Expt. #17: Measurement of effective spring constant of a string-mass assembly using two coupled pendula

1 Aim and principle

The aim of the experiment is to measure the effective spring constant of a string-mass assembly. The string-mass assembly is attached between two nearly identical pendula and act as a spring connecting the two pendula (see figure 1). The natural frequencies of two coupled pendula depend on the spring constant of the coupling spring. Therefore the effective spring constant of the string-mass assembly is determined from the measurements of the natural frequencies of the coupled pendula. Lets denote the spring constant measured as κ_{expt}.

Also, you can estimate the effective spring constant of the same by using the principles of classical mechanics. We denote the spring constant estimated as κ_{estim}.

Finally the two values of the effective spring constant (κ_{expt} and κ_{estim}) are compared.

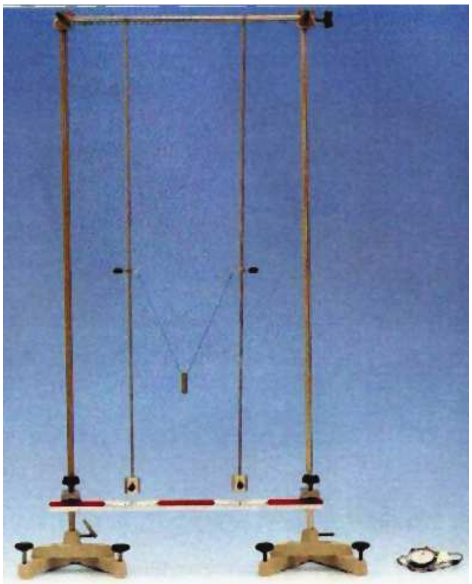

Figure 1: The coupled pendula with the string-mass assembly.

2 Theoretical background

Before we go into the details of the string-mass assembly and the coupled pendula we study in brief how a single pendulum (a member of the aforementioned coupled pendula) oscillates under gravity.

2.1 Single pendulum

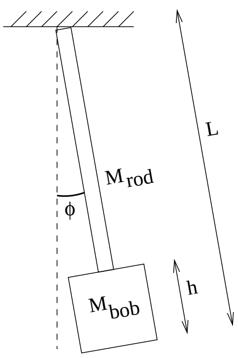

Figure 2: Single pendulum

The schematics in figure 2 shows a single pendulum which we use in our study. The equation of motion for this pendulum can be written as:

$$I\ddot{\phi} + \Gamma = 0 \tag{1}$$

where, I is the moment of inertia of the pendulum, ϕ is angular displacement and

$$\Gamma = M_{\mathrm{bob}} g L \left(1 - \frac{h}{2L}\right) \sin\phi + \frac{1}{2} M_{\mathrm{rod}} g L \sin\phi \tag{2}$$

The moment of inertia of the pendulum can be written as a sum of the moment of inertia of the bob (I_B) and the rod (I_R):

$$I = I_B + I_R \tag{3a}$$
$$= M_{\text{bob}} L^2 \left(1 - \frac{h}{2L}\right)^2 + \frac{1}{3} M_{\text{rod}} L^2 \left(1 - \frac{h}{L}\right)^2 \tag{3b}$$
$$\approx M_{\text{bob}} L^2 \left(1 - \frac{h}{L}\right) + \frac{1}{3} M_{\text{rod}} L^2 \left(1 - \frac{2h}{L}\right) \tag{3c}$$

Under small angle approximation ($\sin\theta \approx \theta$), equation 2 can be simplified to the form:

$$\Gamma \approx M_{\text{bob}} g L \phi + \frac{1}{2} M_{\text{rod}} g L \phi \tag{4}$$

Therefore, the angular frequency of oscillation for the single pendulum is given by:

$$\omega_{\text{s}} = \sqrt{\frac{gL}{I}\left(M_{\text{bob}} + \frac{1}{2} M_{\text{rod}}\right)} \tag{5}$$

Assuming $M_{\text{eff}} = (M_{\text{bob}} + M_{\text{rod}}/2)$ we can write,

$$\omega_{\text{s}} = \sqrt{\frac{gLM_{\text{eff}}}{I}} \tag{6}$$

2.2 Two coupled pendula

The schematics of two pendula coupled by a spring is shown in figure 3(A). Each of the pendula are identical to the single pendulum case discussed in the previous subsection. In addition to regular torque due to gravity we also need to add an additional torque due to the stretching of the spring connecting the pendula.

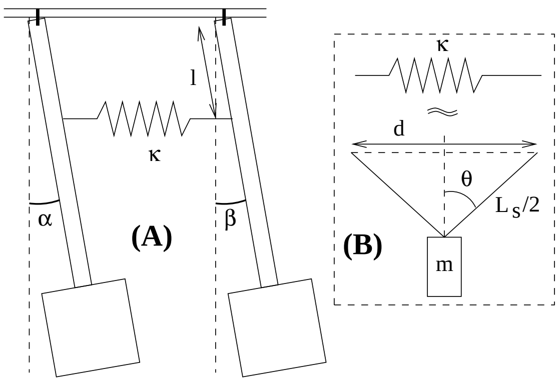

Figure 3: (A) Two pendula coupled by a spring, (B) String-mass assembly modeled as a spring

Including all the terms the equations of motion for the two pendula could be written as:

$$I\ddot{\alpha} = -M_{\text{eff}} g L \sin\alpha - \kappa l^2 (\sin\alpha - \sin\beta) \tag{7a}$$
$$I\ddot{\beta} = -M_{\text{eff}} g L \sin\beta + \kappa l^2 (\sin\alpha - \sin\beta) \tag{7b}$$

Where κ is the spring constant of the spring connecting the pendula, l is the distance from the top of the pendulum to the point of suspension of the spring, α and β are angular displacement of the pendula as shown in figure 3(A).
Under small angle approximations the previous equations take the following form:

$$I\ddot{\alpha} = -M_{\text{eff}} g L \alpha - \kappa l^2 (\alpha - \beta) \tag{8a}$$
$$I\ddot{\beta} = -M_{\text{eff}} g L \beta + \kappa l^2 (\alpha - \beta) \tag{8b}$$

To find the natural frequencies of the system, we take the sum and subtraction of the equations 8(a-b) and we obtain:

$$I(\ddot{\alpha} + \ddot{\beta}) = -M_{\text{eff}} g L (\alpha + \beta) \tag{9a}$$
$$I(\ddot{\alpha} - \ddot{\beta}) = -M_{\text{eff}} g L (\alpha - \beta) - 2\kappa l^2 (\alpha - \beta) \tag{9b}$$

The two equations above are uncoupled and represent the two normal modes of the coupled system. The $\alpha + \beta$ mode or '$+$' mode represent the **in-phase** motion of the pendula where both the pendula are moving with same phase (same direction). The $\alpha - \beta$ mode or '$-$'mode represent the **out-of-phase** motion of the pendula where the pendula are moving with opposite phase (opposite direction).

From the above equations we calculate the natural frequencies of the coupled system as:

$$\omega_+ = \sqrt{\frac{M_{\text{eff}}gL}{I}} \tag{10a}$$

$$\omega_- = \sqrt{\frac{M_{\text{eff}}gL + 2\kappa l^2}{I}} \tag{10b}$$

Since the pendula are moving in phase, the distance (or the angle) between them remains constant during the motion, therefore the spring connecting the two pendula does not exert any additional force between the two pendula. As a result the frequency of in-phase motion ω_+ is identical to that of a single pendulum.

We can express κ in terms of the natural frequencies and other measured quantities as:

$$\kappa = \frac{I}{2l^2}\left(\omega_-^2 - \omega_+^2\right) \tag{11}$$

Since the value of κ could be determined from the experimentally measured quantities we shall refer to the experimentally measured spring constant as κ_{expt}.

2.2.1 Oscillating transfer of energy between the pendula

If one of the pendula is set in motion while the other is at rest, we observe an interesting pattern of oscillation. For the sake of clarity we assume that the left pendulum has been swung while the right pendulum was at rest. We observe that with time the right pendulum picks up speed while the left pendulum's oscillation amplitude slowly diminishes. After a while the left pendulum comes to rest and only the right pendulum oscillates. Such transfer of energy or motion from one member to another is typical of two coupled oscillator. We will learn later that this phenomenon is universal and occurs also in quantum systems. Please note that such transfers happen only when one member is set in motion while the other is at rest. Under such scenario, the amplitude of a give pendulum could be written as a function of time (you can obtain the following result by solving equations 9(a-b) with the initial condition that at $t = 0$, the amplitude and the angular velocity of the first pendulum is non-zero while that of the second pendulum is zero):

$$A_{\text{left}}(t) = A_\circ \cos\left\{(\omega_- - \omega_+)t\right\}\cos\left\{(\omega_- + \omega_+)t\right\} \tag{12a}$$

$$A_{\text{right}}(t) = A_\circ \sin\left\{(\omega_- - \omega_+)t\right\}\sin\left\{(\omega_- + \omega_+)t\right\} \tag{12b}$$

where, we have assumed that the left pendulum has been set in motion (with initial amplitude $A_\circ$) while the right was at rest. The previous equations show that initially the right pendulum had zero amplitude (at rest), also after a time $2\pi/(\omega_- - \omega_+)$ the right pendulum comes back to rest. Therefore, the time period for a complete cycle of motion transfer corresponds to a frequency $(\omega_- - \omega_+)$. Henceforth we refer to this frequency as **beat** frequency (ω_{beat}). Please note that

$$\omega_- = \omega_+ + \omega_{\text{beat}}. \tag{13}$$

2.3 Theoretical estimate of spring constant of a string-mass assembly

If the entire system (pendula and string-mass assembly) is in static equilibrium, the tension of the string (of the string-mass assembly) can be expressed as (refer to figure 3(B):

$$T = \frac{mg}{2\cos\theta} \tag{14}$$

Hence the horizontal component of the tension is:

$$F = \frac{mg\tan\theta}{2} = \frac{mg}{2}\frac{\frac{d}{2}}{\sqrt{\frac{L_s^2}{4} - \frac{d^2}{4}}} \tag{15}$$

where θ is the angle between the vertical axis and the string, m is the mass hanging from the string, d is the horizontal distance between the two end of the string and L_s is the length of the string.

If the system is stretched by a horizontal distance x, let us assume that the angle between the vertical axis and the string changes to a value θ'. For the stretched assembly we can write:

$$F' = \frac{mg\tan\theta'}{2} = \frac{mg}{2}\frac{\frac{d+x}{2}}{\sqrt{\frac{L_s^2}{4} - (\frac{d+x}{2})^2}} \tag{16}$$

From the last two equations we can calculate the spring constant. We find the excess force due to stretching and express this force as a power series of the displacement x.

$$F - F' = \frac{mg\tan\theta}{2} - \frac{mg\tan\theta'}{2} \approx -\frac{mg}{2}\frac{L_s^2 x}{(L_s^2 - d^2)^{\frac{3}{2}}} + \mathcal{O}(x^2, \dots) \tag{17a}$$

$$= -\kappa x + \mathcal{O}(x^2, \dots) \tag{17b}$$

We ignore the terms involving x^2 or higher powers of x (denoted by $\mathcal{O}(x^2, \dots)$) to be small and we approximate the excess force due to stretching as a linear function of x. This approximation allows us to write the spring constant (we call it κ_{estim} henceforth since we estimate the value by using mechanical arguments)

$$\kappa_{\text{estim}} = \frac{mg}{2}\frac{L_s^2}{(L_s^2 - d^2)^{\frac{3}{2}}} \tag{18}$$

3 Procedure

We recall that the aim of the experiment is to compare the values of κ_{expt} and κ_{estim}.

1. **Find** ω_+

   (a) Detach the string-mass assembly from the pendula.

   (b) Remove left pendulum from the overhanging rod.

   (c) Detach the bob from the pendulum (remember not to remove the black screw) and measure the weight (M_{bob}).

   (d) Measure the weight of the rod (M_{rod}) along with the string hook.

   (e) Remove the string hook and measure the length of the rod (L).

   (f) The preceding measurements are sufficient to calculate I, the moment of inertia of the pendulum.

   (g) Replace the string hook. Keep the distance of the string hook from the top of the rod to be 25 cm (l).

   (h) Attach the bob to the rod and then attach the pendulum back to the overhanging rod. The pendulum should be hanging at the position marked with red color on the rod.

   (i) Measure the frequency of oscillation of the single pendulum (remember to keep the string-mass assembly detached). Follow the steps below:

i. Make sure that the whole system is motion-less.
ii. Swing the pendulum by not more than three inches.
iii. Release the pendulum. Let it oscillate for 2/3 times.
iv. Start measuring time period. For consistency you should measure the time for at least 20 oscillation period.
v. Repeat your measurement at least 4 times.
vi. Calculate angular oscillation frequency (rad/s) by using the relation $\omega = 2\pi/T$ where T is the time period of the oscillation.
vii. Tabulate your results. A sample table may look like:

Serial no.	20 T (s)	T (s)	$\omega = 2\pi/T$ (rad/s)

(j) You have measured ω_s which in turn is equal to ω_+. *Can you explain why we do not measure ω_+ by setting both the pendula in synchronized in-phase motion?*

2. **Find** ω_-

(a) Note that ω_- could be determined from ω_+ and ω_{beat}. *Can you explain why we do not measure ω_- by setting both the pendula in synchronized out-of-phase motion?*

(b) To measure ω_{beat}, first make sure the system is at rest.

(c) Set amplitude of one pendulum to not more than 3 cm. Start stopwatch simultaneously.

(d) The second pendulum will slowly pick up speed from the rest position, and after a while, will come back to rest position. Stop the watch when it comes back to rest.

(e) Repeat the measurement at least four times.

(f) Calculate beat frequency. Your table may look like:

Serial no.	T (s)	$\omega = 2\pi/T$ (rad/s)

(g) You have measured ω_{beat}. Use equation 13 to calculate ω_-.

3. **Calculate κ_{expt} using eq. 11.**

4. **Find κ_{estim}.**

(a) Assume $m = 24 \pm 1g$, $L_s = 17.7 \pm 0.1cm$.

(b) Measure d using a scale and noting that the d is defined to be the distance between the points from which the string-mass assembly is hung. Note: d *is not the distance between the rods*.

(c) Use eq. 18 to calculate κ_{estim}.

5. **Repeat steps 1-5 for three different values of l. You may chose $l = 25, 35, 45$ cm.**

6. **Compare the values of κ_{estim} and κ_{expt} for various l. Are they equal within experimental error limit? If not, please provide an explanation.**

7. Equating the relations (11) and (18), we obtain a relation between the beat frequency ω_- and various properties of the coupling mechanism, e.g., the coupling mass m, the string length L_S and the horizontal distance d between the pendula:

$$\omega_-^2 = \frac{l^2 mg}{I} \frac{L_S^2}{(L_S^2 - d^2)^{3/2}} - \omega_+^2 \,. \tag{19}$$

This relation suggests that ω_- should increase as m increases, and that ω_- should decrease as d decreases. Verify these predictions by (a) measuring ω_- while keeping m and L_S fixed but **lowering** d **to two smaller values** other than that employed in steps 1-6 above and (b) measuring ω_- while **increasing the mass** m **to two higher values** other than that employed in steps 1-6 above by adding a small amount of putty onto the coupling mass. Remember to weigh the new coupling mass after having added the putty prior each measurement of ω_-.

Complete your report with an error analysis and discussion section. In the discussion you may write your comments about the setup, suggestions for possible improvements etc.