th

$$= \lim_{z \to \infty} \frac{\cancel{z^2} z^2 \left[3 + 4/z + 18 - 18/z + 27/z^2\right]}{\cancel{z^4}\left[1 - 3/z^3\right]^3}$$

$$= (3+0+18-0+0) / (1-0) = 21$$

Date ______
Page ______

ec

Find the z- transform of following $f^n s$

(i) $k a^k$
(ii) $k^2 a^k$
(iii) $k^3 a^k$

(iv) $t e^{at}, t \geq 0$
(v) $t^2 e^{-at}, t \geq 0$
(vi) $t^3 e^{at}$
(vii) $a^2 t^2, t \geq 0$
(viii) $a^{3t} t^3, t \geq 0$

(ix) $a^k \cos \frac{k\pi}{2}, k \geq 0$

(x) $a^n \sin \frac{n\pi}{2}, n \geq 0$

(i) $k a^k$

we have,

$$Z[a^k] = \frac{z}{z-a}$$

using property of z- transform

$$Z[k a^k] = -z \frac{d}{dz}\left(\frac{z}{(z-a)}\right)$$

$$= -z\left[\frac{(z-a)\cdot 1 - z(1)}{(z-a)^2}\right]$$

$$= -z\left[\frac{z-a-z}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2}$$

(ii) $Z[k^2 a^k]$

$$Z[k^2 a^k] = Z[k\cdot(k a^k)]$$

$$= -z\frac{d}{dz}\left[Z[k a^k]\right]$$

$$= -z\frac{d}{dz}\left[\frac{az}{(z-a)^2}\right]$$

$$= -z\left[\frac{(z-a)^2\cdot a - az\cdot 2(z-a)\cdot 1}{(z-a)^4}\right]$$

$$= -z\left[\frac{\cancel{(z-a)}\left[(z-a)a - 2az\right]}{(z-a)^{\cancel{4}\,3}}\right]$$

$$= -z\left[\frac{za - a^2 - 2az}{(z-a)^3}\right]$$

$$= -z\left[\frac{-az-a^2}{(z-a)^3}\right] = az\left[\frac{z+a}{(z-a)^3}\right]$$

t^n vaako ma

$$Z[t^n] = -Tz \frac{d}{dz}\left[Z[t^{n-1}]\right]$$

For $n=3$

$$Z[t^3] = -Tz \frac{d}{dz}\left[Z(t^2)\right]$$

$$\therefore Z[t^2] = -Tz \frac{d}{dz}\left[\underbrace{Z(t)}_{\frac{Tz}{(z-1)^2}}\right]$$

$$= -T^2 z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

(x) $a^n \sin \frac{n\pi}{2}, n \geq 0$.

we have,

$$Z[\sin n\theta] = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$$

Here,

$$\theta = \pi/2$$

$$Z[\sin n\pi/2] = \frac{z \sin \frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1}$$

$$= \frac{z}{z^2+1}$$

using property of Z- transform

$$Z\left[a^n \sin \frac{n\pi}{2}\right] = \left[\frac{z}{z^2+1}\right]_{z \to z/a}$$

$$= \frac{\frac{z}{a}}{\frac{z^2}{a^2} + 1}$$

$$= \frac{az}{z^2 + a^2}$$

2. Find the Z-transform of $1-e^{-at}$ & hence, calculate /evaluate $x(\infty)$ using Final Value Theorem (F.V.T.)

Given,

function

$$x(t) = 1-e^{-at}$$

$$\Rightarrow x(KT) = 1-e^{-aKT}$$

Then,

$$Z[x(t)] = Z[x(KT)]$$

$$= \sum_{K=0}^{\infty} x(KT)\, z^{-K}$$

$$= \sum_{K=0}^{\infty} (1-e^{-aKT})\, z^{-K}$$

$$= \sum_{K=0}^{\infty} 1 \cdot z^{-K} - \sum_{K=0}^{\infty} e^{-aKT} \cdot z^{-K}$$

$$= \sum_{K=0}^{\infty} z^{-K} - \sum_{K=0}^{\infty} (e^{-aT} z^{-1})^{K}$$

$$= (1+z^{-1}+z^{-2}+\cdots) - [1+(e^{-aT}z^{-1})^{1}+(e^{-aT}z^{-1})^{2}+\cdots]$$

$$= \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}$$

$$= \frac{z}{z-1} - \frac{z}{z-e^{-aT}} = X(z)$$

Using Final Value Theorem,

$$x(\infty) = \lim_{z \to 1} (z-1)\, X(z)$$

$$= \lim_{z \to 1} (z-1) \left[\frac{z}{z-1} - \frac{z}{z-e^{-aT}} \right]$$

$$= \lim_{z \to 1} \left[z - \frac{(z-1)\, z}{z-e^{-aT}} \right]$$

$$= 1 - \frac{(1-1)\, z}{\cancel{z}\ 1-e^{-aT}} \Rightarrow 1-0$$

$$= 1$$

Inverse z-transform

If

$$Z[x(k)] = X(z)$$

Then,

$$Z^{-1}[X(z)] = x(k).$$

we say $x(k)$ is inverse z-transform of complex variable f^n $X(z)$

Then, for discrete time signal f^n $x(t)$

$$Z[x(t)] = Z[x(kT)] = X(z)$$

Then,

$$Z^{-1}[X(z)] = x(kT)$$
$$= x(t)$$

z-transform & Inverse z-transform of f^n in std. form

(1) $Z[1]$ or

$Z[1^k]$ or

$Z[u(k)]$

$$\Rightarrow \frac{z}{z-1}$$

Thus,

$$Z^{-1}\left[\frac{z}{z-1}\right] = 1 \text{ or } 1^k \text{ or } u(k)$$

(2) $Z[a^k] = \frac{z}{z-a}$

$Z^{-1}\left[\frac{z}{z-a}\right] = a^k$

(3) $Z[k] = \frac{z}{(z-1)^2}$

$Z^{-1}\left[\frac{z}{(z-1)^2}\right] = k$

(4) $Z[k a^k] = \frac{az}{(z-a)^2}$

$Z^{-1}\left[\frac{az}{(z-a)^2}\right] = k a^k$

(5) $Z[\sin k\theta] = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

$Z^{-1}\left[\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right] = \sin k\theta$

(6) $Z[\cos k\theta] = \dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

$Z^{-1}\left[\dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right] = \cos k\theta$

(7) $Z[e^{at}] = \dfrac{z}{z - e^{aT}}$

$Z^{-1}\left[\dfrac{z}{z - e^{aT}}\right] = e^{at}$

methods of finding inverse z-transform

1. Long division method
2. partial fraction method
3. calculus of residues or inverse integral method

Date
Page

(2) Partial Fraction Method

let

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}, \quad m < n$$

$$= \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

$$\frac{X(z)}{z} = \frac{f(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Clearly,

$z = p_1, p_2, \cdots p_n$ are simple poles of f^n $X(z)$

Then,

by partial fraction decomposition

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_n}{z-p_n} \rightarrow (*)$$

where,

$$A_1 = \left[(z-p_1)\frac{X(z)}{z}\right]_{z=p_1}$$

$$A_2 = \left[(z-p_2)\frac{X(z)}{z}\right]_{z=p_2}$$

$$A_n = \left[(z - p_n) \frac{X(z)}{z} \right]_{z = p_n}$$

If $\frac{X(z)}{z}$ involve double pole at $z = a$

$$\frac{X(z)}{z} = \frac{F(z)}{(z-a)^2}$$

Then,

by partial fraction

$$\frac{X(z)}{z} = \frac{A_1}{(z-a)^2} + \frac{A_2}{z-a}$$

where,

$$A_1 = \left[(z-a)^2 \frac{X(z)}{z} \right]_{z=a}$$

$$A_2 = \left[\frac{d}{dz} \left[(z-a)^2 \frac{X(z)}{z} \right] \right]_{z=a}$$

similarly,

If

$$\frac{X(z)}{z} = \frac{F(z)}{(z-a)^n}$$

Then,

by partial fraction

$$\frac{X(z)}{z} = \frac{A_1}{(z-a)^n} + \frac{A_2}{(z-a)^{n-1}} + \frac{A_3}{(z-a)^{n-2}}$$

$$+ \cdots + \frac{A_n}{z-a}$$

where,

$$A_1 = \left[(z-a)^n \frac{X(z)}{z}\right]$$

$$A_2 = \frac{1}{(2-1)!}\left[\frac{d^{2-1}}{dz^{2-1}}\left\{(z-a)^n \frac{X(z)}{z}\right\}\right]_{z=a}$$

$$\therefore A_k = \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^n \frac{X(z)}{z}\right\}\right]_{z=a}$$