Chapter 4: Hyperbolas

Learning Outcomes of the Lesson
At the end of the lesson, the student is able to:
(1) define a hyperbola;
(2) determine the standard form of equation of a hyperbola;
(3) graph a hyperbola in a rectangular coordinate system.

Introduction

A **hyperbola** is one of the conic sections that most students have not encountered formally before, unlike circles and parabolas. Its graph consists of two unbounded branches which extend in opposite directions. It is a misconception that each branch is a parabola. This is not true, as parabolas and hyperbolas have very different features. An application of hyperbolas in basic location and navigation schemes are presented in an example and some exercises.

Definition and Equation of a Hyperbola

Consider the points F1(−5, 0) and F2(5, 0) as shown in Figure 1.23. What is the absolute value of the difference of the distances of A(3.75,−3) from F1 and from F2? How about the absolute value of the di↲erence of the distances of B $-5, \frac{16}{3}$from F1 and from F2?

$$|AF_1 - AF_2| = |9.25 - 3.25| = 6$$
$$|BF_1 - BF_2| = \left|\frac{16}{3} - \frac{34}{3}\right| = 6$$

There are other points P such that |PF1 − PF2| = 6. The collection of all such points forms a shape called a hyperbola, which consists of two disjoint branches. For points P on the left branch, PF2 − PF1 = 6; for those on the right branch, PF1 − PF2 = 6.

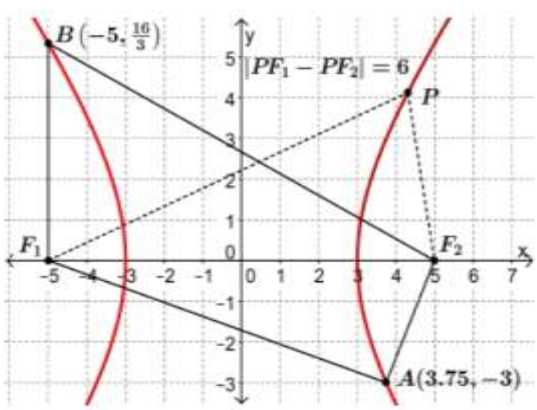


Figure 1.1

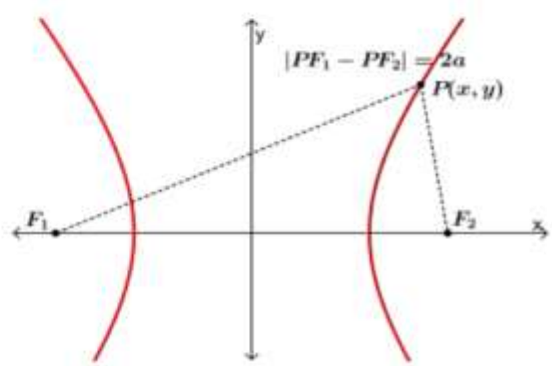


Figure 1.2

Let F_1 and F_2 be two distinct points. The set of all points P, whose distances from F_1 and from F_2 differ by a certain constant, is called a hyperbola. The points F_1 and F_2 are called the foci of the hyperbola.

In Figure 1.2, given are two points on the x-axis, F1(−c, 0) and F2(c, 0), the foci, both c units away from their midpoint (0, 0). This midpoint is the center of the hyperbola. Let P(x, y) be a point on the hyperbola, and let the absolute value of the difference of the distances of P from F_1 *and* F_2, be 2a (the coefficient 2 will make computations simpler). Thus, |PF1 − PF2| = 2a, and so

$$\left|\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\right|=2a.$$

Algebraic manipulations allow us to rewrite this into the much simpler

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad \text{where } b=\sqrt{c^2-a^2}.$$

When we let $b = \sqrt{c^2 - a^2}$, we assumed $c > a$. To see why this is true, suppose that P is closer to F2, so PF1 − PF2 = 2a. Refer to Figure 1.24. Suppose also that P is not on the x-axis, soΔPF_1F_2 is formed. From the triangle inequality F1F2 + PF2 > PF1. Thus, 2c > $PF_1 - PF_2$, = 2a, so c > a.

Now we present a derivation. For now, assume P is closer to F2 so PF1 > PF2, and PF1 − PF2 = 2a.

$$\begin{aligned}
PF_1 &= 2a + PF_2\\
\sqrt{(x+c)^2+y^2} &= 2a+\sqrt{(x-c)^2+y^2}\\
\left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a+\sqrt{(x-c)^2+y^2}\right)^2\\
cx-a^2 &= a\sqrt{(x-c)^2+y^2}\\
(cx-a^2)^2 &= \left(a\sqrt{(x-c)^2+y^2}\right)^2\\
(c^2-a^2)x^2-a^2y^2 &= a^2(c^2-a^2)\\
b^2x^2-a^2y^2 &= a^2b^2 \qquad \text{by letting } b=\sqrt{c^2-a^2}>0\\
\frac{x^2}{a^2}-\frac{y^2}{b^2} &= 1
\end{aligned}$$

We collect here the features of the graph of a hyperbola with standard equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Let $c = \sqrt{a^2 + b^2}$.

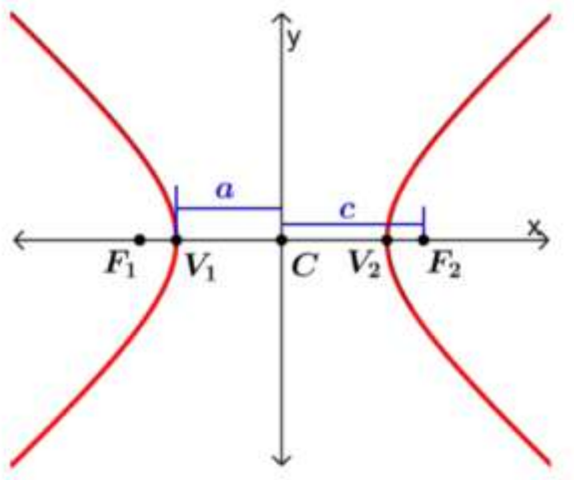


Figure 1.3

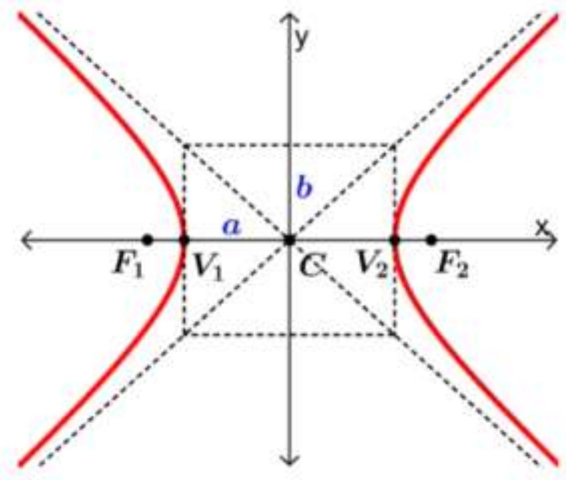


Figure 1.4

(1) center: origin (0, 0)
(2) foci : $F_1(-c, 0)$ and $F_2(c, 0)$

- Each focus is c units away from the center
- For any point on the hyperbola, the absolute value of the difference of its distances from the foci is 2a.

(3) vertices: $V_1(-a, 0)$ and $V_2(a, 0)$

- The vertices are points on the hyperbola, collinear with the center and foci.
- If y = 0, then x = ±a. Each vertex is a units away from the center.
- The segment V_1V_2 is called the transverse axis. Its length is 2a.

(4) asymptotes: $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, the line ℓ_1 *and* ℓ_2 in figure 1.4.

- The asymptotes of the hyperbola are two lines passing through the center which serve as a guide in graphing the hyperbola: each branch of the hyperbola gets closer and closer to the asymptotes, in the direction towards which the branch extends. (We need the concept of limits from calculus to explain this.)
- An aid in determining the equations of the asymptotes: in the standard equation, replace 1 by 0, and in the resulting equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, solve for y.
- To help us sketch the asymptotes, we point out that the asymptotes ℓ_1 *and* ℓ_2 are the extended diagonals of the auxiliary rectangle drawn in Figure 1.4. This rectangle has sides 2a and 2b with its diagonals intersecting at the center C. Two sides are congruent and parallel to the transverse axis V_1V_2. The other two sides are congruent and parallel to the conjugate axis, the segment shown which is perpendicular to the transverse axis at the center, and has length 2b.

Example 1. Determine the foci, vertices, and asymptotes of the hyperbola with equation

$$\frac{x^2}{9} - \frac{y^2}{7} = 1.$$

Sketch the graph, and include these points and lines, the transverse and conjugate axes, and the auxiliary rectangle.

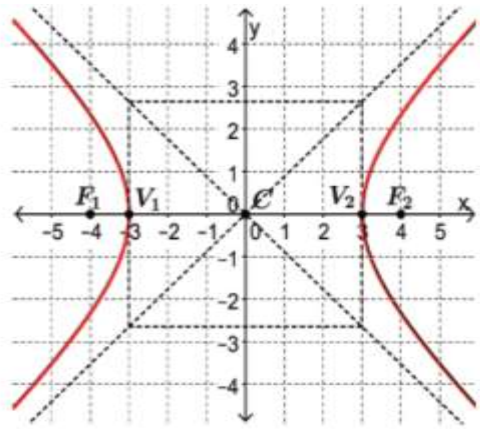
Solution. With $a^2 = 9$ and $b^2 = 7$, we have $a = 3$, $b = \sqrt{7}$, and $c = \sqrt{a^2 + b^2} = 4$.

foci: $F_1(-4, 0)$ and $F_2(4, 0)$

vertices: $V_1(-3, 0)$ and $V_2(3, 0)$

asymptotes: $y = \frac{\sqrt{7}}{3}x$ and $y = -\frac{\sqrt{7}}{3}x$

The graph is shown at the right. The conjugate axis drawn has its endpoints $b = \sqrt{7} \approx 2.7$ units above and below the center. □

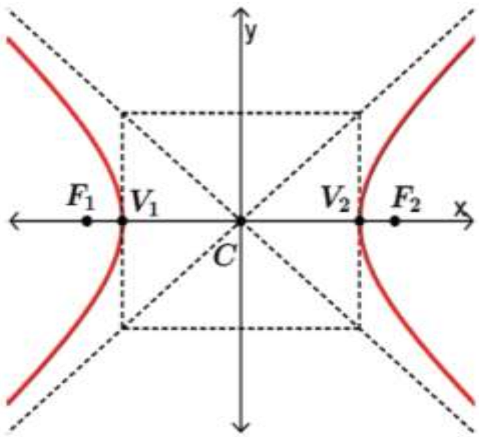
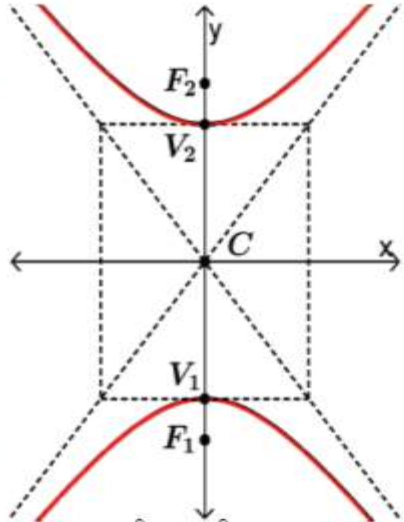
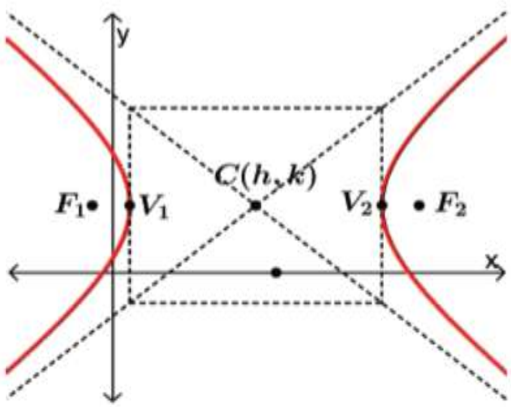
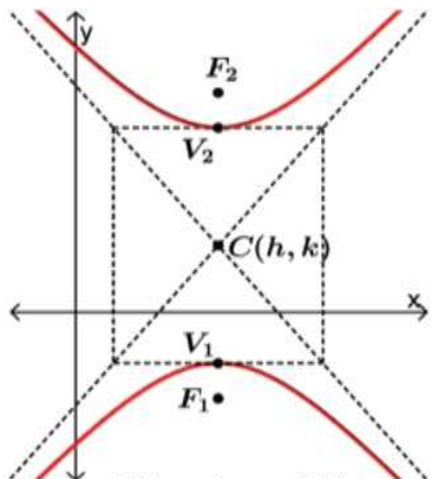


Example2. Find the (standard) equation of the hyperbola whose foci are F1(−5, 0) and F2(5, 0), such that for any point on it, the absolute value of the difference of its distances from the foci is 6.

Solution. We have $2a = 6$ and $c = 5$, so $a = 3$ and $b = \sqrt{c^2 - a^2} = 4$. The hyperbola then has equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$. □

More Properties of Hyperbolas

The hyperbolas we considered so far are “horizontal” and have the origin as their centers. Some hyperbolas have their foci aligned vertically, and some have centers not at the origin. Their standard equations and properties are given in the box. The derivations are more involved, but are similar to the one above, and so are not shown anymore.

Center	Corresponding Graph
(0, 0)	 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
(h, k)	 $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$  $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
	transverse axis: horizontal conjugate axis: vertical transverse axis: vertical conjugate axis: horizontal

In all four cases above, we let c = pa2 + b2. The foci F1 and F2 are c units away from the center C. The vertices V1 and V2 are a units away from the center. The transverse axis V1V2 has length 2a. The conjugate axis has length 2b and is perpendicular to the transverse axis. The transverse and conjugate axes bisect each other at their intersection point, C. Each branch of a hyperbola gets closer and closer to the asymptotes, in the direction towards which the branch extends.

The equations of the asymptotes can be determined by replacing 1 in the standard equation by 0. The asymptotes can be drawn as the extended diagonals of the auxiliary rectangle determined by the transverse and conjugate axes. Recall that, for any point on the hyperbola, the absolute value of the difference of its distances from the foci is 2a.

In the standard equation, aside from being positive, there are no other restrictions on a and b. In fact, a and b can even be equal. The orientation of the hyperbola is determined by the variable appearing in the first term (the positive term): the corresponding axis is where the two branches will open. For example, if the variable in the first term is x, the hyperbola is "horizontal": the transverse axis is horizontal, and the branches open to the left and right in the direction of the x-axis.

Example 3. Give the coordinates of the center, foci, vertices, and asymptotes of the hyperbola with the given equation. Sketch the graph, and include these points and lines, the transverse and conjugate axes, and the auxiliary rectangle.

(1) $\dfrac{(y+2)^2}{25} - \dfrac{(x-7)^2}{9} = 1$

(2) $4x^2 - 5y^2 + 32x + 30y = 1$

Solution. (1) From $a^2 = 25$ and $b^2 = 9$, we have $a = 5$, $b = 3$, and $c = \sqrt{a^2+b^2} = \sqrt{34} \approx 5.8$. The hyperbola is vertical. To determine the asymptotes, we write $\frac{(y+2)^2}{25} - \frac{(x-7)^2}{9} = 0$, which is equivalent to $y + 2 = \pm\frac{5}{3}(x-7)$. We can then solve this for y.

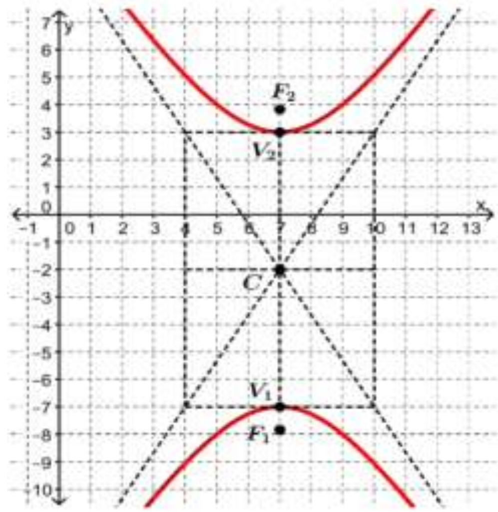
center: $C(7, -2)$

foci: $F_1(7, -2-\sqrt{34}) \approx (7, -7.8)$ and $F_2(7, -2+\sqrt{34}) \approx (7, 3.8)$

vertices: $V_1(7, -7)$ and $V_2(7, 3)$

asymptotes: $y = \frac{5}{3}x - \frac{41}{3}$ and $y = -\frac{5}{3}x + \frac{29}{3}$

The conjugate axis drawn has its endpoints $b = 3$ units to the left and right of the center.



(2) We first change the given equation to standard form.

$$4(x^2+8x)-5(y^2-6y)=1$$
$$4(x^2+8x+16)-5(y^2-6y+9)=1+4(16)-5(9)$$
$$4(x+4)^2-5(y-3)^2=20$$
$$\frac{(x+4)^2}{5}-\frac{(y-3)^2}{4}=1$$

We have $a=\sqrt{5}\approx 2.2$ and $b=2$. Thus, $c=\sqrt{a^2+b^2}=3$. The hyperbola is horizontal. To determine the asymptotes, we write $\frac{(x+4)^2}{5}-\frac{(y-3)^2}{4}=0$ which is equivalent to $y-3=\pm\frac{2}{\sqrt{5}}(x+4)$, and solve for y.

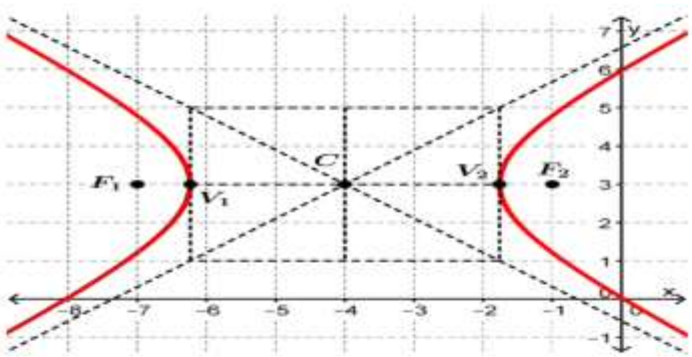
center: $C(-4,3)$

foci: $F_1(-7,3)$ and $F_2(-1,3)$

vertices: $V_1(-4-\sqrt{5},3)\approx(-6.2,3)$ and $V_2(-4+\sqrt{5},3)\approx(-1.8,3)$

asymptotes: $y=\frac{2}{\sqrt{5}}x+\frac{8}{\sqrt{5}}+3$ and $y=-\frac{2}{\sqrt{5}}x-\frac{8}{\sqrt{5}}+3$

The conjugate axis drawn has its endpoints $b=2$ units above and below the center.



Example 4. The foci of a hyperbola are (−5,−3) and (9,−3). For any point on the hyperbola, the absolute value of the difference of its of its distances from the foci is 10. Find the standard equation of the hyperbola.

Solution. The midpoint $(2,-3)$ of the foci is the center of the hyperbola. Each focus is $c = 7$ units away from the center. From the given difference, $2a = 10$ so $a = 5$. Also, $b^2 = c^2 - a^2 = 24$. The hyperbola is horizontal (because the foci are horizontally aligned), so the equation is

$$\frac{(x-2)^2}{25} - \frac{(y+3)^2}{24} = 1. \qquad \square$$

Example 5. A hyperbola has vertices (−4,−5) and (−4, 9), and one of its foci is $(-4, 2-\sqrt{65})$. Find its standard equation.

Solution. The midpoint $(-4,2)$ of the vertices is the center of the hyperbola, which is vertical (because the vertices are vertically aligned). Each vertex is $a = 7$ units away from the center. The given focus is $c = \sqrt{65}$ units away from the center. Thus, $b^2 = c^2 - a^2 = 16$, and the standard equation is

$$\frac{(y-2)^2}{49} - \frac{(x+4)^2}{16} = 1. \qquad \square$$

Reference:
Precalculus Initial Release June 2013 Teachers Guide

Links for Learning:
Conic sections: Intro to hyperbolas | Conic sections | Algebra II | Khan Academy - YouTube
Hyperbolas - Conic Sections - YouTube
Analytic Geometery: Conic Section - Analyzing Hyperbola in Filipino - YouTube
How To Find The Center, Vertices, Foci, and Asymptotes of a Hyperbola - YouTube