8th Dec

Harmonic Function

A real valued fⁿ f is said to be harmonic if it satisfies the Laplace Eqⁿ

i.e. $$\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = 0$$

Harmonic conjugate fⁿ

If $f(z) = u(x, y) + iv(x, y)$ is analytic fⁿ then,

$$\left.\begin{array}{l} \cdot\ u(x, y) \\ \&\ \cdot\ v(x, y) \end{array}\right\}$$ are harmonic conjugate to each other

1 Show that the fⁿ $u = 2x + x^3 - 3xy^2$ is harmonic & determine its harmonic conjugate.

Given, ie $v(x,y)$ nikalne

$$u(x, y) = 2x + x^3 - 3xy^2$$

$$\therefore \frac{\partial u}{\partial x} = 2 + 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

similarly,

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x$$

$$= 0$$

Thus, $u(x,y)$ is a harmonic f^n

- To find its harmonic conjugate

since the f^n $f(z) = u + iv$ is analytic

we have,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Then, from

~~for~~ CR eq^n,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= 2 + 3x^2 - 3y^2 - i6xy$$

Now,

Using Milne's Thompson method

&

we replace

- $x \xrightarrow{\text{by}} z$
- $y \xrightarrow{\text{by}} 0$

$f'(z) = 2 + 3z^2$

Now, integrating on both sides w.r.t

$$f(z) = \int 2\,dz + 3\int z^2\,dz + c$$

$$= 2z + 3\frac{z^3}{3} + c$$

$$= 2z + z^3 + c$$

Now, yeslai $x+iy$ haine

$$u(x,y) + iv(x,y) = 2(x+iy) + (x+iy)^3 + c$$

$$= 2x + 2iy + (x^3 - 3x^2 iy + 3x(iy)^2 - (iy)^3]$$

→ Fig ni banaune sujhyo vane

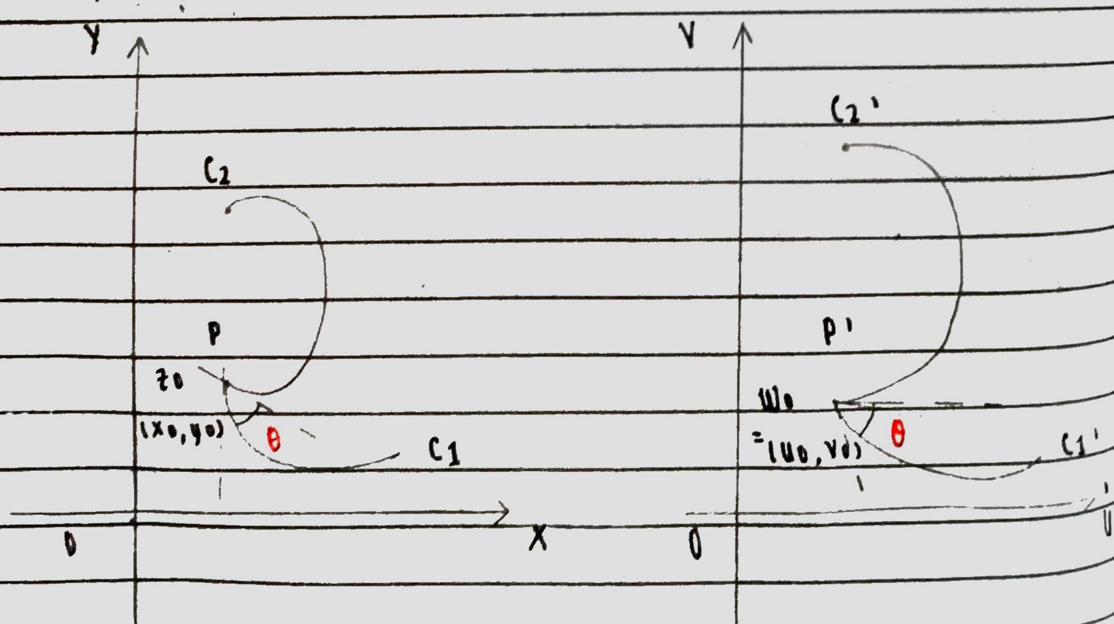
Conformal mapping / conformal transformation

A transformation / mapping $w = f(z)$ is said to be conformal transformation if :

it preserves angle betn oriented curve in

· magnitude

as well as in · sense (dirn)



z - plane

w - plane

· **Note:** If the angle is same in mag. only then,
the transformation $w = f(z)$ is isogonal

Some std. Transformation

(1) Translation

$$w = z + c$$

c = complex constant

let,

$w = u + iv$

$z = x + iy$

$c = c_1 + ic_2$

Then,

$$u + iv = (x + iy) + (c_1 + ic_2)$$
$$= (x + c_1) + i(y + c_2)$$

$$P(x, y) \xrightarrow{w = z + c} P'(x + c_1, y + c_2)$$

② Magnification & Rotation

$w = cz$

$c =$ complex const.

let,

$w = Re^{i\phi}$

$z = re^{i\theta}$ — angle ayera polar ma vayo

$c = \rho e^{i\alpha}$

Then,

$$Re^{i\phi} = \rho e^{i\alpha} \cdot re^{i\theta}$$
$$= \rho r e^{i(\alpha+\theta)}$$

we get,

$R = \rho r$ & $\phi = \theta + \alpha$

$w = cz$

$P(r, \theta) \quad P'(\rho r, \theta + \alpha)$

③ Inversion & Reflection

$$w = \frac{1}{z}$$

let,

$$w = Re^{i\phi}$$
$$z = re^{i\theta}$$

then,

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

we get,

$$R = \frac{1}{r} \quad \& \quad \phi = -\theta$$

$$P(r, \theta) \xrightarrow{w = 1/z} P'\left(\frac{1}{r}, -\theta\right)$$

cartesian ma dexa

so, no polar

1. Find the image of infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$

under the transformation $w = \frac{1}{z}$.

Show the region graphically.

sodhesi gora

Given,

$$w = \frac{1}{z}$$

$$\text{or, } z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$= \frac{1}{u+iv}\left(\frac{u-iv}{u-iv}\right)$$

$$= \frac{u-iv}{u^2+v^2}$$

$$\Rightarrow \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

Equating real & imag part, we get

$$x = \frac{u}{u^2+v^2} \quad \& \quad y = \frac{-v}{u^2+v^2}$$

When $y = \frac{1}{4}$

$$\frac{1}{4} = \frac{-v}{u^2+v^2}$$

or, $u^2+v^2+4v = 0$

or, $(u-0)^2 + (v+2)^2 = 2^2$

This is the image of circle in w-plane w. center at $(0,-2)$ & radius 2.

When $y = \frac{1}{2}$

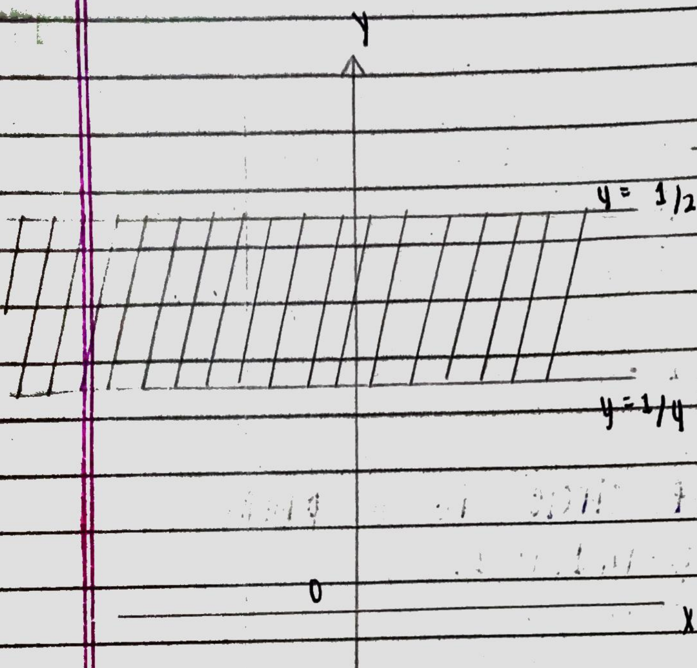
$$\frac{1}{2} = \frac{-v}{u^2+v^2}$$

$\Rightarrow u^2+v^2+2v = 0.$

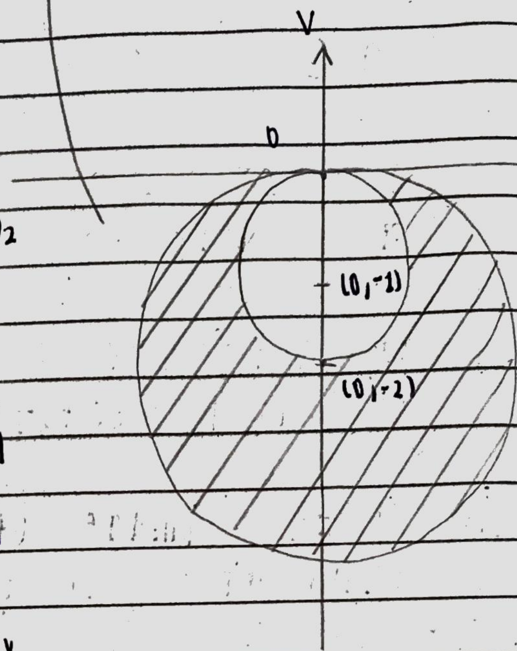
$\Rightarrow (u-0)^2 + (v+1)^2 = 1$

This is the image of circle in w-plane w. center at $(0,-1)$ & radius 1.

compass chalau



z- plane



w-plane

Hence, the ∞ strip $\frac{1}{4} \leq y \leq \frac{1}{2}$ in z-plane transform the region bounded by 2 circles in w-plane

2. Show that the image of hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscate $\rho^2 = \cos 2\phi$, where $w = \rho e^{i\phi}$

Soln

Given,

$$W = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\text{or, } x + iy = \frac{1}{\rho e^{i\phi}}$$

$$= \frac{1}{\rho} e^{-i\phi}$$

$(\cos\phi - i\sin\phi)$

$$= \frac{1}{\rho}(\cos\phi - i\sin\phi)$$

$$= \frac{1}{\rho}\cos\phi + i\left(-\frac{1}{\rho}\sin\phi\right)$$

Equating real & imag parts, we get

$$x = \frac{1}{\rho}\cos\phi$$

$$y = -\frac{1}{\rho}\sin\phi$$

Given hyperbola:

$$x^2 - y^2 = 1$$

$$\Rightarrow \frac{1}{\rho^2}\cos^2\phi - \frac{1}{\rho^2}\sin^2\phi = 1$$

$$\Rightarrow \frac{1}{\rho^2}(\underbrace{\cos^2\phi - \sin^2\phi}_{\cos 2\phi}) = 1$$

$$\Rightarrow \frac{1}{\rho^2}\cos 2\phi = 1$$

$$\Rightarrow \cos 2\phi = \rho^2$$

$$\therefore \cos 2\phi = \rho^2$$

5. Discuss the transformation $w = z^2$

Given,

$$w = z^2$$

$$\therefore \frac{dw}{dz} = 2z, \quad z \neq 0$$

Show transformation $w = z^2$ is conformal

Now,

$$w = z^2$$

$$\Rightarrow u + iv = (x + iy)^2$$

$$= (x^2 - y^2) + i(2xy)$$

Equating real & imag parts

- $u = x^2 - y^2$
- & $v = 2xy$

Eliminating x & y seperately from above 2 relations.

$u = x^2 - \frac{v^2}{4x^2}$ [from 2nd relⁿ]

or, $4x^2 u = 4x^4 - v^2$

or, $v^2 = 4x^4 - 4x^2 u$

or, $v^2 = -4x^2 (u - x^2)$

$u = \frac{v^2}{4y^2} - y^2$

or, $4y^2 u = v^2 - 4x^2 y^2$

or, $v^2 = 4y^2 u + 4x^2 y^2$

or, $v^2 = 4y^2 (u + x^2)$

when $x = a$

$v^2 = -4a^2 (u - a^2)$ ↳ (*)

when $x = b > a$

$v^2 = -4b^2 (u - b^2)$ ↳ (* *)

such is enough

is most

when $y = c$

$v^2 = 4c^2 (u + c^2)$ ↳ (* * *)

when $y = d > c$

$v^2 = 4d^2 (u + d^2)$ ↳ (* * * *)

- **Note:** $(y-k)^2 = 4a(x-h)$ → std. eqⁿ of parabola
- vertex = (h, k)
- focus = $(h+a, k)$

For parabola given by eq^n (*)

- Focus

$= a^2 + (-a^2), 0$

$= (0, 0)$

- vertex

$= (a^2, 0)$

For eq^n (* *)

- Focus

$= b^2 + (-b^2), 0$

$= (0, 0)$

- vertex

$= (b^2, 0)$

For eq^n (* * *)

- Focus

$= -c^2 + c^2, 0$

$= (0, 0)$

- vertex

$= (-c^2, 0)$

For eq^n (* * * *)

- Focus

$= -d^2 + d^2, 0$

$= (0, 0)$

$a \le x \le b$

$c \le y \le d$

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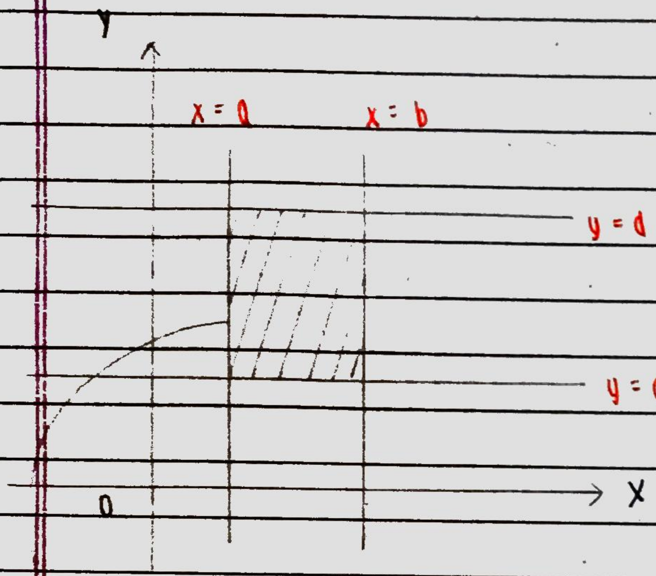
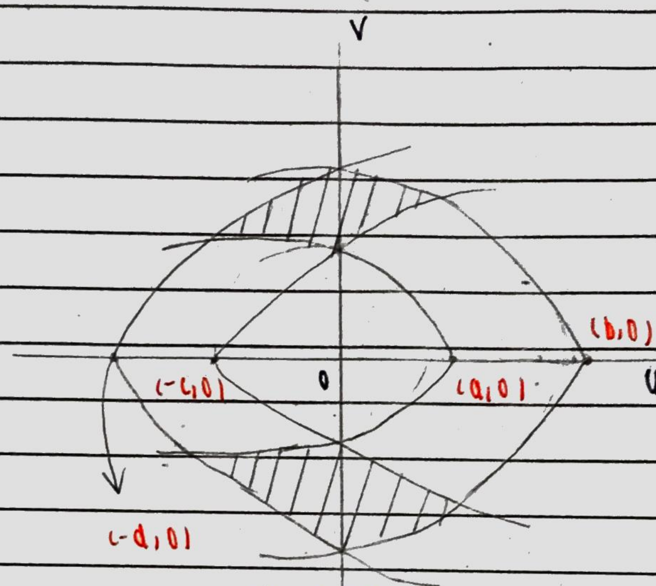
· vertex

$= (-d^2, 0)$





written as:

$a \le x \le b$

& $c \le y \le d$

Hence the rectangular region $a \le x \le b$ & $c \le y \le d$ in z-plane transform the region bounded by 4 parabolas