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# Chaos

## 1 Introduction

Chaos is a word that many of us are familiar with, but not necessarily within the realm of physics and mathematics. To physicists chaos means something much deeper than the mess on top of your desk, but explains simple annoyances like why a 100% chance of rain could end up being a perfectly sunny day. Chaos, as it is understood today, is a fundamental property of any dynamical system that displays extreme sensitivity to initial conditions. The study of Chaos is just one part in the broader subject of *dynamics*, which at the most principal level is the study of change. "Whether the system in question settles down to equilibrium, keeps repeating in cycles, or does something more complicated, it is dynamics that we use to analyze the behavior" (Strogatz, 1994).

The study of dynamics is deeply entwined with the mathematical study of differential equations, which appear in many fields of study (to name a few: physics, mathematics, engineering, or even biology). Newton's famous second law $\vec{F} = m\vec{a}$, Maxwell's equations, and Schrödinger's equation are all examples of differential equations that are used extensively in Classical Mechanics, Electricity and Magnetism, and Quantum Mechanics respectively. (See Appendix A for a brief introduction to solving a simple linear differential equation). In the early 20th century, nonlinear dynamics became a popular field of study - founded on the work of Isaac Newton and Henri Poincaré, and greatly expanded by physicists and mathematicians such as Balthasar van der Pol, Stephen Smale, George Birkhoff, Mary Cartwright, and John Littlewood. Several hundred years after the publication of Newton's *Principia*, who would have thought there was still so much to uncover in the field of dynamics?

With the invention of the computer in the 1950's, progress made in the field of nonlinear dynamics began to accelerate. Computers allowed scientists and mathematicians to work both more accurately and more efficiently, and so in the years to follow a mathematician by the name of Edward Lorenz stumbled upon some very peculiar behavior in what was considered a simple deterministic model of basic weather patterns (Lorenz, 1963). What he discovered, by chance, is that nonlinear systems can display extreme sensitivity to initial conditions. Solutions to his model, for certain values of parameters (see section 5), would never settle down to predictable motion and would instead move in some chaotic manner. What was even more surprising, was that the trajectories, even though they were chaotic, were bounded to some region of phase space - now known as a *strange attractor*. We will encounter this type of behavior in the study of a nonlinear (forced, damped) pendulum - section 3 - and later in a discussion of the Lorenz equations themselves - section 5. In order to get a feel for the types of behavior we could expect in the more complicated cases of the pendulum or the Lorenz equations, the first dynamic model we will consider is the Logistic Map.

# 2 The Logistic Map

A very simple model of the population $N$ of a species is given by the logistic equation $\frac{dN}{dt} = \mu N(1-N)$, where $\mu$ is the growth rate of the species such that $\mu > 0$. The logistic map $x_{n+1} = \mu x_n(1-x_n)$ is a discretized version of this logistic equation popularized by the biologist Robert May in his 1976 paper *Simple Mathematical Models with Very Complicated Dynamics* (May, 1976). This mapping is a 2nd order (so, nonlinear) mapping, and is considered a characteristic example of a simple dynamical model that displays chaotic behavior for certain values of the parameters (in this case, just $\mu$).

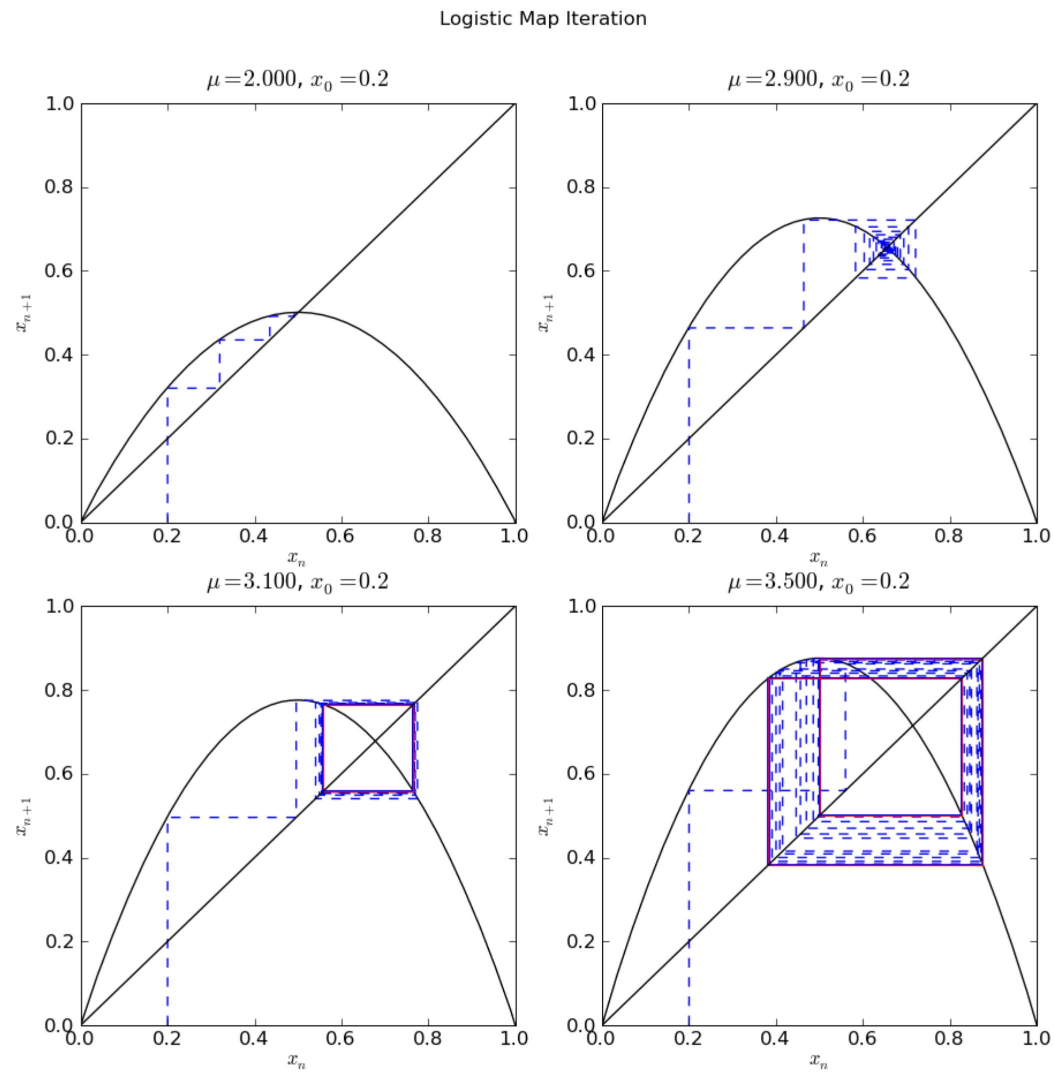



Figure 1: Iteration of the Logistic Map from initial point $x_0 = 0.2$ for varying values of $\mu$. Solid red lines indicate steady-state behavior of the system.

Figure 1 shows in black the curve $\mu x_n(1 - x_n)$ for the displayed values of $\mu$, with the vertical axis as $x_{n+1}$. For consistency I use the initial condition $x_0 = 0.2$ on all plots, but vary the parameter $\mu$ from plot to plot. The dashed blue line represents a graphical solution to the logistic equation, and it shows the various possible long-term outcomes of the system. In the top-left plot of figure 1, the system simply reaches equilibrium at a fixed point after a short number of iterations. Moving to the top-right plot of the same figure, $\mu$ is increased to 2.9 and the system still converges on a fixed point, however it takes many more iterations to reach this point (this may be an indicator that something interesting is about to happen). We call this kind of fixed point a stable point (or *attractor*), because if we start iterating at any number near it, they end up 'falling in' on this point. This is akin to watching a sticky substance slide down the inside of a bowl, but stops when it reaches the bottom. If you move it back up the side of the bowl, it will always be attracted to the bottom of the bowl due to the effect of gravity. Other types of motion are possible, as shown by the bottom two plots in figure 1. When $\mu = 3.1$, something peculiar happens (marked by the red line). Now instead of reaching a single fixed point, and staying at that point - the system appears to be oscillating between *two* points. We can now say that the blue dashed line represents the transient behavior of the system, and the red solid line shows the steady-state behavior. After some initial jumping around, the system settles down to a steady-state oscillation between two points, and stays in this cycle so long as nothing perturbs the system. If we increase $\mu$ further to 3.5, the system again stabilizes into a cycle - but this time it oscillates between 4 points. For obvious reasons, this is called a 4-cycle where the former is called a 2-cycle. The phenomena of a dynamical system spontaneously changing periodic behavior after tuning some parameters is dubbed period doubling, and is characteristic of some dynamical systems on their way to chaotic behavior.

# 3 Period Doubling

In order to visualize a more 'real-life' example of the concept of period doubling, our discussion will now move to a slightly more complicated case: the periodically forced, damped pendulum. The equation of motion for such a pendulum ends up being a second order differential equation with trigonometric functions - a nasty beast - and so we decompose it into the following system of first order equations (for a derivation of these formulas, please refer to Appendix B):

$$\frac{d\theta}{dt} = \omega \tag{1}$$

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \beta\cos(\phi) - \frac{1}{Q}\omega - \sin\theta; \qquad \phi = \Omega\tau + \phi_0 \tag{2}$$

$$\frac{d\phi}{dt} = \omega_D \tag{3}$$

In order to try to get a feel for the kind of motion described by these equations, it is useful to consider some limiting cases that simplify them.

Phase space (in our case, the phase plane) is a space in which each point represents a distinct state of a system. In our case, the two dynamical variables we have are $\theta$ and $\omega$, so we will do some analysis in phase space plots of $\omega$ vs. $\theta$. First let us consider a situation in which the amplitude of the periodic forcing, $\beta$, is negligible, $\beta \approx 0$. Under this limit, if we start the pendulum at some initial point above $\theta = 0$, the pendulum will oscillate around the equilibrium point but intuitively we know that friction will cause the amplitude to decay with time. A time series plot of the solutions for different values of $Q$ (the damping parameter) is shown below, and a phase space plot of $\omega$ vs. $\theta$ at the same intervals of $Q$. Note how in phase space, the state of the system follows a spiral trajectory, and converges upon the *point attractor* $(\theta, \omega) = (0, 0)$.

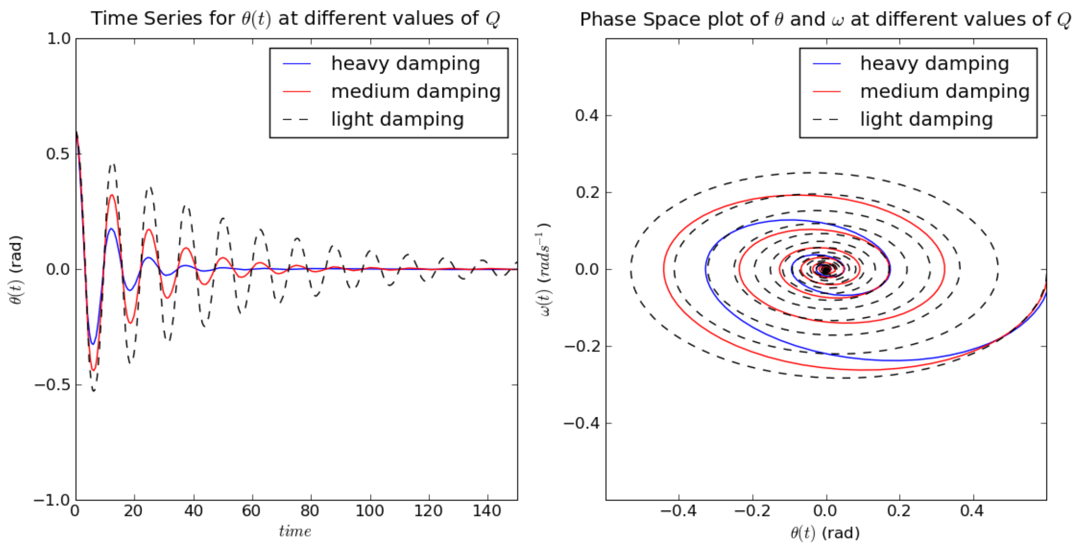
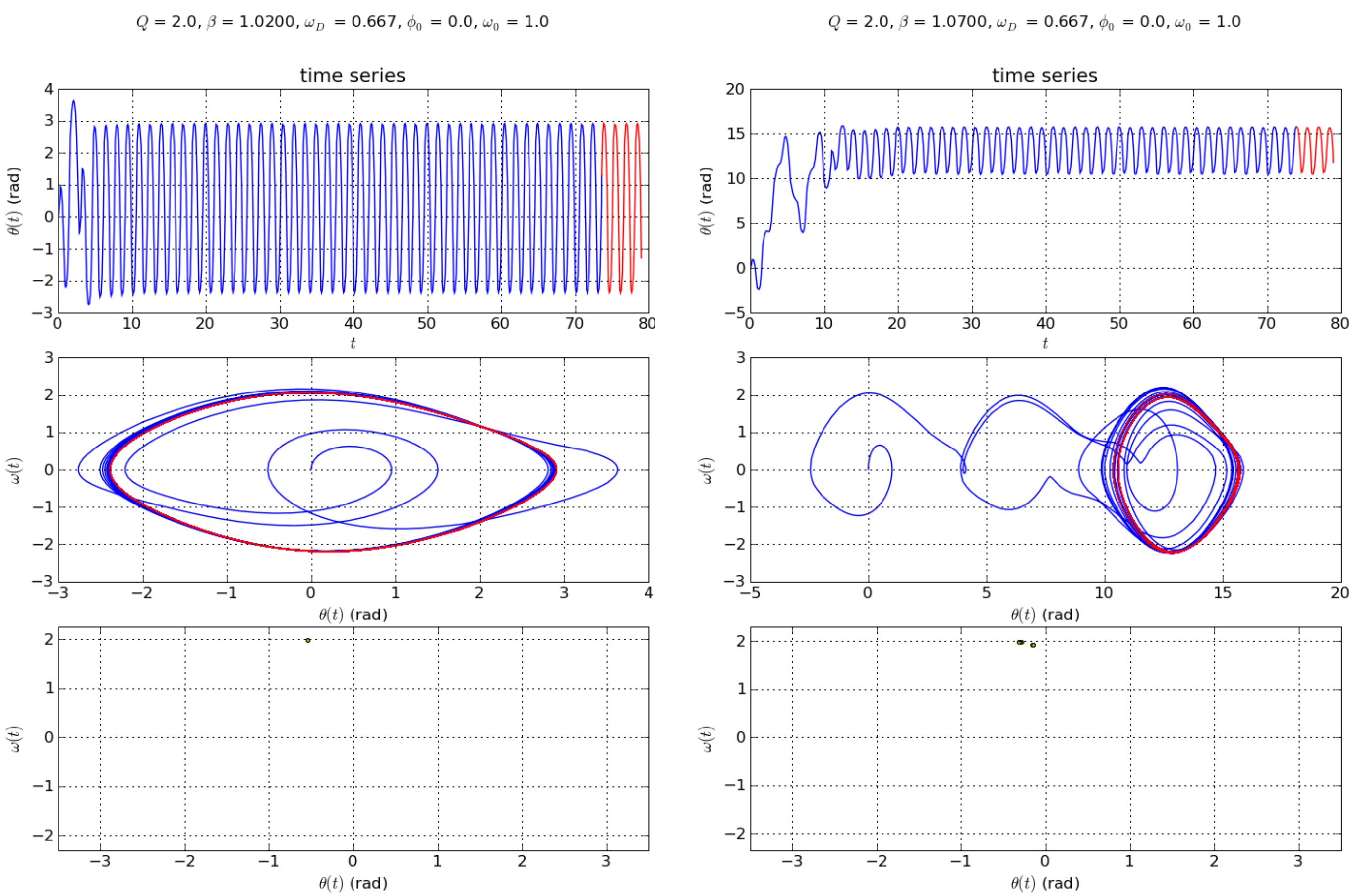



Figure 2: Time series and phase space plots for the damped but not forced pendulum.

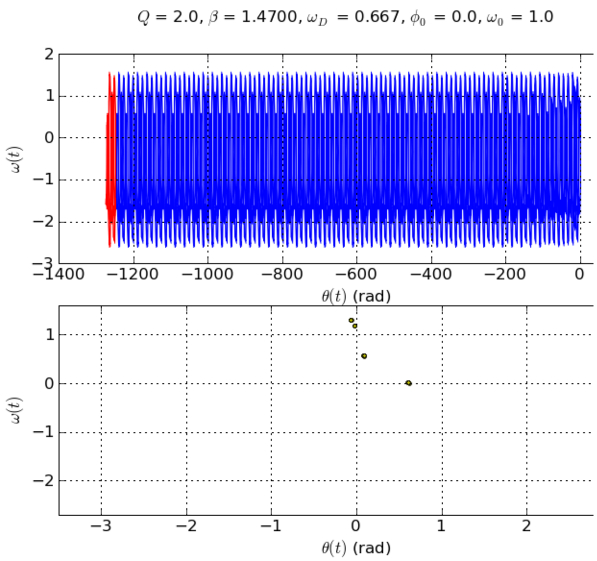
There is not much more to say about this limiting case, but if we put the periodic forcing term back into the expression, things get much more interesting. Now that we have some continuous forcing of the pendulum, there is no reason to assume that the motion of the pendulum will decay to some fixed point. The forcing term allows another type of solution, periodic solutions, to appear. In the next few figures there are three plots on each - the first (figure 3(a), top) being a time series (like plot (a) of figure 2) but you'll notice that now, after some initial *transient* behavior, the motion settles down to *steady-state* periodic motion. The middle plot is just a phase space plot of $\omega$ vs. $\theta$, but the bottom plot is something new. The bottom plot is a so-called Poincaré section or Poincaré plot - which is a plot of points $(\theta, \omega)$ at times $t_0$, $t_0 + T_D$, $t_0 + 2T_D$, etc., where $t_0$ is some arbitrary time *after* the system has reached steady-state, and $T_D$ is the drive period. If there is one point on the Poincaré section, we say it is a 1-cycle (figure 3(a)), if it has two points it is a 2-cycle (figure 3(b)), and so on. These are analogous to the 'motion' we saw in the logistic map, where we experienced similar behavior.

Poincaré sections make it very easy to identify chaotic behavior, as the next few plots will show (for the next four plots, I have dropped the time series plots because they often end up too small to really add anything to the figures). Much like in the logistic map, period doubling is a tell-tale sign that something interesting is about to happen. In figure 4(a), the period doubles from two to four, and then again in figure 4(b) to be an 8-cycle. Something surprising, however, occurs in figure 4(c), our first encounter with chaos in the nonlinear pendulum. If we focus our attention to the Poincaré section, it appears as though there is some order in the chaotic motion! The points here trace out what is known as a *strange attractor*, a topic that we will discuss in much more depth in the context of the Lorenz equations (section 5). It is worth briefly noting that while the strange attractor in figure 4(c) looks like a one-dimensional line, it actually has fine-scale structure if we were to zoom in very close. The last of the four figures shows that even tuning the forcing by a mere 0.01 units will bring the system out of chaos, into a 5-cycle!

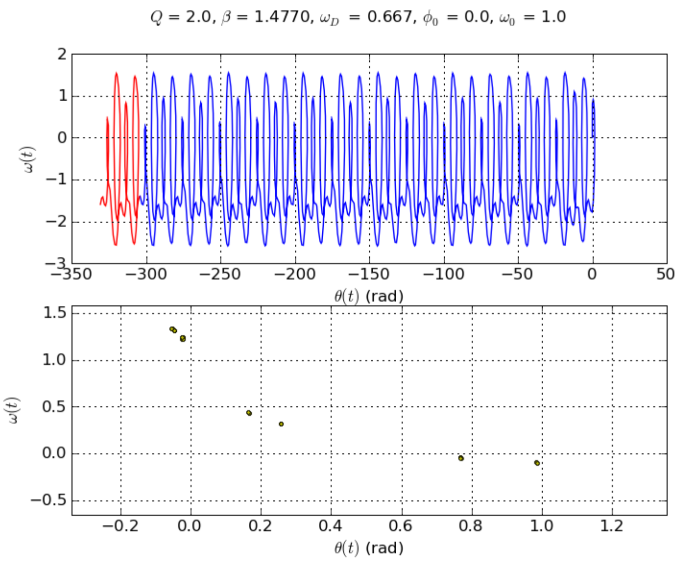



(a) (b)

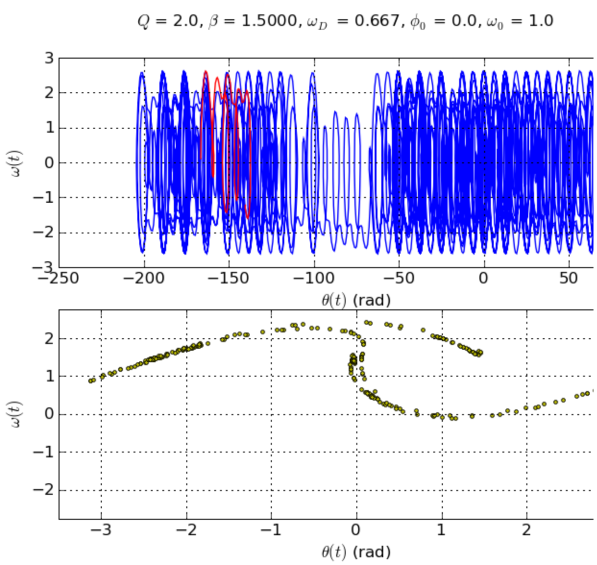
Figure 3: Time series, phase space, and Poincaré section plots for the damped, forced pendulum. Parameters are shown on the top, and initial conditions are $\theta = 0.0$, $\omega = 0.0$.



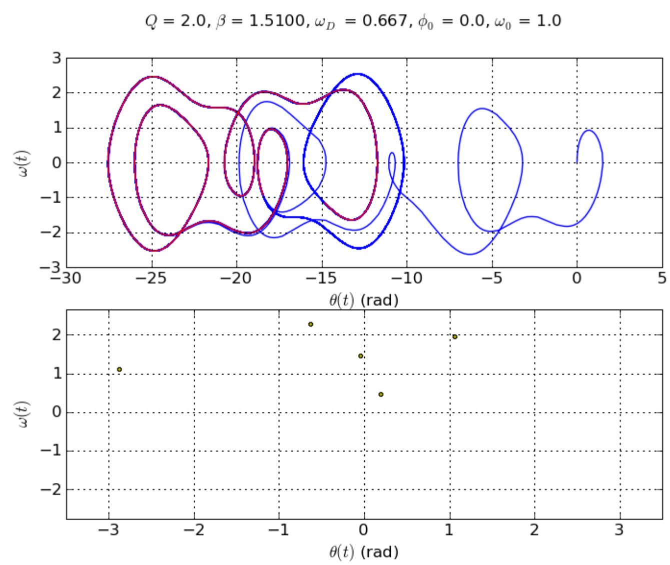

(a)




(b)




(c)




(d)

Figure 4: Phase space, and Poincaré section plots for the damped, forced pendulum. Parameters are shown on the top, and initial conditions are $\theta = 0.0$, $\omega = 0.0$.

Before considering the Lorenz equations, there is one more graphical tool that is worth discussing; it shows the behavior of a system as some parameter in the system is varied. This type of figure is called a *Bifurcation Diagram*, and it is very useful for demonstrating the long-term values of a system as a function of some parameter that you wish to change. As I mentioned before, sometimes what seems like a small change in a parameter can lead to drastically different behavior, and these diagrams are great for visualizing this sensitivity. Figure 5 is an example of a bifurcation diagram for the pendulum we just analyzed, where the parameter we're tuning is $\beta$, the forcing amplitude.

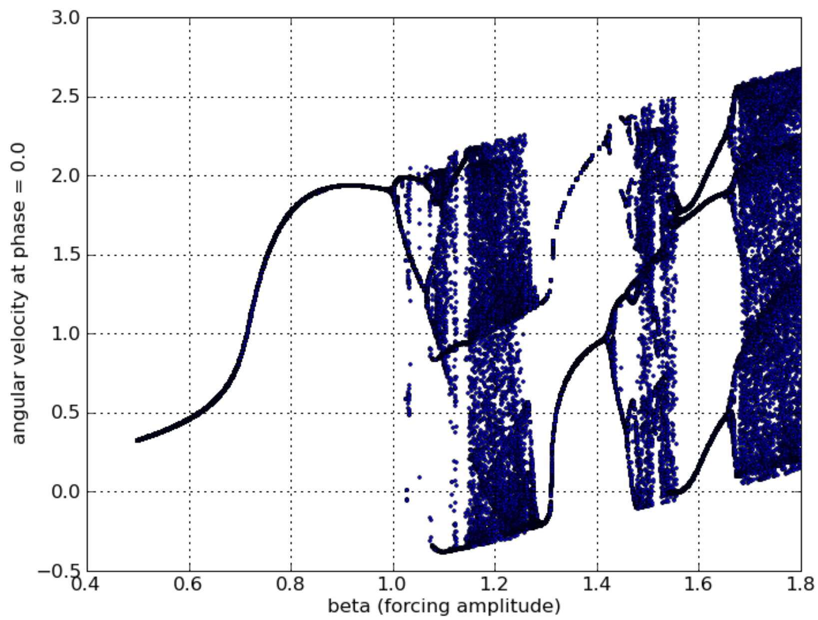
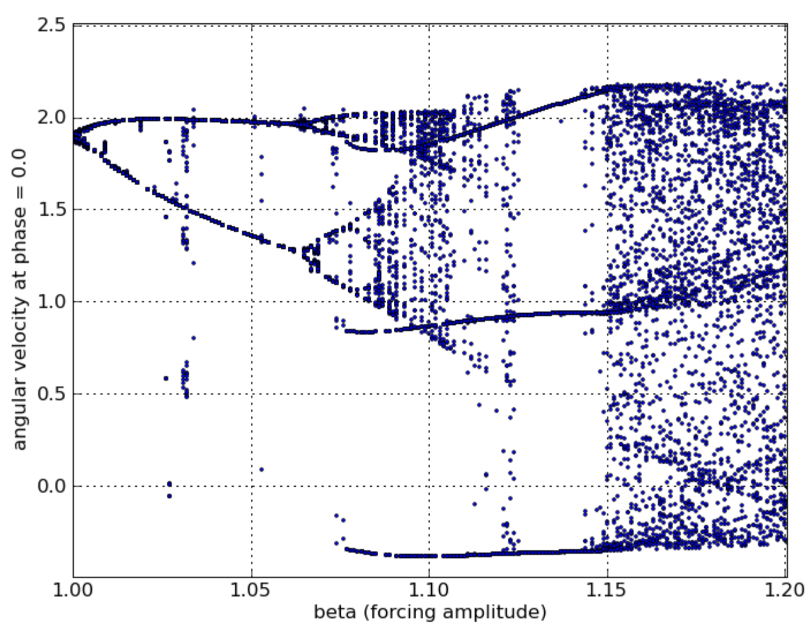
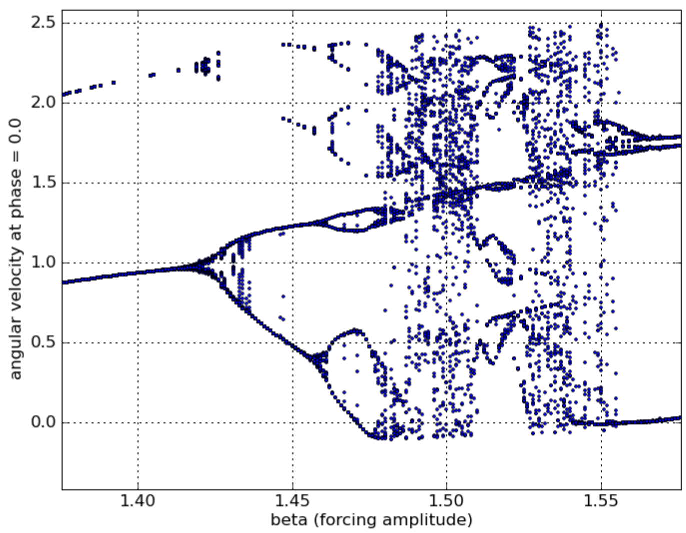



Figure 5: Bifurcation Diagram of long term $\omega$ as a function of the forcing amplitude $\beta$.

Notice that for small $\beta$, or $\beta < 1.0$, there is a single curve which around $\beta \approx 1.0$ splits (*bifurcates*) into two main branches (with some other 'noisy' points involved). This branching is precisely the transition from a 1-cycle to a 2-cycle, where it takes two full periods to get back to the initial position in the pendulum's orbit. If we zoom in on the region around $\beta = 1.1$, we can see that there is more intricate structure on fine-scale.



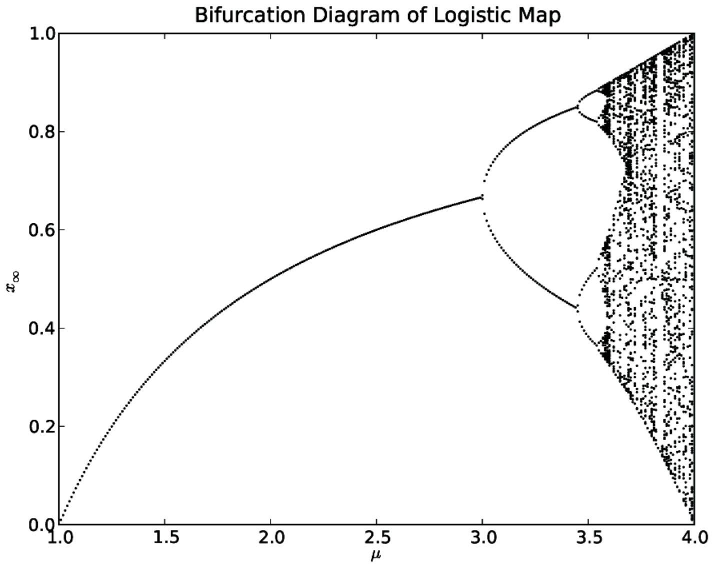
In these diagrams, the number of branches per value of $\beta$ represent the periodicity of the pendulum's motion, where the regions that look 'noisy' represent chaotic behavior. Here is one more region of the diagram, around the values that I picked in figure 4. Referring back to this figure, we can see that for a value around $\beta = 1.47$, there are four branches of the diagram that are clearly defined. If we could zoom in more, we'd see around $\beta = 1.477$ that these four branches would each go through a period doubling, and would result in the 8-cycle of figure 4(b). At $\beta = 1.5$ there is clearly chaotic behavior, corresponding to the motion exhibited in figure 4(c).



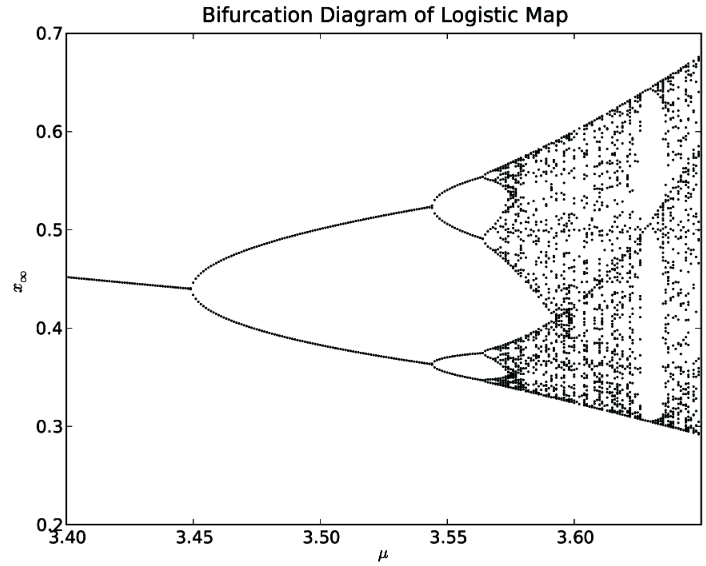

# 4 Feigenbaum's Number

The bifurcation diagrams for the Logistic Map look simpler, and so just to draw an analogy to the simpler case - I have included figures 6(a) through 6(d). The different panels are just different 'zoom levels,' so pay attention to the scale of the axes. Notice that no matter how close we zoom in on a region of period doubling, there are always smaller regions of period doubling - the bifurcation diagram is *self-similar*, and the system goes through an infinite number of period doublings on its way to chaos! We can construct a table of the precise values for which these period doublings occur, to some finite precision (Strogatz, 1994):

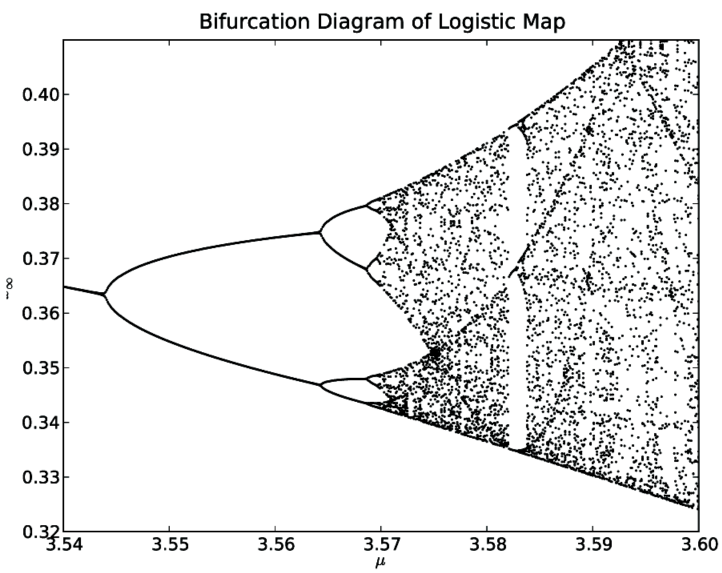
| $\mu_n$ | periodicity |
|---|---|
| $\mu_1 = 3.0$ | 2 |
| $\mu_2 = 3.449...$ | 4 |
| $\mu_3 = 3.54409...$ | 8 |
| $\mu_4 = 3.5644...$ | 16 |
| $\mu_5 = 3.568759...$ | 32 |
| $\vdots$ | $\vdots$ |
| $\mu_\infty = 3.569946...$ | $\infty$ |



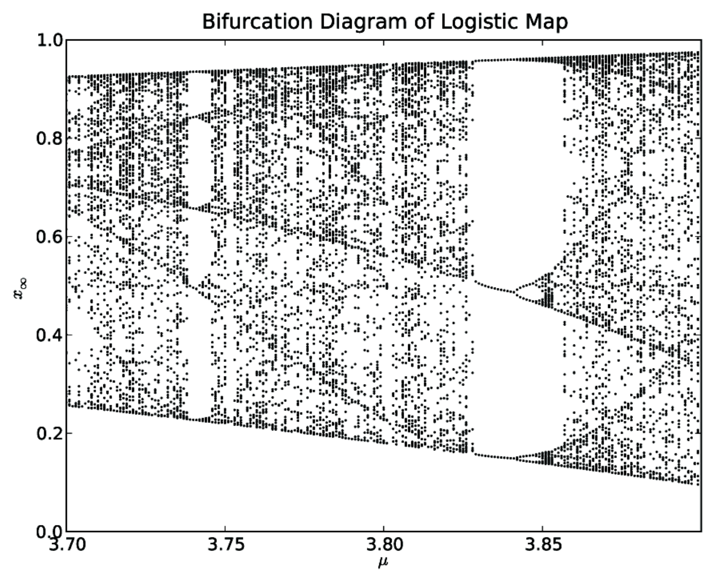

(a)




(b)




(c)




(d)

The rate at which bifurcations happen seems to accelerate as we increase $\mu$, but eventually converges to some value $\mu_\infty$ where the system has an *infinite* period: chaos! In 1975, a mathematical physicist named Mitchell Feigenbaum (using a basic desk calculator) discovered that the ratio of points at which period doubling occurs in the logistic map converges geometrically to a constant irrational number: 4.66920... Feigenbaum tried to fit this number to all mathematical constants he knew, but had little luck. He decided to look at a different mapping - the sine map $x_{n+1} = r \sin(\pi x_n)$ - for which he knew period doubling occurred, and found again that the values for which period doubling occurred followed some convergence. Amazingly, he recalls, "...the rate was the same 4.669 that I remembered [from the logistic map]" (Strogatz, 1994). He took this finding further, to note that this same convergence rate, Feigenbaum's number, is found in any unimodal, iterated map (a map that takes some general interval $I$ and maps it onto $I$). The number is calculated by taking the following limit:

$$\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n+1}}{\mu_{n+1} - \mu_n} = 4.669201... \tag{4}$$

where the $\mu_n$'s are the values of $\mu$ for which period doubling occurs.

# 5 Strange Attractors

Edward Lorenz, one of the pioneers of Chaos Theory, derived a system of equations in the 1960's for a simple deterministic system that displays extremely erratic dynamics (Lorenz, 1963). The Lorenz Equations describe a three-dimensional system that is an extremely simple model of convection rolls in the atmosphere, as mentioned in the introduction. The equations (5) are these Lorenz Equations, where $\sigma$, $r$, and $b$ are parameters under the constraint $\sigma$, $r$, $b > 0$.

$$\dot{x} = \sigma(y - x) \tag{5a}$$

$$\dot{y} = rx - y - xz \tag{5b}$$

$$\dot{z} = xy - bz \tag{5c}$$

What Lorenz found, was that if you plotted the three dimensional trajectories (solutions) to this set of equations, for certain values of the parameters, they remain contained in some complicated bounded region - but would describe chaotic behavior. For small $r$, the system simply decays to one of two fixed points - but as the *Rayleigh number*, $r$, is varied - we stumble upon chaotic behavior at $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$.

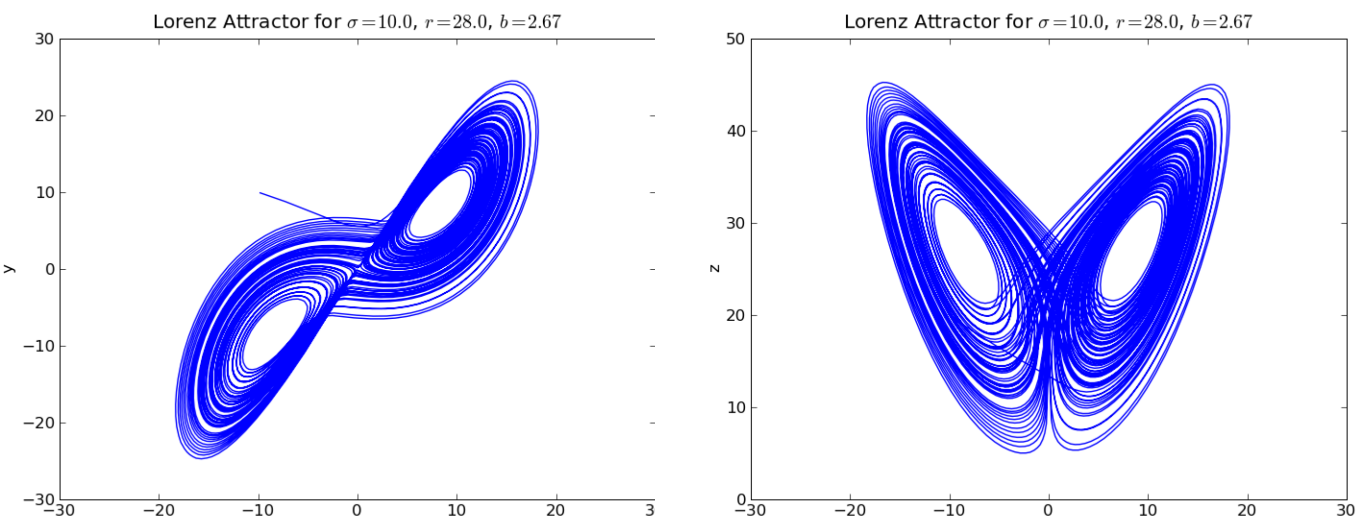



Figure 6: Time series for the lorenz equations; 3-dimensional trajectories projected onto the X-Y and X-Z planes.

Figure 6 shows time series for one value of initial conditions, but it is more interesting to look at the progression of two separate trajectories. In figure 7, the red and blue lines represent two separate time series, started only 0.01 units away from each other. We can see that initially the trajectories are extremely close, but eventually they trace out two distinct paths. It is more obvious in the bottom panel, where the line in the middle of the plot shows the first 100 time steps of both lines—it is very hard to distinguish between the two because they are so close. The disjointed red and blue curves are taken from a sample between the 500th to 600th time steps of each curve - showing that even though they started extremely close, they eventually get quite far from each other and trace out unique orbits.

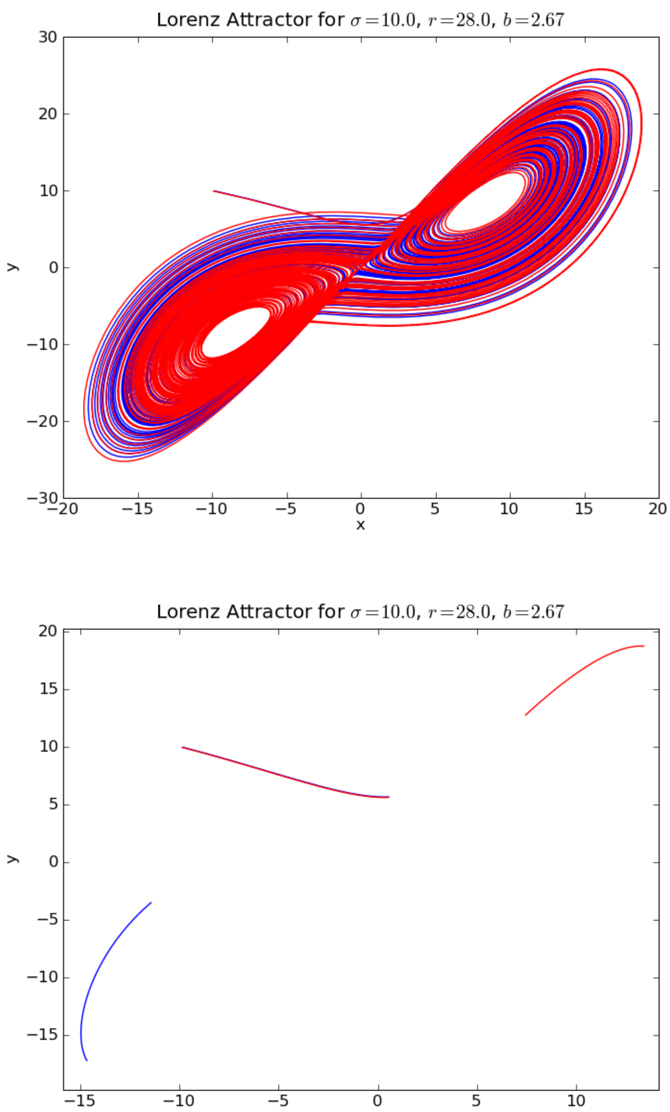



Figure 7: *Top*: Time series plot of two trajectories; red was started at (-10.01,10), blue at (-10,10). *Bottom*: Same as top, except only displaying 0th to 100th time steps, and 500th to 600th time steps.

If we increase the Rayleigh number even more, we eventually get to a stable orbit around $r = 150.0$, shown in figure 8, and then delve back into chaos around $r = 160$; the point being, this complicated example has all the same tricks that we've encountered in the nonlinear pendulum, and in the logistic map previously. The same tools that we used to analyze the dynamics of those simple systems can be applied here to the Lorenz equations, with some slight sophistication, to study the behavior of much more complicated models.

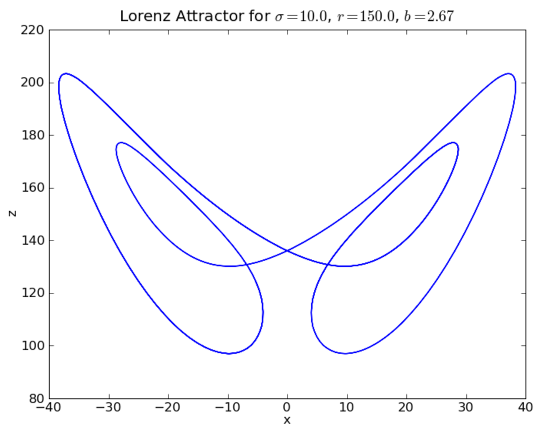



Figure 8: Stable orbit of the Lorenz equations.

# 6 Concluding Remarks

Chaos Theory is still very much a progressing branch of science and mathematics, where new discoveries are fueled by progress in the technology sector as well as in pure mathematics. As computers get more powerful, finding solutions and analyzing complicated systems will become easier and faster. In the years to come I suspect that it will lead us to more answers and many more questions—like any scientific discovery should. Chaos theory has many applications: in computer science, physics, biology, mathematics, and meteorology to name a few, and it is still a very active area of research across all of these disciplines. I see the discovery of chaos as an entirely new way to look at the universe. Humans have a tendency to see beauty in order, and to look for patterns where there may otherwise be no structure. Chaos theory tells us that beauty can come from disorder, and perhaps there is no theory of everything—the natural world is just fundamentally complex.

> *There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.*
>
> *There is another which states that this has already happened.*
>
> \- Douglas Adams

## Acknowledgements

I thank Professor David Pine for introducing me to the numerical methods used to generate the many plots included in this document. I also would like to thank Professor Engelbert Schucking and Professor John Rinzel, and finally Greg Rae for useful discussion.

## References

Lorenz, E. 1963, Journal of Atmospheric Science, 20, 130

May, R. M. 1976, Nature, 261, 459

Strogatz, S. H. 1994, Nonlinear Dynamics And Chaos: With Applications To Physics, Biology, Chemistry, And Engineering (Studies in nonlinearity), 1st edn., Studies in nonlinearity (Perseus Books Group)

# A Appendix

Here is a solution to a common differential equation, Newton's second law, often seen as $\vec{F} = m\vec{a}$ - which I will write in the form

$$\vec{F} - m\ddot{\vec{x}} = 0 \tag{6}$$

where $\ddot{x} \equiv \frac{d^2x}{dt^2}$. For this example we will consider an ideal mass on an ideal spring, with no friction or dissipative forces in the system.

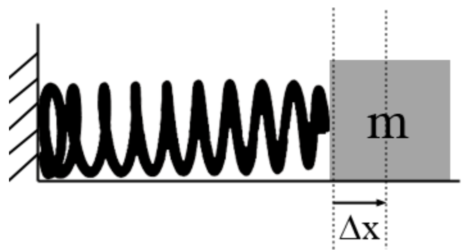



Figure 9: $\Delta x$ is the displacement of the mass $m$ from the equilibrium point.

We call $x$ our axis in this simple one dimensional oscillator, with the origin defined at the equilibrium point of the block, and write the force due to the spring as $F$ =-$kx$. We can now plug in to equation (6) to get:

$$-kx - m\ddot{x} = 0 \tag{7}$$

If we define a quantity $\omega^2 = \frac{k}{m}$, equation (7) can be rewritten as:

$$\ddot{x} = -\omega^2 x \tag{8}$$

We need to find a function whose 2nd derivative is (within a constant) a *negative* multiple of itself. We could consider a complex exponential, such as $e^{ix}$ (this greatly simplifies the algebra in more complicated examples) but for our purposes we can use sin and cos in a superposition $x(t) = A\sin\omega t + B\cos\omega t$. $A$ and $B$ are simply constants that are fixed by initial conditions. Notice that we started with a 2nd order differential equation, and so we have two integration constants, which require two initial conditions to be fixed - such as the value of $x(0)$ and $\dot{x}(0)$. For example, if I said $x(0) = 0.01m$ and $\dot{x}(0) = 0ms^{-1}$, this would set the values of $A$ and $B$.

$$\dot{x}(t) = A\omega\cos\omega t - B\omega\sin\omega t \tag{9}$$
$$\dot{x}(0) = A\omega = 0 \tag{10}$$
$$A = 0 \tag{11}$$
$$x(t) = B\cos\omega t \tag{12}$$
$$x(0) = 0.01m = B \tag{13}$$
$$x(t) = (0.01m)\cos\omega t \tag{14}$$

The quantity $\omega$ turns out to be the frequency of oscillation of the mass.

# B Appendix

A more challenging, but more illuminating, dynamical system is that of the forced, damped pendulum. We shift our attention to a pendulum that is driven by some periodic force $f_D = A\cos\omega_D t$ (where the subscript $D$ stands for drive), and is subject to some dissipative force (e.g. friction) that is proportional to the velocity of the pendulum, $F_{damping} = \Gamma\frac{d\theta}{dt}$. The pendulum also feels the effect of gravity, which after working out the geometry we get $F_{grav} = mgL\sin\theta$. The system then looks like this:

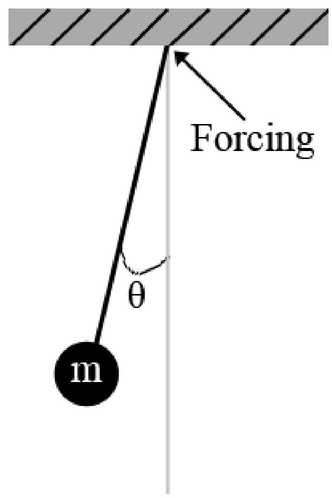



Figure 10: $\theta$ is the displacement of the mass $m$ from the equilibrium point.

We can write down Newton's $F = m\ddot{x}$ now as, where $\phi_0$ is some arbitrary initial phase in the forcing of the pendulum:

$$f_D = A\cos(\omega_D t + \phi_0) = mL^2\frac{d^2\theta}{dt^2} + \Gamma\frac{d\theta}{dt} + mgL\sin\theta \tag{15}$$

$$\frac{A}{mL^2}\cos(\omega_D t + \phi_0) = \frac{d^2\theta}{dt^2} + \frac{\Gamma}{mL^2}\frac{d\theta}{dt} + \frac{g}{l}\sin\theta \tag{16}$$

$$\tag{17}$$

If we now redefine our constants such that $B = \frac{A}{ml^2}$, $\gamma = \frac{\Gamma}{mL^2}$, and $\omega_0^2 = \frac{g}{l}$, the equation becomes much nicer:

$$B\cos(\omega_D t + \phi_0) = \frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2\sin\theta \tag{18}$$

$\omega_0$ is simply the natural frequency of the pendulum, the frequency it would oscillate at if there were no external forcing. It is common here to adopt dimensionless units, such that $\tau = \omega_0 t$, $Q = \frac{\omega_0}{\gamma}$, $\beta = \frac{B}{\omega_0^2}$, and $\Omega = \frac{\omega_D}{\omega_0}$. Equation 18 then becomes

$$\beta\cos(\Omega\tau + \phi_0) = \frac{d^2\theta}{d\tau^2} + \frac{1}{Q}\frac{d\theta}{d\tau} + \sin\theta \tag{19}$$

The process to follow will be a necessary step in order to solve this equation using numerical methods. What we do is rewrite the second order differential equation as a system of first order differential equations. We start by defining

$$\frac{d\theta}{dt} = \omega \tag{20}$$

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \beta\cos(\phi) - \frac{1}{Q}\omega - \sin\theta; \qquad \phi = \Omega\tau + \phi_0 \tag{21}$$

$$\frac{d\phi}{dt} = \omega_D \tag{22}$$

This decomposition puts the system into what is said to be the "autonomous form," and in general any $n$-order differential equation can be broken up into $n$ first-order differential equations. These equations can now be numerically integrated to find solutions, using an algorithm such as the Runge-Kutta method.