15th Dec

Bilinear / Mobius / Linear Fnctional Transformation

A transformation of the form $w = \dfrac{az+b}{cz+d}$

where,

a, b, c & d are complex constants

& $ad - bc \neq 0$

is called bilinear transformation.

Cross Ratio

If z_1, z_2, z_3 & z_4 are any 4 points in z-plane then cross ratio is denoted by (z_1, z_2, z_3, z_4) & is defined by:

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

• **Note:**

If z_1, z_2, z_3 be any three points in z-plane

&

w_1, w_2, w_3 be any three points in w-plane resp.

Then,

the associated bilinear transformation can be found by using

$$\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)}$$

→ Theorem DIY

Fixed / Invariant / critical points

The point that coincides w. their transformation under the linear fractional transformation is called fixed points.

· To find fixed points

replacing

$\cdot\ w \xrightarrow{\text{by}} z$

1. Prove that the linear fractional transformation preserves cross ratio of 4 points

i.e.

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

Given,

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

$$\cdot\ w_i = \frac{az_i+d}{cz_i+d}$$

$w_1 = ?$ $\quad$ $w - w_1 = ?$

$w_2 = ?$ $\quad$ $w_2 - w_3 = ?$

$w_3 = ?$ $\quad$ $w_1 - w_2 = ?$

$w_3 - w = ?$

2. Find the bilinear transformation which ~~ma~~ maps the points $z = 1, i, -1$ into the points $w = i, -1, -i$

Soln

Let $z_1 = 1, \; z_2 = i, \; z_3 = -1$

$w_1 = i, \; w_2 = -1, \; w_3 = -i$

We have,

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\Rightarrow \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-i)(-1+i)}{(i+1)(-i-w)}$$

$$\Rightarrow \cancel{-}\left(\frac{z-1}{z+1}\right)\left(\frac{i+1}{1-i}\right) = \cancel{-}\left(\frac{w-i}{w+i}\right)\left(\frac{i-1}{i+1}\right)$$

$$\Rightarrow \left(\frac{z-1}{z+1}\right)\left(\frac{i+1}{1-i}\right)\left(\frac{1+i}{1+i}\right) = \left(\frac{w-i}{w+i}\right)\left(\frac{i-1}{i+1}\right)\left(\frac{i-1}{i-1}\right)$$

$$\Rightarrow \left(\frac{z-1}{z+1}\right)\left(\frac{2i}{2}\right) = \left(\frac{w-i}{w+i}\right)\left(\frac{-2i}{-2}\right)$$

$$\Rightarrow \frac{z-1}{z+1} = \frac{w-i}{w+i}$$

$$\Rightarrow wz - w + iz - i = wz + w - iz - i$$

$$\Rightarrow 2w = 2iz$$

$$w = iz$$

$$\boxed{\therefore w = iz}$$

as in z_3 common zero katne

finite banauna parxa

3 Find the bilinear transformation which maps the points $z = 0, 1, \infty$ into the points $w = -5, -1, 3$ & hence find the fixed points.

We know,

$z_1 = 0$ $\quad w_1 = -5$

$z_2 = 1$ $\quad w_2 = -1$

$z_3 = \infty$ $\quad w_3 = 3$

Then,

$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$= \frac{-\cancel{z_3}(z-z_1)\left(1-\frac{z_2}{z_3}\right)}{\cancel{z_3}(z_1-z_2)\left(1-\frac{z}{z_3}\right)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

or, $$\frac{-(z-0)\left(1-\frac{1}{\infty}\right)}{(0-1)\left(1-\frac{z}{\infty}\right)} = \frac{(w+5)(-1-3)}{(-5+1)(3-w)}$$

or, $$z = \frac{w+5}{3-w}$$

or, $w+5 = 3z - wz$

or, $(w + wz) = 3z - 5$

$w(1+z) = 3z - 5$

$\Rightarrow w = \dfrac{3z-5}{1+z}$

- To find fixed points

$\hookrightarrow$ replacing $w \xrightarrow{by} z$

Now,

$$z = \frac{3z-5}{1+z}$$

or, $z(1+z) = 3z - 5$

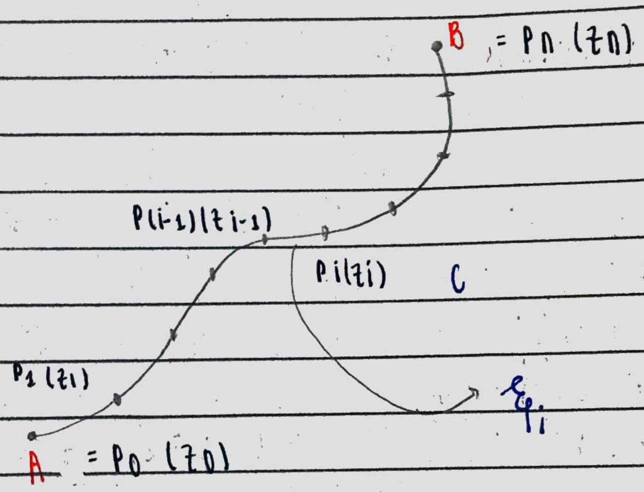
or, $z^2 + z - 3z + 5 = 0$

or, $z^2 - 2z + 5 = 0$ $\therefore z = 1 \pm 2i$

$\therefore$ Hence, $z = 1+2i$ & $z = 1-2i$ are fixed points

Complex Integration

Consider a continuous fn $f(z)$ of a complex variable $z = x+iy$ defined at all points of a curve C having points A & end B.



Divide the curve 'C' into 'n' equal parts with points $P_0(z_0)$, $P_1(z_1)$ .... $P_{i-1}(z_{i-1})$, $P_i(z_i)$, ... $P_n(z_n)$

let,

$\delta z_i = z_i - z_{i-1}$

& ξ_i be any point on the arc $P_{i-1}P_i$

The sum of limit $\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$

as $\cdot\ n \to \infty$

&

$\cdot$ each $\delta z_i \to 0$ if it exist then the line integral of $f(z)$ is denoted by :

$$\int_C f(z)\, dz$$

along the curve C.

If P_0 & P_n are coincide then the open curve reduce closed curve & it is denoted by :

$$\oint_C f(z)\, dz$$

- **Note :**

If $f(z) = u(x,y) + i v(x,y)$

&

$dz = dx + i dy.$

Then,

$$\int_C f(z) dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$