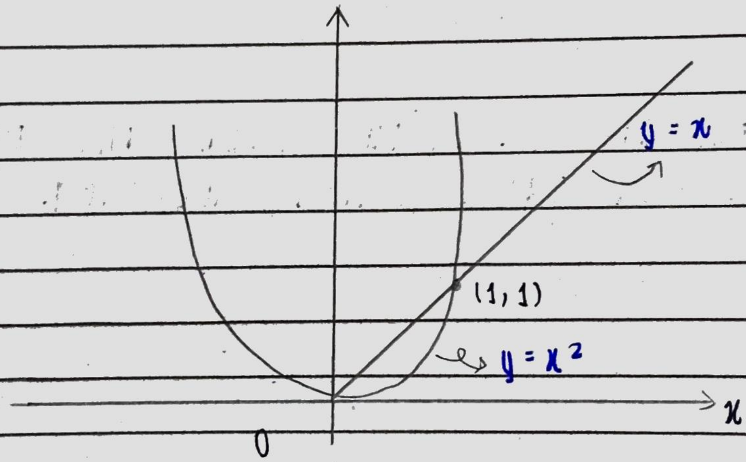
22nd Dec

1. Evaluate $\int_0^{1+i} (x^2+iy)\,dz$ along the path

ⓐ $y = x$

ⓑ $y = x^2$

Soln.



ⓐ $y = x$

$dy = dx$

Now,

$$dz = dx + i\,dy$$
$$= dx + i\,dx$$
$$= (1+i)\,dx$$

$\int_0^{1+i}$ → $x+iy$ huda $x = 1$

→ $x+iy$ huda $x=0$

$0 \leq x \leq 1$

So,

$$\int_0^{1+i} (x^2+iy)\,dz = (1+i)\int_0^1 (x^2+ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} + i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left(\frac{1}{3} + \frac{i}{2}\right)$$

$$= \frac{1}{3} + \frac{i}{2} + \frac{i}{3} - \frac{1}{2}$$

$$= -\frac{1}{6} + \frac{5}{6}i$$

//

· limit lida

$1 \le x \le 2$

$-1 \le y \le 1$] see limit ma coeff. acc. to $x+iy$

2. Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the path

(i) $x = t+1$, $y = 2t^2 - 1$

(ii) the st. line joining $(1-i)$ & $(2+i)$

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Soln

(i) $x = t+1$ $\quad y = 2t^2 - 1$

$dx = dt$ $\quad dy = 4t\,dt$

Then,

$dz = dx + i\,dy$

$= dt + i(4t\,dt)$

$= (1 + 4it)\,dt$

Then,

· for limit of x $(1 \le x \le 2)$

$x = 1,\ t = 0$

$x = 2,\ t = 1$] same aauna parxa

· limit of y $(-1 \le y \le 1)$

$y = -1,\ t = 0$

$y = 1,\ t = 1$

Thus,

limit of t , $0 \le t \le 1$

$$\int_{1-i}^{2+i} (2x + iy + 1)\, dz = \int_0^1 (2(t+1) + i(2t^2 - 1) + 1)(1 + 4it)\, dt$$

$$= \int_0^1 [2t + 2 + i2t^2 - i + 1][1 + 4it]\, dt$$

imp imp imp

3. Evaluate $\oint_C \log z\, dz$ where C is the circle $|z| = 1$

$|z| = r$

$z = re^{i\theta}$

soln

Given circle

$$|z| = 1$$

$$\therefore z = e^{i\theta}$$

$$dz = ie^{i\theta}\, d\theta$$

&

$0 \le \theta \le 2\pi$ → limit for circle

Now,

$$\log z = \log(e^{i\theta})$$

$$= i\theta$$

So,

$$\oint_C \log z\, dz = \int_0^{2\pi} i\theta i e^{i\theta}\, d\theta$$

$$= -\int_0^{2\pi} \theta e^{i\theta}\, d\theta$$

$$= -\left[\theta\left[\frac{e^{i\theta}}{i}\right] - \int 1\cdot\left[\frac{e^{i\theta}}{i}\right]\right]_0^{2\pi}$$

$$= -\left[\frac{\theta e^{i\theta}}{i} - \frac{e^{i\theta}}{i^2}\right]_0^{2\pi}$$

$$= -\left[\frac{\theta e^{i\theta}}{i} + e^{i\theta}\right]_0^{2\pi}$$

$$= -\left[e^{i\theta}\left(\frac{\theta}{i} + 1\right)\right]_0^{2\pi}$$

$$= -\left[e^{i2\pi}\left(\frac{2\pi}{i} + 1\right) - 1\right]$$

$$= 1 - e^{i2\pi}\left(\frac{2\pi}{i} + 1\right)$$

$$\cos 2\pi + i\sin 2\pi$$

$$= 1 - (\cos 2\pi + i\sin 2\pi)\left(\frac{2\pi}{i} + 1\right)$$

$$= 1 - (1+0)\left(\frac{2\pi}{i} + 1\right)$$

$$= 1 - \frac{2\pi}{i} - 1$$

$$= -\frac{2\pi}{i} \times \frac{i}{i}$$

$$= 2\pi i$$

most imp

asked

asked

theory

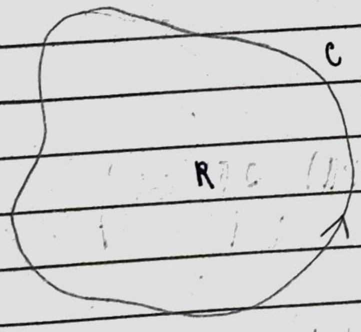
Cauchy's Integral Theorem

- **Statement :**

If $f(z)$ is an analytic f^n & $f'(z)$ is continuous at each point within & on a simple closed curve c then,

$$\oint_c f(z)\,dz = 0$$

- **Proof :**



Let 'R' be the region bounded by the simple closed curve 'c'.

We have,

$$f(z) = u + iv$$

&

$$dz = dx + idy$$

Now,

$$\oint_c f(z)\,dz = \oint_c (u+iv)(dx+idy)$$

$$= \oint_C (\overset{F_1}{u}\,dx - \overset{F_2}{v}\,dy) + i \oint_C (\overset{F_1}{v}\,dx + \overset{F_2}{u}\,dy) \quad \text{(*)}$$

since,

$f'(z)$ is continuous then,

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are also continuous in a region 'R'.

using Green's theorem in (*), we get:

$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{(**)}$$

Also,

since $f(z)$ is analytic at each point of a region 'R', then from C-R eqns in cartesian form,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Using these relations in (**),

$$\oint_C f(z)\,dz = \iint_R \left(+ \frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx\,dy$$

$$+ i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \right) dx\,dy$$

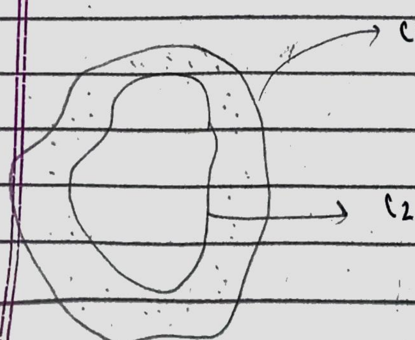
$$= 0$$

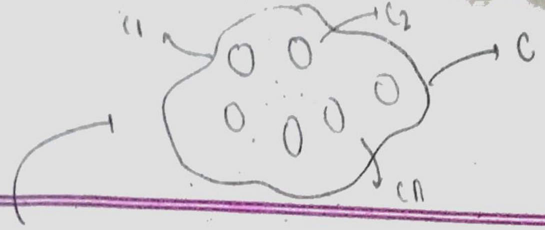
$$\therefore \oint_C f(z)\,dz = 0$$

Extension of Cauchy's Integral Theorem

If $f(z)$ is analytic f^n in the region bet^n closed curves C_1 & C_2 then,

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$





Note :

If the closed curves $C_1, C_2, \cdots, C_n$ lies within C (closed curve)

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots \oint_{C_n} f(z)\,dz$$

imp asked

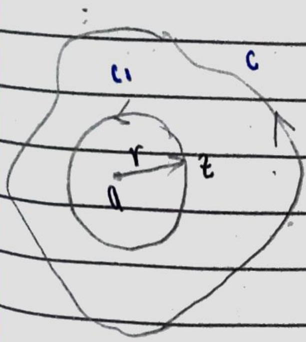
Cauchy's Integral Formula

· Statement :

If $f(z)$ is analytic within & on a closed curve 'C' & 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz$$

· Proof :



consider a f^n $\boxed{\frac{f(z)}{z-a}}$, which is analytic at each points within C except at $z = a$

Draw a small circle C_1 within C w.

· center = a & · radius = r

Thus,

$\frac{f(z)}{z-a}$ is analytic in the region betⁿ C & C_1

By,

extension of Cauchy's Integral Theorem

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{— } (*)$$

Now,

eqⁿ of circle C_1 is

$|z-a| = r$

$\Rightarrow z-a = re^{i\theta}$

$\therefore dz = ire^{i\theta}\,d\theta$

&

$0 \le \theta \le 2\pi$

So,

from $(*)$

$$\oint_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta})\,d\theta$$

In the limiting form, as the circle c_1 shrinks to point 'a'

then $r \to 0$

so,

$$\oint_c \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a)\,d\theta$$

$$= i f(a) \int_0^{2\pi} d\theta$$

$$= i f(a) [\theta]_0^{2\pi}$$

$$= i f(a) * 2\pi$$

$$= 2\pi i f(a)$$

$$\therefore \oint_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz$$

Derivatives of Cauchy's Integral Formula

We have,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

Diff. B.S. w.r.t. 'a' by using Leibnitz's rule for diff. under integral sign.

Then,

$$\frac{d}{da}[f(a)] = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{(z-a)}\right] dz$$

or, $$f'(a) = \frac{1!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz$$

Diff B.S. w.r.t. 'a' by using Leibnitz's rule for diff under integral sign

$$\frac{d}{da}[f'(a)] = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{(z-a)^2}\right] dz$$

$$= \frac{1}{2\pi i} \oint_C \left[\frac{(z-a)^2 \cdot 0 - f(z)\, 2\cancel{(z-a)} \cdot (-1)}{(z-a)^{\cancel{4}\,3}}\right] dz$$

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3}\, dz$$

In general,

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$$\therefore \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^n(a)$$