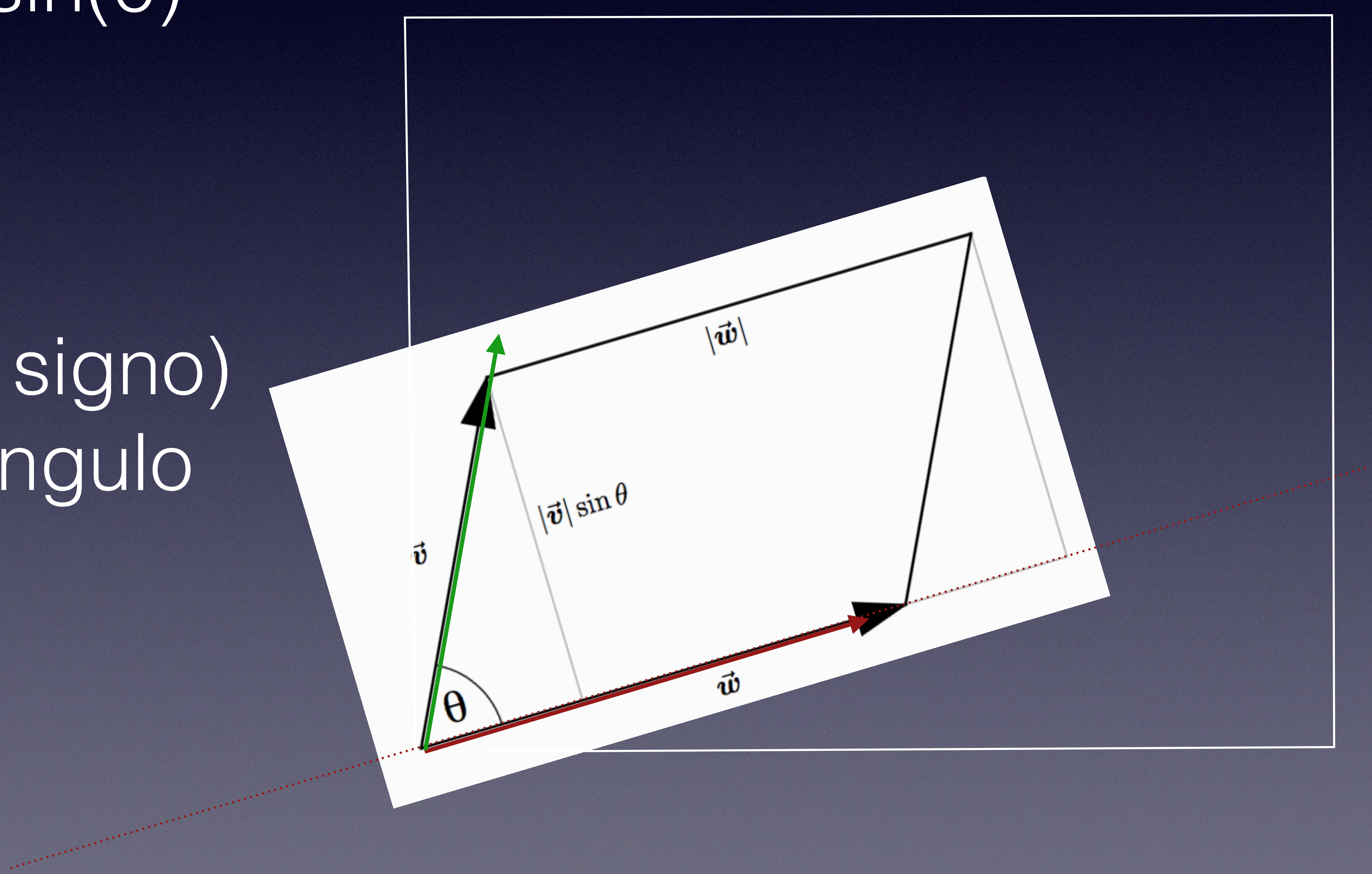
$$v \times w = \{(y_v{*}z_w)-(y_w{*}z_v),\\ (z_v{*}x_w)-(z_w{*}x_v),\\ (x_v{*}y_w)-(x_w{*}y_v)\}$$

Magnitud Cross Prod

|v x w| = |v||w|sin(θ)

significa:

|v x w| = area (y signo)
0.5*|v x w| = triángulo



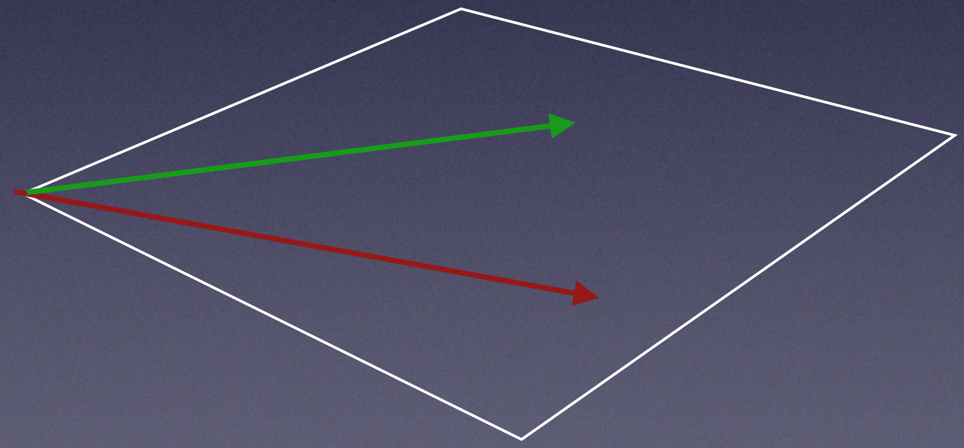
Producto Vectorial

Ejercicio: explicar lookAt geométricamente:

```
void lookAtToAxes(const Vector3& pos, const Vector3& target, const Vector3& upDir,
                  Vector3& left, Vector3& up, Vector3& forward)
{
    // compute the forward vector
    forward = target - pos;
    forward.normalize();

    // compute the left vector
    left = upDir.cross(forward);  // cross product
    left.normalize();

    // compute the orthonormal up vector
    up = forward.cross(left);     // cross product
    up.normalize();
}
```



TRANSFORMACIONES 2X2

$$b = Ax$$

TRANSFORMACIONES 2X2

$$b = \begin{bmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{bmatrix} x$$

TRANSFORMACIONES 2X2

b(0)	=	a(0,0)	a(0,1)		x(0)
b(1)		a(1,0)	a(1,1)		x(1)

TRANSFORMACIONES 2X2

$$b(0) = a(0,0) \cdot x(0) + a(0,1) \cdot x(1)$$

$$b(1) = a(1,0) \cdot x(0) + a(1,1) \cdot x(1)$$

TRANSFORMACIONES 2X2

$$
\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}
$$

TRANSFORMACIONES 2X2

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & 0 \\ 0 & a(1,1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

Escalado

TRANSFORMACIONES 2X2

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & 0 \\ 0 & a(1,1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

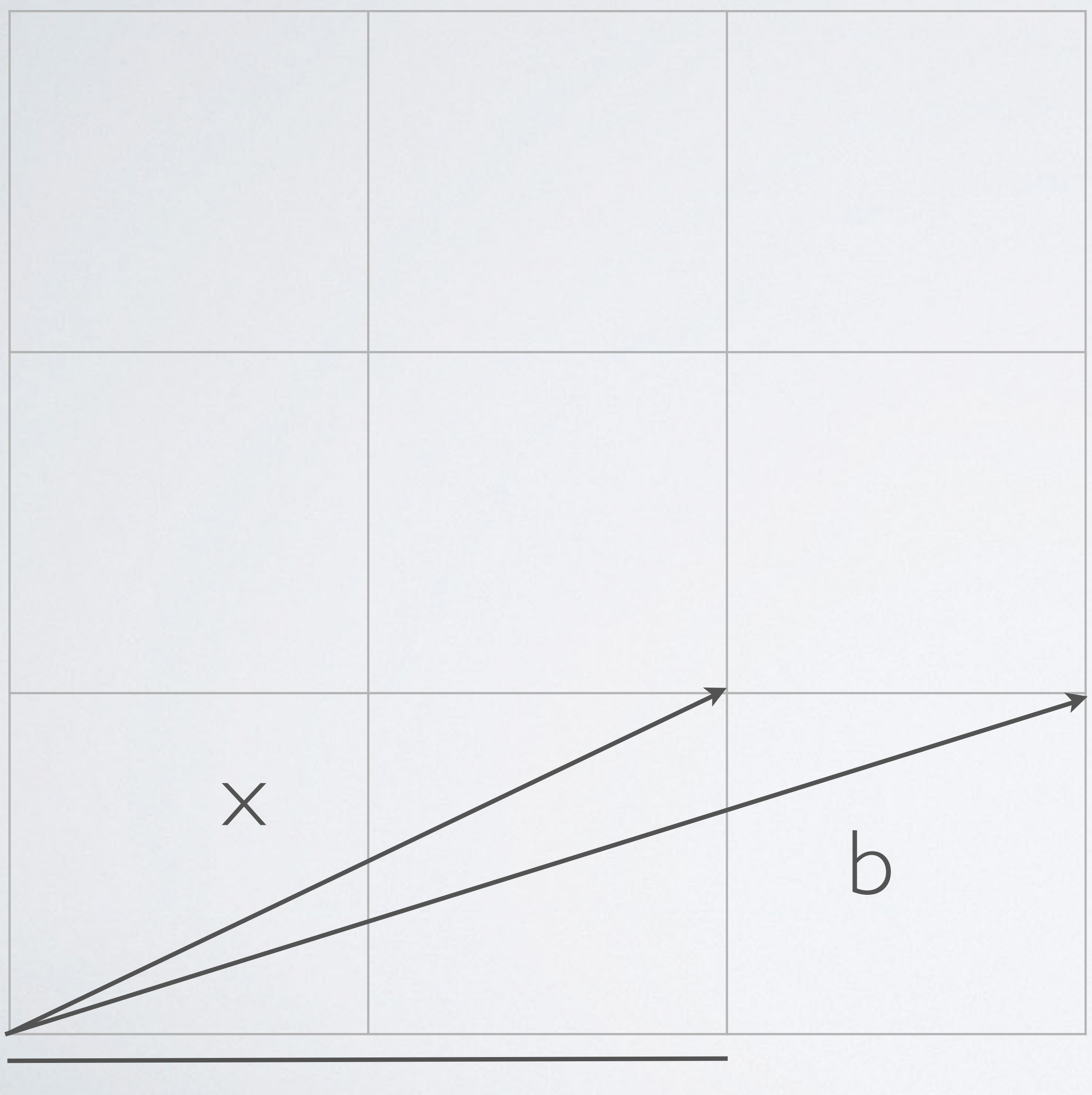
Escalado

TRANSFORMACIONES 2X2

$$\begin{pmatrix} b(0) \\ b(1) \end{pmatrix} = \begin{pmatrix} a(0,0) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \end{pmatrix}$$

Escalado

TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

Escalado

b(0) es x(0) escalado

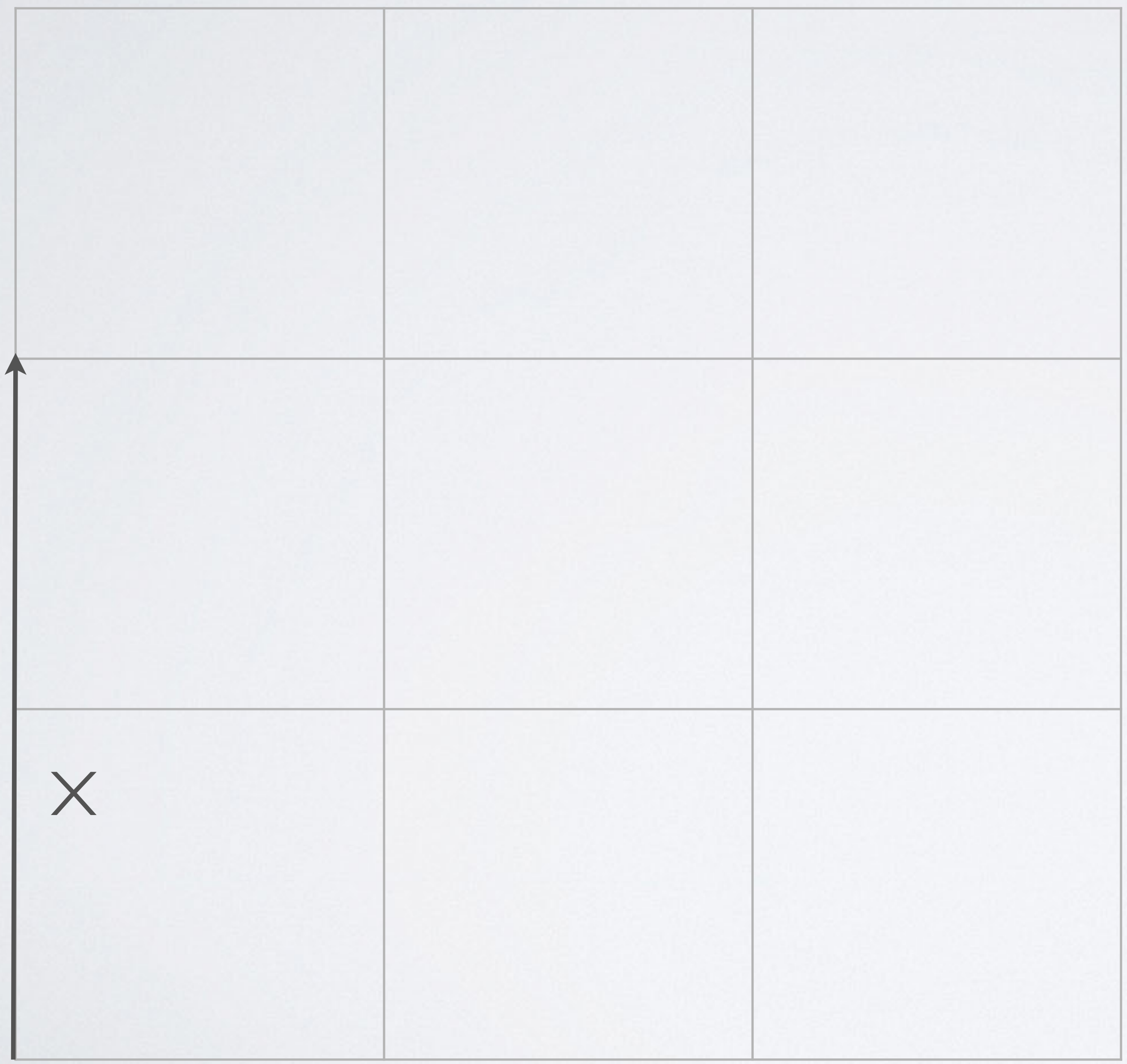
TRANSFORMACIONES 2X2

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

TRANSFORMACIONES 2X2

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} 1 & a(0,1) \\ a(1,0) & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

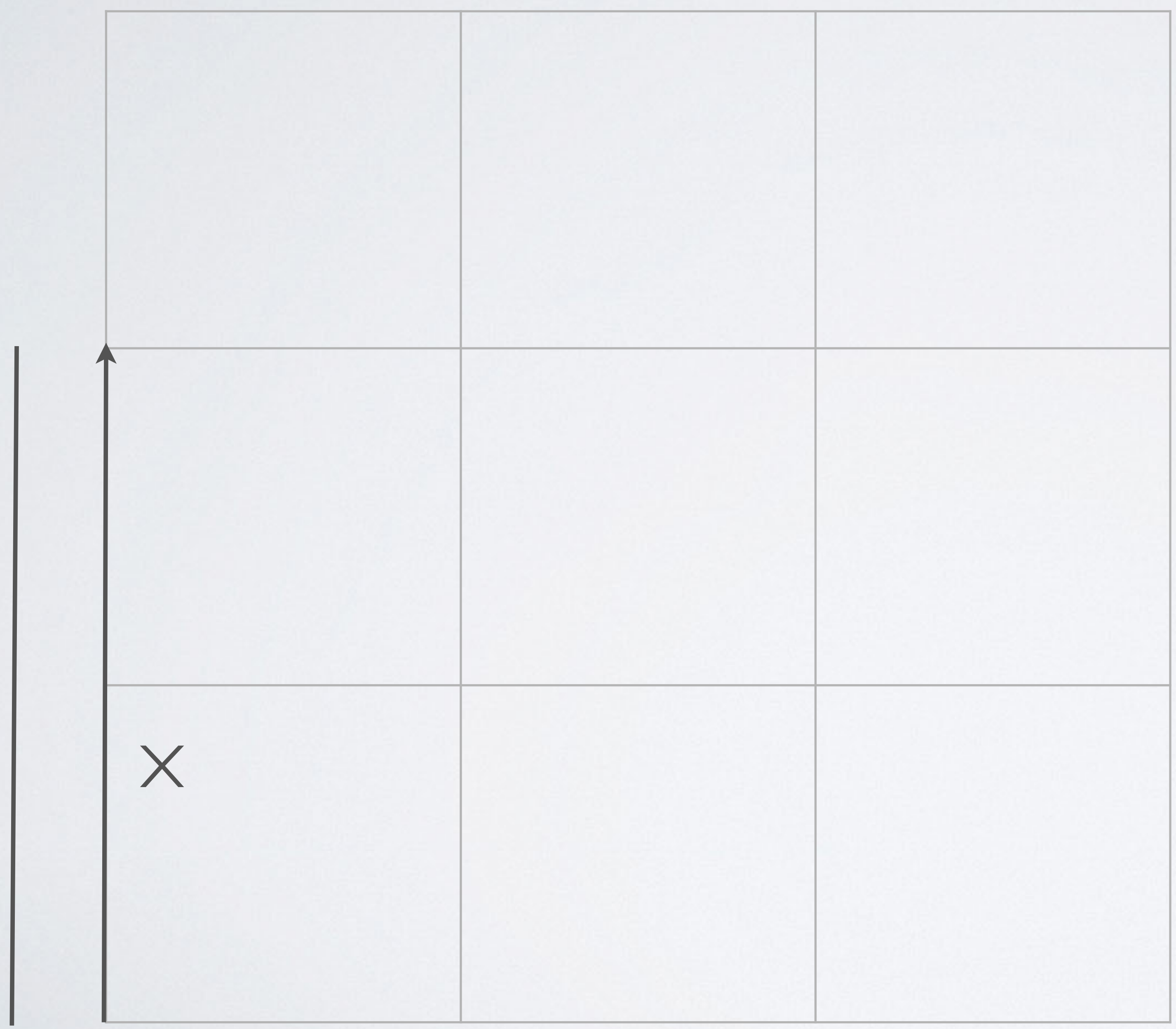
TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} 1 & a(0,1) \\ a(1,0) & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

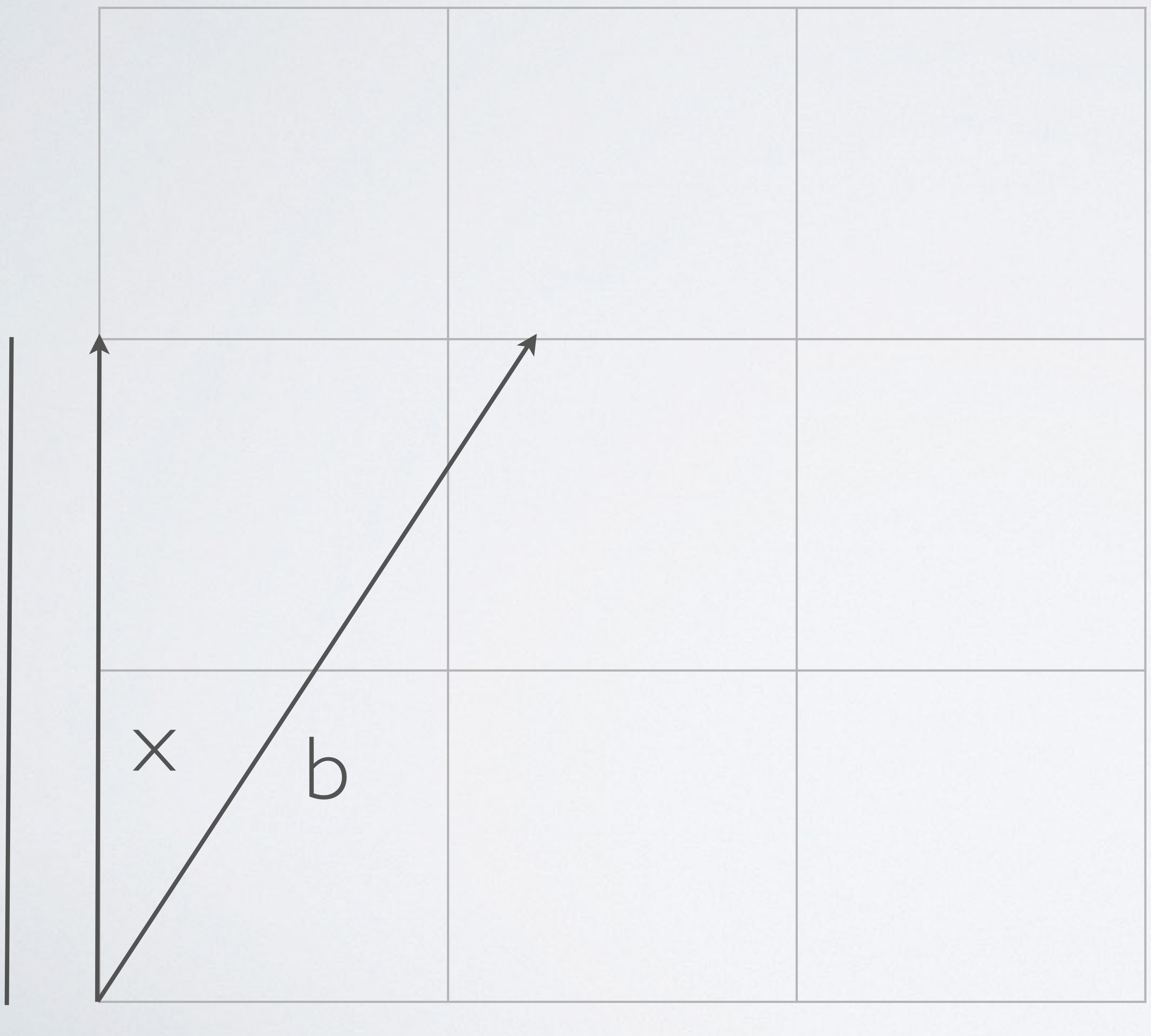
Sesgo (Shear)

TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} 1 & a(0,1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} 1 & a(0,1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

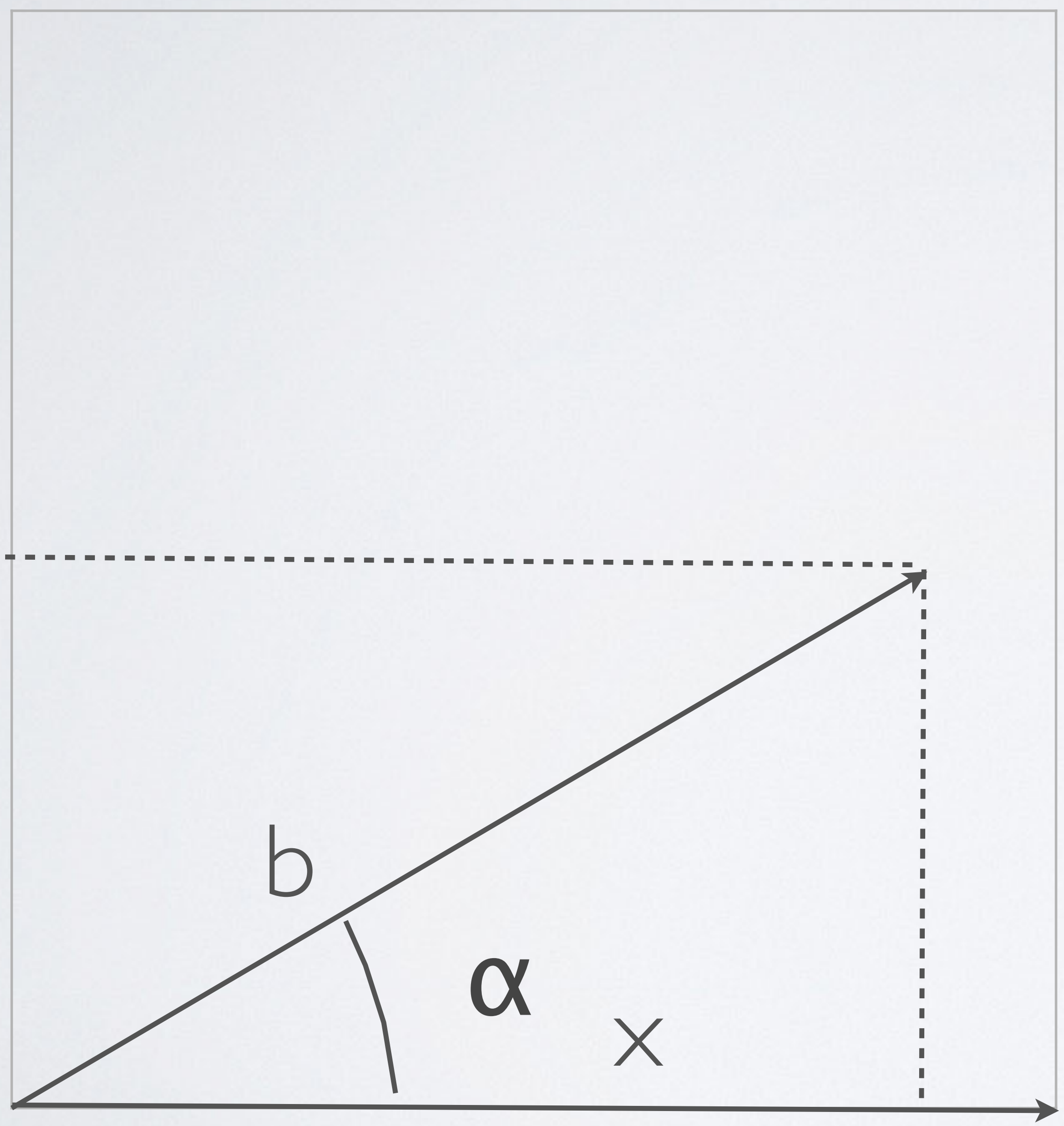
Pasamos algo de x(1) a b(0)

TRANSFORMACIONES 2X2

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

Rotación

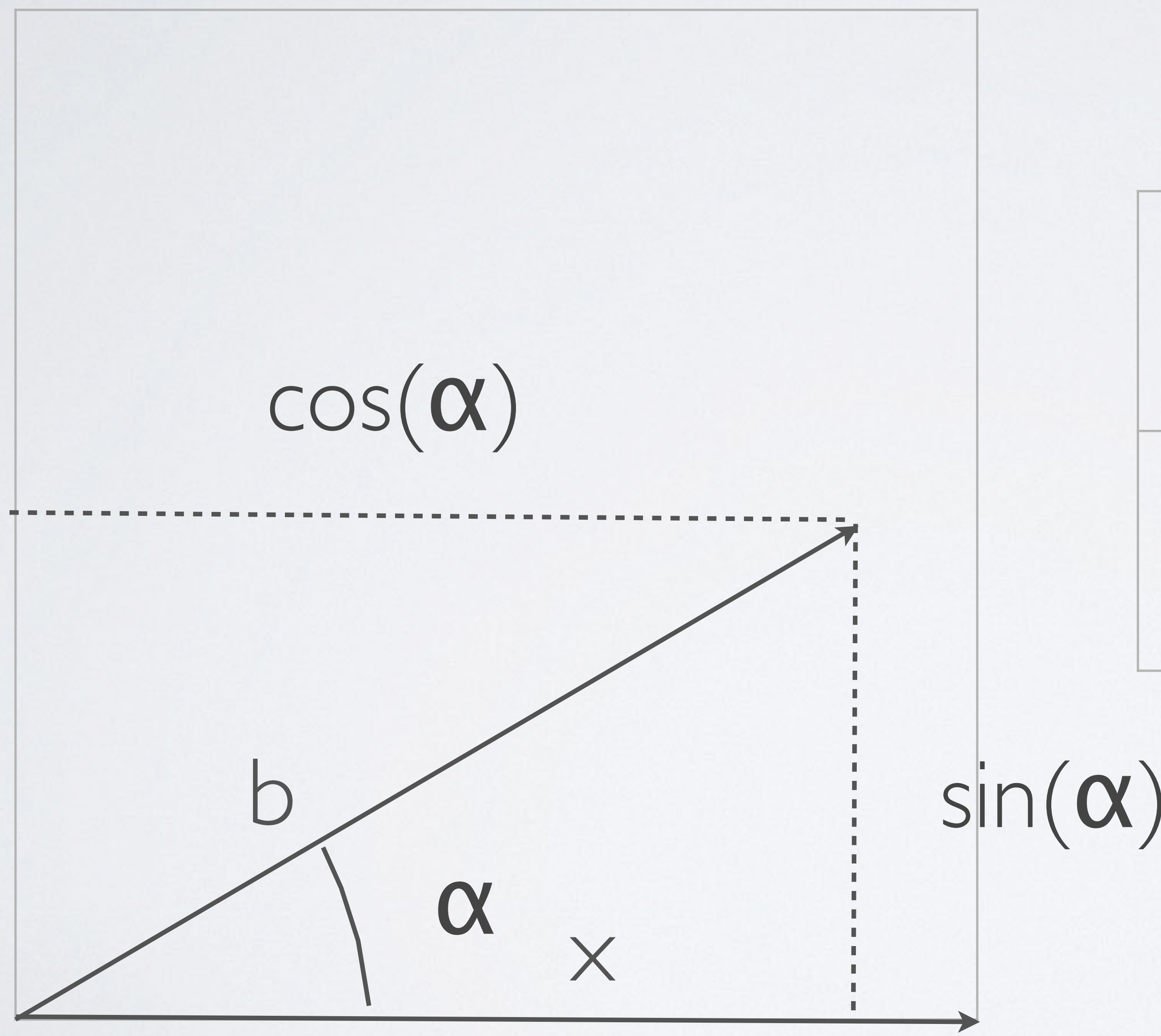
TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rotación

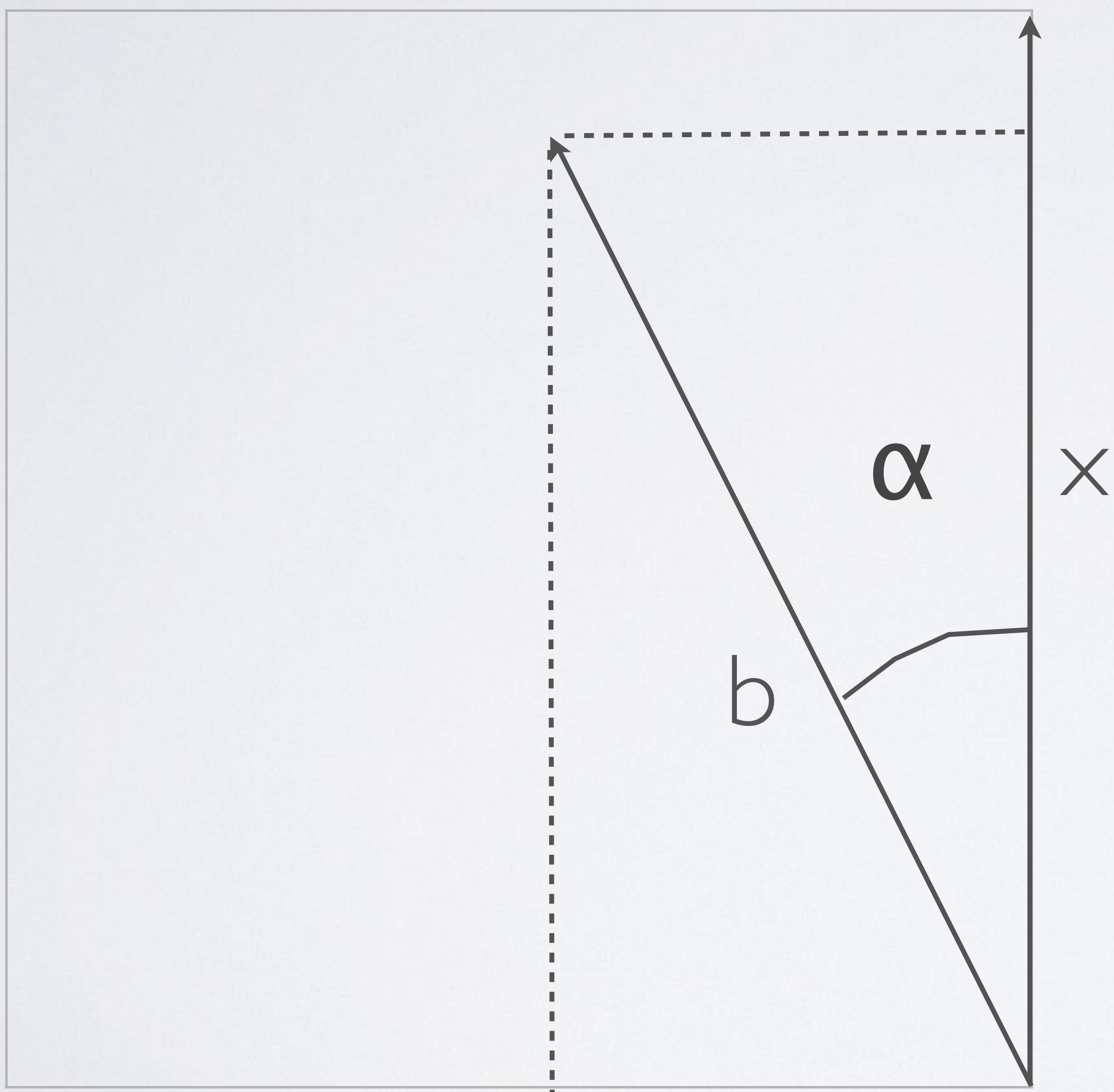
TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & a(0,1) \\ \sin(\alpha) & a(1,1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rotación

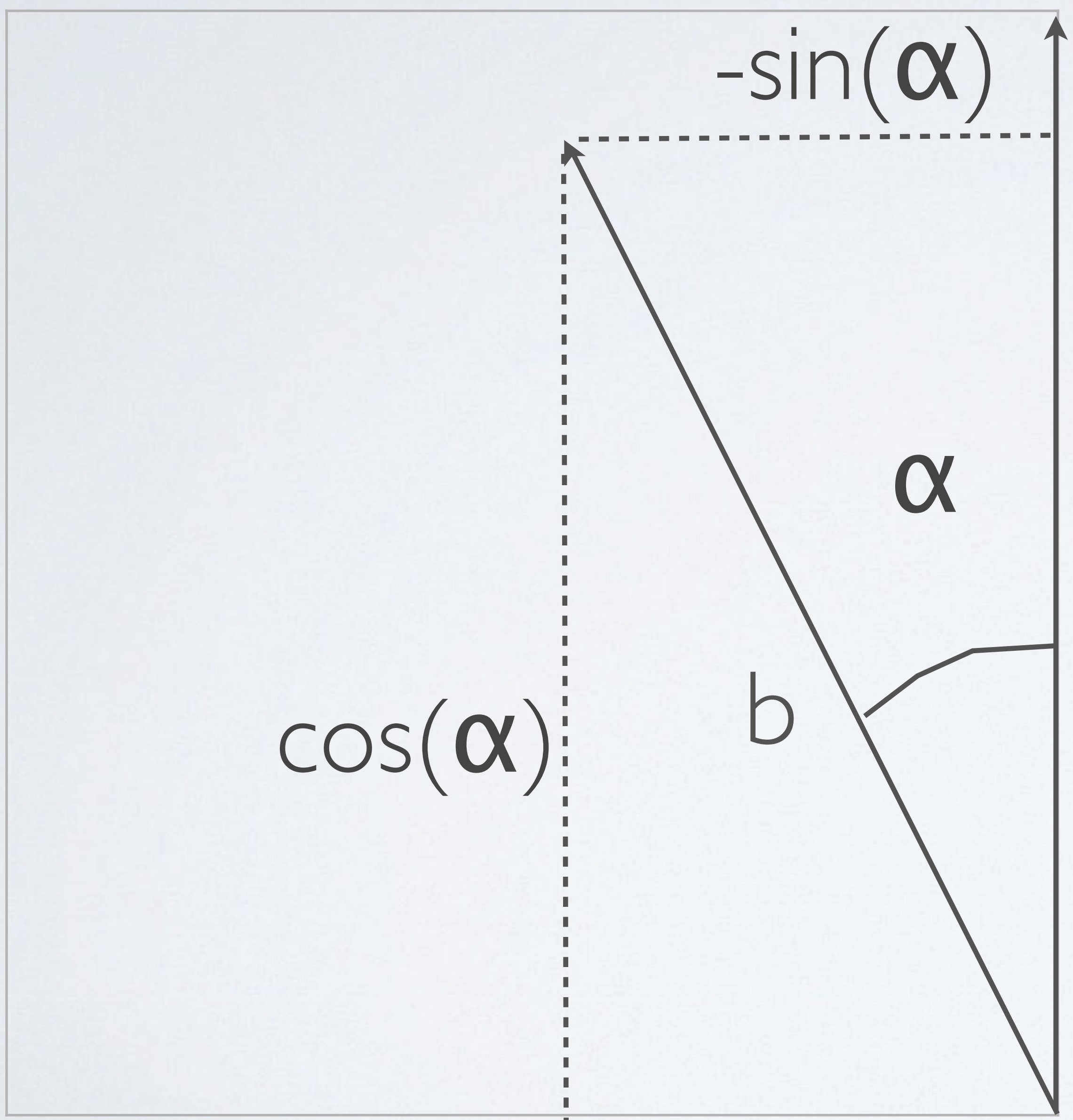
TRANSFORMACIONES 2X2



$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} \cos(\boldsymbol{\alpha}) & a(0,1) \\ \sin(\boldsymbol{\alpha}) & a(1,1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rotación

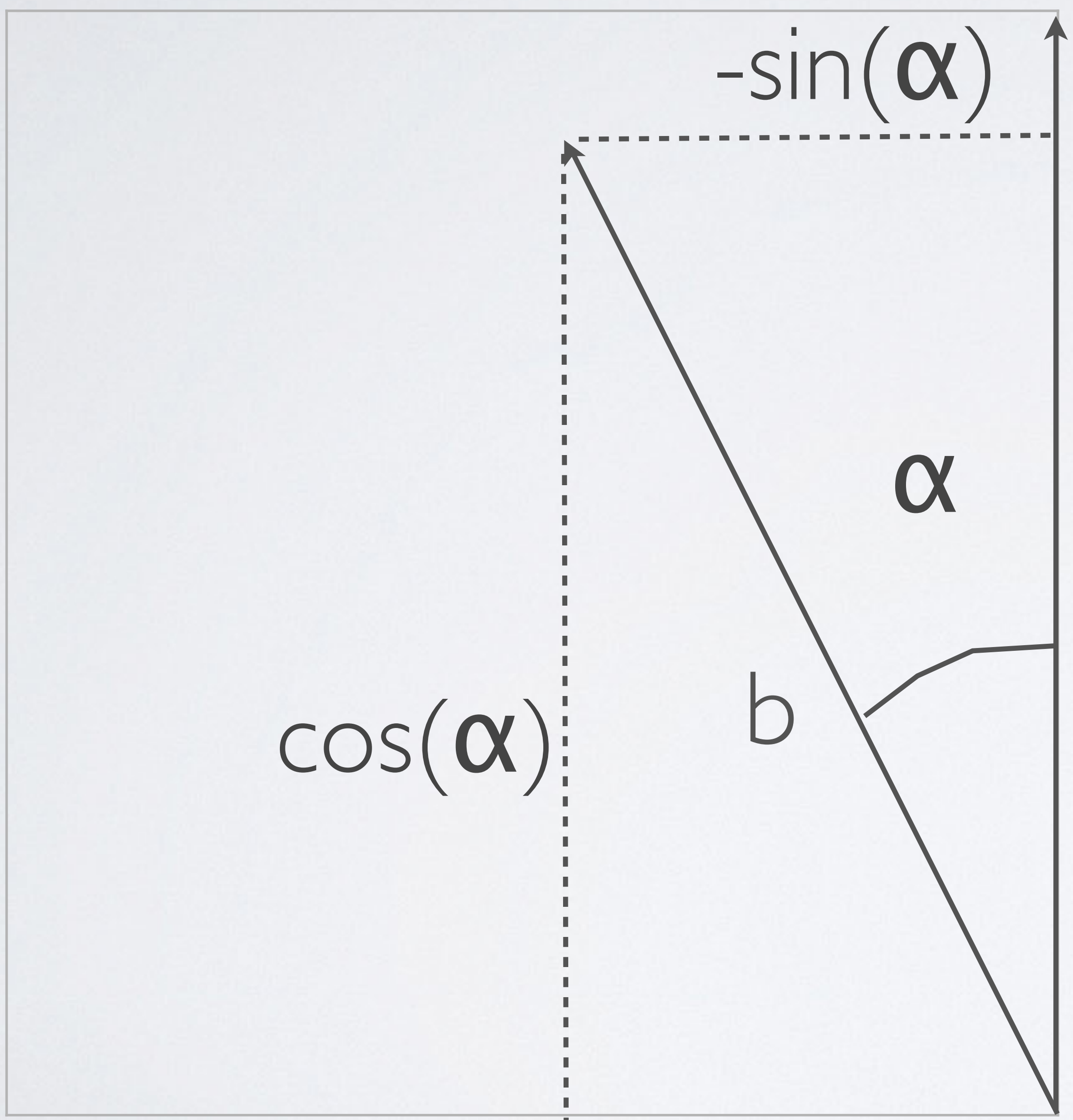
TRANSFORMACIONES 2X2



$$\times \begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rotación

TRANSFORMACIONES 2X2



×

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

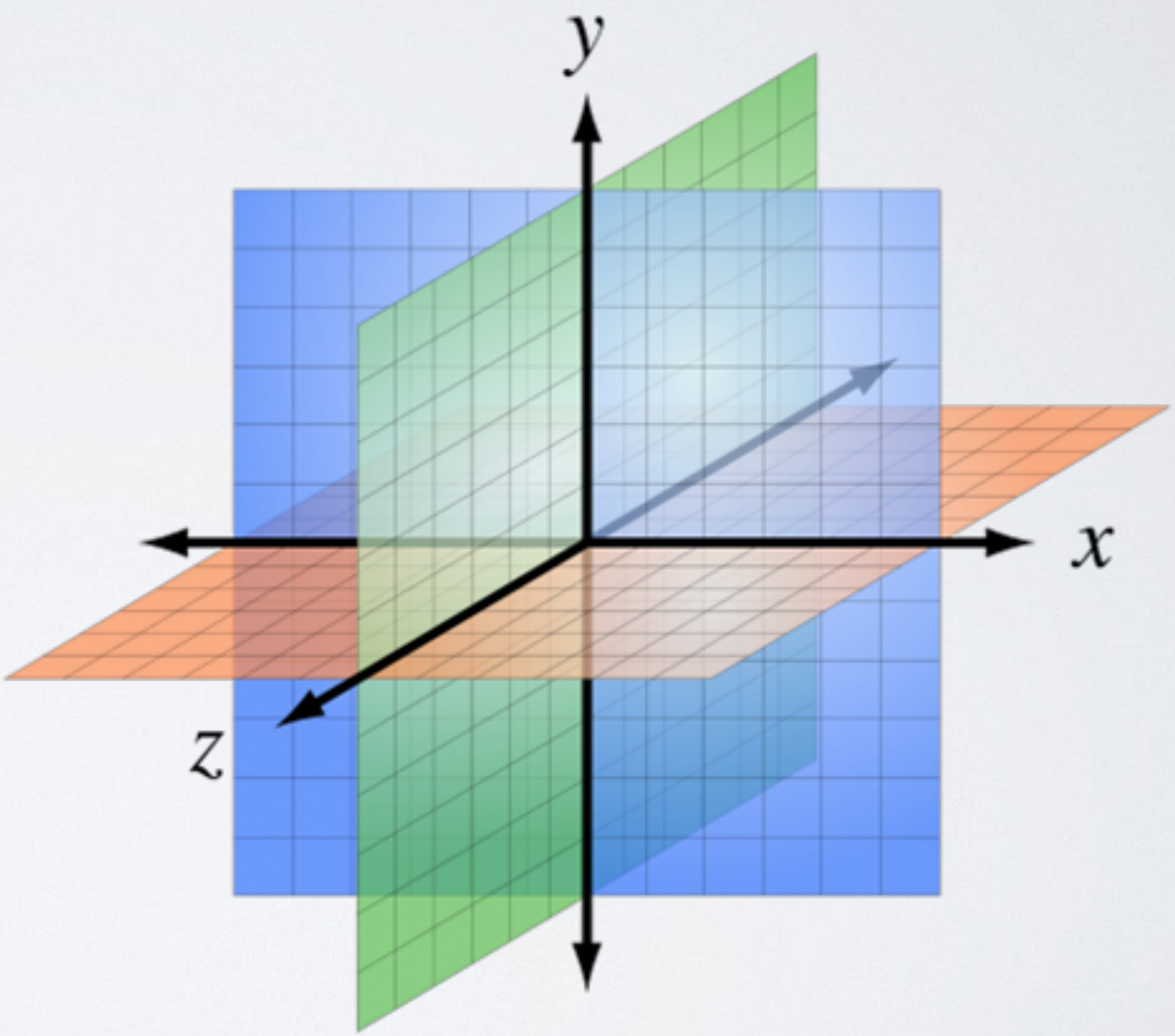
Rotación

TRANSFORMACIONES 2X2

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



TRANSFORMACIONES AFINES

$$b = Ax + t$$

TRANSFORMACIONES 2X2

$$\begin{bmatrix} b(0) \\ b(1) \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) \\ a(1,0) & a(1,1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} + \begin{bmatrix} t(0) \\ t(1) \end{bmatrix}$$

TRANSFORMACIONES AFINES

$$\begin{bmatrix} b(0) \\ b(1) \\ 1 \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) & t(0) \\ a(1,0) & a(1,1) & t(1) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ 1 \end{bmatrix}$$

TRANSFORMACIONES AFINES

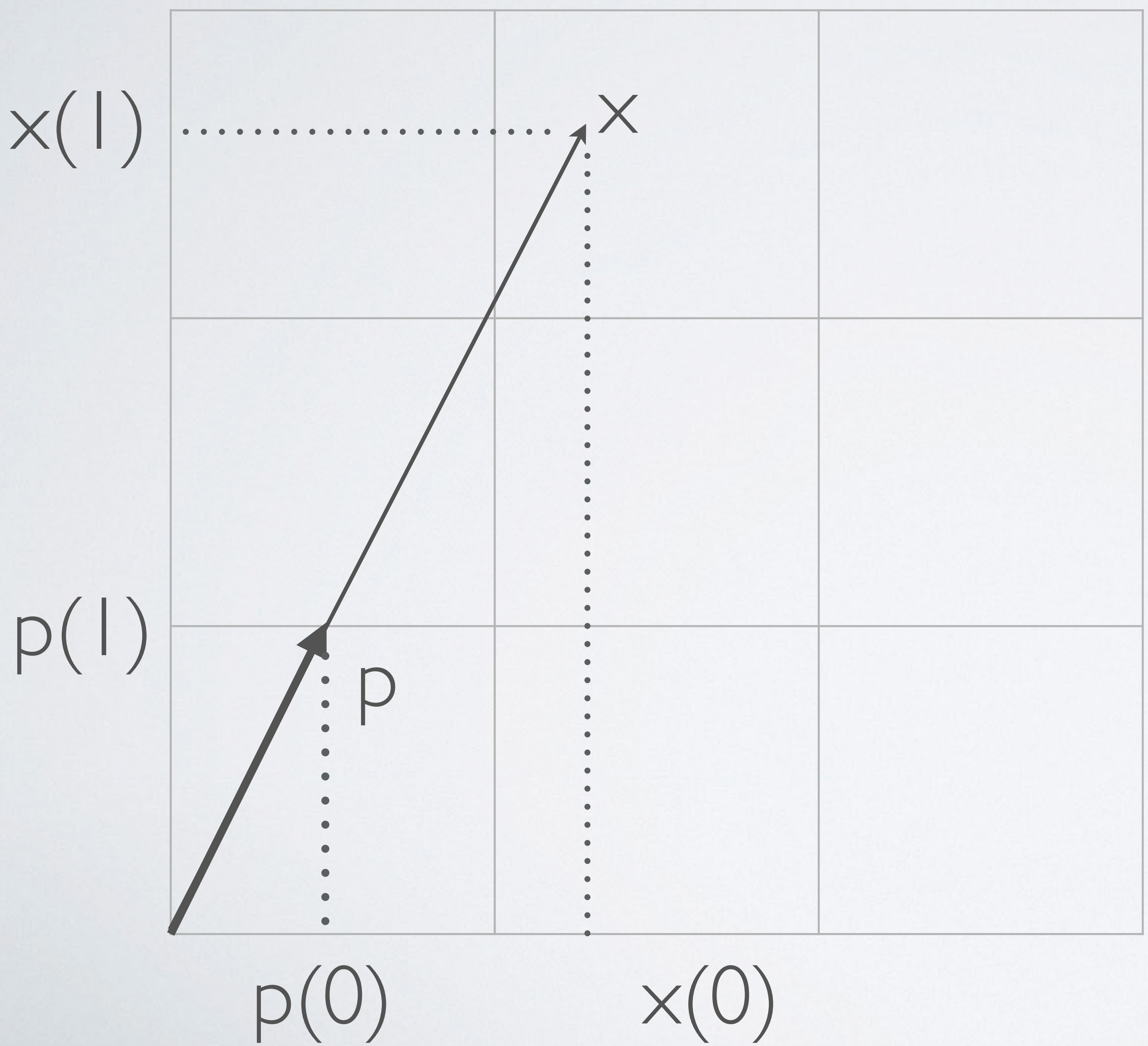
$$b(0) = a(0,0) \cdot x(0) + a(0,1) \cdot x(1) + t(0)$$

$$b(1) = a(1,0) \cdot x(0) + a(1,1) \cdot x(1) + t(1)$$

PROYECCIONES

$$\begin{bmatrix} b(0) \\ b(1) \\ \mathbf{b(2)} \end{bmatrix} = \begin{bmatrix} a(0,0) & a(0,1) & 0 \\ a(1,0) & a(1,1) & 0 \\ 0 & \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ 1 \end{bmatrix}$$

PROYECCIONES



$$x(0)/x(1) = p(0)/p(1)$$

$$p(0) = x(0)/x(1)$$

$$p(1) = 1$$

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PROYECCIONES

$$
\begin{bmatrix} b(0) \\ b(1) \\ \mathbf{b(2)} \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mathbf{?} & \mathbf{0} \end{bmatrix}
\begin{bmatrix} x(0) \\ x(1) \\ 1 \end{bmatrix}
$$

PROYECCIONES

$$\begin{bmatrix} b(0) \\ b(1) \\ \mathbf{b(2)} \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ x(1) \end{bmatrix}$$

b(2) no es igual a 1 !!!

PROYECCIONES

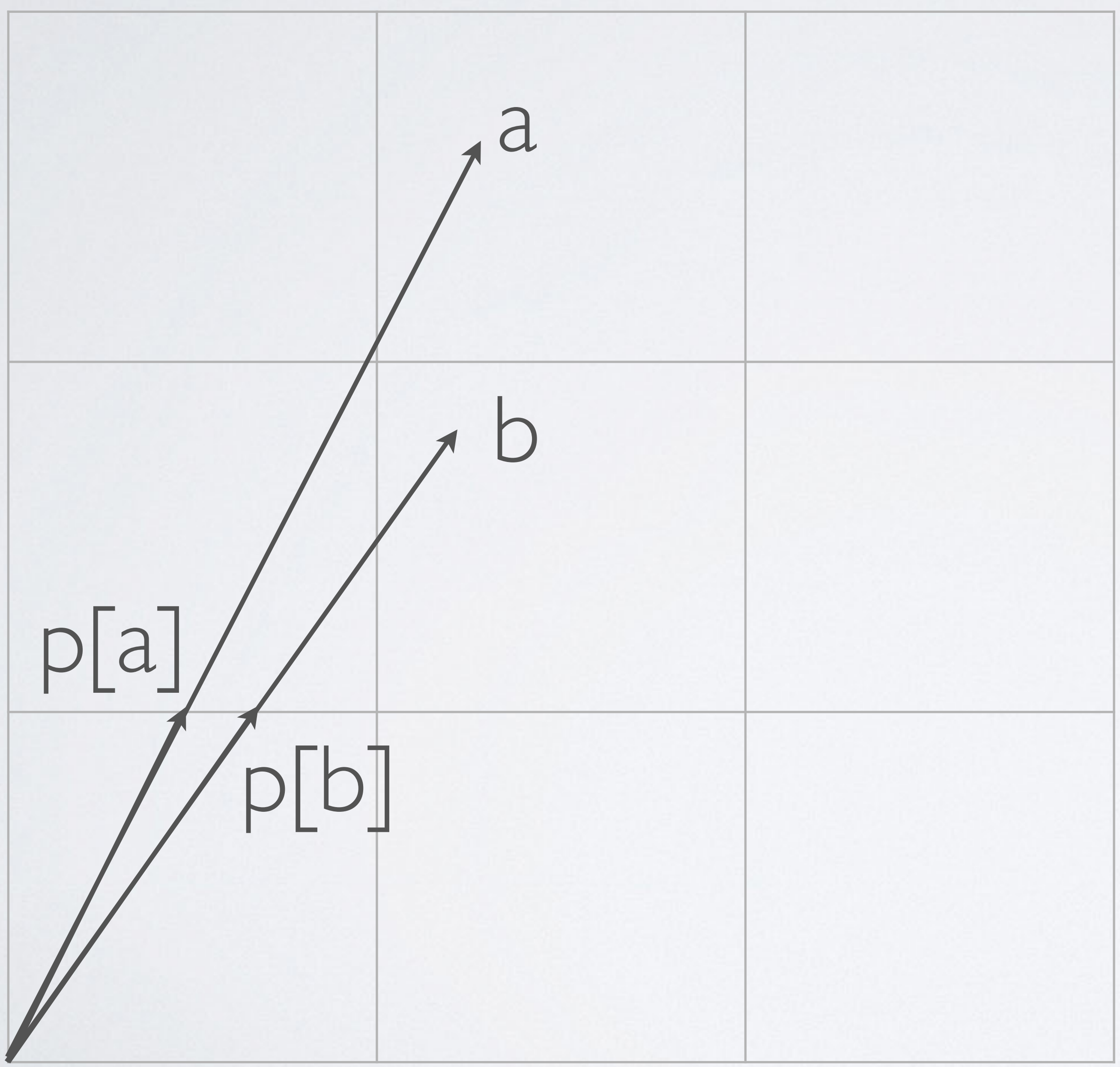
$$\begin{bmatrix} p(0) \\ p(1) \\ 1 \end{bmatrix} = \begin{bmatrix} b(0)/b(2) \\ b(1)/b(2) \\ b(2)/b(2) \end{bmatrix}$$

Homogeneización: división entre b(2)

PROYECCIONES

$$\begin{bmatrix} p(0) \\ p(1) \\ 1 \end{bmatrix} = \begin{bmatrix} x(0)/x(1) \\ x(1)/x(1) \\ x(1)/x(1) \end{bmatrix}$$

PROYECCIONES

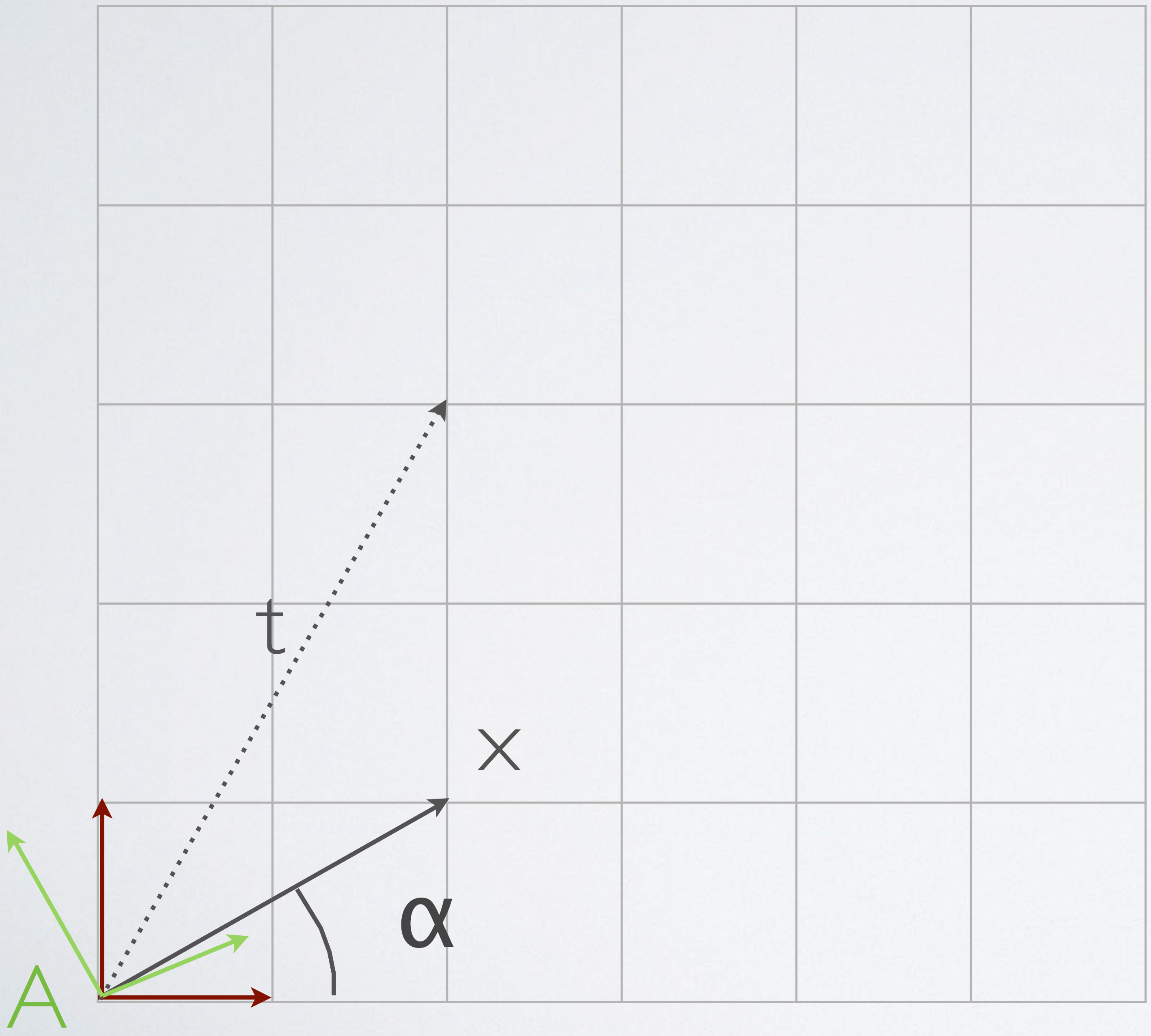


$$\begin{pmatrix} p(0) \\ p(1) \\ 1 \end{pmatrix} = \begin{pmatrix} x(0)/x(1) \\ x(1)/x(1) \\ 1 \end{pmatrix}$$

CAMBIO DE BASE

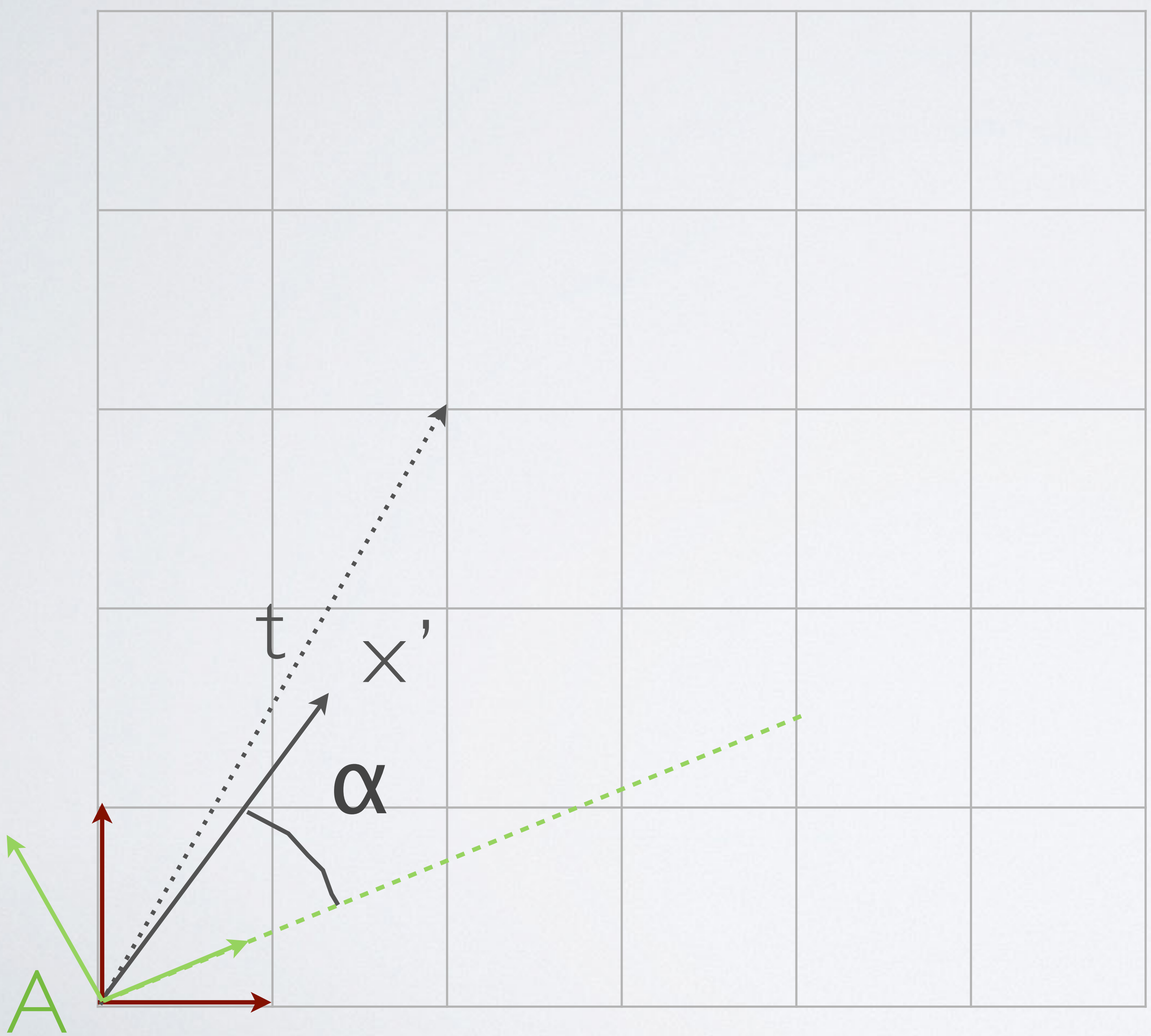
$$b = Ax + t$$

CAMBIOS DE BASE



$$b = Ax + t$$

CAMBIOS DE BASE

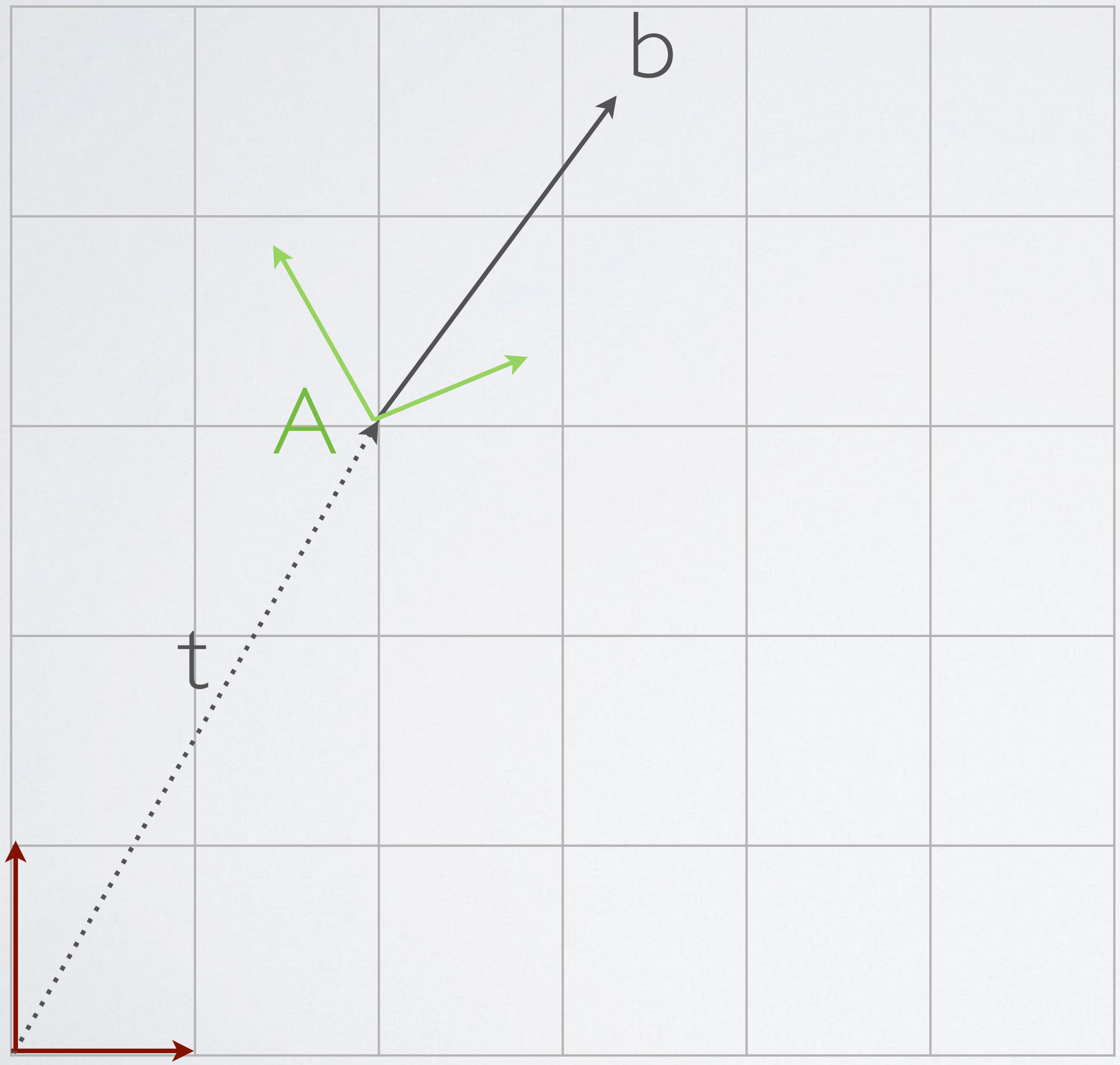


$$b = \underbrace{Ax}_{x'} + t$$

CAMBIOS DE BASE

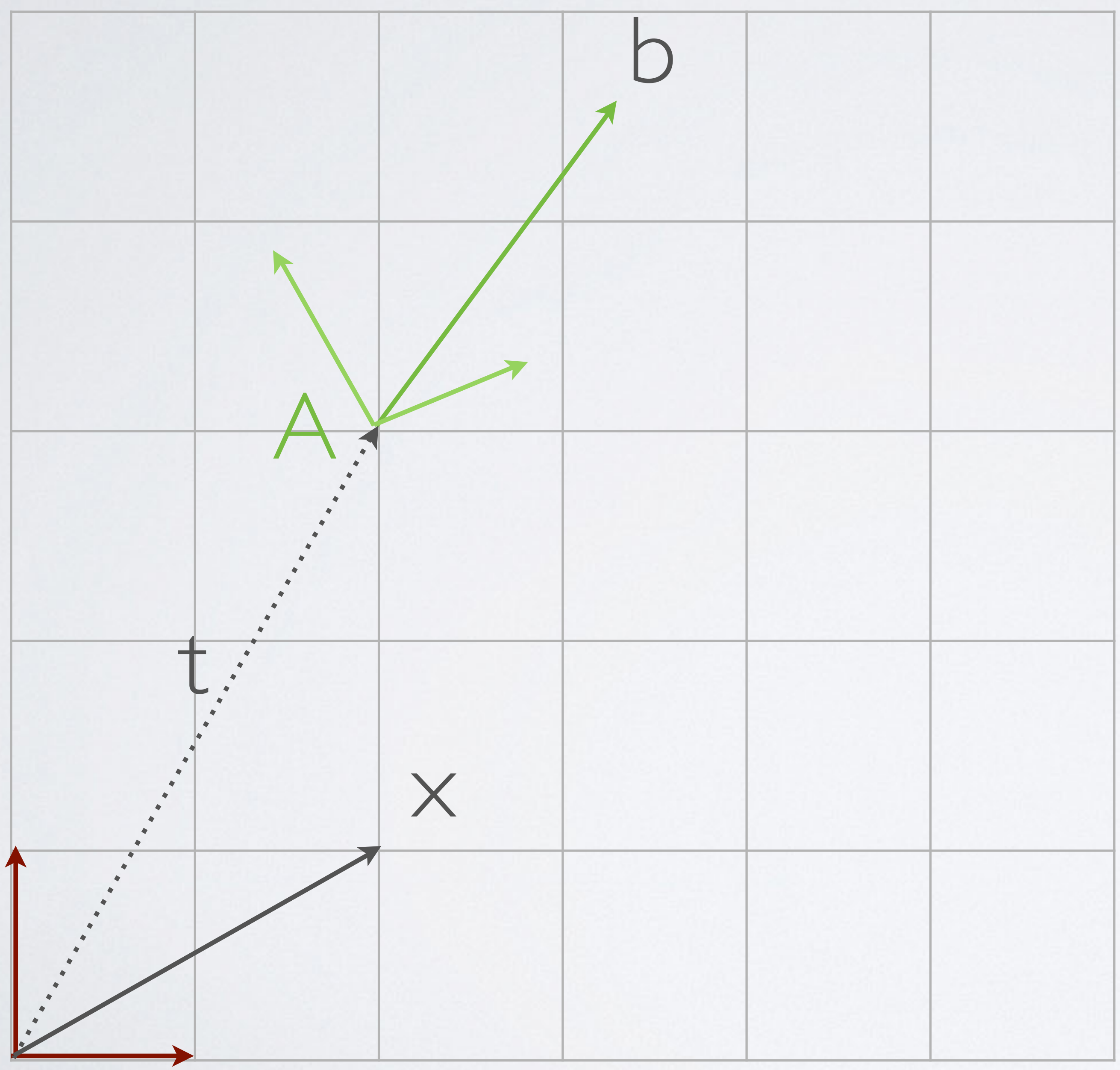
$$b = \underbrace{Ax + t}_{x''}$$

CAMBIOS DE BASE



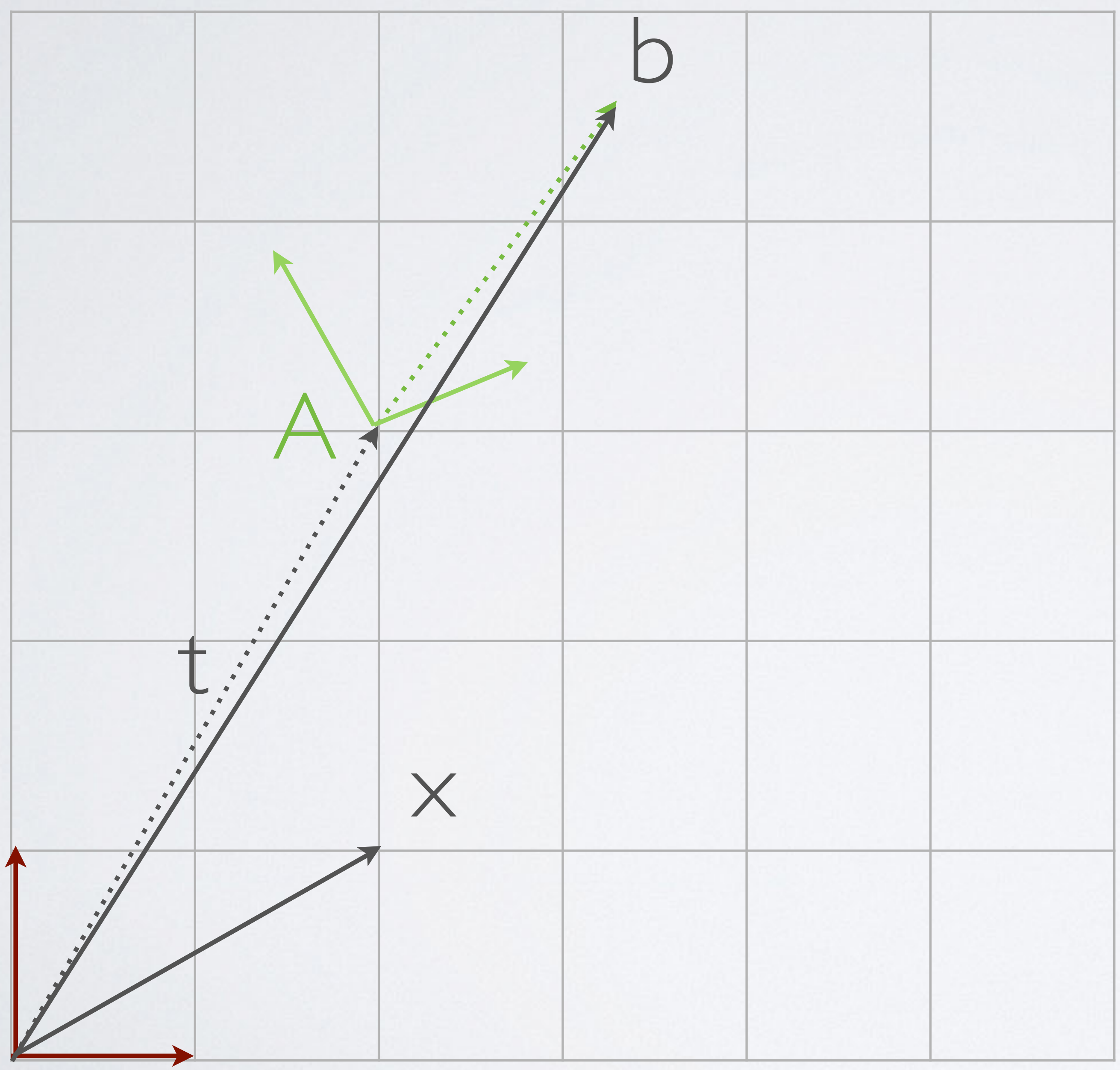
$$b = Ax + t$$

CAMBIOS DE BASE



$$b = Ax + t$$

CAMBIOS DE BASE



$$b = Ax + t$$

INVERSAS

$$b = Ax$$

INVERSAS

$$A^{-1}b = A^{-1}A\,x$$

INVERSAS

$$A^{-1}b = x$$

INVERSAS

$$b = Ax + t$$

INVERSAS

$$A^{-1}b = A^{-1}(Ax + t)$$

INVERSAS

$$A^{-1}b = A^{-1}Ax + A^{-1}t$$

INVERSAS

$$A^{-1}b - A^{-1}t = A^{-1}Ax$$

INVERSAS

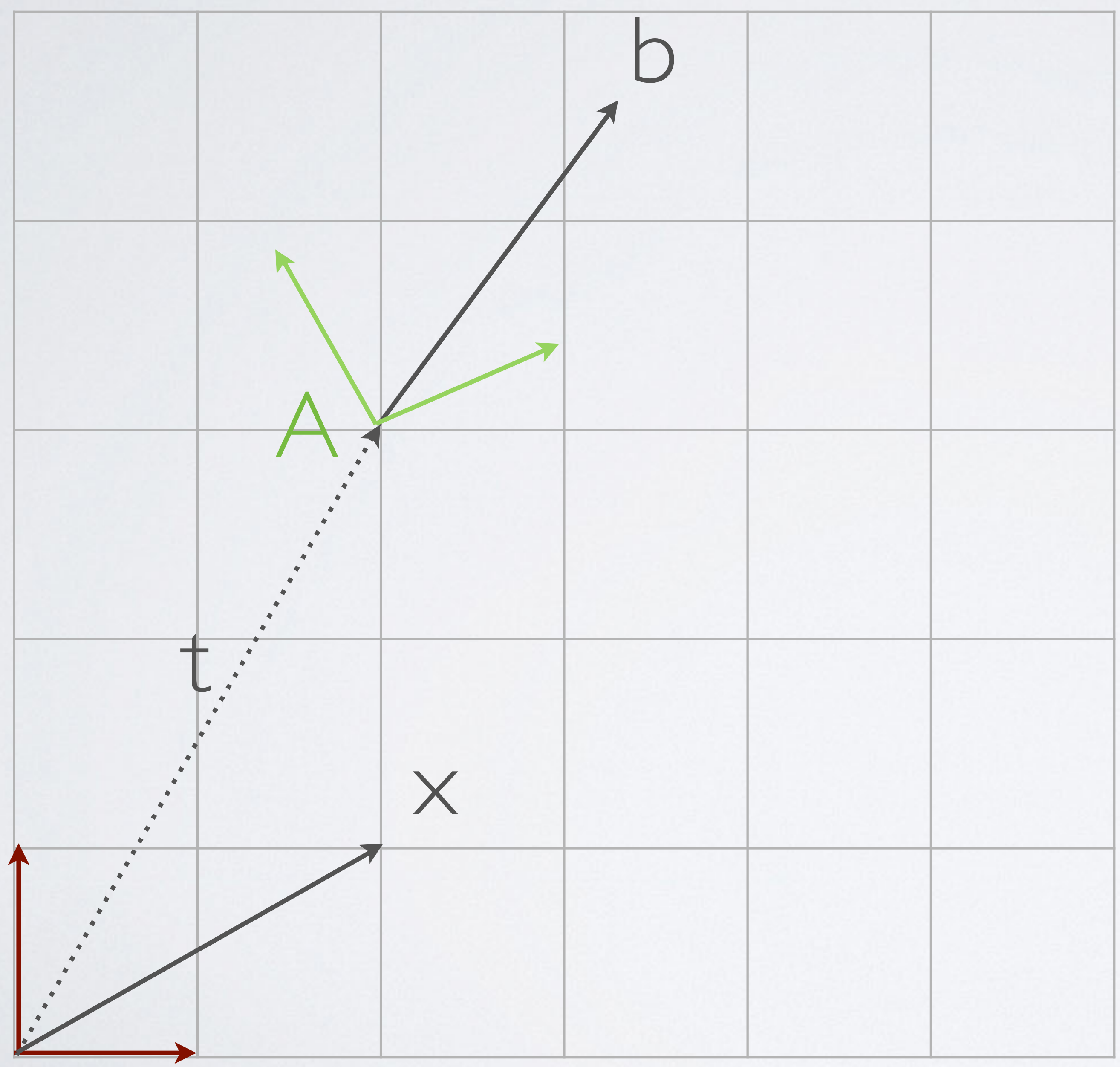
$$A^{-1}b - A^{-1}t = x$$

INVERSAS

$$A^T b - A^T t = x$$

Sólo si A es una rotación!

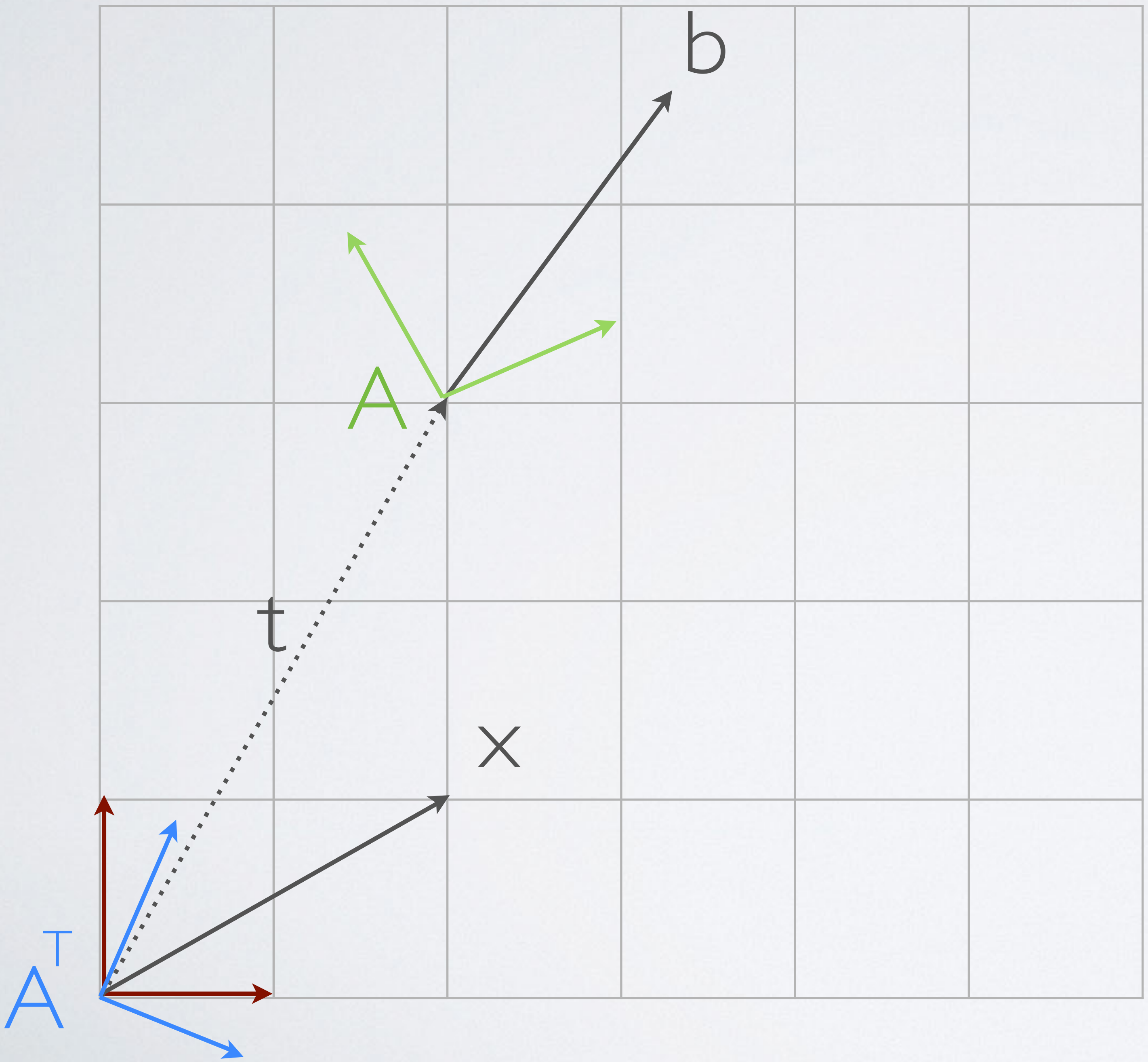
INVERSA



$$b = Ax + t$$

$$x = A^T b - A^T t$$

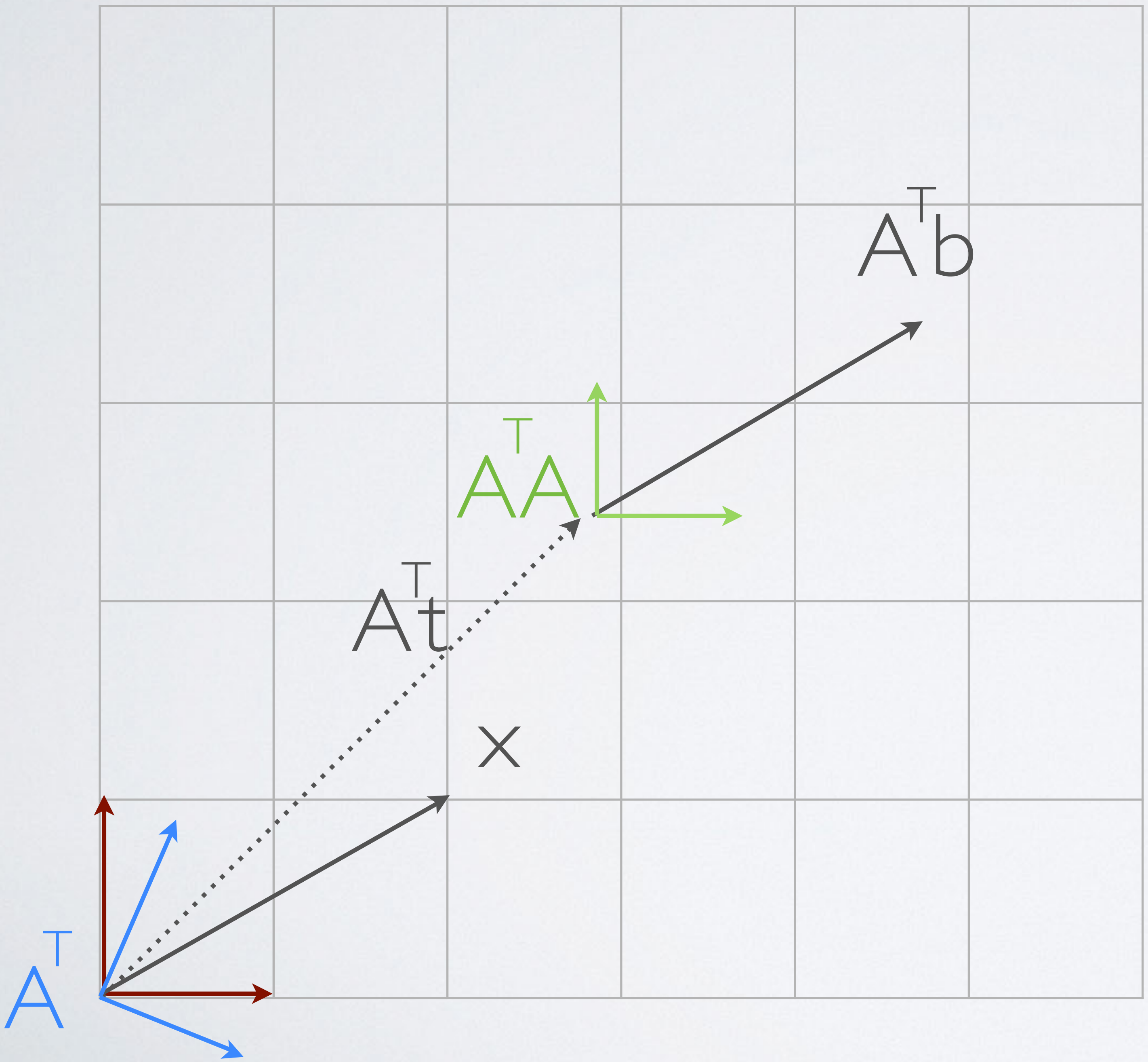
INVERSA



$$b = Ax + t$$

$$x = A^T b - A^T t$$

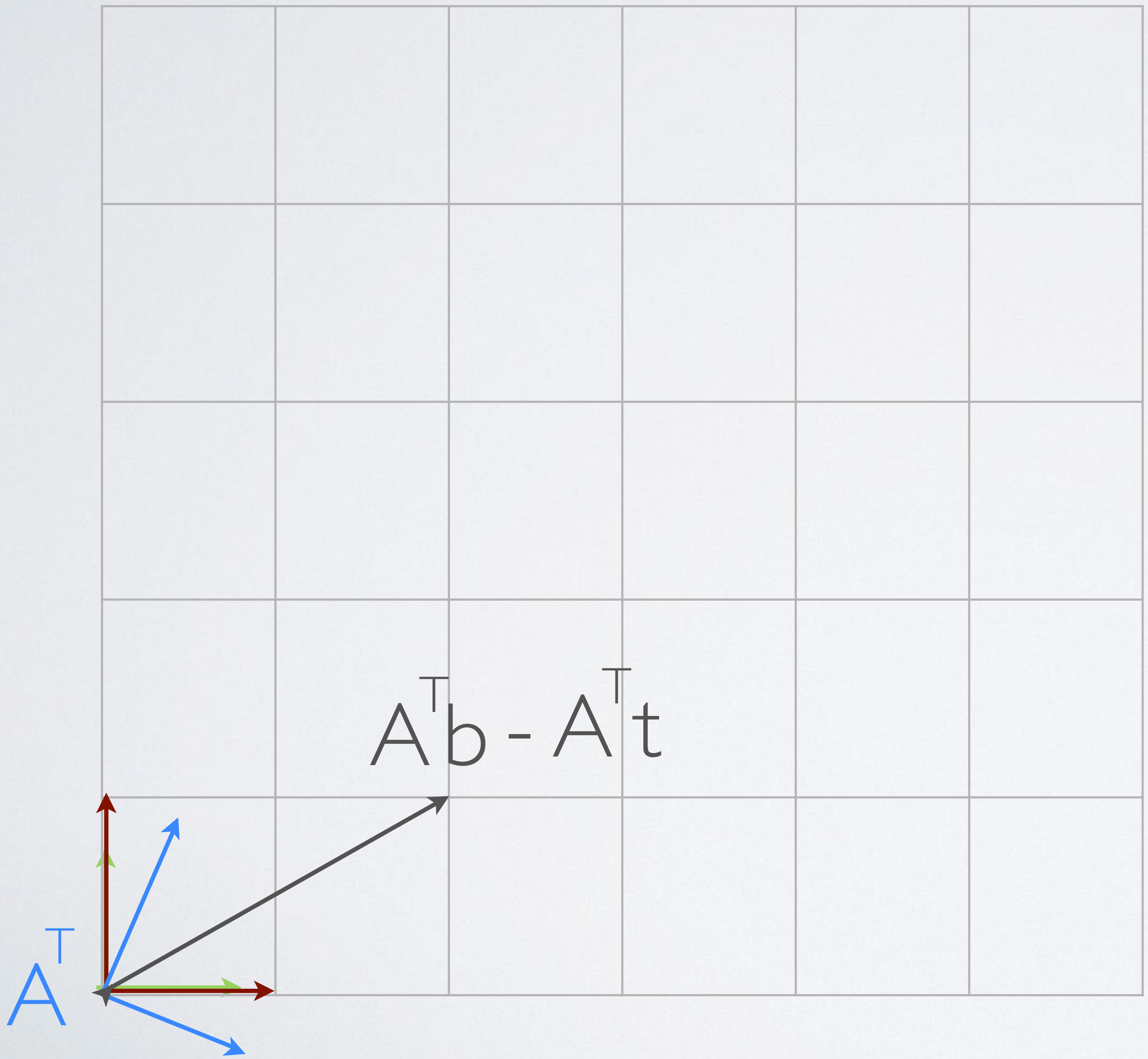
INVERSA



$$b = Ax + t$$

$$x = A^T b - A^T t$$

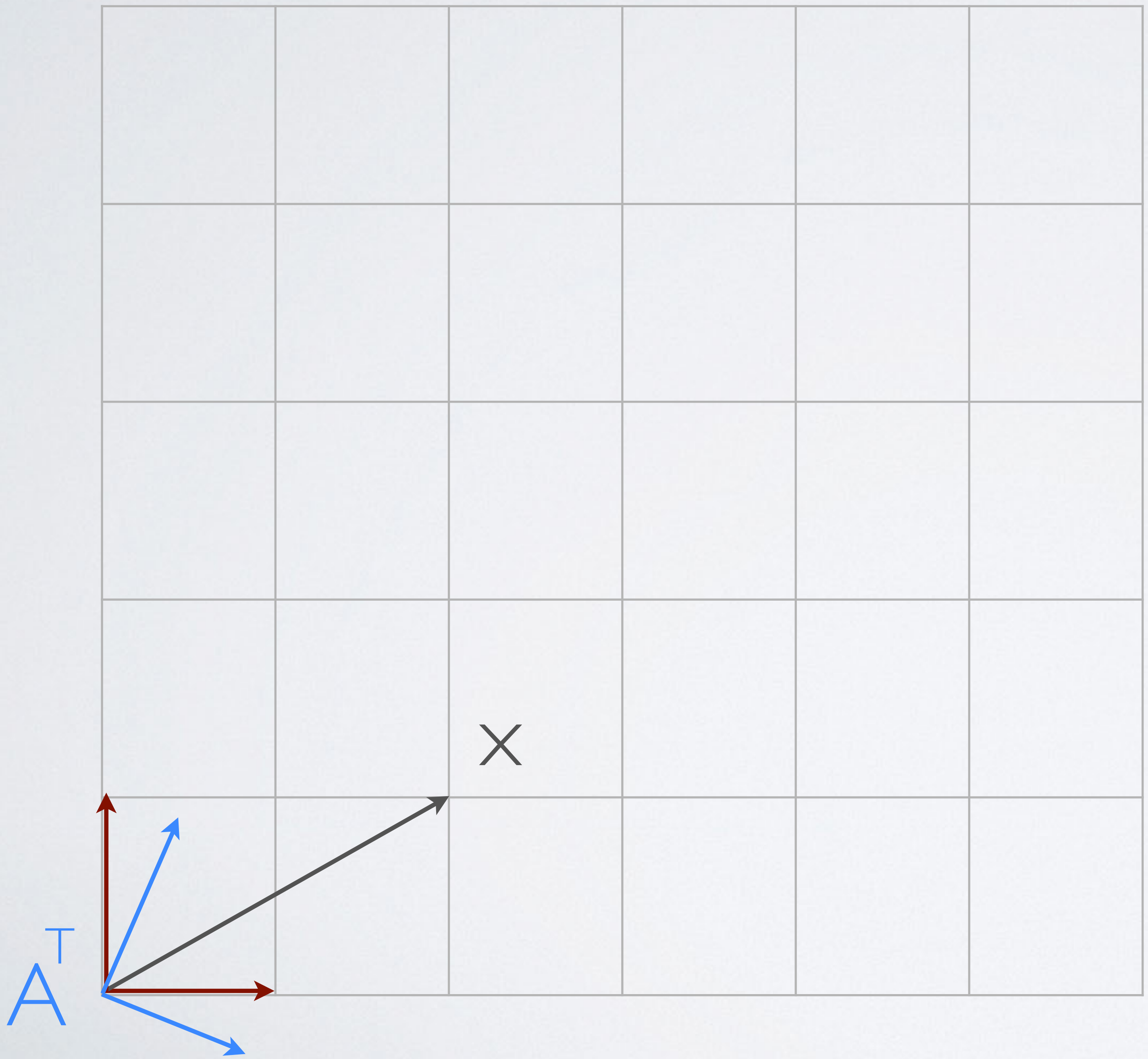
INVERSA



$$b = Ax + t$$

$$x = A^T b - A^T t$$

INVERSA



$$b = Ax + t$$

$$x = A^T b - A^T t$$