Math Challenge Solutions

By Adrian Ochoa
For Scientific Games Interactive

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Problem 1. Say you have three vectors:
[B, C, E, F, A, D, F, C, G, A, A, C, B, D, E, F],
[G, C, E, F, B, D, F, A, A, A, B, C, E, D, E, F],
[E, F, B, D, A, E, A, C, E, D, E, F, A, B, F, G].
Randomly choose three sequential elements from each vector. This makes nine total items chosen. What is the probability that among the nine there are three A's?

Solution. There are 14 possible sequential lists of three in each of the three vectors. Therefore, by the fundamental counting principle, there are $14 \cdot 14 \cdot 14 = 2744$ different possible outcomes for choices of nine items. We need now find how many of these outcomes contain three A's. We find from an exhaustive search in `problem1.py` that 843 of these outcomes contain three A's. Therefore

$$P(\text{A occurs three times}) = \frac{843}{2744} \approx 0.3072157434. \tag{1}$$

□

Problem 2. You have a bag containing 1 red ball, 1 blue ball, and 1 yellow ball. You remove two balls. The second ball you remove, color it the same as the first, and put the two balls back in the bag. What is the average number of times you do this before all balls are the same color?

Solution. For simplicity, label the original state of the bag as a list where an integer corresponds to each color:

$$A_0 = [1, 2, 3]. \tag{2}$$

If we execute the above replacement a single time, the bag will clearly be in a state of having precisely two balls of the same color, thus eliminating one of the colors from future iterations. For example,

$$A_1 = [1, 1, 3]. \tag{3}$$

We now see that the next state will either be functionally equivalent to the previous (i.e. having precisely two balls of the same color) or will terminate to the desired state of uniform color.

The possibilities are as follows:

$$A_2 = [1, 1, 1]$$

or

$$A_2 = [1, 1, 3]$$

or

$$A_2 = [3, 3, 1]. \tag{4}$$

From here, the probability that it takes k more trials before reaching a state of uniform color is clearly the geometric distribution,

$$P(X = k) = (1 - p)^{k-1} p \tag{5}$$

where p is the probability of the next state being uniform in color. The expected value of this probability distribution is

$$E(X) = \frac{1}{p}. \tag{6}$$

So we now need only find the probability that the next state is of uniform color. Returning to A_1, we see by the fundamental counting principle there are $3 \cdot 2 = 6$ ways for any of the possible A_2 states to occur and there are $2 \cdot 1 = 2$ possible ways for a state of uniform color to occur, which is when either of the majority color balls is chosen and subsequently the minority color ball is chosen. That is to say that

$$p = \frac{2}{6} = \frac{1}{3}$$

and

$$E(X) = 3. \tag{7}$$

We therefore expect that, on average, the bag will contain three balls of the same color after **four** replacements. The simulation offered in `problem2.py` estimates this probability.

□

Problem 3. A person has two blank 6-sided dice. They have 2 of every number $(1 - 6)$ available to randomly distribute on the faces of the dice. Considering all possible combinations, what is the probability of rolling a seven?

Solution. The solution is via the script written in `problem3.py`. The code finds all combinations of dice and calculates the number of ways to roll a seven for each set of dice. Summing all of these and dividing by the number of combinations and the number of possible rolls, we find the desired probability:

$$p = \frac{\text{Total possible seven's}}{36 \cdot \binom{12}{6}} = \frac{6048}{36 \cdot 924} = \frac{2}{11} = 0.181818\ldots. \tag{8}$$

By simulating dice rolls with randomly generated dice, we find an estimate for this probability in the code. We also observe from the plot `problem3convergence.png` that the estimate improves with increasing number of trials.

□