Equating real and imaginary part we get -

$$\boxed{\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial v}{\partial \sigma} &= -\frac{\partial v}{\partial \theta} \end{aligned}}$$

which is the cauchy Reimann eq^n in Polar form.

* Formulas

$\cosh z = \frac{e^{z}+e^{-z}}{2}$ $\qquad$ $\sinh z = \frac{e^{z}-e^{-z}}{2}$

$\cos z = \frac{e^{iz}+e^{-iz}}{2}$ $\qquad$ $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

$\cos iz = \frac{e^{-z}+e^{z}}{2}$ $\qquad$ $\sin iz = \frac{e^{-z}-e^{z}}{2i}$

$= \frac{e^{z}+e^{-z}}{2}$ $\qquad$ $= -\left(\frac{(e^{z} - e^{-z})}{2i}\right) \times \frac{i}{i}$

$= \cosh x$ $\qquad$ $=$

$= i\left(\frac{e^{z}-e^{-z}}{2}\right)$

$= i\sinh z$

* Show that the function $\log z$, $x \neq 0$ is analytic and hence find its derivatives.

-> Sol^n ->

Let $f(x) = \log z$

or, $u(x,y) + i v(x,y) = \log(x+iy)$

substitute $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore x+iy = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

So,

$$u + iv = \log(re^{i\theta})$$
$$= \log r + \log e^{i\theta}$$
$$= \log r + i\theta$$
$$= \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)$$

equating real and imaginary part we get:-

$$u = \log\sqrt{x^2+y^2}$$

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

then,

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot y \cdot \left(\frac{-1}{x^2}\right)$$

$$= -\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2/x^2} \cdot \frac{1}{x}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

clearly $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are continuous function of x and y & satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

So, the f^n $\log z$, $z \neq 0$ is analytical.

We have $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$$= \frac{x}{x^2+y^2} + i\left(-\frac{y}{x^2+y^2}\right)$$

$$= \frac{x - iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

$$= \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{1}{x+iy}$$

$$= \frac{1}{z}$$

$$\frac{d}{dz}[\log z] = \frac{1}{z}, \quad z \neq 0$$

* Show that the f^n $\sin z$ is analytic and hence find it's derivative

→ Sol^n

let $f(z) = \sin z$

or, $u(x,y) + iv(x,y) = \sin(x+iy)$

$\because \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

$\therefore u + iv = \sin x \cosh y + i \cos x \sinh y$

equating real and imaginary part

$u = \sin x \cosh y$

$v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial u}{\partial y} = \sin x \sinh y$

$\frac{\partial v}{\partial x} = -\sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cosh y$

clearly $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous f^n of x and y

VVVIMP

* Show that the $f^n = \sqrt{|xy|}$ is not regular at origin even though C.R. eq^n's are satisfied.

→ Sol^n

Given,

$f(z) = \sqrt{|xy|}$

$\Rightarrow u + iv = \sqrt{|xy|} + i \times 0$

Equating real and imaginary part we get

$u = \sqrt{|xy|}$ and $v = 0$

We have

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h}$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(x,y+k) - u(x,y)}{k}$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(x+h,y) - v(x,y)}{h}$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(x,y+k) - v(x,y)}{k}$$

At $(0,0)$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence the f^n $f(z) = \sqrt{|xy|}$ satisfies C.R. eq^n's.

At $(0,0)$,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

As $z \to 0$ along the line $y = mx$ then,

$$f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x + imx}$$

$$= \lim_{x \to 0} \frac{\sqrt{|m|}}{1 + im}$$

This limit is not unique since it depends on value of m. So $f'(0)$ does not exist. Hence the f^n $f(z) = \sqrt{|xy|}$ is not regular at origin.

* Construction of Analytic function

Case I :- When real part $u(x,y)$ or imaginary part $v(x,y)$ is given :-

1.) Calculate $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when $u(x,y)$ is given. Calculate $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ then $v(x,y)$ is given.

2.) Using $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

3.) Using C.R. eq^n's in cartesian form on the basis of given f^n.

4.) Using Milne Thomson method, replacing x by z and y by zero and then intregate on both side we get the reqd. analytical f^n

Case II : When real part $u(r,\theta)$ or imaginary part $v(r,\theta)$ is given.

1.) Calculate $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ when $u(r,\theta)$ is given, caculate $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ then $v(r,\theta)$ is given.

2.) Using $f'(z) = \frac{\partial u}{\partial r} + \frac{1}{e^{i\theta}}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$

3.) Using CR eq^n in polar for on the basis of given f^n.

4.) Using Milne Thomson method, relpacing r by z and θ by zero and then intregrate on B.S. we get the reqd analytic f^n.

Q.) Determine the Analytic f^n whose real part is $e^x(x\cos y - y\sin y)$ and hence find its imaginary part.

→ Sol^n →

Given,

$$u(x,y) = e^x(x\cos y - y\sin y)$$

$$\therefore \frac{\partial u}{\partial x} = e^x\cos y + e^x(x\cos y - y\sin y)$$

$$= e^x(\cos y + x\cos y - y\sin y)$$

$$\therefore \frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

$$= -e^x(x\sin y + y\cos y + \sin y)$$

We have

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \left[\text{using CR eqn's } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

$$= e^x(\cos y + x\cos y - y\sin y) + ie^x(x\sin y + y\cos y + \sin y)$$

Using Milne Thompson's method, replacing x by z and y be 0, we get

$$= e^z(1+z) + ie^z \times 0$$

$$= e^z(1+z)$$

integrate on both side w.r.t z we get

$$f(z) = \int (1+z)e^z dz + C$$

$$f(z) = (1+z)e^z - e^z + C$$

$$= e^z + e^z z - e^z + C$$

$$f(z) = e^z z + C$$

$$\boxed{\therefore f(z) = ze^z + C}$$

which is the reqd analytic f^n

Here,

$$f(z) = ze^z + C$$

$$u(x,y) + iv(x,y) = (x+iy)e^{x+iy} + C$$

$$= e^x(x+iy)e^{iy} + C$$

$$= e^x(x+iy)(\cos y + i\sin y) + C$$

$$= e^x(x\cos y + ix\sin y + iy\cos y - y\sin y) + C$$

$$= e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y) + C$$

~~$\therefore f(z) = ze^z + C$~~

Imaginary part

$$v(x,y) = e^x(x\sin y + y\cos y) + C$$

Q.) Determine the analytic whose imaginary part $\sinh x\cos y$ and hence find its real part.

Given,

$$v(x,y) = \sinh x\cos y$$

$$\therefore \frac{\partial v}{\partial x} = -\cosh x\cos y$$

$$\therefore \frac{\partial v}{\partial y} = -\sinh x\sin y$$

We have,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \left[\text{From CR rel}^n \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$$

~~= cosh x cos y~~

$= -\sinh x \sin y + i \cosh x \cos y$

Using Milne Thompson's method, replacing x by z & y by 0, we get

$f'(z) = \cancel{\sinh z}\ 0 - i\cosh z$

$f'(z) = -i\cosh z$

Intregating on both sides we get

$$f(z) = -\int (i\cosh z)\,dz + C$$

$$= +i\sinh z + C$$

$$f(z) = i\sinh z + C$$

Here,

$f(z) = i\sinh z + C$

$u(x,y) + iv(x,y) = \cancel{(x+iy)}\ i\sinh(x+iy) + C$

~~$= i\sinh x - \sinh y + C$~~

$= i\sin(x+iy) + C$

$= i[\sin x \cos iy + \cancel{\cos iy}\ \cos x \sin iy] + C$

~~$= i\sin x\, i\cos iy + i\cos x \sin iy + C$~~

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Real part $= \sin x \cos iy + \cos x \sin iy + C$

$= \sin x \cosh y + \cos x \sin iy + C$

Q.) If $f(z) = u + iv$ is an analytic f^n, find $f(z)$ if $u - v = (x-y)(x^2+4xy+y^2)$

→ Sol^n ⇒

Given

$f(z) = u + iv$ — (*)

$if(z) = iu - v$ — (**)

Adding (*) & (**) we get,

$(1+i)f(z) = (u-v) + i(u+v)$

or, $u - v = U$

or, $u + v = V$

or, $f(z) = U + V$

Here

$U = u - v$

$= (x-y)(x^2+4xy+y^2)$

$\frac{\partial U}{\partial x} = (x^2+4xy+y^2) + (x-y)(2x+4y)$

$= x^2+4xy+y^2 + 2x^2 + 4xy - 2xy - 4y^2$

$= 3x^2 + 6xy - 3y^2$

$\frac{\partial U}{\partial y} = -(x^2+4xy+y^2) + (x-y)(4x+2y)$

$= -x^2 - 4xy - y^2 + 4x^2 + 2xy - 4xy + 4y^2$

$= 3x^2 - 6xy + 3y^2$

$f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \quad \left[\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}\right]$

$= (x^2+4xy+y^2) + (x-y)(2x+4y) + i(x^2+4xy+y^2) + i(x-y)(4x+2y)$

$f(z) = (1-i)z^3 + C$

$= \cancel{(x^2+4xy+y^2)} + \cancel{(x-y)(1+i)} + \cancel{(1+i)[(x-y)(2x+4y)}$

$= (x^2+4xy+y^2)(1+i) + (x-y)[2x+4y + i(4x+2y)]$

$= 3x^2+6xy-3y^2 - i(3x^2-6xy+3y^2)$

From Milne Thompson put $x=z$ & $y=0$

$= 3z^2 - 3z^2 i$ $\quad 3z^2(1-i)$

Intregating

$F(z) = 3\dfrac{z^3}{3}(1-i) + C$

$F(z) = z^3(1-i) + C$

then,

$F(z) = (1-i)z^3 + C$

or, $(1+i)f(z) = (1-i)z^3 + C$

or, $f(z) = \dfrac{(1-i)z^3}{(1-i)} + \dfrac{C}{1+i}$

$f(z) = -iz^3 + K$ where $K = \dfrac{C}{1+i}$