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Applied Data Science Program

TIME SERIES

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Learning Time-Series

- Time Series are everywhere
  - Finance, weather, control systems

- Questions:
  - Modeling
  - Forecasting
  - Decisions and policy

Learning Time-Series: Outline

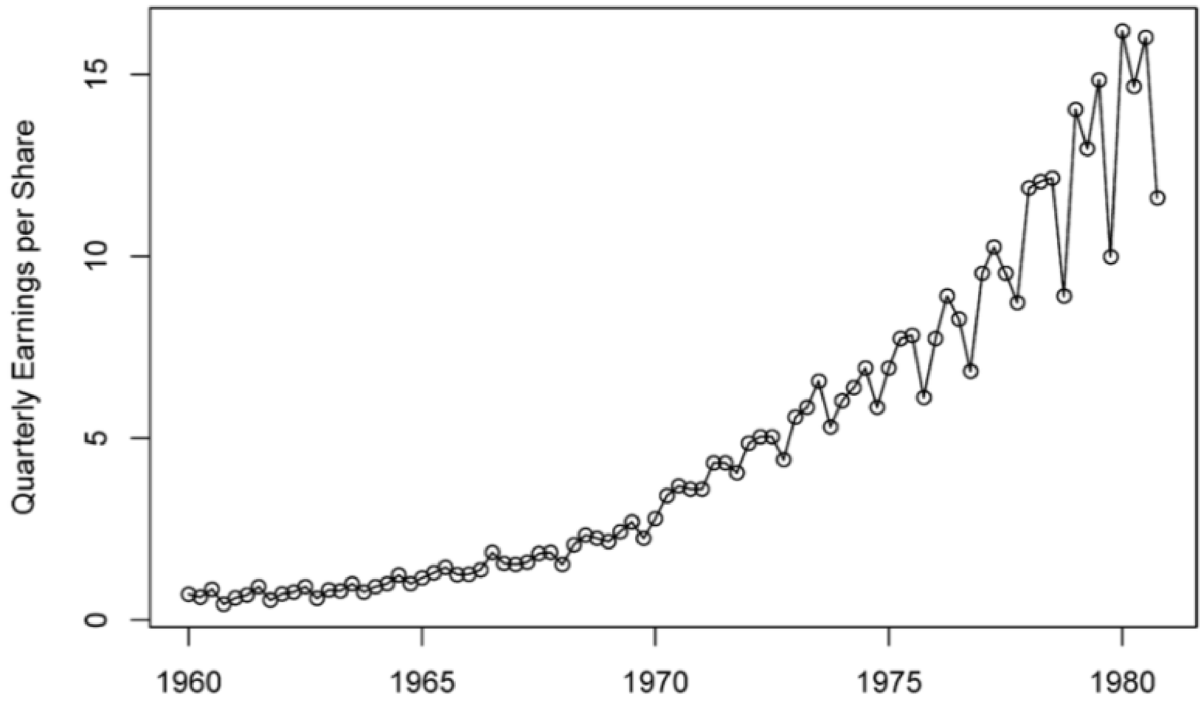
- Part I: Introduction to Time Series
  - Stationarity
  - Trends

- Part II: Models of Time Series

- Part 3: Learning Time Series

Introduction

Johnson & Johnson quarterly earnings per share



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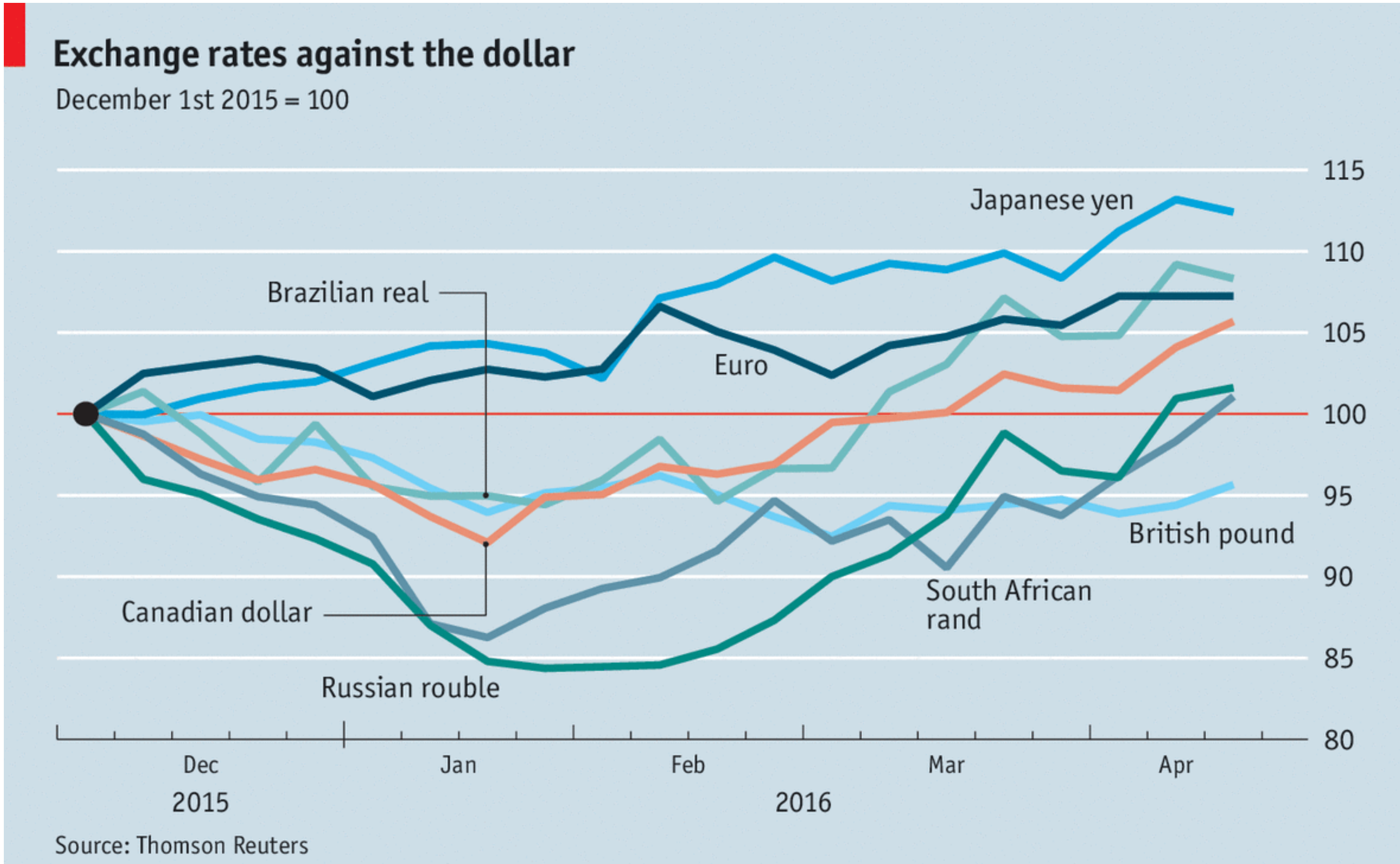
Exchange Rates Against the Dollar



Economist.com

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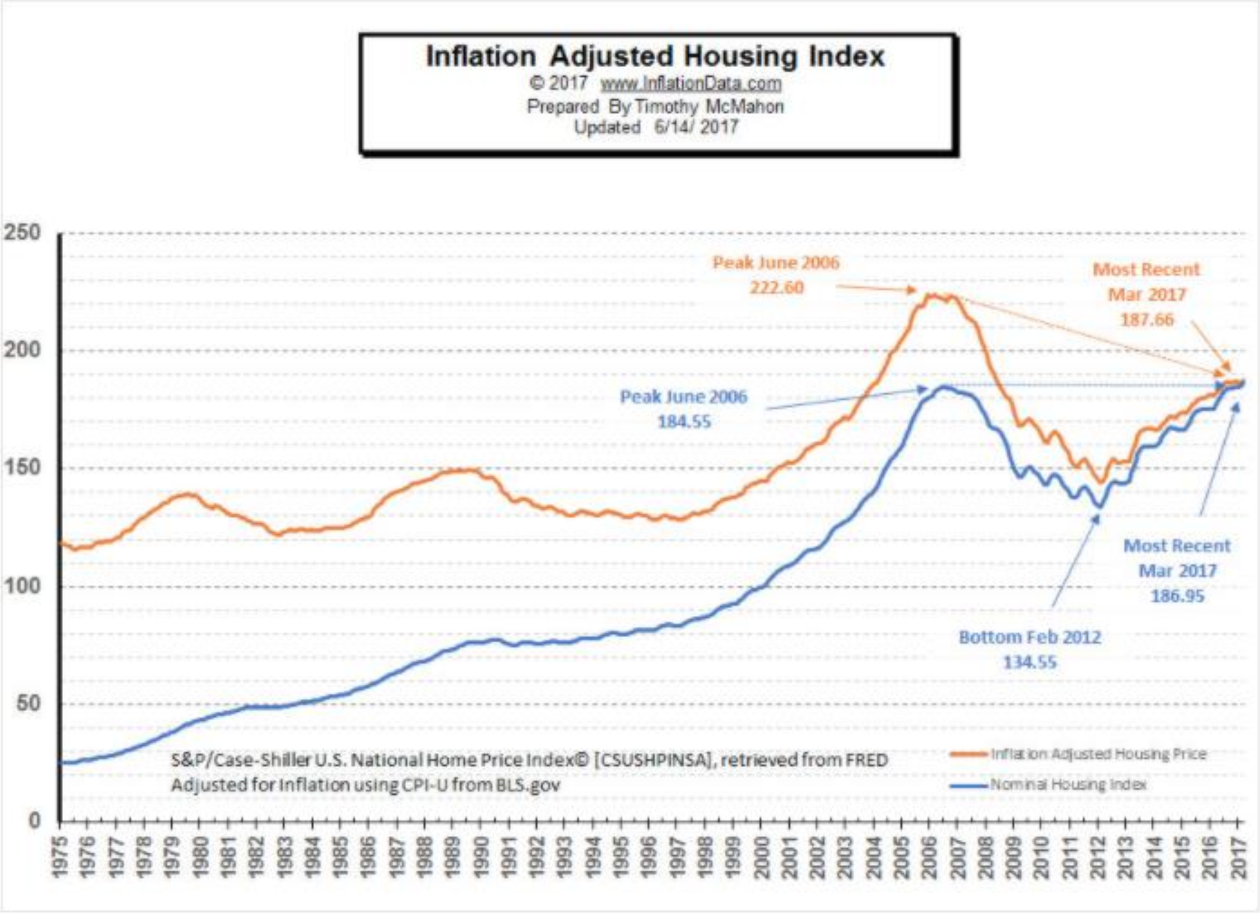
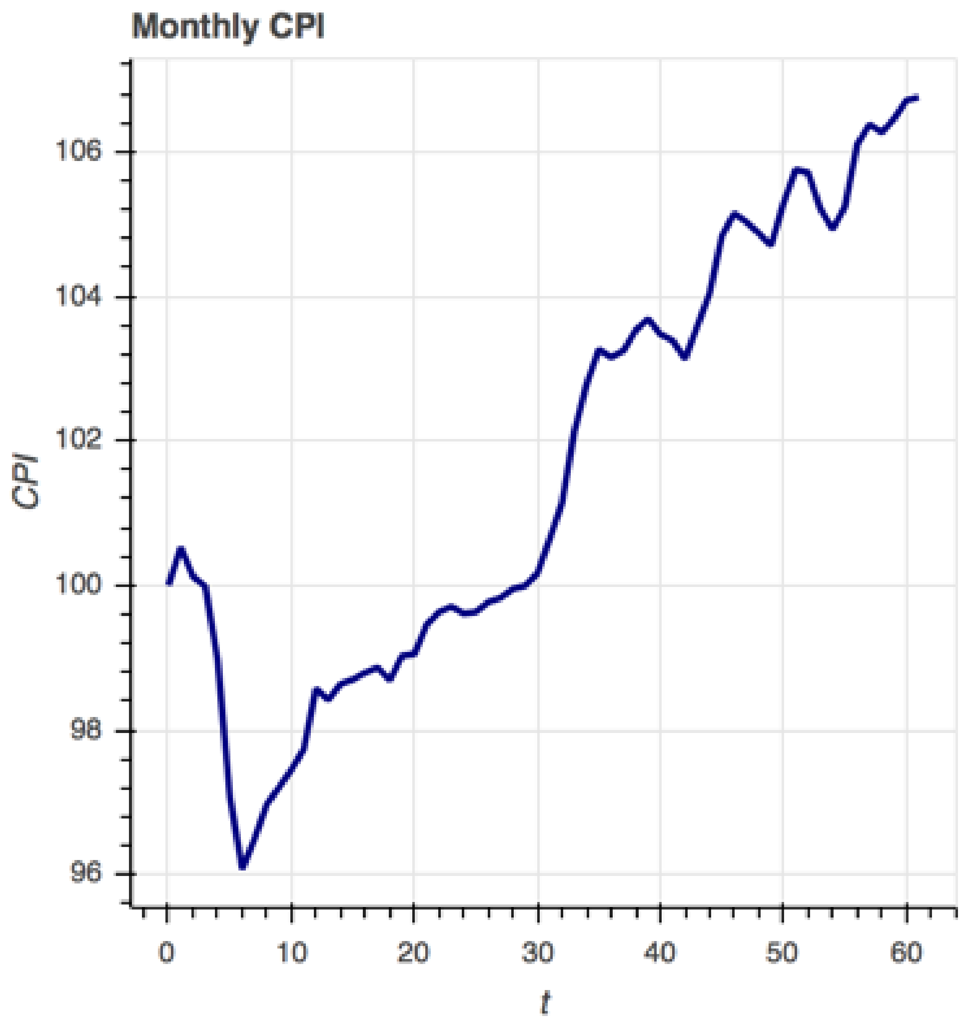
House Prices



Consumer Price Index (billion Prices)



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…What is new here?

- What you measure today depends on yesterday
  - Could be simple or complicated dependence
  - Memory can be high
  - Memory is typically unknow
- The variation in the data can be due to a time-varying average: trends
  - Linear, quadratic (deterministic)
  - Periodic (seasonal)
- Transformation of the data may help

Stationarity

- Time series need to have some structure
- Stationarity: Some variables are ‘constant’ over time
  - Mean
  - Covariance
- What happens if such variables are changing
  - Transform to a stationary series
  - Assume slow variation

Mean and Autocovariance: Stationarity

- Mean of process is constant over time

$$\mathbf{E}(X_t) = \mu$$

- Sample Mean (constant over. λ)

$$\hat{\mu} = \frac{1}{N-\lambda}\sum_{i=\lambda}^{N-1} X_i$$

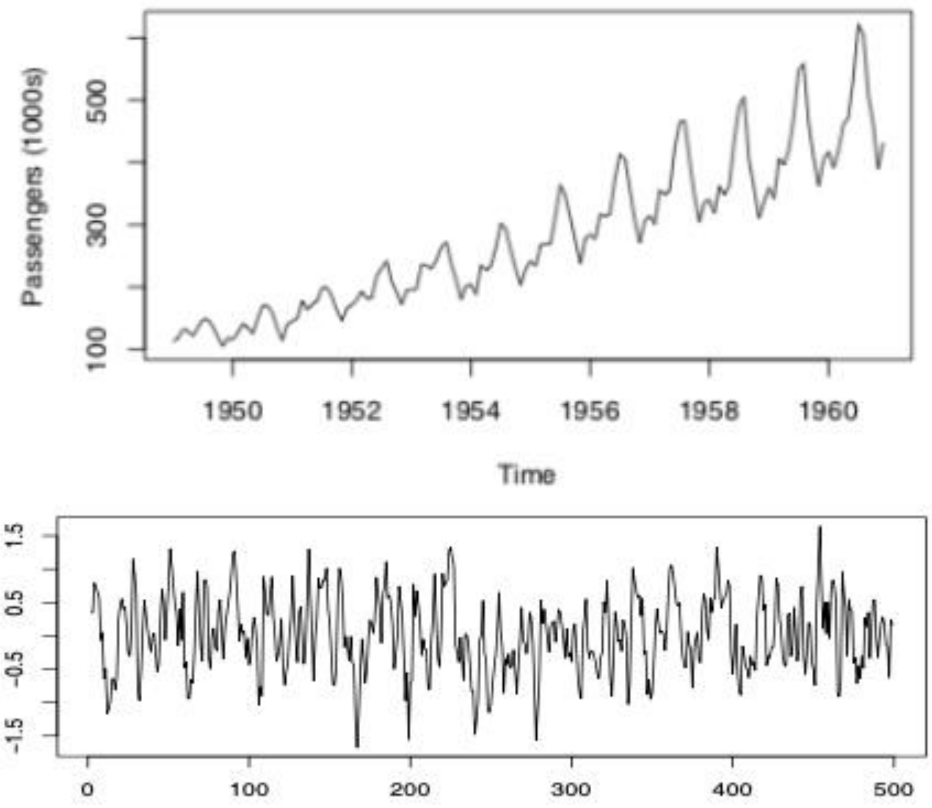
- Autocovariance is a function of the time difference

$$R_X(t_1 - t_2) = R_X(t_2 - t_1)$$

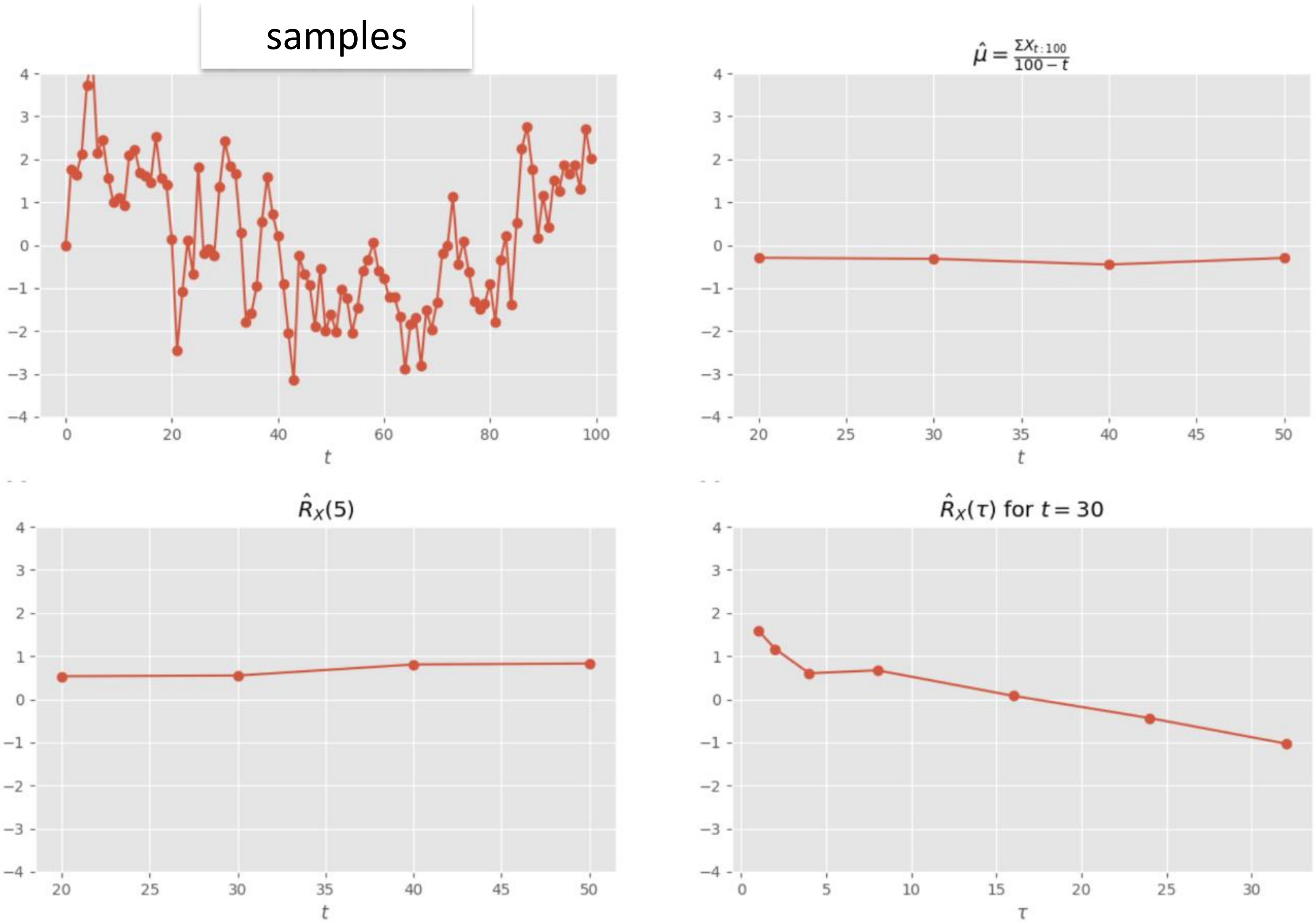
- Sample Autocovariance

$$\hat{R}_X(\tau) = \frac{1}{N-\lambda}\sum_{i=\lambda}^{N-1}(X_i - \hat{\mu})(X_{i+\tau} - \hat{\mu})$$

Testing Stationarity



Demonstrate: 100 samples



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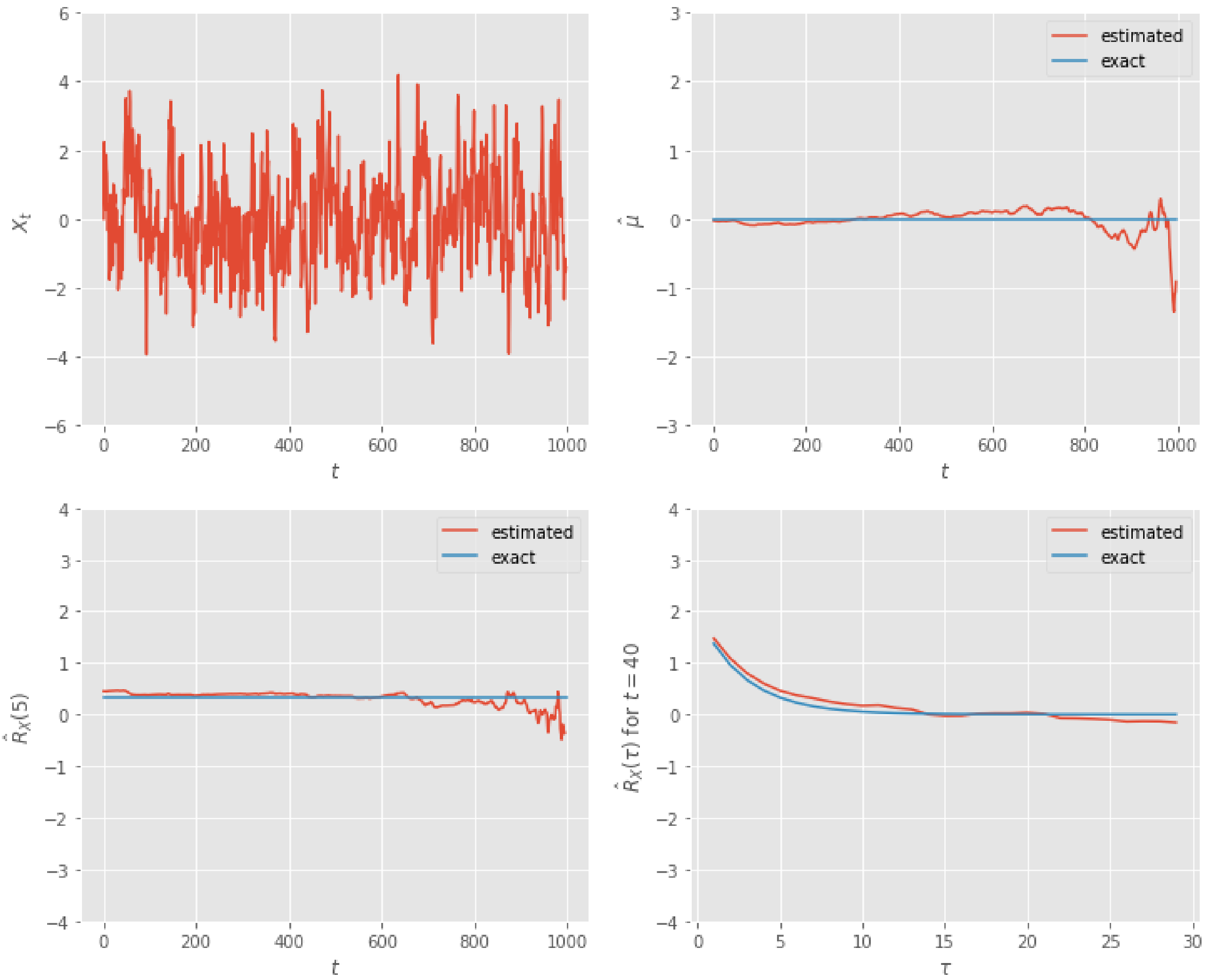
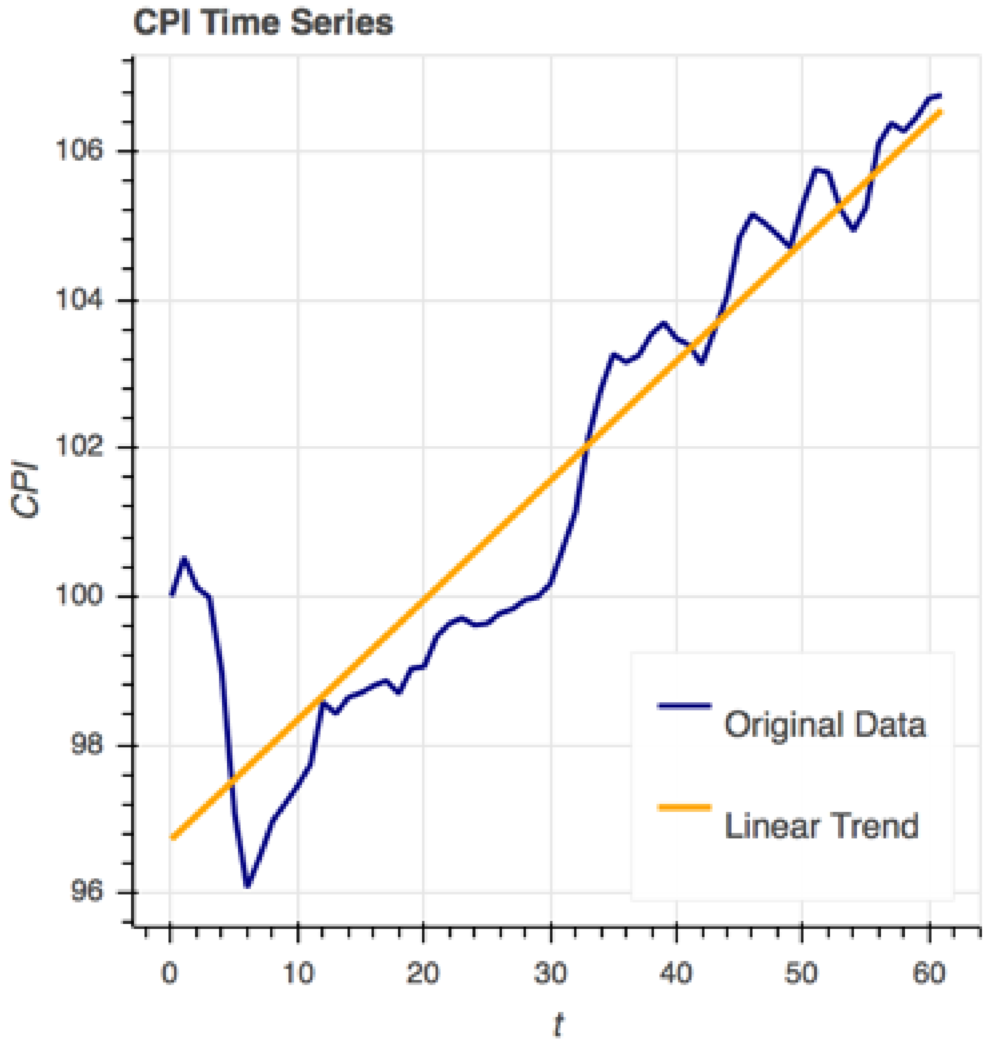
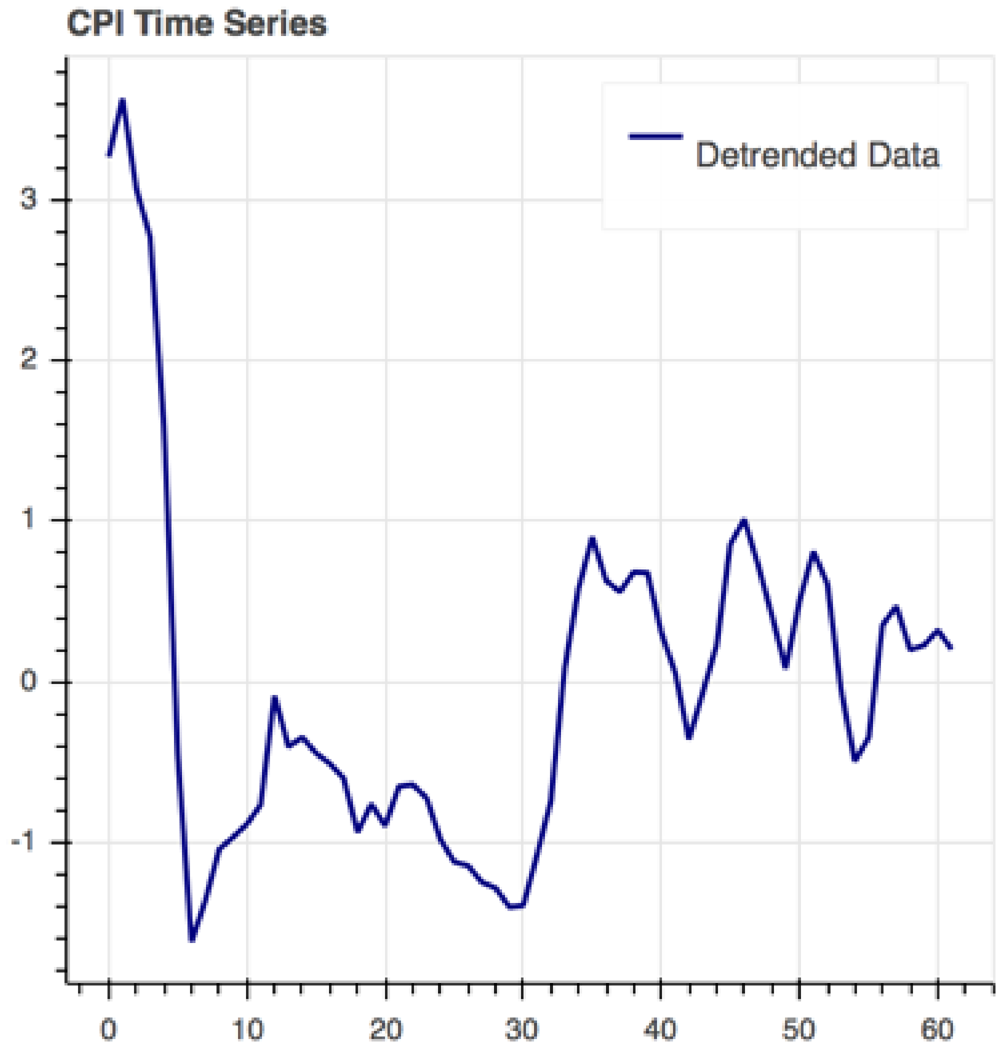
Demonstrate: 1000 samples



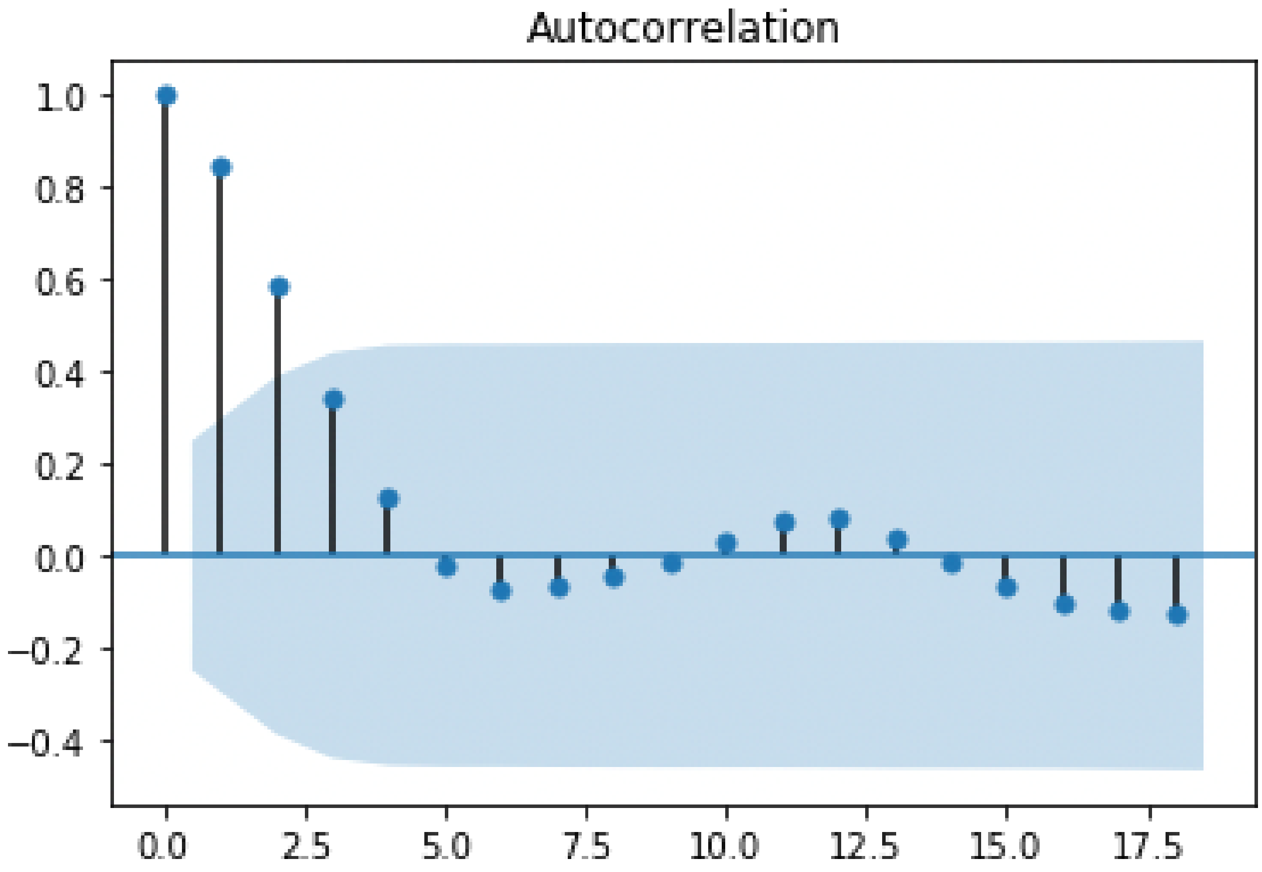
Trends in the Consumer Price Index (CPI) data



De-trended

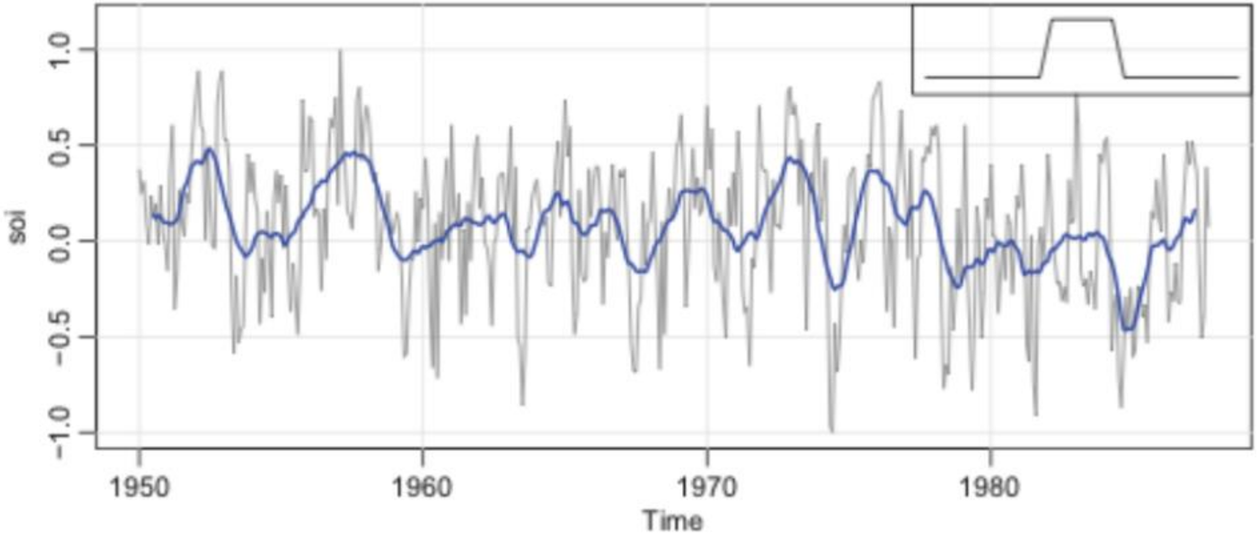


Empirical Autocovariance



Removing Seasonality

- $X_t = S_t + Y_t$
- Estimate S_t using a periodic regression
- Smoothing: $\hat{Y}_t = \sum_{h=-k}^{k} \gamma_h X_{t+h}$.



End

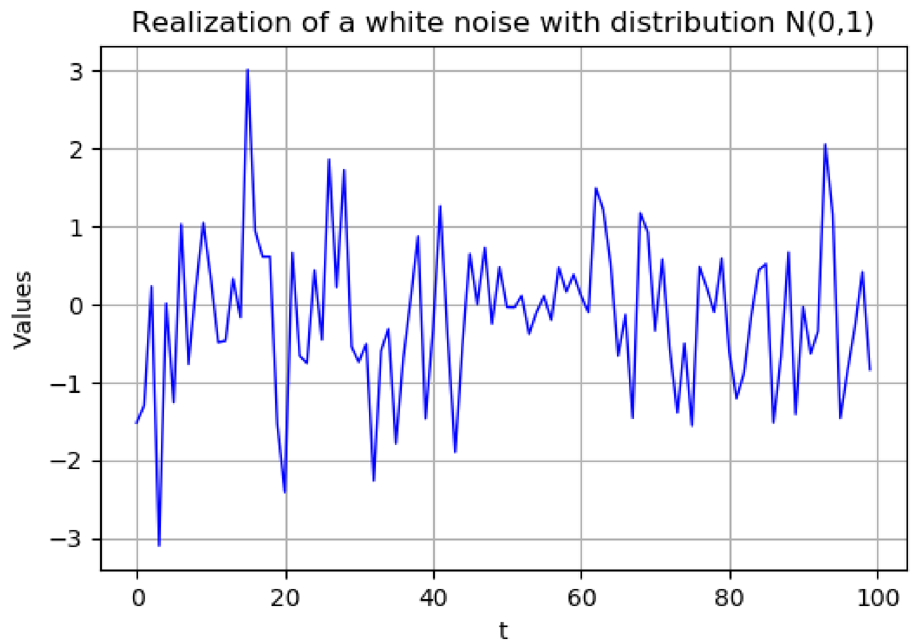
Models of time series

Models of Time-Series

- White noise
- Auto-regressive (AR)
- Moving Average (MA)
- Auto-regressive moving average (ARMA)
- Autocovariance of these examples

White noise

- Process $X_t = w$ $\mathbf{E}(w_t) = 0$

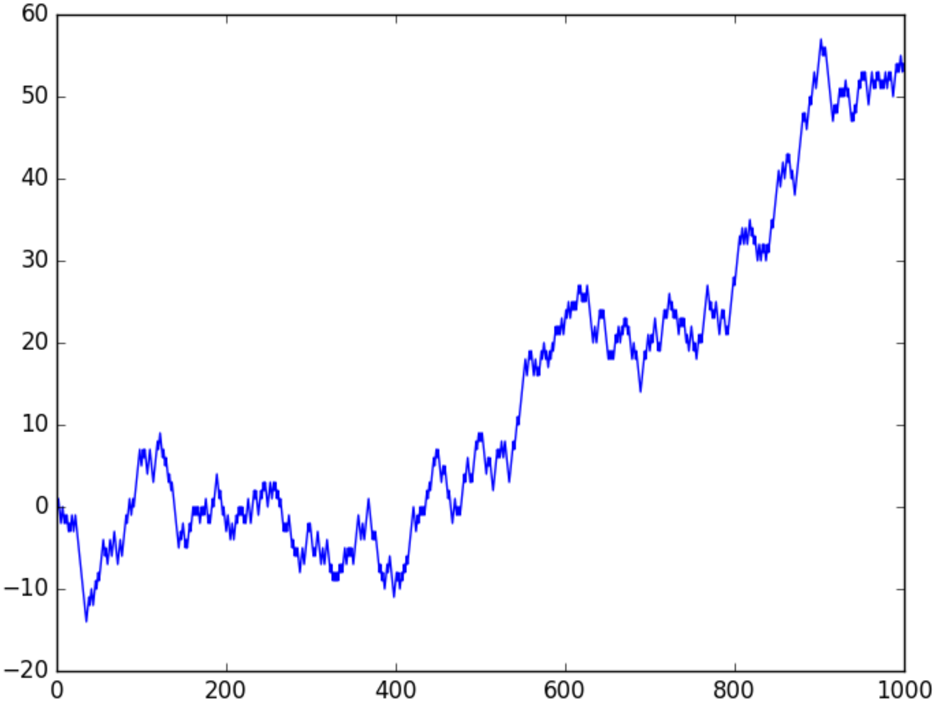


- Autocovariance =

$$\mathbf{E}(w_{t_1} w_{t_2}) = \sigma^2 \, \delta(t_1 - t_2)$$

Random Walk

$$X_t = X_{t-1} + w_t$$



Autoregressive Process AR(p)

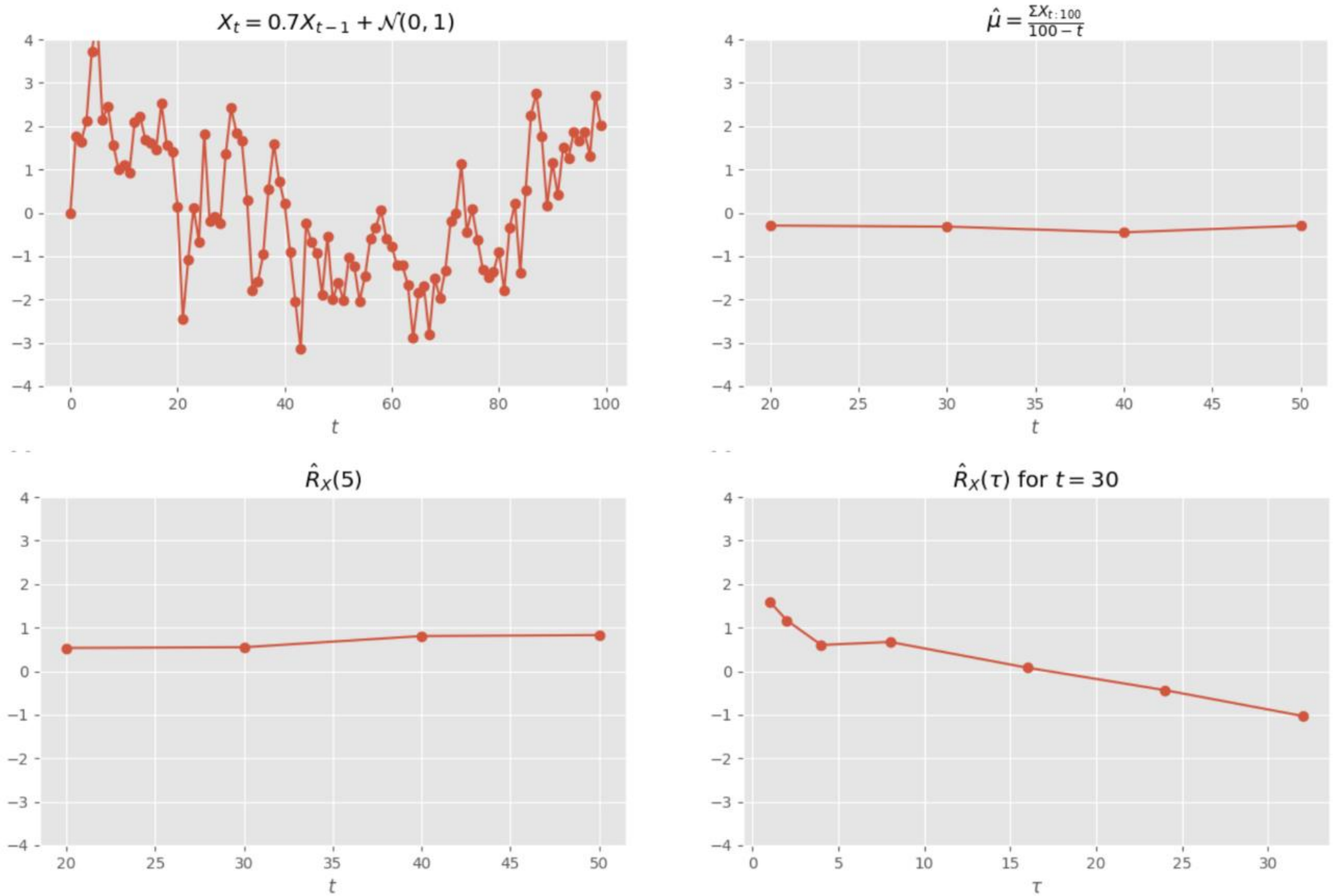
- Process

$$X_t = \sum_{i=1}^{p} a_i X_{t-i} + w_t$$

- w_t is white

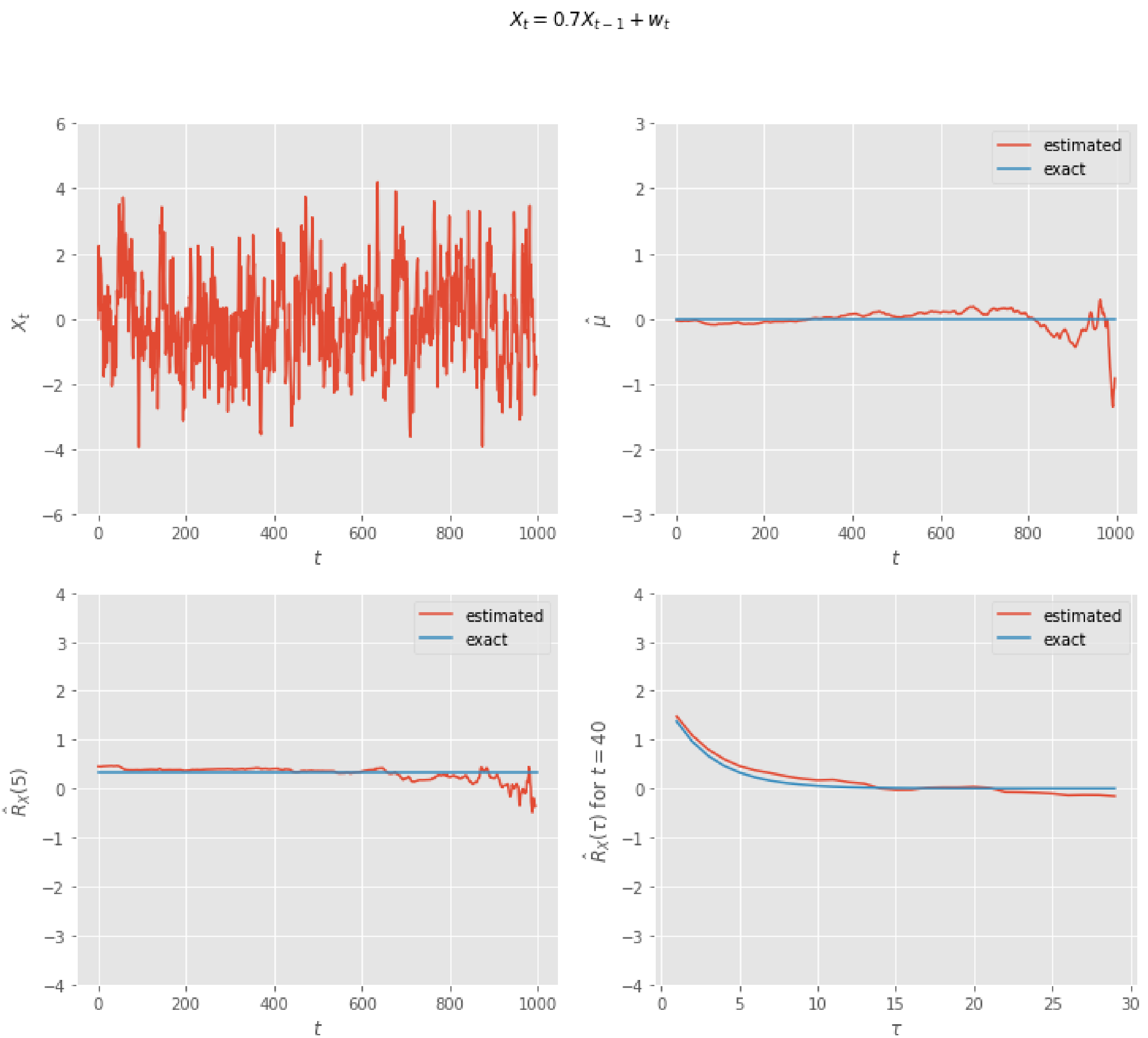
- Interpretation

Example AR(1)



$X_t = 0.7X_{t-1} + \mathcal{N}(0,1)$
$\hat{\mu} = \frac{\Sigma X_{t:100}}{100-t}$
$\hat{R}_X(5)$
$\hat{R}_X(\tau)$ for $t = 30$
t
τ

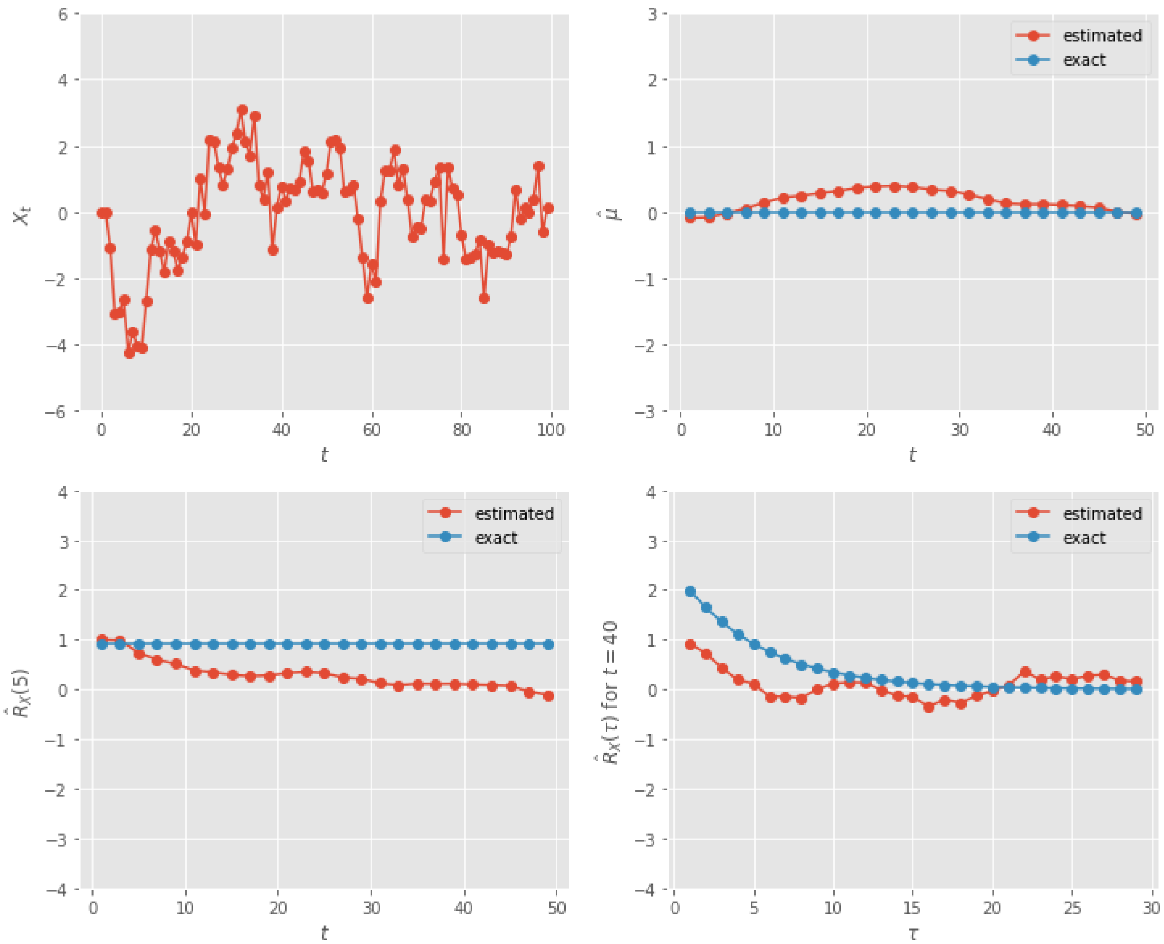
Example: AR(1)



$X_t = 0.7X_{t-1} + w_t$
X_t
t
$\hat{\mu}$
t
estimated
exact
$\hat{R}_X(5)$
t
estimated
exact
$\hat{R}_X(\tau)$ for $t=40$
τ
estimated
exact

Example: AR(2)

$$X_t = 0.7X_{t-1} + 0.1X_{t-2} + w_t$$



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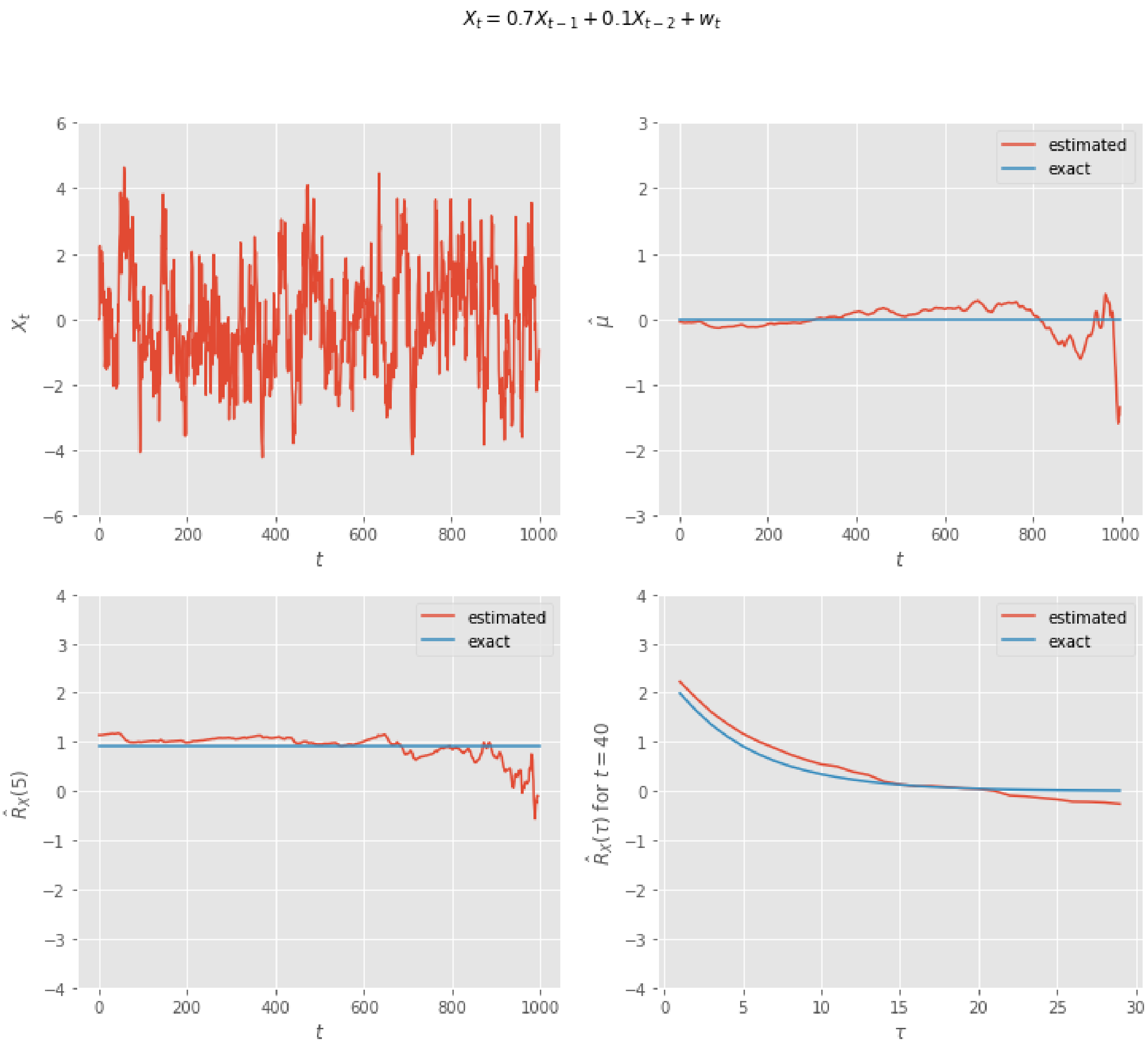
Example: AR(2)



$X_t = 0.7X_{t-1} + 0.1X_{t-2} + w_t$
X_t
t
$\hat{\mu}$
estimated
exact
$\hat{R}_X(5)$
$\hat{R}_X(\tau)$ for $t = 40$
τ

Example: AR(2)—argue it is an exponential

- Recall: $R_X(\tau) = R_X(-\tau)$

- Solve for $R_X(0), R_X(1), R_X(2)$ from

$$R_X(0) = a_1 R_X(1) + a_2 R_X(2) + \sigma^2$$

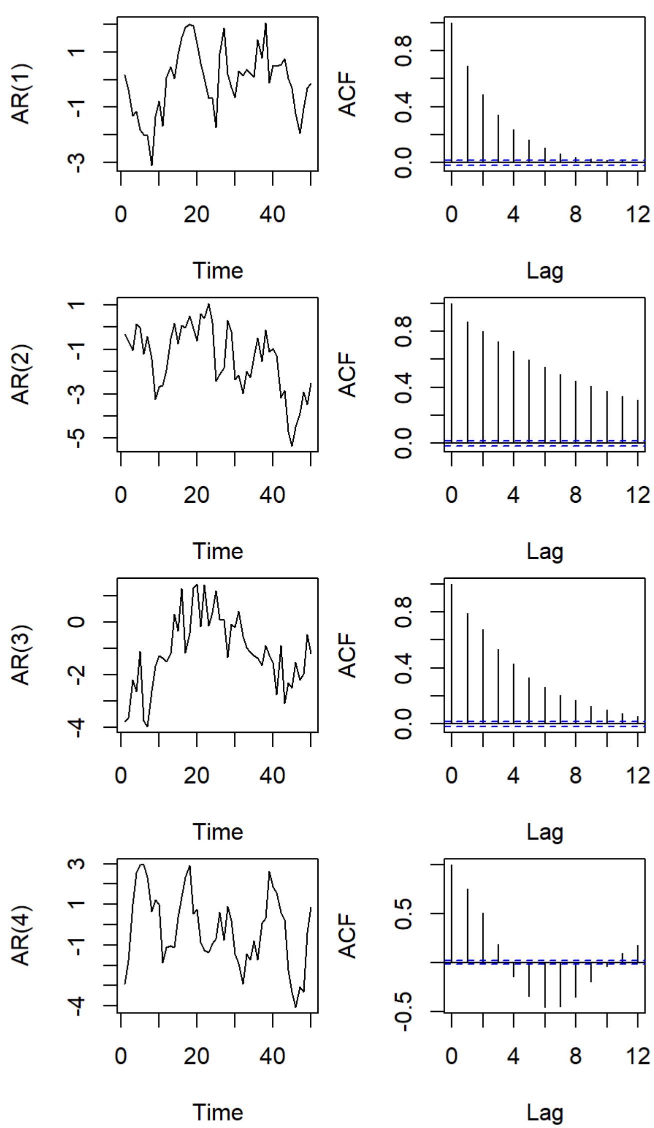
$$R_X(1) = a_1 R_X(0) + a_2 R_X(1)$$

$$R_X(2) = a_1 R_X(1) + a_2 R_X(0)$$

- Compute the rest

$$R_X(\tau) = \sum_{i=1}^{p} a_i R_X(\tau - i), \tau \geq 3$$

Geometric Shape of ACF for AR(p)



Moving Average MA(q)

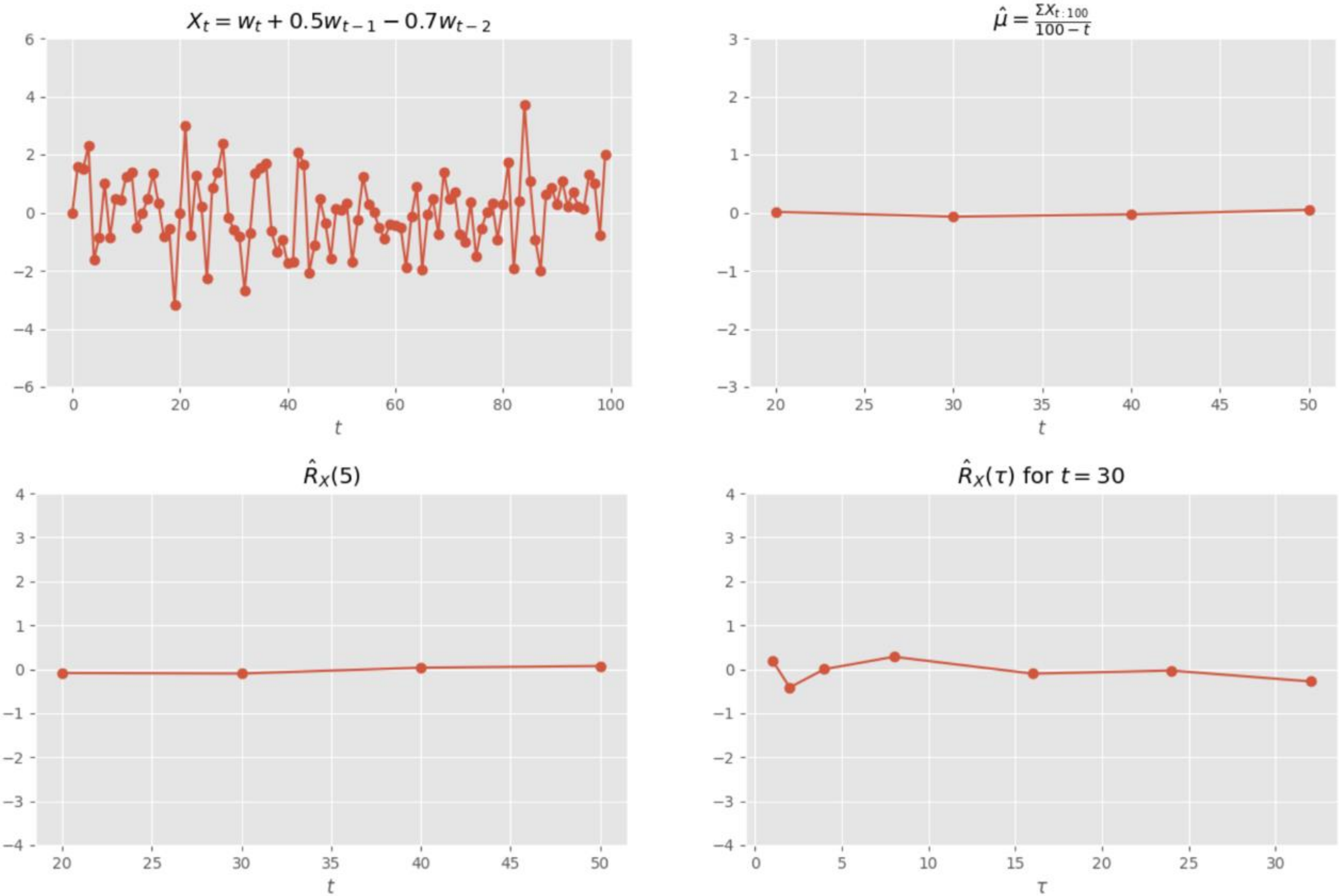
- Process

$$X_t = \sum_{i=0}^{q} b_i w_{t-i}$$

- w_t is white

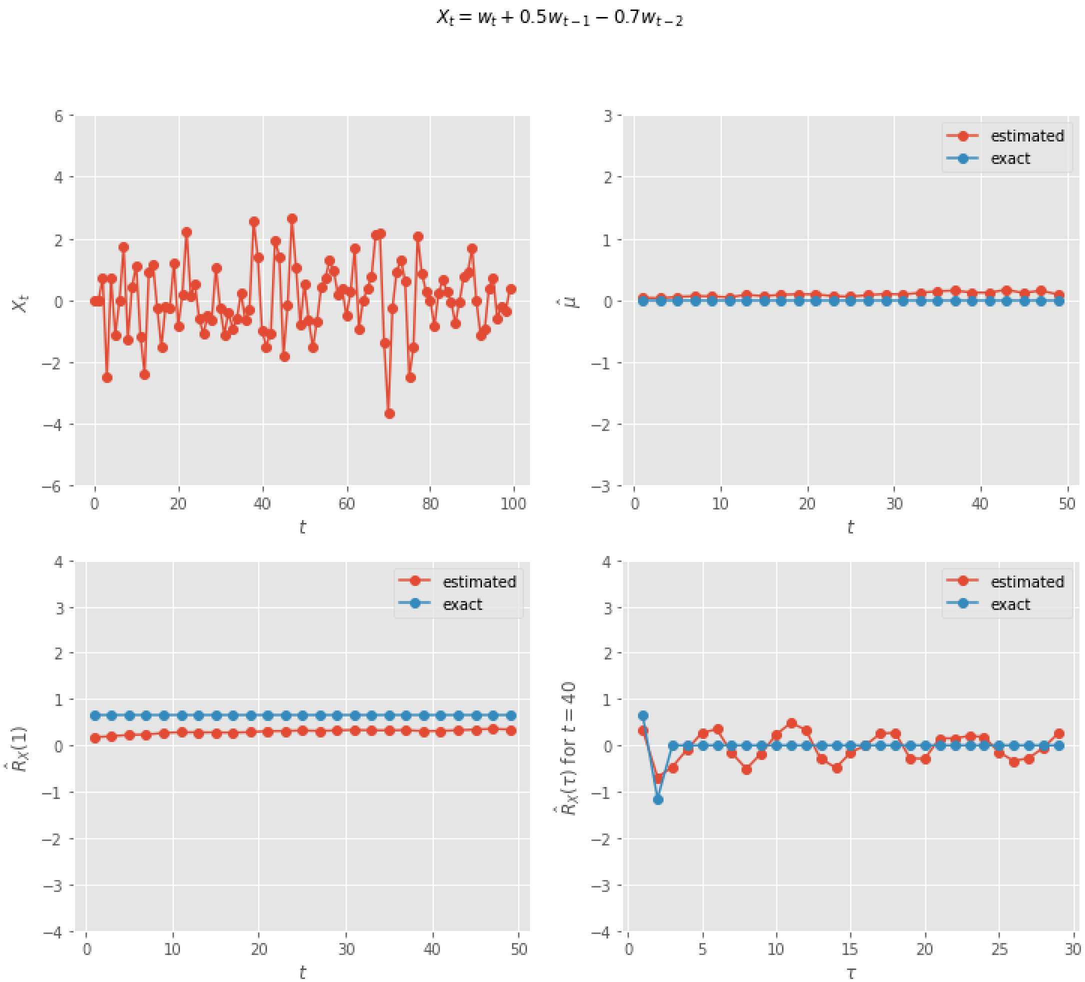
- Interpretation

Example MA(2)



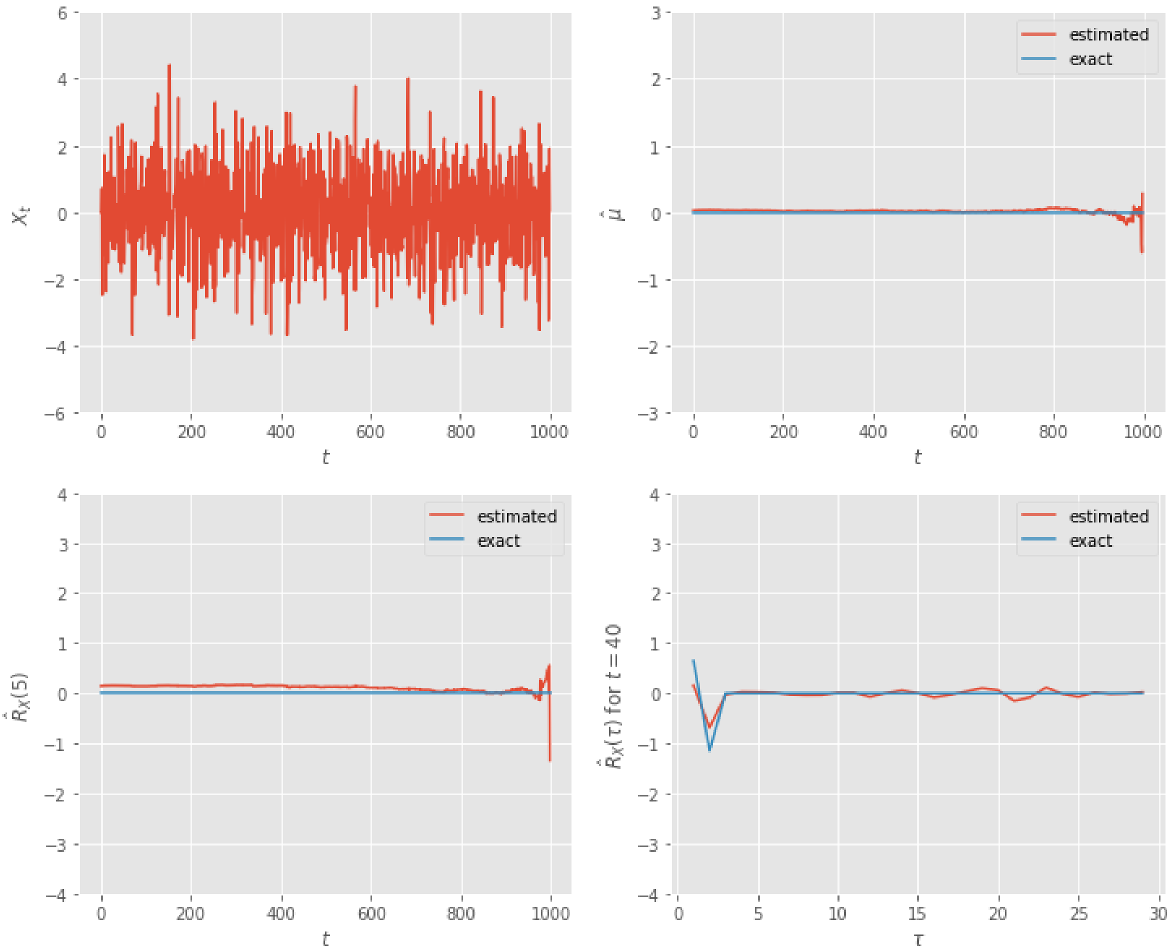
$X_t = w_t + 0.5w_{t-1} - 0.7w_{t-2}$
$\hat{\mu} = \frac{\Sigma X_{t:100}}{100-t}$
$\hat{R}_X(5)$
$\hat{R}_X(\tau)$ for $t = 30$
t
τ

Sample vs Actual



More Samples

$$X_t = w_t + 0.5w_{t-1} - 0.7w_{t-2}$$



Autocovariance of MA—finite (remove proof)

- How do you compute the exact Autocovariance of

$$X_t = \sum_{i=0}^{q} b_i w_{t-i}$$

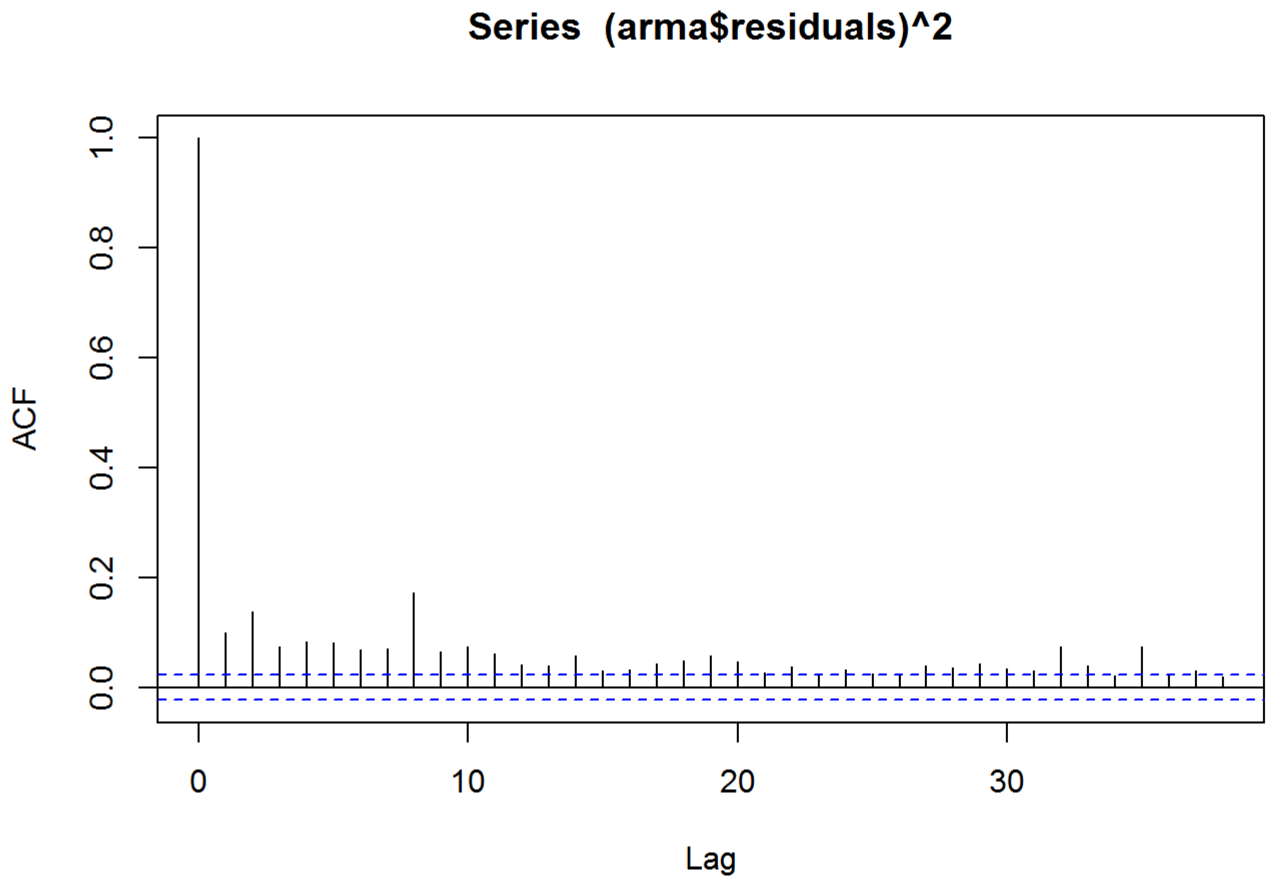
- It follows that:

$$\mathbf{E}(X_t X_{t+\tau}) = \sum_{i=0}^{q} \sum_{j=0}^{q} b_i b_j \mathbf{E}(w_{t-i} W_{t+\tau-j})$$

- Conclusion: simple convolution with finite support

$$R_X(\tau) = \sigma^2 \sum_{j=0}^{q} b_j b_{j-\tau}$$

Finite lag Shape of ACF for MA



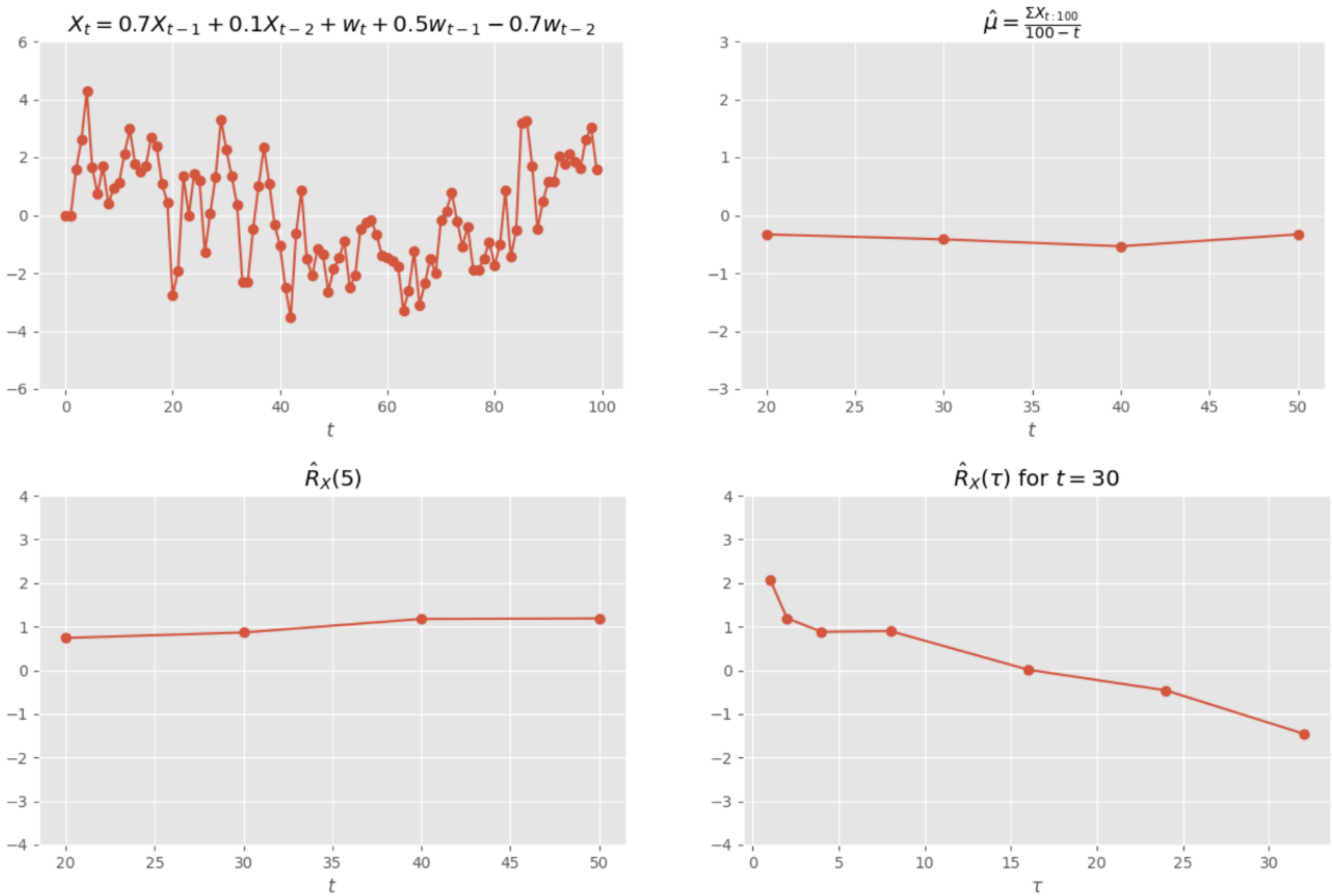
Autoregressive Moving Average Process (ARMA)

- We can combine the two processes

$$X_t = \sum_{i=1}^{p} a_i X_{t-i} + \sum_{j=0}^{q} b_j w_{t-j}$$

- Superposition of different processes

Example: ARMA



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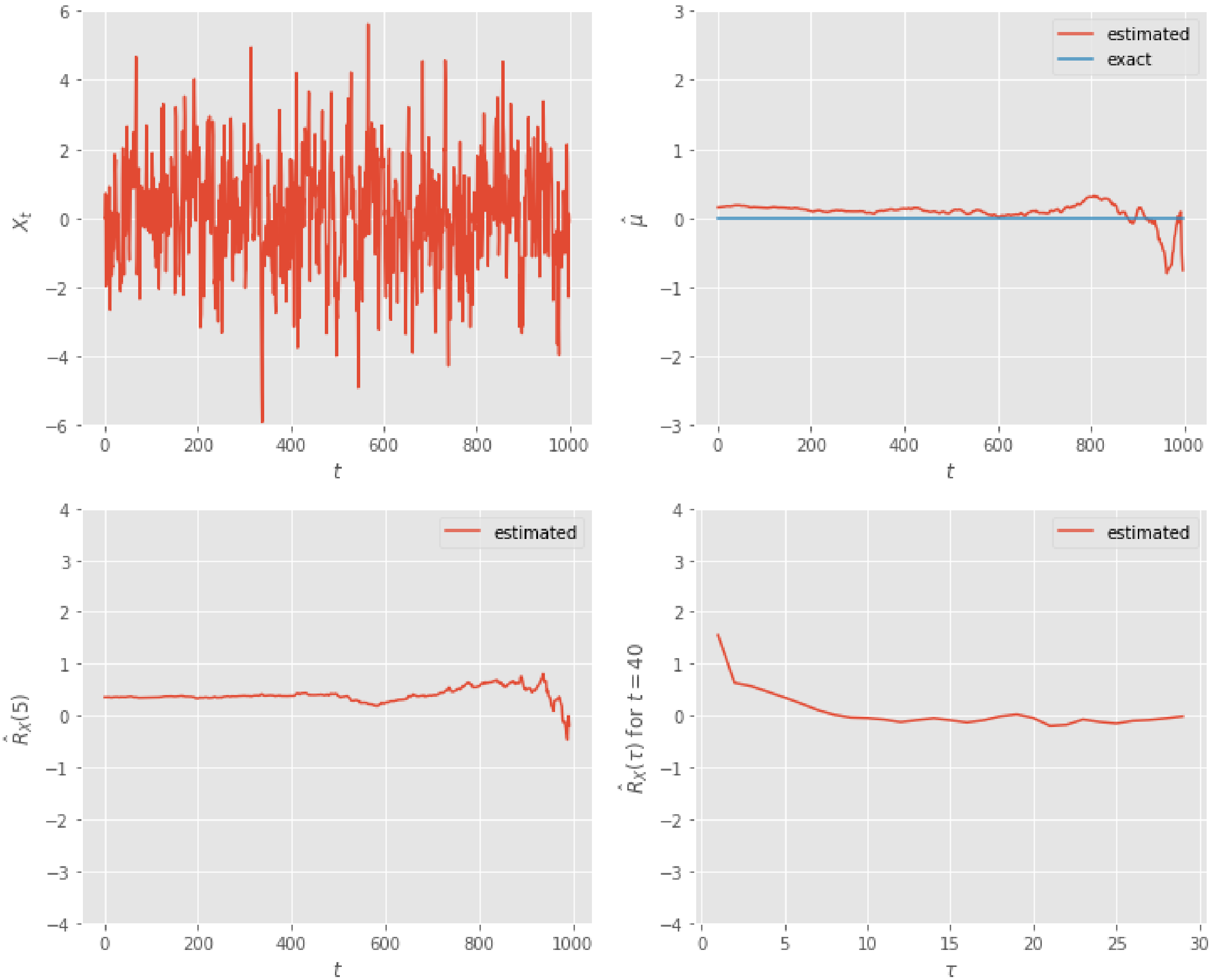
ARMA: More Samples

$$X_t = 0.7X_{t-1} + 0.1X_{t-2} + w_t + 0.5w_{t-1} - 0.7w_{t-2}$$



End

Learning Time Series

Learning a Time Series Model

- Data to Models
  - Autoregressive (AR) models
  - Moving Average (MA)
- AR learning Looks like a standard Least Squares
  - What's the catch?
  - What can we learn?
- How about other models
- Example: Consumer Price Data

Example

- Generate 100 data points according to

$$x_t = .7\, x_{t-1} + .1\, x_{t-2} + \mathrm{w_t}$$

- Fit an AR(1) model
  – Answer: $a = .8704$

- Fit an AR(2) model
  – Answer $a_1 = 0.7448 \;\; and \; a_2 = \; .1742$

- Observations?

Order Estimation

- Example highlights the importance of order estimation
- Derive multiple estimates and choose based on error on cross validation data
- Add penalty to model complexity (MDL, AIC)
- Use the Autocovariance function (ACF) as a guidance!
  - Can we do better than ACF?

Partial Autocovariance Function (PACF)

- $X_t \;\; | X_{t+1}, X_{t+2}, \ldots., X_{t+k-1} | \; X_{t+k}$

- Project X_{t+1} and X_{t+k} on the variables in between:

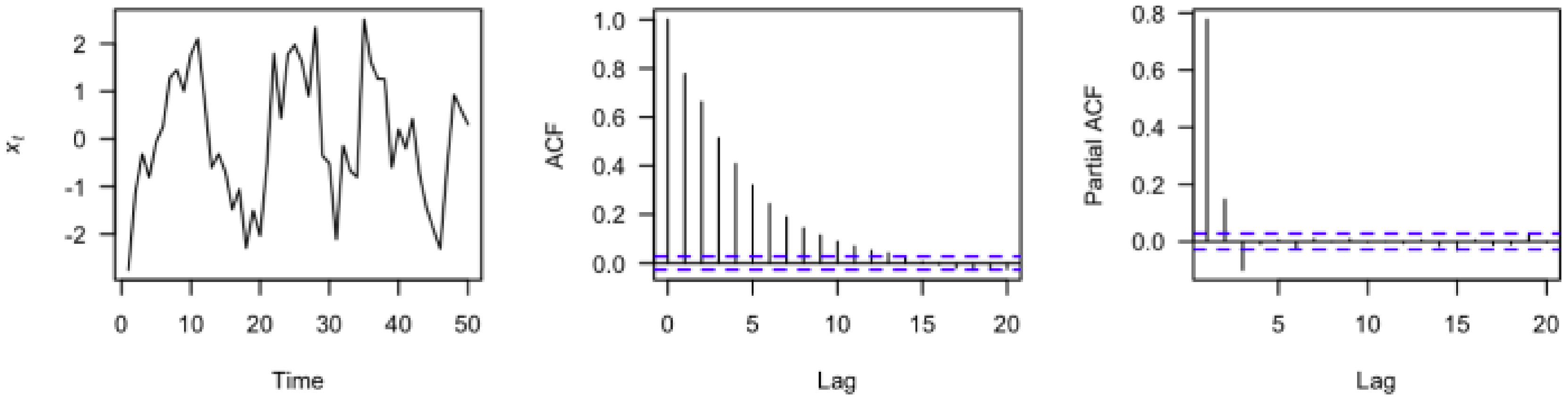
$$P_t \quad and \quad P_{t+k}$$

- In the case of ARX, the new PACF is zero outside p

- *Use to estimate order!*

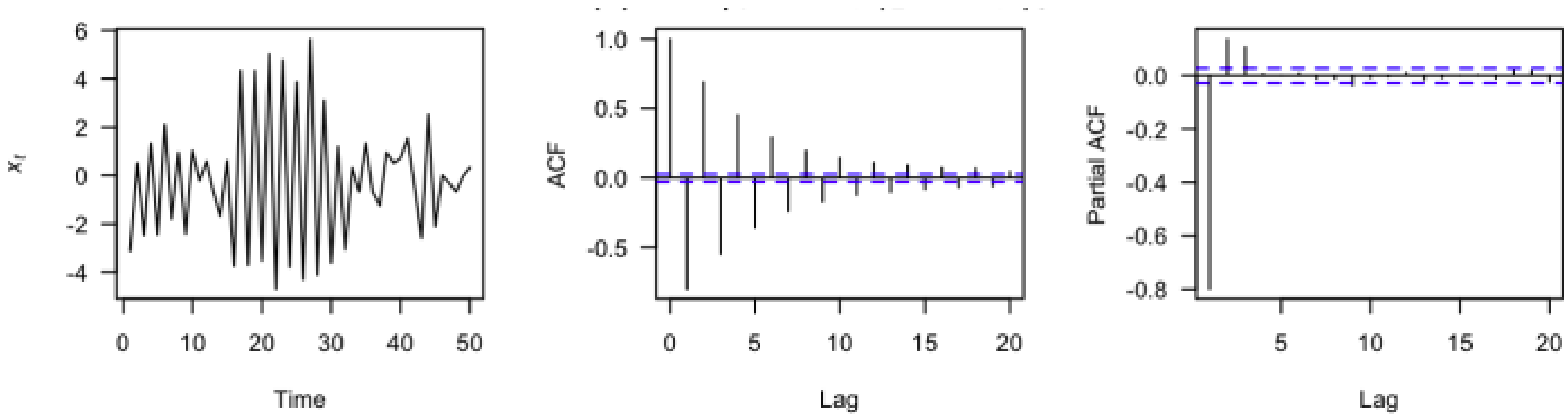
Example: AR(3)

$$X_t = .7\,X_{t-1} + .2\,X_{t-2} - .1\,X_{t-3}$$

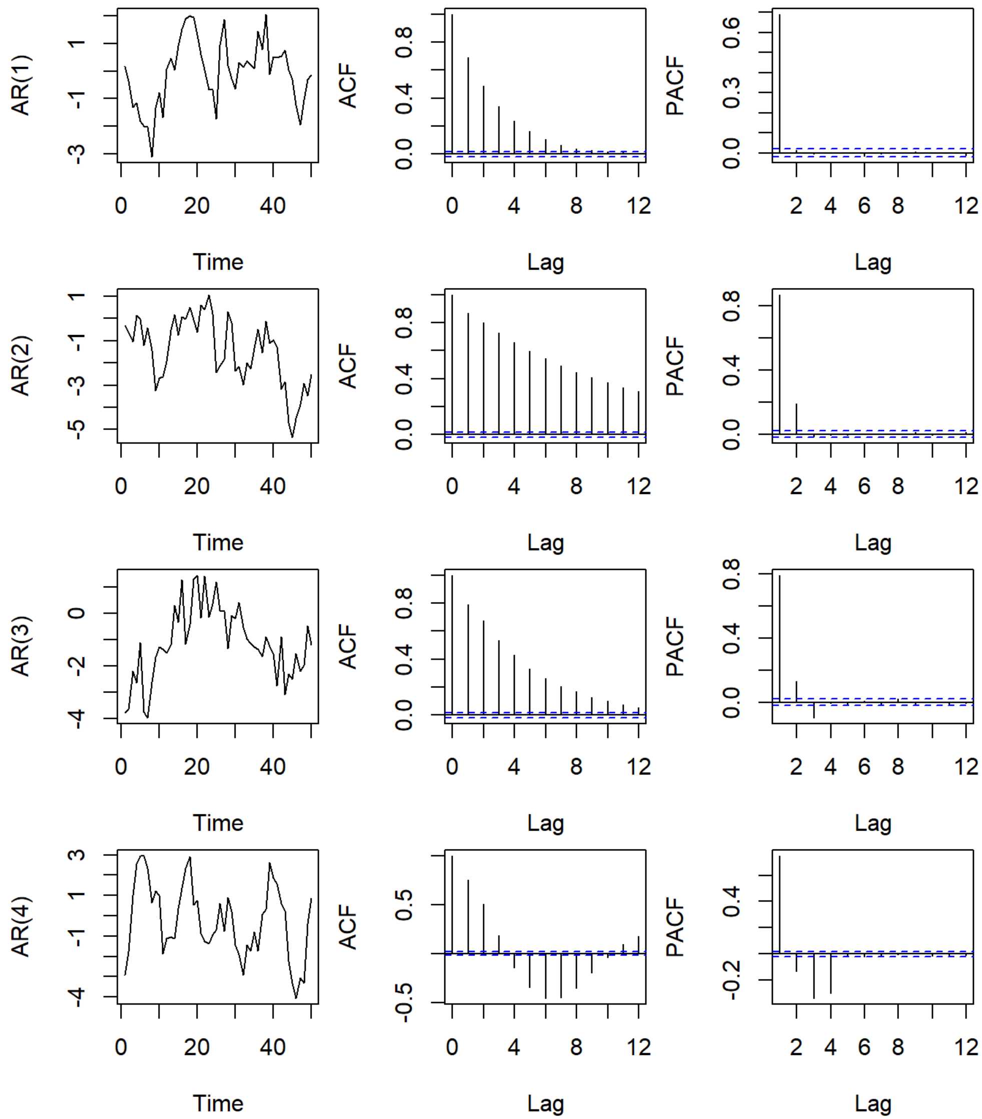


Example: AR(3)

$$X_t = -.7\,X_{t-1} + \;.2\,X_{t-2} + .1\,X_{t-3}$$



Geometric Shape of ACF for AR(p)



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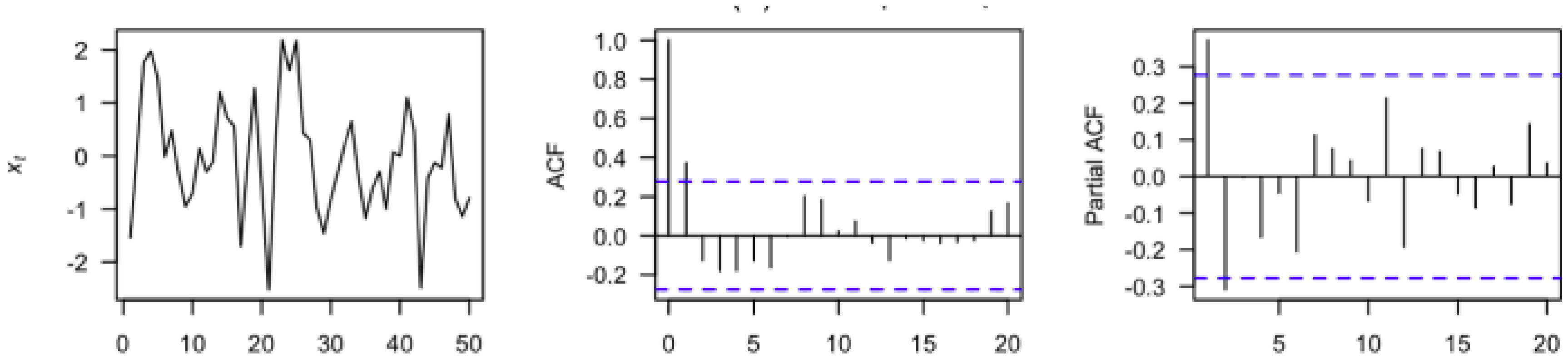
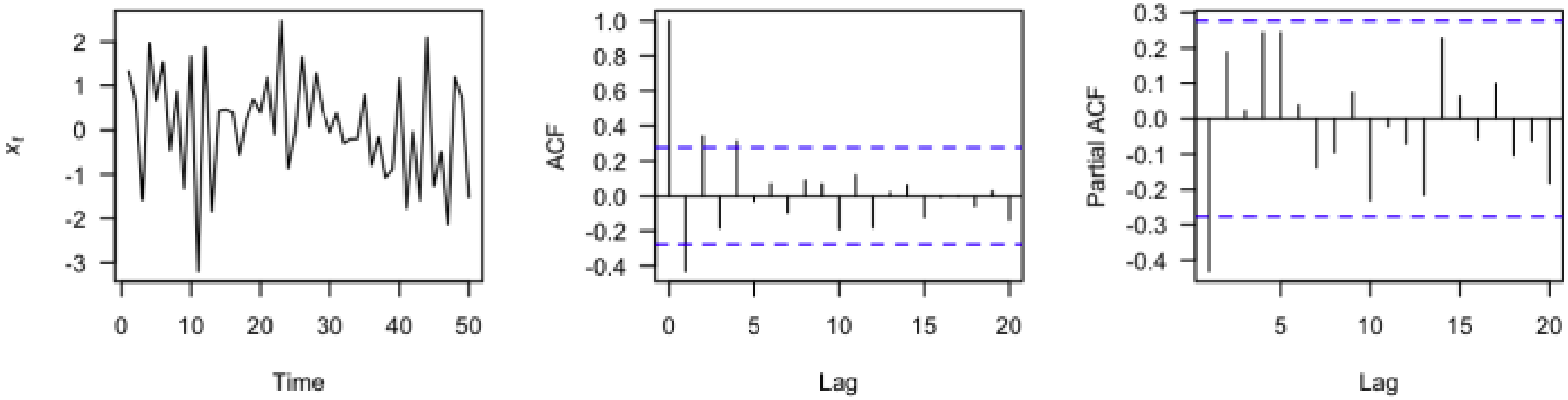
Example: MA(2)

$$X_t = .7\,W_{t-1} + W_t$$



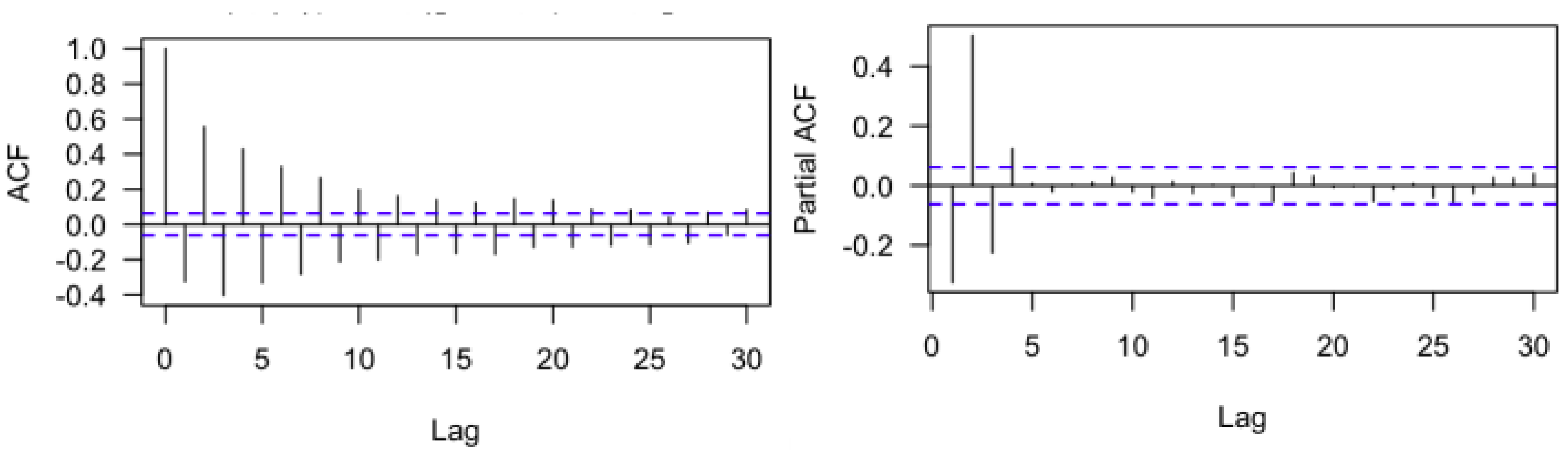
Example: MA(3)

$$X_t = W_t - .7\, W_{t-1} + \; .2\, W_{t-2} + .1 \;\; W_{t-3}$$



Example: ARMA(3)

$$X_t = .7\,X_{t-1} + \;.2\,X_{t-2} + .7\,W_t + \;.2\,W_{t-1}$$



ACF vs PACF

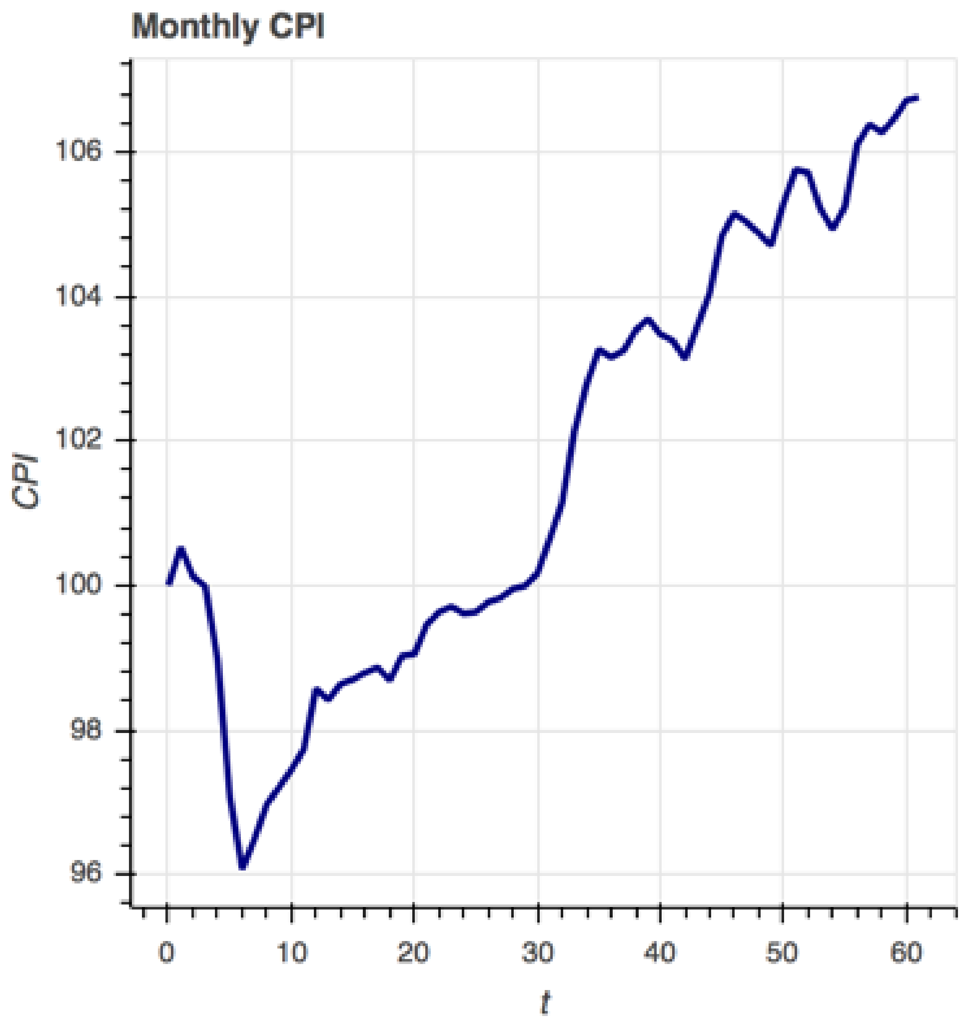
	ACF	PACF
AR(p)	decays	zero for $h > p$
MA(q)	zero for $h > q$	decays
ARMA(p,q)	decays	decays

Billion Price Data

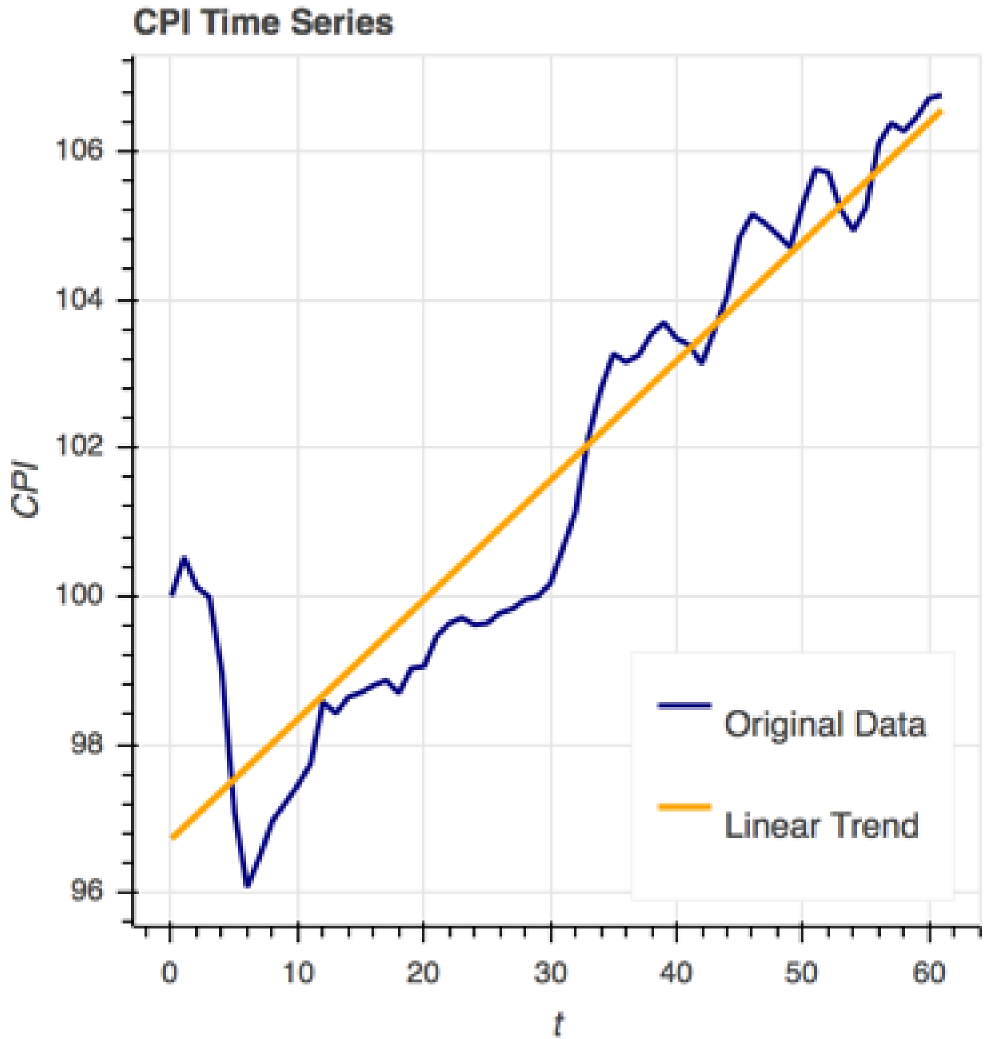
- The goal of this problem is to analyze the PriceStats data from the MIT Billion Prices Project, provided by Professor Rigobon.

- Consumer Price Index Data: (consumer price index, the price of a ”market basket of consumer goods and services” - a proxy for inflation)

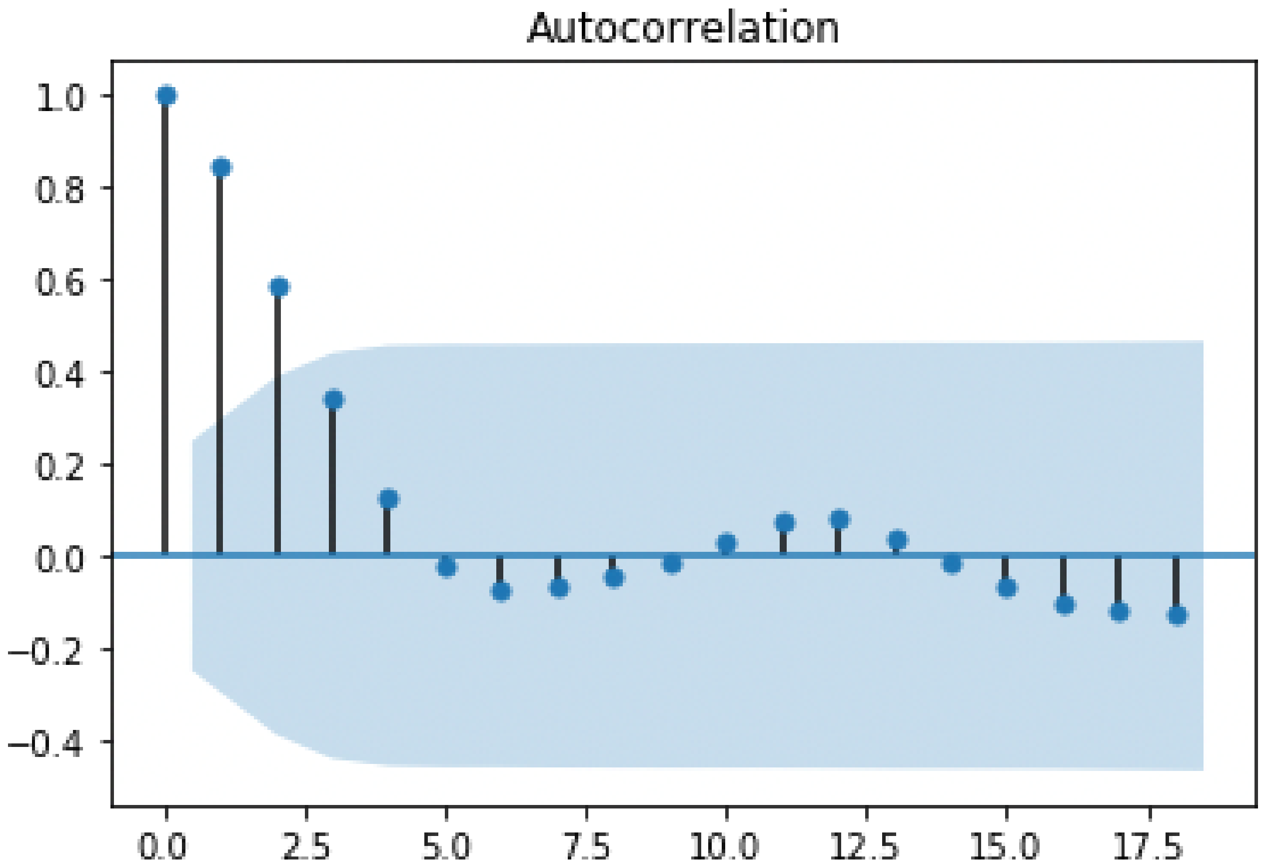
Consumer Price Index



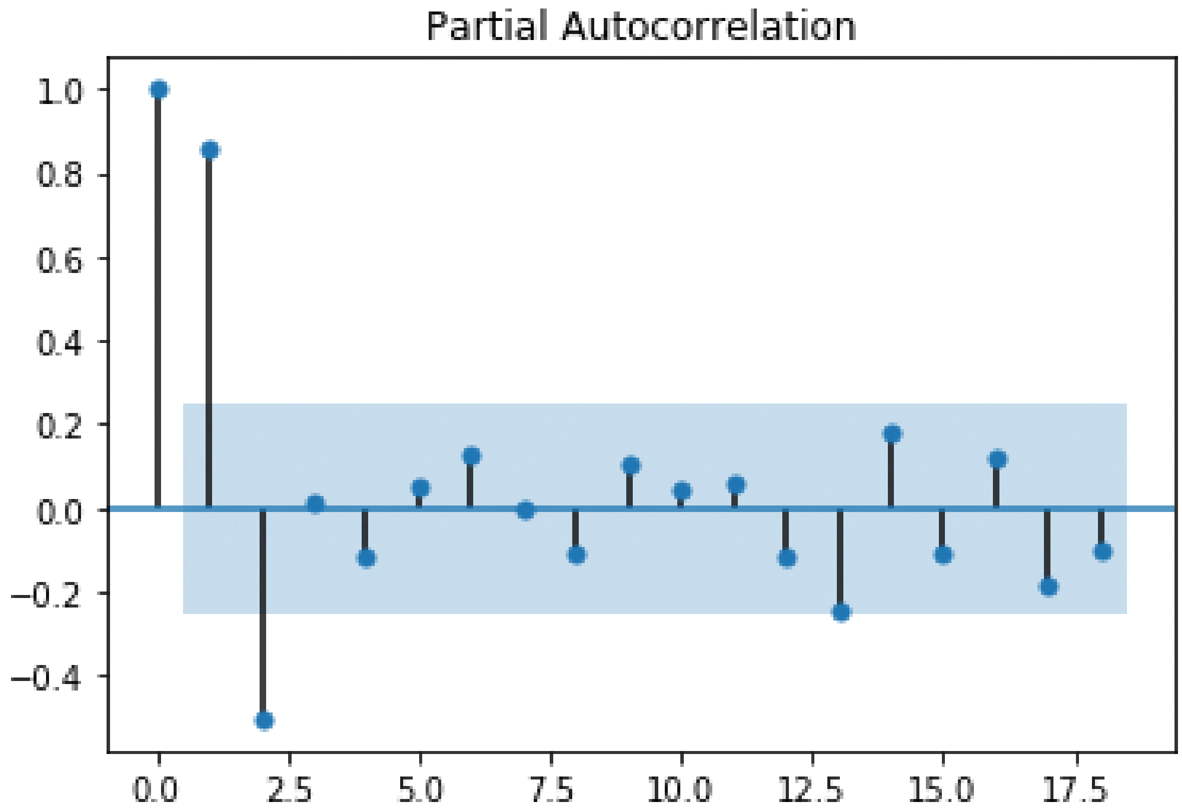
Identify Trends in the CPI data



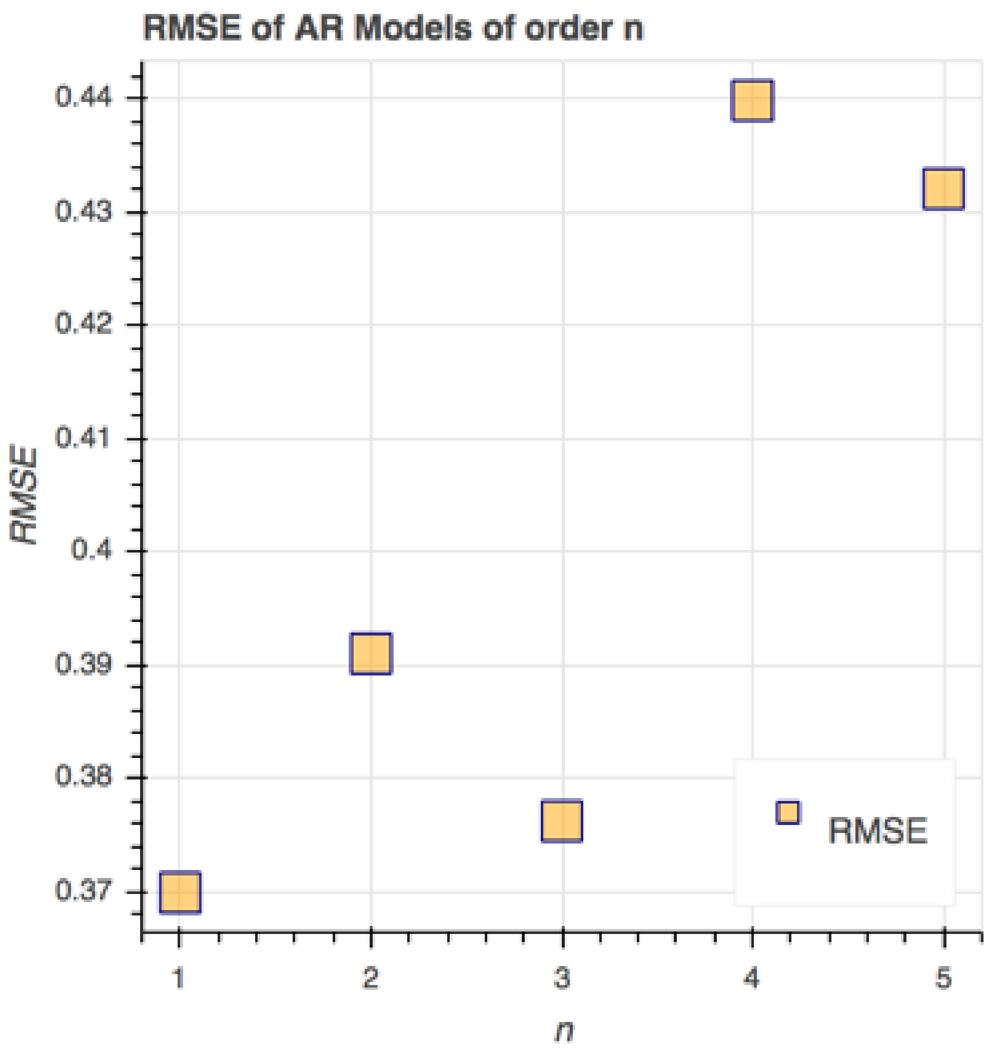
Empirical Autocovariance



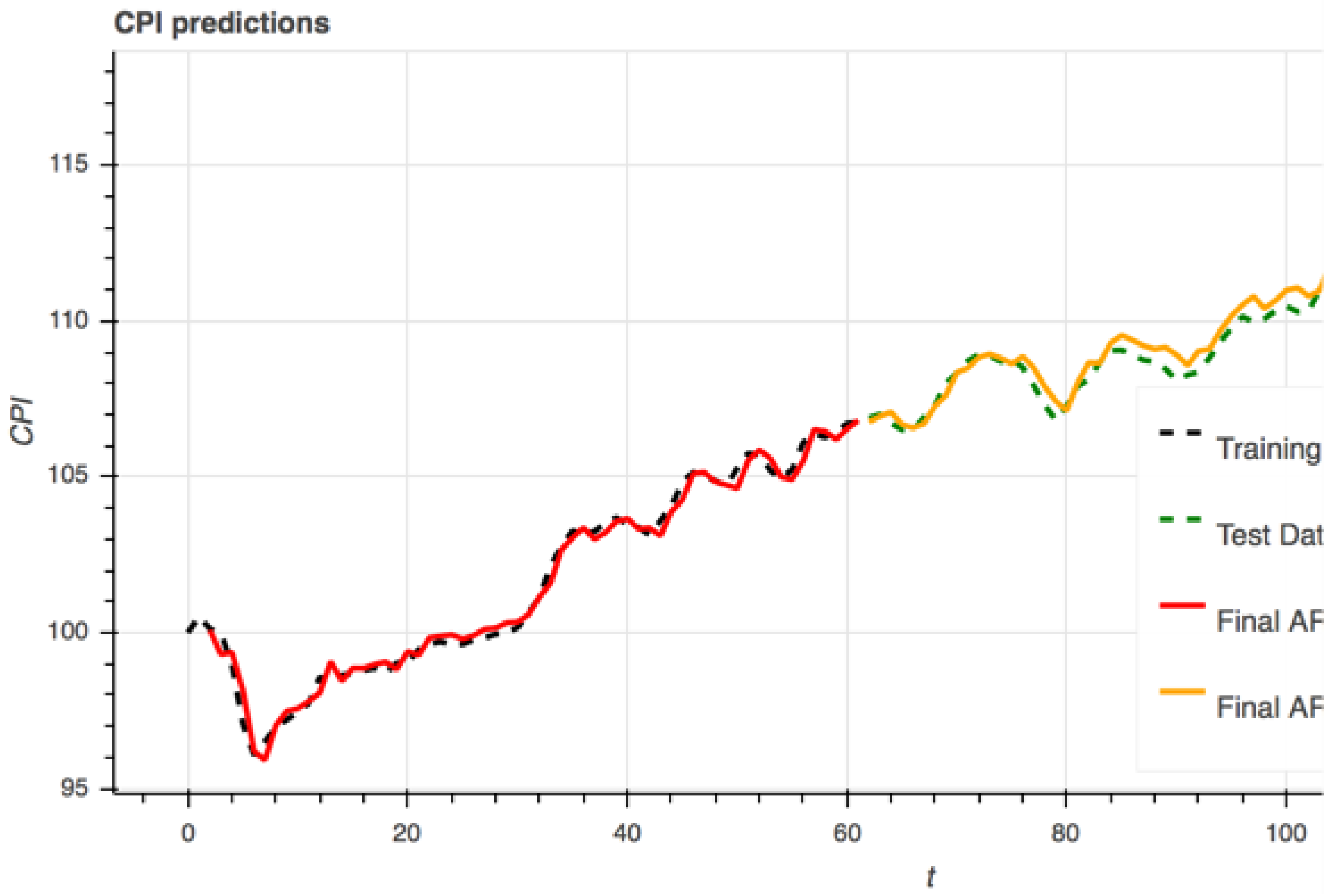
Empirical Partial Autocovariance



AR(1) or AR(2)


RMSE of AR Models of order n
RMSE
0.44
0.43
0.42
0.41
0.4
0.39
0.38
0.37
1
2
3
4
5
n
RMSE

Prediction of AR(2)



Thank You

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