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Find the z-transform of following sequences

1) $x(k) = \{3, 5, 2, \underset{\uparrow}{4}, 1, 0, 9, -12\}$

this symbol indicates 0^{th} index element

i.e $x(0) = 4$

Soln,

the symbol '↑' is used to indicate the zeroth position element of sequence

So,

$x(0) = 4$

$x(1) = 1$

$x(2) = 0$

$x(3) = 9$

$x(4) = -12$

and

$x(-1) = 2$

$x(-2) = 5$

$x(-3) = 3$

then, by definition of z-transform,

$$Z[x(k)] = \sum_{k=-3}^{4} x(k) z^{-k}$$

$$= x(-3)z^3 + x(-2)z^2 + x(-1)z^1 + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 3z^3 + 5z^2 + 2z + 4 + \frac{1}{z} + 0 + \frac{9}{z^3} - \frac{12}{z^4}$$

2) $f(k) = \{5, 6, -3, -2, 8, 10\}$

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3) $f(K) = \begin{cases} 5^K & \text{for } K < 0 \\ 2^K & \text{for } K \geq 0 \end{cases}$

soln,

$$Z[f(K)] = \sum_{K=-\infty}^{\infty} f(K) z^{-K}$$

$$= \sum_{K=-\infty}^{-1} f(K) z^{-K} + \sum_{K=0}^{\infty} f(K) z^{-K}$$

$$= \sum_{K=-\infty}^{-1} 5^K z^{-K} + \sum_{K=0}^{\infty} 2^K z^{-K}$$

$$= \sum_{K=-\infty}^{-1} \left(\frac{5}{z}\right)^K + \sum_{K=0}^{\infty} \left(\frac{2}{z}\right)^K$$ not needed

$$= (\cdots + 5^{-3} z^3 + 5^{-2} z^2 + 5^{-1} z^1) + (1 + 2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + \cdots)$$

$$= \left[\left(\frac{z}{5}\right)^1 + \left(\frac{z}{5}\right)^2 + \left(\frac{z}{5}\right)^3 + \cdots\right] + \left[1 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots\right]$$

$$= \frac{\frac{z}{5}}{1 - \frac{z}{5}}, \left|\frac{z}{5}\right| < 1 + \frac{1}{1 - \frac{2}{z}}, \left|\frac{2}{z}\right| < 1$$

$$= \frac{z}{5 - z} + \frac{z}{z - 2}, \quad |z| < 5 \text{ and } |z| > 2$$

Properties of Z-transform

① Linearity

If $x_1(K)$ and $x_2(K)$ are two sequences, then,

$$Z[a x_1(K) + b x_2(K)] = a Z[x_1(K)] + b Z[x_2(K)]$$

where,

a and b are constant.

② Multiplication by a^K

If $x(K)=0$ for $K<0$ and $Z[x(K)]=X(z)$ for $K\geq 0$,

then,

$$Z[a^K x(K)] = X\left(\frac{z}{a}\right)$$

Proof:

$$\text{let } Z[x(K)] = \sum_{K=0}^{\infty} x(K) z^{-K} = X(z)$$

then,

$$Z[a^K x(K)] = \sum_{K=0}^{\infty} \{a^K x(K)\} z^{-K}$$

$$= \sum_{K=0}^{\infty} x(K) (a^K z^{-K})$$

$$= \sum_{K=0}^{\infty} x(K) \left(\frac{z}{a}\right)^{-K}$$

$$= \sum_{K=0}^{\infty} x(K) z_1^{-K} \quad \text{where } z_1 = \frac{z}{a}$$

$$= X(z_1)$$

$$= X\left(\frac{z}{a}\right)$$

$$\boxed{\therefore Z[a^K x(K)] = X\left(\frac{z}{a}\right)}$$

③ Multiplication by K

If $x(K) = 0$ for $K < 0$ and $Z[x(K)] = X(z)$ for $K \geq 0$.

Then,

$$Z[Kx(K)] = -z \frac{d}{dz} X(z)$$

Proof:

We have,

$$Z[x(K)] = \sum_{K=0}^{\infty} x(K) z^{-K} = X(z)$$

then,

$$Z[Kx(K)] = \sum_{K=0}^{\infty} \{Kx(K)\} z^{-K}$$

$$= \sum_{K=0}^{\infty} x(K)(K z^{-K})$$

$$= \sum_{K=0}^{\infty} x(K)(-K)z^{-K}(-1)$$

$$= \sum_{K=0}^{\infty} x(K)\{(-K)z^{-K-1}\}(-z)$$

$$= -z\sum_{K=0}^{\infty} x(K)\frac{d}{dz}(z^{-K})$$

$$= -z\frac{d}{dz}\sum_{K=0}^{\infty} x(K)z^{-K}$$

$$= -z\frac{d}{dz} Z[x(K)]$$

$$\boxed{\therefore Z[Kx(K)] = -z\frac{d}{dz}X(z)}$$