Assignment 1

Geometric Motion Planning

Name

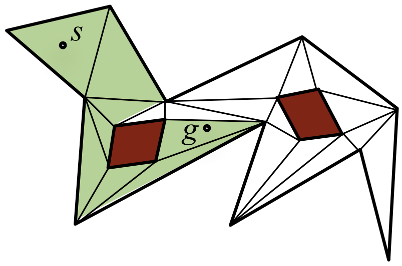
Matriculation Number

Instructions

- Due date: **14 Feb 2023** in class.
- Type up your solutions or write **neatly**.
- Submit your work in hard copy. Make sure to include the cover page.
- The starred questions provide extra-credits.

Problem	Max Points	Points
1	10	
2	10	
3	10	
4	15	
5	15	
6	15*	
7	25	
8	10	
9	15	
10	10	
11	10 + 5* (proof)	
Total	**130+20***	

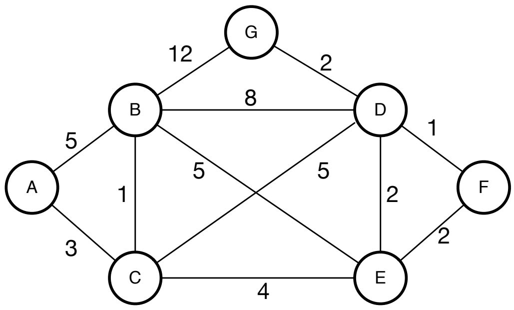
1. For a point robot in a 2-D polygonal environment with obstacles, its free space is decomposed into and represented as a set of triangles. A *channel* is a sequence of adjacent triangles that contains a collision-free path connecting a start position s and a goal position g.



   (a) Define a search graph G over the triangles so that we can use G to find a channel σ that connects s and g. Describe what the nodes and edges of G represent.

   (b) Given a channel σ that connects s and g, describe a method to generate a collision-free path from s to g. Draw an example to illustrate your solution.

2. Find the shortest path to node A from every other node in the weighted graph below.



   (a) Apply the backward dynamic programming algorithm. Let $V^*(s)$ be the shortest-path length from node s to A and $V_i(s)$ be the estimated shortest-path length in the i'th iteration of the dynamic programming algorithm. Show the values for $V_0(s)$, $V_1(s)$, and $V_2(s)$ as well as $V^*(s)$ for all nodes in the graph.

	A	B	C	D	E	F	G
V_0							
V_1							
V_2							
V^*							

   (b) Apply the Dijkstra's algorithm. The shortest paths form a tree. Draw the shortest-path tree.

3. Let M be a 3×3 orthonormal matrices with determinant $+1$. We have discussed in the class that every such matrix corresponds to a rotation in 3-D space, and vice versa. This implies that M has only 3 *independent* degrees of freedom (DOFs); however, M contains 9 parameters.

   (a) What are the constraints on the 9 parameters that reduce the number of DOFs of M to 3?

   (b) M is required to have determinant $+1$. Does this constraint reduce the number of DOFs of M? Why or why not?

4. Give the dimension of the configuration space for the following systems. Briefly justify your answer.

   (a) An articulated robot in a 2D plane with a fixed base and two revolute joints.

   (b) Two mobile robots freely translate and rotate in the plane.

   (c) An aerial manipulator consisting of two manipulators attached to an unmanned aerial vehicle (UAV). Each manipulator has 6 revolute joints.

5. This problem explores the configuration space of lines and that of line segments.
   (*a*) Consider an infinite line ℓ that translates and rotates freely in 3-D space. Give two different parameterizations of the configuration space C for ℓ: one that makes use of angles and one that makes no use of angles.
   (*b*) What is the dimension of C?
   (*c*) Consider a straight-line segment s that translates and rotates freely in 3-D space. What is the dimension of the configuration space for s? Can you use the two parameterizations in part (*a*) for s? What modifications would be needed if any?

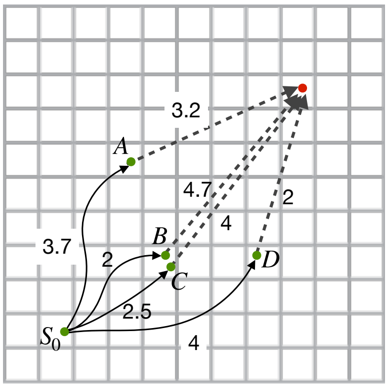
6. This problem examines the relationship between distance in the configuration space and distance in the workspace. Specifically, if a robot moves by a certain amount in the configuration space, how much can a point on the robot move in the workspace? Consider a planar robot arm with n sequential links. Each link is a straight-line segment of length L. One endpoint of the link is called the *origin*, and the other is called the *extremity*. The origin of the first link is fixed. The origin of the ith link ($2 \le i \le n$) coincides with the extremity of the $(i-1)$th link at a point called a *joint*. A link can rotate freely about the joint.
   (*a*) A configuration q of this robot can be represented by the joint angles $(\theta_1, \theta_2, \ldots, \theta_n)$. The metric d_c in the robot's configuration space is defined as
   $$d_c(q, q') = \max_{1 \le i \le n} |\theta_i - \theta'_i|$$
   for two configurations q and q'. Suppose that the robot moves from a configuration $q = (\theta_1, \theta_2, \ldots, \theta_n)$ to a configuration $q' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ along the straight-line segment joining q and q' in the Cartesian space R^n. In other words, the robot moves along the path $(1-\lambda)q + \lambda q'$ for $0 \le \lambda \le 1$. Show that no point on the robot traces a path longer than $\alpha d_c(q, q')$ for some positive constant α. Give a bound of α in terms of L, the link length, and n, the number of links.
   (*b*) Let $d_w(q, B)$ denote the minimum distance between the robot placed at a configuration q and a (workspace) obstacle B, *i.e.*, the distance between the closest pair of points on the robot placed at q and B. Using the result from part (*a*), calculate the radius ρ of the neighborhood
   $$N(q) = \{q' \mid d_c(q, q') \le \rho\}$$
   in which the robot is guaranteed to move freely without colliding with B. Express your answer in terms of α and $d_w(q, B)$.

7. Describe how to sample points uniformly from each set below. For part (*a*) and (*b*), use polar-coordinates parametrization.
   (*a*) A circle with center c and radius r: $S = \{p \in \mathrm{R}^2 \mid \|p - c\| = r\}$.
   (*b*) A disc B_2 with center c and radius r: $B_2 = \{p \in \mathrm{R}^2 \mid \|p - c\| \le r\}$.
   (*c*) Alternatively, sample the disc B_2 using the rejection method. Sample uniformly from the bounding box of B containing B_2 and reject the samples outside of the disc. Which method would you choose, polar-coordinates parametrization or rejection sampling? Why?
   (*d*) An N-dimensional sphere with center c and radius r: $B_N = \{p \in \mathrm{R}^N \mid \|p - c\| \le r\}$. If using rejection sampling, discuss the rejection rate and how it scales with the dimension of the space, N.
   (*e*) Can you think of a way to sample B_N without rejection for arbitrary large N?

8. Suppose that the configuration space $\mathcal{C}$ is the unit square $[0,1] \times [0,1]$. The multi-query PRM algorithm, first samples n collision-free configurations and then tries to connect these milestones by calling LINK.
   (*a*) If the algorithm calls LINK for every pair of roadmap nodes, give an asymptotic upper bound on the number of calls to LINK.
   (*b*) Suppose that the algorithms calls LINK only if the Euclidean distance between two milestones is smaller than a threshold t. Give an asymptotic bound on the number of calls to LINK when $t = O(1/\sqrt{n})$. You may assume that the milestones are distributed roughly uniformly in $\mathcal{C}$.

9. Among the motion planning algorithms discussed in the class (dynamic programming, A*, PRM, EST, and RRT), choose the best algorithm for the robot motion planning tasks below. Justify your choice by considering the configuration space dimensions, environment characteristics, computational efficiency, ...
   (*a*) A robot car trying to park itself in an open parking slot.
   (*b*) Because of the COVID-19 pandemic, a cleaning robot must move around and disinfect the busy areas of the hospital every 2 hours.
   (*c*) A self-reconfigurable modular robot consists of many identical modules and can reconfigure its shape to fit a task. See the video for an example.

10. Consider the hybrid A* algorithm described in the class and apply it to plan the motion of an autonomous robot car. Suppose that the A* search starts at the node S, shown in the figure below. By applying four candidate actions, it reaches new nodes: A, B, C, and D. The cost-to-come and the heuristic estimate of the cost-to-go for all the new nodes are shown in the figure.



   (*a*) What is the dimension of the grid used in the hybrid A* search?
   (*b*) What does the priority queue contain at the stage of the hybrid A* search illustrated in the figure? For each item, in the priority queue, specify the node and its associated f-value.

11. A key step in applying the A* algorithm in practice is to design a good heuristic function. Suppose that we want to apply A* to a shortest-path problem and have two heuristic functions $h_1(x)$ and $h_2(x)$, both of which are admissible.
   (*a*) Show that $h(x) = \max\{h_1(x), h_2(x)\}$ is also admissible.
   (*b*) Of the three heuristic functions, $h_1(x), h_2(x)$, and $h(x)$, which one would you use? Why? Give a proof if you can.