DEC

Find the inverse z-transform of f^n

$$X(z) = \frac{z^2}{(z+1)(z-1)^2}$$

by using partial fraction method.

Soln

Here, the given f^n

$$X(z) = \frac{z^2}{(z+1)(z-1)^2}$$

The poles of f^n is obtained by

$$(z+1)(z-1)^2 = 0$$

$\Rightarrow z+1 = 0$

$\therefore z = -1$ is simple pole

$\Rightarrow (z-1)^2 = 0$

$\Rightarrow z = 1$ is multiple pole of order two

Then,

$$\frac{X(z)}{z} = \frac{z}{(z+1)(z-1)^2}$$

By partial fraction decomposition

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2} \quad \text{(i)}$$

Where, $\left.\begin{array}{l} \cdot A_1 \\ \cdot A_2 \\ \cdot A_3 \end{array}\right]$ are constants simple pole

- To find A_1

$$A_1 = \left[(z+1) \frac{X(z)}{z} \right]_{z=-1}$$

$$= \left[(z+1) \frac{z}{(z+1)(z-1)^2} \right]_{z=-1}$$

$$= \frac{-1}{(-1-1)^2} \quad = \frac{-1}{(-2)^2}$$

$$= \frac{1}{4}$$

$$\therefore A_1 = -\frac{1}{4}$$

multiple poles

- To find A_3

$(z-1)^2$

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$$A_3 = \left[(z-1)^2 \frac{X(z)}{z} \right]_{z=1}$$

$$= \left[(z-1)^2 \frac{z}{(z+1)(z-1)^2} \right]_{z=1}$$

$$= \frac{1}{1+1} \quad = \frac{1}{2}$$

$$\therefore A_2 = 1/2$$

To find A_2

$$A_2 = \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}\left\{(z-1)^2 \frac{X(z)}{z}\right\}\right]_{z=1}$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left\{(z-1)^2\frac{z}{(z+1)(z-1)^2}\right\}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left\{\frac{z}{z+1}\right\}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+1)-z\cdot 1}{(z+1)^2}\right]$$

$$= \lim_{z\to 1}\left[\frac{1}{(z+1)^2}\right]$$

$$= \frac{1}{(2)^2} = \frac{1}{4}$$

$$\therefore A_2 = \frac{1}{4}$$

In eqn (i)

$$\frac{X(z)}{z} = \frac{-1/4}{z+1} + \frac{1/4}{z-1} + \frac{1/2}{(z-1)^2}$$

$$\Rightarrow X(z) = -\frac{1}{4}\,\frac{z}{z+1} + \frac{1}{4}\,\frac{z}{z-1} + \frac{1}{2}\,\frac{z}{(z-1)^2}$$

Now,
taking inverse of z-transform on B.S.

We get,

$$Z^{-1}[X(z)] = -\frac{1}{4}\,Z^{-1}\underbrace{\left[\frac{z}{z+1}\right]}_{(-1)^k} + \frac{1}{4}\,Z^{-1}\underbrace{\left[\frac{z}{z-1}\right]}_{(1)^k}$$

$$+ \frac{1}{2}\,Z^{-1}\underbrace{\left[\frac{z}{(z-1)^2}\right]}_{k}$$

$$\therefore x(k) = -\frac{1}{4}(-1)^k + \frac{1}{4}(1)^k + \frac{1}{2}k$$

Which is reqd. soln

$$\therefore Z^{-1}[X(z)] = x(k) = -\frac{1}{4}(-1)^k + \frac{1}{4}(1)^k + \frac{1}{2}k$$

Calculus of residues
or 'Inversion integral method

In this method,
we obtain inverse z-transform by summing the residues of fn $\{X(z) \cdot z^{k-1}\}$ at all poles of $X(z)$

Mathematically,

$$z^{-1}[X(z)] = x(k) = \sum_{\substack{\text{all poles} \\ \text{of } X(z)}} \text{residues of } \{X(z) \cdot z^{k-1}\}$$

• Residues formulas

① At simple pole
$z = z_i$

$$\underset{z = z_i}{\text{Res}} = \left[(z - z_i)\, X(z)\, z^{k-1} \right]_{z = z_i}$$

$z \xrightarrow{\text{lim}} z_i$

⑪ At multiple poles of order 'r'
$z = z_i$

$$\underset{z = z_i}{\text{Res}} = \lim_{z \to z_i} \frac{1}{(r-1)!} \left[\frac{d^{r-1}}{dz^{r-1}} \{ (z - z_i)^r X(z) z^{k-1} \} \right]$$

1. Find inverse Z-transform of f^n

$$X(z) = \frac{3z^3 + 2z}{(z-3)^2 (z-2)}$$ using

calculus of residues method (inversion & integral method)

Soln

Here,

given f^n

$$X(z) = \frac{3z^3 + 2z}{(z-3)^2 (z-2)}$$

The poles of f^n $X(z)$ is obtained by

$$(z-3)^2 (z-2) = 0$$

$\Rightarrow$ $z - 2 = 0$

$z = 2$ is simple pole

$\Rightarrow$ $(z-3)^2 = 0$

$z = 3$ is multiple pole of order two

Now,

• Residue of f^n at $z = 2$

simple pole

$$\underset{z=2}{\text{Res}} = \lim_{z \to 2} \left[(z-2)\, X(z)\, z^{k-1} \right]$$

$$= \lim_{z \to 2} \left[\frac{(\cancel{z-2})(3z^3+2z)}{(z-3)^2(\cancel{z-2})} z^{k-1} \right]$$

$$= \lim_{z \to 2} \left[\frac{3z^{k+2}+2z^k}{(z-3)^2} \right]$$

$$= \frac{3(2)^{k+2}+2(2)^k}{(2-3)^2} \qquad = (2)^k \left[\frac{3 \cdot 2^2+2}{(-1)^2} \right]$$

$$= 2^k \cdot 14$$

$$\therefore \underset{z=2}{\text{Res}} = 14\,(2)^k$$

Again,

- Residue of f^n at $z=3$

multiple poles of order two

$$\underset{z=3}{\text{Res}} = \lim_{z \to 3} \frac{1}{(2-1)!} \left[\frac{d}{dz} \left\{ (z-3)^2 \times f(z)\, z^{k-1} \right\} \right]$$

$$= \lim_{z \to 3} \left[\frac{d}{dz} \left\{ (z-3)^2 \frac{(3z^3+2z)}{(z-3)^2(z-2)} z^{k-1} \right\} \right]$$

$$= \lim_{z \to 3} \left[\frac{d}{dz} \left\{ \frac{(3z^{k+2} + 2z^{k})}{(z-2)} \right\} \right]$$

$$= \lim_{z \to 3} \left[\frac{(z-2)\{3(k+2)z^{k+1} + 2z^{k-1}k\} - (3z^{k+2} + 2z^{k}) \cdot 1}{(z-2)^2} \right]$$

$$= \frac{(3-2)\{(3k+6)(3)^{k+1} + 2k(3)^{k-1}\} - 3(3)^{k+2} - 2(3)^{k}}{(3-2)^2}$$

$$= (3k+6)(3)^{k+1} + 2 \cdot k \cdot 3^{k-1} - 3(3)^{k+2} - 2(3)^{k}$$

$$= (3)^{k} \left[9k + 18 + \frac{2k}{3} - 27 - 2 \right]$$

$$\therefore \underset{z=3}{\text{Res}} = (3)^{k} \left[\frac{29k}{3} - 11 \right]$$

We have,

$Z^{-1}[X(z)]$ = sum of all residues at given poles

$$x(k) = 14(2)^{k} + (3)^{k} \left[\frac{29k}{3} - 11 \right]$$ be inverse Z-transform of given f^n

Application of z-transform to diff. eqⁿ

- **Difference eqⁿ**

An eqⁿ which connects the various differences of unknown function is called diff eqⁿ

- **Eg:**

$$y_{k+2} - y_{k+1} + y_k = 2k$$

$$x(k+1) - x(k) = k \quad \text{etc.}$$

- **Note:** from right shifting theorem

(i) $Z[x(k+3)] = z^3 [X(z) - x(0) - x(1)z^{-1} - x(2)z^{-2}]$

(ii) $Z[x(k+2)] = z^2 [X(z) - x(0) - x(1)z^{-1}]$

(iii) $Z[x(k+1)] = z[X(z) - x(0)]$

1. Solve the diff. eq^n $x(k+2) - 3x(k+1) + 2x(k) = 4^k$, $x(0) = 0$, $x(1) = 1$ using Z-transform method

Sol^n

Here, given diff. eq^n is:

$$x(k+2) - 3x(k+1) + 2x(k) = 4^k$$

where, $x(0) = 0$

$x(1) = 1$

Taking Z-transform on both sides of above eq^n, we get

$$Z[x(k+2)] - 3Z[x(k+1)] + 2Z[x(k)] = Z[(4)^k]$$

or, $z^2[X(z) - x(0) - x(1) \cdot z^{-1}] - 3 \cdot z[X(z) - x(0)]$

$$+ 2X(z) = \frac{z}{z-4}$$

using conditions,

- $x(0) = 1$
- $x(1) = 1$

or, $z^2[X(z) - 0 - 1 \cdot z^{-1}] - 3z[X(z) - 0] + 2X(z)$

$$= \frac{z}{z-4}$$

or, $(z^2 - 3z + 2)X(z) = \frac{z}{z-4} + z$

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or, $(z^2 - 3z + 2)\,X(z) = \dfrac{z + z^2 - 4z}{(z-4)}$

or, $(z^2 - 3z + 2)\,X(z) = \dfrac{z^2 - 3z}{(z-4)}$

$$\therefore X(z) = \frac{z^2 - 3z}{(z-4)(z^2 - 3z + 2)}$$

$$= \frac{z^2 - 3z}{(z-4)(z-1)(z-2)}$$

Also find $Z^{-1}[X(z)] = x(k)$

$$\frac{X(z)}{z} = \frac{z-3}{(z-4)(z-1)(z-2)}$$

By applying partial fraction decomposition rule $\dfrac{X(z)}{z}$ can be expanded as:

$$\frac{z-3}{(z-4)(z-1)(z-2)} = \frac{\overset{-2/3}{A_1}}{z-1} + \frac{\overset{1/2}{A_2}}{z-2} + \frac{\overset{1/6}{A_3}}{z-4}$$