19th Nov

F^ns of complex variable

If x & y are real no, then $z = x + iy$ is called complex variable.

If corresponding to each value of complex variable $z(x+iy)$ in a region R, there corresponds one or more value of another complex variable $w (= u + iv)$.

Then, w is called f^n of complex variable z.

& is denoted & defined by

$$w = f(z) = u(x, y) + i v(x, y)$$

where,

u = real part of w

v = imaginary part of w

Eg:

$$f(z) = z^2$$

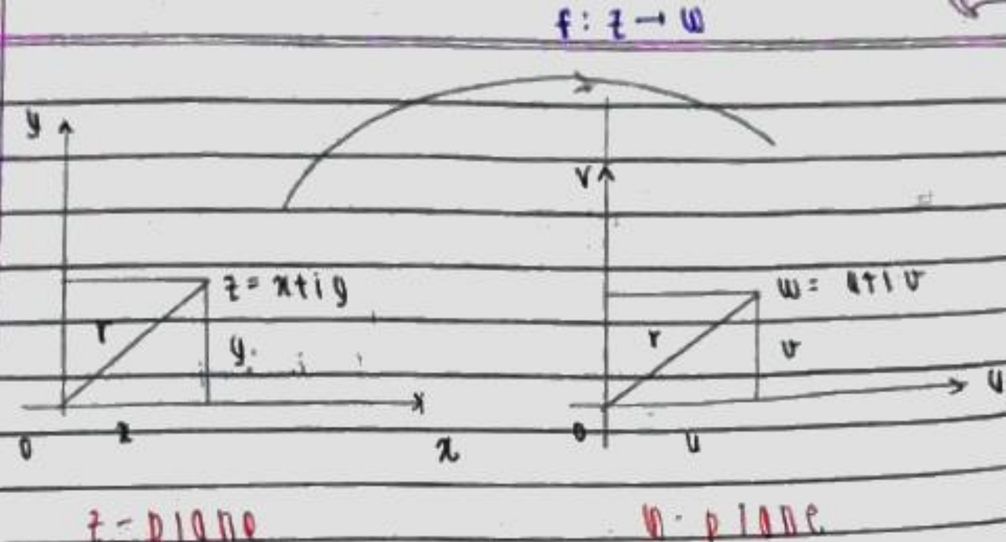
$$\Rightarrow u(x, y) + i v(x, y) = (x + iy)^2$$

$$= x^2 + i 2xy - y^2$$

$$= (x^2 + y^2) + i(2xy)$$

Real part (u) is (x^2-y^2)

Imag part (v) is $2xy$



Analytic f^n / Regular f^n / Holomorphic f^n

If a single valued f^n $f(z)$ possess a unique derivative at each point of a region R, then $f(z)$ is called analytic / regular / holomorphic f^n of z in R

Eg:

$$f(z) = \frac{z}{z-1}, \quad z \neq 1$$

is analytic f^n since

$$f'(z) = \frac{(z-1)\cdot 1 - z(1)}{(z-1)^2} = \frac{-1}{(z-1)^2}, \quad z \neq 1$$

Necessary & sufficient condⁿ for a function f(z) to be analytic

The necessary & sufficient condⁿ for the fⁿ $f(z) = u(x,y) + i v(x,y)$ in the region R, are:

(i) 1st order partial derivatives $\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)$ are continuous fⁿ of x & y in region R

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

• **Note**: The relⁿ

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

is known as Cauchy Riemann (C.R.) eqⁿ in cartesian form

Cauchy Riemann (C.R.) eq^n in cartesian form

(Necessary cond^n)

Statement: If the f^n $f(z) = u(x,y) + iv(x,y)$ is differential at $z = x+iy$ then, 1st order partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are exist &

satisfy $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Proof:

Since $f(z) = u(x,y) + iv(x,y)$ is differential at point $z = x+iy$.

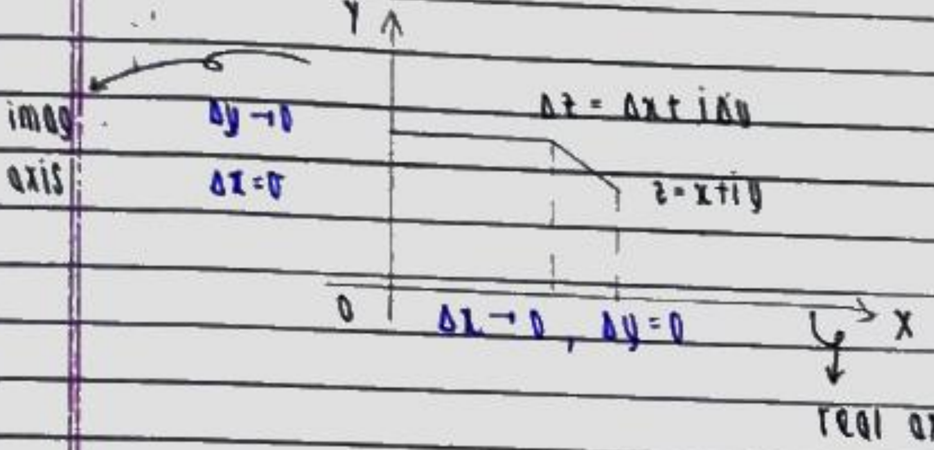
Then,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided the limit exists

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y)] - [u(x,y) + i v(x,y)]}{\Delta x + i \Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i \Delta y}$$



As,

$\Delta z \to 0$ along real axis then,

$\cdot \ \Delta x \to 0$

$\& \cdot \ \Delta y = 0$

So,

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)] + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \underbrace{\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}}_{\frac{\partial u}{\partial x}} + i \underbrace{\lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}}_{\frac{\partial v}{\partial x}}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (*)$$

Also, as $\Delta z \to 0$ along imaginary axis then,

$\cdot \Delta y \to 0$

&

$\cdot \Delta x = 0$

So,

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

yo denumerator ma hunna

so, $\frac{1}{i} \times \frac{i}{i}$ garne

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \to (**)$$

Clearly, $f'(z)$ exist & hence equating real & imaginary part from $(*)$ & $(**)$,

we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

&

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C.R. eq^n in polar form

Derivation: Let (r,θ) be the polar co-ordinates of the point whose cartesian co-ordinates are (x,y) then

$x = r\cos\theta$ & $y = r\sin\theta$

Now,

$$z = x+iy$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta}$$

Then,

$$w = f(z)$$

$$\Rightarrow u+iv = f(re^{i\theta}) \quad \text{--- (i)}$$

Diff. (i) partially w.r.t. 'r', we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})e^{i\theta}$$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \to (*)$

Also,

Diff. (i) partially w.r.t. 'θ' we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot r \cdot e^{i\theta} \cdot i$$

$\hookrightarrow (**)$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{(from (*))}$$

$$\text{or, } \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{i}{r}\frac{\partial v}{\partial \theta} = i\frac{\partial u}{\partial r} - \frac{\partial v}{\partial r}$$

Equating real & imag part, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

which is the C.R. eq^n in polar form.

- Formulas

(i) $\cosh z = \dfrac{e^z + e^{-z}}{2}$

(ii) $\sinh z = \dfrac{e^z - e^{-z}}{2}$

(iii) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

(iv) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

(v) $\cos iz = \dfrac{e^{-z}+e^{z}}{2}$

$= \dfrac{e^{z}+e^{-z}}{2}$

$= \cosh z$

$\cos iz = \cosh z$

(vi) $\sin iz = \dfrac{e^{-z}-e^{z}}{2i}$

$= -\left(\dfrac{e^{z}-e^{-z}}{2i}\right) \times \dfrac{i}{i}$

$= i\left(\dfrac{e^{z}-e^{-z}}{2}\right)$

$= i\sinh z$

$\sin iz = i\sinh z$