School of Computing
National University of Singapore
CS5340: Uncertainty Modeling in AI
Semester 1, AY 2022/23

Exercise 2

Question 1

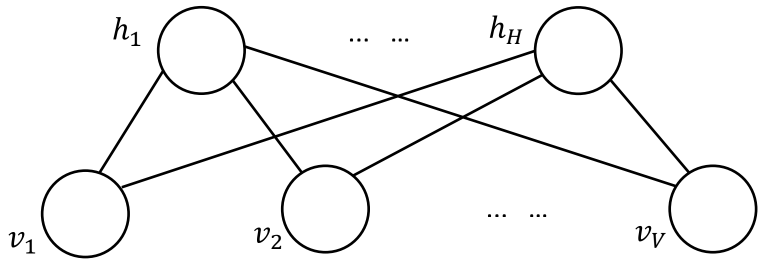


Fig. 1.1

The restricted Boltzmann machine is a Markov Random Field (MRF) defined on a bipartite graph as shown in Fig. 3.1. It consists of a layer of visible variables $\boldsymbol{v} = [v_1, \dots, v_V]^T$ and hidden variables $\boldsymbol{h} = [h_1, \dots, h_H]^T$, where all variables are binary taking states $\{0,1\}$. The joint distribution of the MRF is given by:

$$p(\boldsymbol{v}, \boldsymbol{h}) = \frac{1}{Z(\boldsymbol{W}, \boldsymbol{a}, \boldsymbol{b})} \exp(\boldsymbol{v}^T \boldsymbol{W} \boldsymbol{h} + \boldsymbol{a}^T \boldsymbol{v} + \boldsymbol{b}^T \boldsymbol{h}),$$

where $\theta = \{\boldsymbol{W}_{V\times H}, \boldsymbol{a}_{V\times 1}, \boldsymbol{b}_{H\times 1}\}$ are the parameters of the potential functions, and $Z(.)$ is the partition function.

a) Given that:

$$p(h_i = 1 \mid \boldsymbol{v}) = \sigma\big(b_i + \textstyle\sum_j W_{ji} v_j\big),$$

where $\sigma(x) = \frac{e^x}{1+e^x}$ is the sigmoid activation function. Show that the distribution of hidden units conditioned on the visible units factorizes as:

$$p(\boldsymbol{h} \mid \boldsymbol{v}) = \textstyle\prod_i p(h_i \mid \boldsymbol{v}).$$

Show all your workings clearly.

Answer:

Using product rule, we have:

$$
\begin{aligned}
p(\boldsymbol{h}|\boldsymbol{v}) &= \frac{p(\boldsymbol{h}, \boldsymbol{v})}{\sum_{\boldsymbol{h}} p(\boldsymbol{h}, \boldsymbol{v})} \\
&= \frac{\frac{1}{Z} exp\{\boldsymbol{v}^T \boldsymbol{W} \boldsymbol{h} + \boldsymbol{a}^T \boldsymbol{v} + \boldsymbol{b}^T \boldsymbol{h}\}}{\frac{1}{Z} \sum_{\boldsymbol{h}} exp\{\boldsymbol{v}^T \boldsymbol{W} \boldsymbol{h} + \boldsymbol{a}^T \boldsymbol{v} + \boldsymbol{b}^T \boldsymbol{h}\}} \\
&= \frac{exp\{(\boldsymbol{v}^T \boldsymbol{W} + \boldsymbol{h}^T)\boldsymbol{h}\} exp\{\boldsymbol{a}^T \boldsymbol{v}\}}{\sum_{(\boldsymbol{h}} exp\{\boldsymbol{v}^T \boldsymbol{W} + \boldsymbol{h}^T)\boldsymbol{h}\} exp\{\boldsymbol{a}^T \boldsymbol{v}\}} \\
&= \frac{exp\{(\boldsymbol{v}^T \boldsymbol{W} + \boldsymbol{h}^T)\boldsymbol{h}\}}{\sum_{(\boldsymbol{h}} exp\{\boldsymbol{v}^T \boldsymbol{W} + \boldsymbol{h}^T)\boldsymbol{h}\}}
\end{aligned}
$$

Let $\boldsymbol{m}^T = \boldsymbol{v}^T \boldsymbol{W} + \boldsymbol{b}^T$ and since $\boldsymbol{h} = [h_1, h_2, ...h_H]^T$, we have:

$$
\begin{aligned}
p(\boldsymbol{h}|\boldsymbol{v}) &= \frac{exp\{[m_1, m_2 ... m_H][h_1, h_2 ... h_H]^T\}}{\sum_{\boldsymbol{h}} exp\{[m_1, m_2 ... m_H][h_1, h_2 ... h_H]^T\}} \\
&= \frac{exp\{m_1 h_1, m_2 h_2 ... m_H h_H\}}{\sum_{\boldsymbol{h}} exp\{m_1 h_1, m_2 h_2 ... m_H h_H\}} \\
&= \frac{exp(m_1 h_1) exp(m_2 h_2) ... exp(m_H h_H)}{\sum_{h_1} \sum_{h_2} ... \sum_{h_H} exp(m_1 h_1) exp(m_2 h_2) ... exp(m_H h_H)} \\
&= \frac{exp(m_1 h_1)}{\sum_{h_1} exp(m_1 h_1)} \frac{exp(m_2 h_2)}{\sum_{h_2} exp(m_2 h_2)} \cdots \frac{exp(m_H h_H)}{\sum_{h_H} exp(m_H h_H)} \\
&= \Pi_i \frac{exp(m_i h_i)}{\sum_{h_i} exp(m_i h_i)} \\
&= \Pi_i \frac{exp(m_i h_i)}{exp(m_i h_i = 0) + exp(m_i h_i = 1)} \\
&= \Pi_i \frac{exp(m_i h_i)}{1 + exp(m_i h_i = 1)} \\
&= \Pi_i p(h_i|\boldsymbol{v})
\end{aligned}
$$

b) Assuming that the restricted Boltzmann machine consists of only 2 visible and 1 hidden variables, and the joint distribution of the MRF is given by:

h	v_1	v_2	$\exp(\boldsymbol{v}^T \boldsymbol{W} h + \boldsymbol{a}^T \boldsymbol{v} + bh)$
0	0	0	1.00
0	0	1	2.13
0	1	0	4.65
0	1	1	9.90
1	0	0	3.65
1	0	1	8.66
1	1	0	4.22
1	1	1	10.01

Find the unknown parameters, i.e. $\theta = \{\boldsymbol{W}_{2\times 1}, \boldsymbol{a}_{2\times 1}, b\}$.

Answer:

$$exp\{[v_1\ v_2] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} h + [a_1\ a_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + bh\}$$

Case 1: $h = 0, v_1 = 0, v_2 = 1$

$$exp\{[0\ 1] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} 0 + [a_1\ a_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\ 0\} = 2.13$$

$$\Rightarrow exp(a_2) = 2.13 \Rightarrow a_2 = 0.756$$

Case 2: $h = 0, v_1 = 1, v_2 = 0$

$$exp\{[0\ 1] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} 0 + [a_1\ a_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\ 0\} = 4.65$$

$$\Rightarrow exp(a_1) = 4.65 \Rightarrow a_1 = 1.537$$

Case 3: $h = 1, v_1 = 0, v_2 = 0$

$$exp\{[0\ 0] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} 1 + [a_1\ a_2] \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b\ 1\} = 3.65$$

$$\Rightarrow exp(b) = 3.65 \Rightarrow b = 1.2947$$

Case 4: $h = 1, v_1 = 0, v_2 = 1$

$$exp\{[0\ 1] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} 1 + [a_1\ a_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\ 1\} = 8.66$$

$$\Rightarrow exp(w_2 + a_2 + b) = 8.66 \Rightarrow exp(w_2 + 0.756 + 1.2947) = 8.66$$

$$\Rightarrow w_2 + 2.0507 = 2.1587 \Rightarrow w_2 = 0.1080$$

Case 5: $h = 1, v_1 = 1, v_2 = 0$

$exp\{[1\ 0]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} 1 + [a_1\ a_2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\ 1\} = 4.22$

$\Rightarrow exp(w_1 + a_1 + b) = 4.22 \Rightarrow exp(w_1 + 1.537 + 1.2947) = 4.22$

$\Rightarrow w_1 + 2.8317 = 1.4398 \Rightarrow w_1 = -1.3919$

Verifications:

Case 1: $h = 0, v_1 = 0, v_2 = 0 \Rightarrow exp(0) = 1.00$

Case 2: $h = 0, v_1 = 1, v_2 = 1 \Rightarrow exp(a_1 + a_2) = exp(1.537 + 0.756) = 9.90$

Case 3: $h = 1, v_1 = 1, v_2 = 1$

$\Rightarrow exp(v_1 W_1 h + v_2 W_2 h + a_1 v_1 + a_2 v_2 + bh) = exp(w_1 + w_2 + a_1 + a_2 + b)$

$= exp(-1.3919 + 0.1080 + 1.537 + 0.756 + 1.2947) = 10.0122$

Question 2

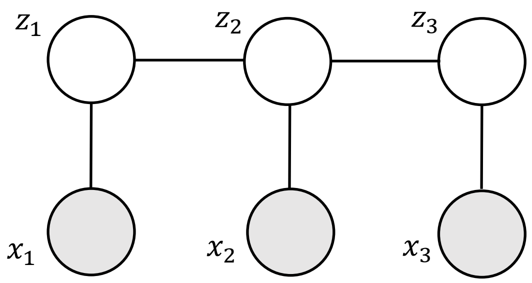


Fig. 2.1

Fig. 4.1 shows a Markov Random Field (MRF) representation of a Hidden Markov Model (HMM) over three time steps. The hidden variables z_1, z_2, z_3 are discrete random variables that take three possible states $z_n \in \{F, H, M\}$, and x_1, x_2, x_3 are the observed variables that take on real values $x_n \in \mathbb{R}$. The joint distribution is given by:

$$p(z_1, z_2, z_3, x_1, x_2, x_3) = \frac{1}{Z} \prod_{n=2}^{3} \psi_t(z_n, z_{n-1}) \prod_{n=1}^{3} \psi_e(x_n, z_n),$$

where Z is the partition function, and the transition potential $\psi_t(z_n, z_{n-1})$ and the emission potentials $\psi_e(x_n, z_n)$ are given by:

$\boldsymbol{\psi_t(z_n, z_{n-1})}$	$z_n = F$	$z_n = H$	$z_n = M$
$z_{n-1} = F$	2.0	3.0	5.0
$z_{n-1} = H$	1.0	6.0	3.0
$z_{n-1} = M$	4.5	2.0	2.5

z_1	$\psi_e(x_1, z_1)$
F	1.0
H	8.0
M	1.0

z_2	$\psi_e(x_2, z_2)$
F	7.0
H	1.0
M	2.0

z_3	$\psi_e(x_3, z_3)$
F	2.0
H	3.0
M	5.0

Decode the message that corresponds to the states of the hidden variables that give the maximal probability. Show all your workings clearly.

Answer:

The solution can be evaluated as:

$$\max_{z_1,z_2,z_3} \psi(z_3,x_3)\psi(z_2,z_3)\psi(z_2,x_2)\,\psi(z_1,z_2)\psi(z_1,x_1) =$$

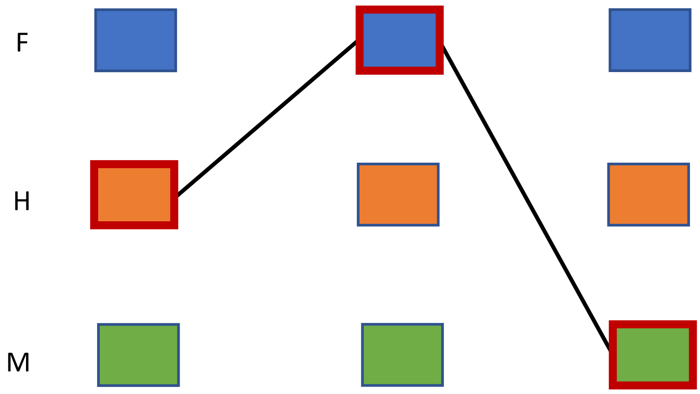
$$\max_{z_3}\psi(z_3,x_3)\max_{z_2}\psi(z_2,z_3)\psi(z_2,x_2)\max_{z_1}\psi(z_1,z_2)\psi(z_1,x_1)$$

$\boldsymbol{Z_2}$	$\max\limits_{z_1}\boldsymbol{\psi(z_1,z_2)\psi(z_1,x_1) = z_2^{max}(z_1)}$	$\boldsymbol{\delta^{max}(z_1)}$
F	$\max(2.0\times 1.0, 1.0\times 8.0, 4.5\times 1.0) = \max(2.0,8.0,4.5) = 8.0$	H
H	$\max(3.0\times 1.0, 6.0\times 8.0, 2.0\times 1.0) = \max(3.0,48.0,2.0) = 48.0$	H
M	$\max(5.0\times 1.0, 3.0\times 8.0, 2.5\times 1.0) = \max(5.0,24.0,2.5) = 24.0$	H

$\boldsymbol{Z_3}$	$\max\limits_{z_2}\boldsymbol{\psi(z_2,z_3)\psi(z_2,x_2)\,z_2^{max}(z_1) = z_3^{max}(z_2)}$	$\boldsymbol{\delta^{max}(z_2)}$
F	$\max(2.0\times 7.0\times 8.0, 1.0\times 1.0\times 48.0, 4.5\times 2.0\times 24.0)$ $= \max(112.0,48.0,216.0) = 216.0$	M
H	$\max(3.0\times 7.0\times 8.0, 6.0\times 1.0\times 48.0, 2.0\times 2.0\times 24.0)$ $= \max(168.0,288.0,96.0) = 288.0$	H
M	$\max(5.0\times 7.0\times 8.0, 3.0\times 1.0\times 48.0, 2.5\times 2.0\times 24.0)$ $= \max(280.0,144.0,120.0) = 280.0$	F

$\max\limits_{z_3}\boldsymbol{\psi(z_3,x_3)\,z_3^{max}(z_2)}$	$\boldsymbol{\delta^{max}(z_3)}$
$\max(216\times 2.0, 288.0\times 3.0, 280\times 5.0)$ $= \max(432.0,864.0,1400.0) = 1400.0$	M

Backtracking:



The code is: HFM

Question 3

Fig. 3.1 shows a Bayesian network of the mixture of Bernoulli Distribution. X_n is a binary random variable, i.e. $x_n \in \{0,1\}$. N is the total number of observations. Z_n is the 1-of-k indicator random variable, $z_{nk} = 1 \Rightarrow z_{n,j\neq k} = 0$ indicates the assignment of the random variable x to the k^{th} Bernoulli density. $z_{nk} \in \{0,1\}$ and $\sum_k z_{nk} = 1$.

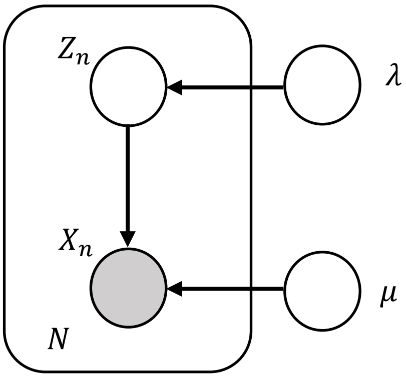


Fig. 3.1

Given the expressions for the Bernoulli distribution:

$$p(\mathrm{x} \mid \mu) = \prod_{n=1}^{N} \mu^{x_n}(1-\mu)^{(1-x_n)},$$

and marginal distribution of Z_n, which is a categorical distribution specified in terms of the mixing coefficients λ_k:

$$p(\mathrm{z_n}) = \prod_{k=1}^{K} \lambda_k^{z_{nk}} = \mathrm{cat}_{\mathrm{z_n}}[\lambda], \text{ where } 0 \leq \lambda_k \leq 1 \text{ and } \sum_k \lambda_k = 1.$$

(a) Show that the mixture of Bernoulli distribution is given by:

$$p(\mathrm{x} \mid \mu, \lambda) = \prod_{n=1}^{N} \sum_{k=1}^{K} \lambda_k \mu_k^{x_n}(1-\mu_k)^{(1-x_n)}.$$

(b) Derive the responsibility $\gamma(z_{nk}) = p(z_{nk} = 1 \mid \mathrm{x})$, and show that the updates for the unknown parameters μ and λ in the maximization step of the EM algorithm are given by:

$$\mu_k = \frac{1}{N_k} \sum_{n=1}^{N} \gamma(z_{nk}) \mathrm{x}_n,$$
$$\lambda_k = \frac{N_k}{N}, \text{ where } N_k = \sum_{n=1}^{N} \gamma(z_{nk}).$$

Show all your workings clearly.

Answer:

Refer to Section 9.3.3 in “Pattern Recognition and Machine Learning”, Christopher Bishop.

Question 4

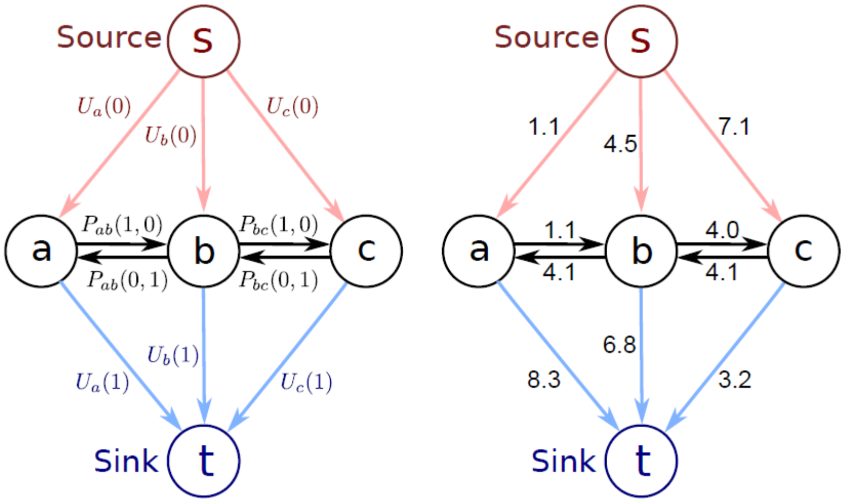


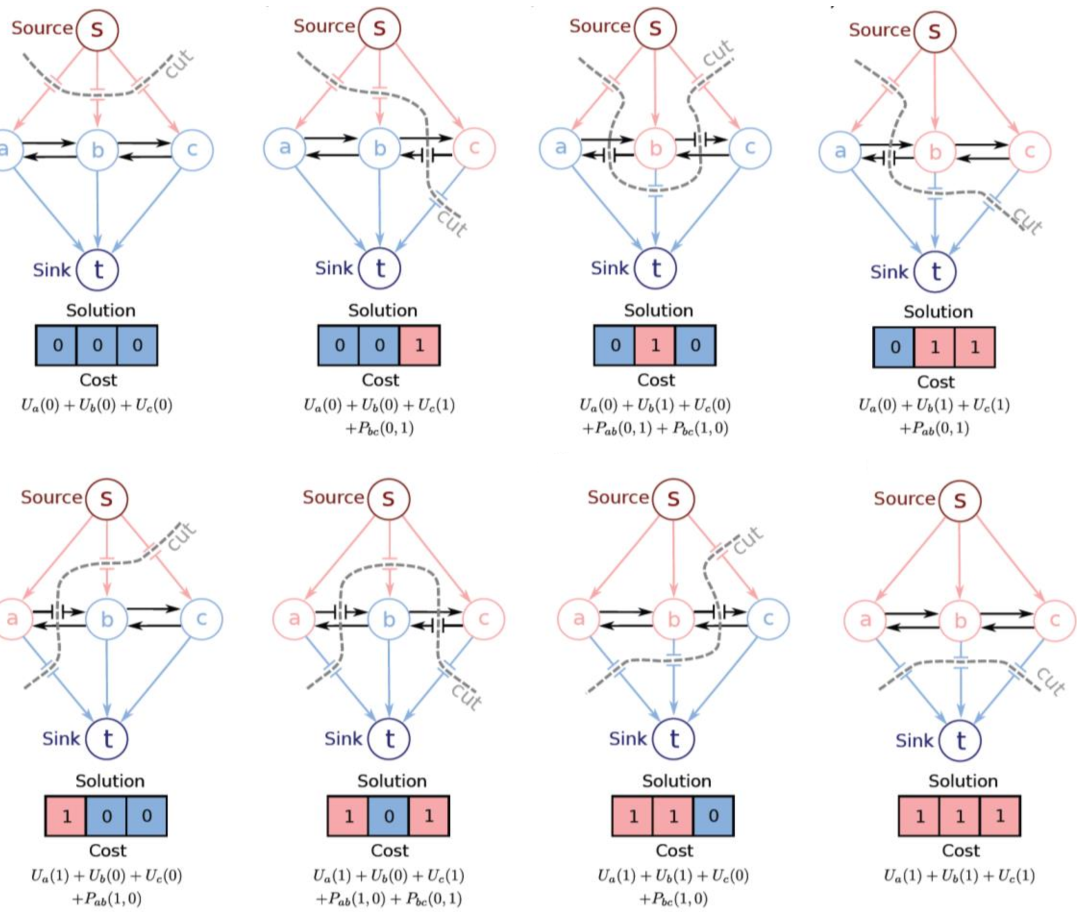
Fig 4.1

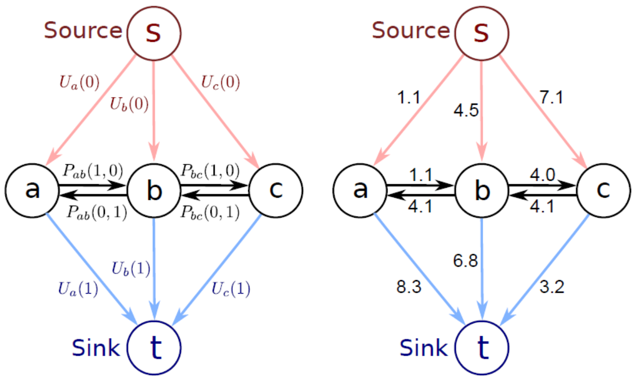
(Image source: "Computer Vision: Models, Learning and Inference", Simon Prince)

Compute the **MAP solution** to the three-pixel graph cut problem in Fig. 4.1 by

(i) computing the cost of all eight possible solutions explicitly and finding the one with the minimum cost, and

Answer:





$\boldsymbol{U_a(0) + U_b(0) + U_c(0) = 1.1 + 4.5 + 7.1 = 12.7}$

$U_a(0) + U_b(0) + U_c(1) + P_{bc}(0,1) = 1.1 + 4.5 + 3.2 + 4.1 = 12.9$

$U_a(0) + U_b(1) + U_c(0) + P_{ab}(0,1) + P_{bc}(1,0) = 1.1 + 6.8 + 7.1 + 4.1 + 4.0 = 23.1$

$U_a(0) + U_b(1) + U_c(1) + P_{ab}(0,1) = 1.1 + 6.8 + 3.2 + 4.1 = 15.2$

$U_a(1) + U_b(0) + U_c(0) + P_{ab}(1,0) = 8.3 + 4.5 + 7.1 + 1.1 = 21$

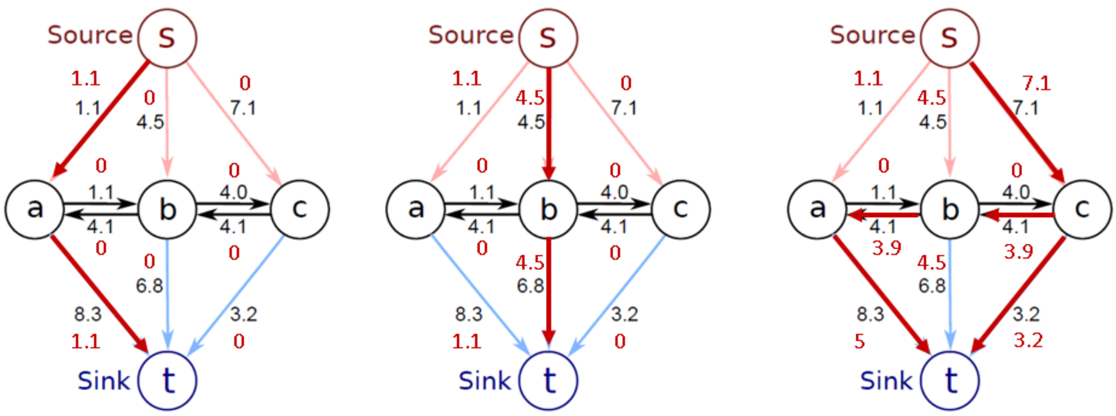
$U_a(1) + U_b(0) + U_c(1) + P_{ab}(1,0) + P_{bc}(0,1) = 8.3 + 4.5 + 3.2 + 1.1 + 4.1 = 21.2$

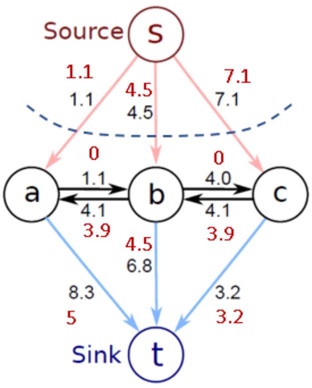
$U_a(1) + U_b(1) + U_c(0) + P_{bc}(1,0) = 8.3 + 6.8 + 7.1 + 4.0 = 26.2$

$U_a(1) + U_b(1) + U_c(1) = 8.3 + 6.8 + 3.2 = 18.3$

(ii) running the augmenting paths algorithm on this graph by hand and interpreting the minimum cut.

Answer:





<u>Question 5</u>

Consider the simple 3-node graph shown in Fig. 5.1 in which the observed node X is given by a Gaussian distribution $\mathcal{N}(x|\mu, \tau^{-1})$ with mean μ and precision τ. Suppose that the marginal distributions over the mean and precision are given by $\mathcal{N}(\mu|\mu_0, s_0)$ and $\mathrm{Gam}(\tau|a, b)$, where $\mathrm{Gam}(.|.,.)$ denotes a gamma distribution. Write down expressions for the conditions distributions for the conditional distributions $p(\mu|x, \tau)$ and $p(\tau|x, \mu)$ that would be required to apply Gibbs sampling to the posterior distribution $p(\mu, \tau \mid x)$.

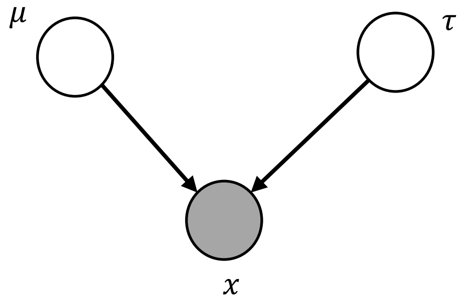


Fig. 5.1

Answer:

$$p(\mu|x,\tau) = \frac{p(\mu, x, \tau)}{\int p(\mu, x, \tau) d\mu} = \frac{p(\mu)p(\tau)p(x|\mu,\tau)}{\int p(\mu)p(\tau)p(x|\mu,\tau)d\mu} = \frac{p(\mu)p(x|\mu,\tau)}{\int p(\mu)p(x|\mu,\tau)d\mu}$$

$$p(x \mid \mu, \tau) = C_x \exp\{-0.5\tau(x-\mu)^2\}$$
$$p(\mu \mid \mu_0, s_0) = C_\mu \exp\{-0.5 s_0(\mu_0 - \mu)^2\}$$
$$p(\mu)p(x|\mu,\tau) = C_x C_\mu \exp\{-0.5\left[\mu^2(\tau + s_0) - 2\mu(\tau x - s_0\mu_0) + (\tau x^2 + s_0\mu_0^2)\right]\}$$

$$p(\mu|x,\tau) = \frac{p(\mu)p(x|\mu,\tau)}{\int p(\mu)p(x|\mu,\tau)d\mu}$$

$$= \frac{\exp\{-0.5\left[\mu^2(\tau+s_0) - 2\mu(\tau x - s_0\mu_0) + (\tau x^2 + s_0\mu_0^2)\right]\}}{\int \exp\{-0.5\left[\mu^2(\tau+s_0) - 2\mu(\tau x - s_0\mu_0) + (\tau x^2 + s_0\mu_0^2)\right]\}d\mu}$$

$$= \frac{\exp\{-0.5\left[\mu^2(\tau+s_0) - 2\mu(\tau x - s_0\mu_0)\right]\}}{\int \exp\{-0.5\left[\mu^2(\tau+s_0) - 2\mu(\tau x - s_0\mu_0)\right]\}d\mu}$$

$$= \frac{\exp\{-\alpha\mu^2 + \beta\mu\}}{\int \exp\{-\alpha\mu^2 + \beta\mu\}d\mu}, \quad \text{where} \quad \alpha = 0.5(\tau + s_0) \text{ and } \beta = \tau x - s_0\mu_0 \, .$$

Since $\int_{-\infty}^{+\infty} \exp\{-\alpha x^2 + \beta x\}\, dx = \sqrt{\frac{\pi}{\alpha}} \exp\left\{\frac{\beta^2}{4\alpha}\right\}$,

$$p(\mu|x,\tau) = \frac{\exp\{-\alpha\mu^2 + \beta\mu\}}{\sqrt{\frac{\pi}{\alpha}} \exp\left\{\frac{\beta^2}{4\alpha}\right\}}$$

$$p(\tau|x,\mu) = \frac{p(\mu,x,\tau)}{\int p(\mu,x,\tau)d\tau} = \frac{p(\mu)p(\tau)p(x|\mu,\tau)}{\int p(\mu)p(\tau)p(x|\mu,\tau)d\tau} = \frac{p(\tau)p(x|\mu,\tau)}{\int p(\tau)p(x|\mu,\tau)d\tau}$$

$p(x \mid \mu,\tau) = C_x \exp\{-0.5\tau(x-\mu)^2\}$
$p(\tau \mid a,b) = C_\tau \tau^{a_0-1}\, exp(-b_0\tau)$
$p(\tau)p(x|\mu,\tau) = C_x C_\tau \tau^{a_0-1} \exp\{\tau[-0.5(x-\mu)^2 - b_0]\}$

$$p(\tau|x,\mu) = \frac{p(\tau)p(x|\mu,\tau)}{\int p(\tau)p(x|\mu,\tau)d\tau} = \frac{\tau^{a_0-1}\exp\{\tau[-0.5(x-\mu)^2 - b_0]\}}{\int \tau^{a_0-1}\exp\{\tau[-0.5(x-\mu)^2 - b_0]\}\,d\tau}$$

$$= \frac{\tau^n \exp\{-\alpha\tau\}}{\int \tau^n \exp\{-\alpha\tau\}d\tau}, \quad \text{where } n = a_0 - 1 \text{ and } \alpha = 0.5(x-\mu)^2 + b_0.$$

$$\text{Since } \int_0^\infty x^n \exp\{-\alpha x\}\,dx = \begin{cases} \frac{\Gamma(n+1)}{\alpha^{n+1}}, & (n > -1, \alpha > 0) \\ \frac{n!}{a^{n+1}}, & (n = 0,1,2,\dots,\alpha > 0) \end{cases},$$

$$p(\tau|x,\mu) = \frac{\tau^n \exp\{-\alpha\tau\}}{\frac{\Gamma(n+1)}{\alpha^{n+1}}}, \quad \text{since } a_0 > 0.$$

Question 6

Figure 6.1 shows a Markov Random Field (MRF) with two random variables X_1 and X_2, where $x_i \in \{0,1\}$. Furthermore, let $\phi_1(x_1)$ and $\phi_2(x_2)$ denote the unary potentials, and $\psi_{12}(x_1,x_2)$ denotes the pairwise potential. Given the observations over 14 trials as shown in Table 6.1, find the unknown value of $\psi_{12}(x_1 = 0, x_2 = 0)$ in the potential tables shown in Table 6.2. Show all your workings clearly.

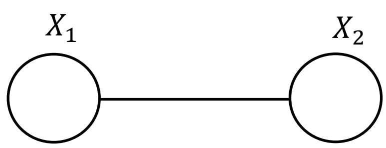


Figure 6.1

Trial Number	Outcomes	
	X_1	X_2
1	0	0
2	1	0
3	1	1
4	1	0
5	0	0
6	0	1
7	1	1
8	0	0
9	1	0
10	1	1
11	0	0
12	0	0
13	1	0
14	1	1

Table 6.1

X_1	$\phi_1(x_1)$
0	2
1	1

X_2	$\phi_2(x_2)$
0	1
1	2

X_1	X_2	$\psi_{12}(x_1, x_2)$
0	0	$\psi_{12}(x_1 = 0, x_2 = 0)$
0	1	1
1	0	2
1	1	2

Table 6.2

Answer:

Joint probability:

$$p(x_1, x_2) = \prod_n \frac{1}{Z_p} \phi_1(x_{1,n})\phi_2(x_{2,n})\psi_{12}(x_{1,n}, x_{2,n}),$$

where

$$Z_p = \sum_{x_1} \sum_{x_2} \phi_1(x_1)\phi_2(x_2)\psi_{12}(x_1, x_2).$$

$$\ln p(x_1, x_2) = \sum_n \ln \phi(x_{1,n}) + \sum_n \ln \phi_2(x_{2,n}) + \sum_n \ln \psi_{12}(x_{1,n}, x_{2,n}) - N \ln Z_p \ ,$$

where $N =$ # observations.

Note that:

$\sum_n \ln\phi(x_{1,n}) = \ln\phi(x_{1,1}) + \ln\phi(x_{1,2}) + \ \dots\ + \ln\phi(x_{1,n}) = \sum_{x_1} N(x_1)\ln\phi(x_1)$,

where $N(x_1)$ is # times x_1 takes a state, e.g. $x_1 = 0$.

Hence,

$\ln p(x_1, x_2)$

$= \sum_n \ln\phi(x_{1,n}) + \sum_n \ln\phi_2(x_{2,n}) + \sum_n \ln\psi_{12}(x_{1,n}, x_{2,n}) - N\ln Z_p$

$= \sum_{x_1} N(x_1)\ln\phi(x_1) + \sum_{x_2} N(x_2)\ln\phi(x_2) + \sum_{x_1}\sum_{x_2} N(x_1, x_2)\ln\psi(x_1, x_2) - N\ln Z_p$

To find: $\underset{\psi_{12}}{\operatorname{argmax}} \ln p(x_1, x_2)$,

$$\Rightarrow \frac{\partial \ln p(x_1,x_2)}{\partial\psi(x_1=0,x_2=0)} = \sum_{x_1}\sum_{x_2}\frac{N(x_1,x_2)}{\psi(x_1=0,x_2=0)} - \frac{N}{Z_p}\frac{\partial Z_p}{\partial\psi(x_1=0,x_2=0)}$$

$$= \frac{N(x_1=0,x_2=0)}{\psi(x_1=0,x_2=0)} - \frac{N\phi_1(x_1=0)\phi_2(x_2=0)}{\sum_{x_1}\sum_{x_2}\phi_1(x_1)\phi_2(x_2)\psi_{12}(x_1,x_2)} = 0$$

$$\frac{N(x_1 = 0, x_2 = 0)}{\psi(x_1 = 0, x_2 = 0)} - N\frac{\phi_1(x_1 = 0)\phi_2(x_2 = 0)}{k} = 0$$

$$k = \phi_1(x_1 = 0)\phi_2(x_2 = 0)\psi_{12}(x_1 = 0, x_2 = 0) + \phi_1(x_1 = 0)\phi_2(x_2 = 1)\psi_{12}(x_1 = 0, x_2 = 1) + \phi_1(x_1 = 1)\phi_2(x_2 = 0)\psi_{12}(x_1 = 1, x_2 = 0) + \phi_1(x_1 = 1)\phi_2(x_2 = 1)\psi_{12}(x_1 = 1, x_2 = 1)$$

From Table 1.1,

$N(x_1 = 0, x_2 = 0) = 5$,

$N(x_1 = 0, x_2 = 1) = 1$,

$N(x_1 = 1, x_2 = 0) = 4$,

$N(x_1 = 1, x_2 = 1) = 4$.

Now, we have:

$$\frac{5}{\psi(x_1 = 0, x_2 = 0)} - 14 \times \frac{(2)(1)}{(2)(1)\psi_{12}(x_1 = 0, x_2 = 0) + (2)(2)(1) + (1)(1)(2) + (1)(2)(2)} = 0$$

$$\Rightarrow \frac{5}{\psi_{12}(x_1 = 0, x_2 = 0)} - \frac{28}{2\psi_{12}(x_1 = 0, x_2 = 0) + 10} = 0$$

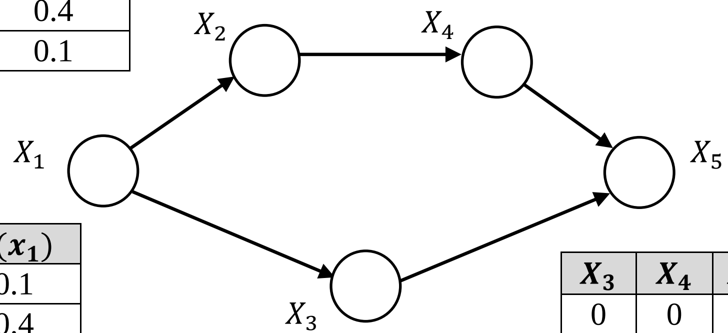
$$\Rightarrow \psi_{12}(x_1 = 0, x_2 = 0) = \frac{25}{9}$$

Question 7

The Bayesian network shown in Figure 7.1 has five random variables X_1, X_2, X_3, X_4, X_5, where $x_i \in \{0,1,2\}$ for $i = 1, 2$ and $x_i \in \{0,1\}$ for $i = 3, 4, 5$.

X_1	X_2	$p(x_2\|x_1)$
0	0	0.1
1	0	0.4
2	0	0.8
0	1	0.2
1	1	0.4
2	1	0.1

X_2	X_4	$p(x_4\|x_2)$
0	0	0.2
1	0	0.5
2	0	0.7



X_1	$p(x_1)$
0	0.1
1	0.4

X_3	X_4	X_5	$p(x_5\|x_3,x_4)$
0	0	0	0.6
0	1	0	0.3
1	0	0	0.1
1	1	0	0.8

X_1	X_3	$p(x_3\|x_1)$
0	0	0.3
1	0	0.4
2	0	0.1

Figure 7.1

(a) Given the following numbers drawn from a uniform distribution $u \sim \mathcal{U}(0,1)$:

$$u = [\ 0.4387 \quad 0.4898 \quad 0.7513 \quad 0.4984 \quad 0.2760\ ],$$

generate one set of samples from the joint distribution $p(x_1, x_2, x_3, x_4, x_5)$ using Gibbs sampling. Use $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$ as the initialization. Show all your workings clearly.

(b) Table 7.1 shows 10 sets of samples drawn from Gibbs sampling. Ignoring the burn-in effect and initialization, find the approximation for the following probabilities using the generated samples:

i. $p(x_2)$
ii. $p(x_3, x_5)$
iii. $p(x_3, x_4 = 1, x_5 = 1)$
iv. $p(x_3|x_2 = 1)$

Sample #	X_1	X_2	X_3	X_4	X_5
0	0	0	0	0	0
1	2	0	1	1	0
2	2	0	0	1	0
3	0	0	0	1	1
4	1	1	1	0	0
5	2	2	1	1	0
6	2	0	1	0	1
7	1	2	0	0	0
8	2	1	0	0	0
9	1	0	1	1	0
10	1	0	1	1	1

Table 7.1

Answer:

(a) Write down the expressions for the conditional probabilities:

$$p(x_1|x_2,x_3,x_4,x_5) = \frac{p(x_1,x_2,x_3,x_4,x_5)}{p(x_2,x_3,x_4,x_5)} = \frac{p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_3,x_4)}{p(x_4|x_2)p(x_5|x_3,x_4)\sum_{x_1} p(x_1)p(x_2|x_1)p(x_3|x_1)}$$

$$\propto p(x_1)p(x_2|x_1)p(x_3|x_1)$$

$$p(x_2|x_1,x_3,x_4,x_5) = \frac{p(x_1,x_2,x_3,x_4,x_5)}{p(x_1,x_3,x_4,x_5)} = \frac{p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_3,x_4)}{p(x_1)p(x_3|x_1)p(x_5|x_3,x_4)\sum_{x_2} p(x_2|x_1)p(x_4|x_2)}$$

$$\propto p(x_2|x_1)p(x_4|x_2)$$

$$p(x_3|x_1,x_2,x_4,x_5) = \frac{p(x_1,x_2,x_3,x_4,x_5)}{p(x_1,x_3,x_4,x_5)} = \frac{p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_3,x_4)}{p(x_1)p(x_2|x_1)p(x_4|x_2)\sum_{x_3} p(x_3|x_1)p(x_5|x_3,x_4)}$$

$$\propto p(x_3|x_1)p(x_5|x_3,x_4)$$

$$p(x_4|x_1,x_2,x_3,x_5) = \frac{p(x_1,x_2,x_3,x_4,x_5)}{p(x_1,x_3,x_4,x_5)} = \frac{p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_3,x_4)}{p(x_1)p(x_2|x_1)p(x_3|x_1)\sum_{x_4} p(x_4|x_2)p(x_5|x_3,x_4)}$$

$$\propto p(x_4|x_2)p(x_5|x_3,x_4)$$

$$p(x_5|x_1,x_2,x_3,x_4) = \frac{p(x_1,x_2,x_3,x_4,x_5)}{p(x_1,x_3,x_4,x_5)} = \frac{p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)p(x_5|x_3,x_4)}{p(x_1)p(x_2|x_1)p(x_3|x_1)p(x_4|x_2)\sum_{x_5} p(x_5|x_3,x_4)}$$

$$\propto p(x_5|x_3,x_4)$$

Gibbs sampling:

t	X_1	X_2	X_3	X_4	X_5
0	0	0	0	0	0
1	1	1	0	0	0

Iteration 1:

$$p(x_1 = 0|x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0) \propto p(x_1 = 0)p(x_2 = 0|x_1 = 0)p(x_3 = 0|x_1 = 0)$$
$$= (0.1)(0.1)(0.3) = 0.003$$

$$p(x_1 = 1|x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0) \propto p(x_1 = 1)p(x_2 = 0|x_1 = 1)p(x_3 = 0|x_1 = 1)$$
$$= (0.4)(0.4)(0.4) = 0.064$$

$$p(x_1 = 2|x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0) \propto p(x_1 = 2)p(x_2 = 0|x_1 = 2)p(x_3 = 0|x_1 = 2)$$
$$= (0.5)(0.8)(0.1) = 0.04$$

Normalization,

$$p(x_1 = 0|x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0) = \frac{0.003}{0.107} = 0.028$$

$$p(x_1 = 1|x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0) = \frac{0.064}{0.107} = 0.598$$

$$p(x_1 = 2|x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0) = \frac{0.04}{0.107} = 0.374$$

$$\boldsymbol{u = 0.4387 \Rightarrow x_1 = 1.}$$

$$p(x_2 = 0|x_1 = 1, x_3 = 0, x_4 = 0, x_5 = 0) \propto p(x_2 = 0|x_1 = 1)p(x_4 = 0|x_2 = 0)$$
$$= (0.4)(0.2) = 0.08$$

$$p(x_2 = 1|x_1 = 1, x_3 = 0, x_4 = 0, x_5 = 0) \propto p(x_2 = 1|x_1 = 1)p(x_4 = 0|x_2 = 1)$$
$$= (0.4)(0.5) = 0.20$$

$$p(x_2 = 2|x_1 = 1, x_3 = 0, x_4 = 0, x_5 = 0) \propto p(x_2 = 2|x_1 = 1)p(x_4 = 0|x_2 = 2)$$
$$= (0.2)(0.7) = 0.14$$

Normalization,

$$p(x_2 = 0|x_1 = 1, x_3 = 0, x_4 = 0, x_5 = 0) = \frac{0.08}{0.42} = 0.191$$

$$p(x_2 = 1|x_1 = 1, x_3 = 0, x_4 = 0, x_5 = 0) = \frac{0.20}{0.42} = 0.476$$

$$p(x_2 = 2|x_1 = 1, x_3 = 0, x_4 = 0, x_5 = 0) = \frac{0.14}{0.42} = 0.333$$

$\boldsymbol{u = 0.4898 \Rightarrow x_2 = 1}$

$$p(x_3 = 0|x_1 = 1, x_2 = 1, x_4 = 0, x_5 = 0) \propto p(x_3 = 0|x_1 = 1)p(x_5 = 0|x_3 = 0, x_4 = 0)$$
$$= (0.4)(0.6) = 0.24$$

$$p(x_3 = 1|x_1 = 1, x_2 = 1, x_4 = 0, x_5 = 0) \propto p(x_3 = 1|x_1 = 1)p(x_5 = 0|x_3 = 1, x_4 = 0)$$
$$= (0.6)(0.1) = 0.06$$

Normalization,

$$p(x_3 = 0|x_1 = 1, x_2 = 1, x_4 = 0, x_5 = 0) = \frac{0.24}{0.30} = 0.8$$

$$p(x_3 = 1|x_1 = 1, x_2 = 1, x_4 = 0, x_5 = 0) = \frac{0.06}{0.30} = 0.2$$

$\boldsymbol{u = 0.7513 \Rightarrow x_3 = 0}$

$$p(x_4 = 0|x_1 = 1, x_2 = 1, x_3 = 0, x_5 = 0) \propto \ p(x_4 = 0|x_2 = 1)p(x_5 = 0|x_3 = 0, x_4 = 0)$$
$$= (0.5)(0.6) = 0.3$$

$$p(x_4 = 1|x_1 = 1, x_2 = 1, x_3 = 0, x_5 = 0) \propto \ p(x_4 = 1|x_2 = 1)p(x_5 = 0|x_3 = 0, x_4 = 1)$$
$$= (0.5)(0.3) = 0.15$$

Normalization,

$$p(x_4 = 0|x_1 = 1, x_2 = 1, x_3 = 0, x_5 = 0) = \frac{0.30}{0.45} = 0.667$$

$$p(x_4 = 1|x_1 = 1, x_2 = 1, x_3 = 0, x_5 = 0) = \frac{0.15}{0.45} = 0.333$$

$\boldsymbol{u = 0.4984 \Rightarrow x_4 = 0}$

$$p(x_5 = 0|x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0) \propto p(x_5 = 0|x_3 = 0, x_4 = 0)$$
$$= 0.6$$

$$p(x_5 = 1|x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0) \propto p(x_5 = 1|x_3 = 0, x_4 = 0)$$
$$= 0.4$$

No need for normalization.

$\boldsymbol{u = 0.2760 \Rightarrow x_5 = 0}$

(b)

i. $p(x_2 = 0) = \frac{6}{10} = 0.6$
$p(x_2 = 1) = \frac{2}{10} = 0.2$
$p(x_2 = 2) = \frac{2}{10} = 0.2$

ii. $p(x_3 = 0, x_5 = 0) = \frac{3}{10} = 0.3$
$p(x_3 = 0, x_5 = 1) = \frac{1}{10} = 0.1$
$p(x_3 = 1, x_5 = 0) = \frac{4}{10} = 0.4$
$p(x_3 = 1, x_5 = 1) = \frac{2}{10} = 0.2$

iii. $p(x_3 = 0, x_4 = 1, x_5 = 1) = \frac{1}{10} = 0.1$

iv. $p(x_3 = 0 \,|x_2 = 1) = \frac{1}{2} = 0.5$
$p(x_3 = 1 \,|x_2 = 1) = \frac{1}{2} = 0.5$

Question 8

a. Figure 8.1 shows a homogeneous hidden Markov Model (HMM) over three time steps. The latent random variables are Y_1, Y_2, Y_3, where $Y_n \in \{0, 1, 2\}$, and the observed random variables are X_1, X_2, X_3, where $X_n \in \mathbb{R}$.

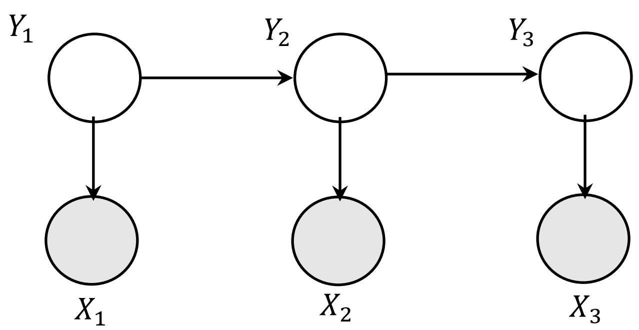


Figure 2.1

The prior probability of the random variable Y_1 is $p(Y_1 \mid \pi) = \prod_k \pi_k^{y_{1k}}$, where $\pi = \{0.2, 0.5, 0.3\}$. Furthermore, the transition probability is given by:

$$p(Y_n \mid Y_{n-1}, A) = \prod_k \prod_j A_{jk}^{y_{n-1,j}y_{nk}}, \text{ where } A = \begin{bmatrix} 0.2 & \alpha & \beta \\ 0.1 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}, \text{ and}$$

the emission probabilities of the respective observed random variables X_n are shown in Table 8.1.

	$k = 0$	$k = 1$	$k = 2$
X_1	0.3	0.6	0.4
X_2	0.5	0.4	0.4
X_3	0.3	0.8	0.5

Table 8.1

Given that the minimum probability of the joint distribution $p(Y_1, Y_2, Y_3, X_1, X_2, X_3)$ is 0.000216 and occurs at $Y_1 = 0, Y_2 = 1, Y_3 = 0$, find the unknown values α and β in the transition probability.

b. Figure 8.2 shows an undirected graphic model with six random variables X_1, X_2, X_3, X_4, X_5 and X_6, where $X_i \in \{0,1,2\}$. The potential $\psi(X_i, X_j)$ between any pair of nodes X_i and X_j, where $i < j$ is given in Table 2.2. Given $X_1 = 0, X_3 = 1$ and $X_5 = 2$, find the states of X_2, X_4 and X_6 that maximizes the joint distribution $p(X_1, X_2, X_3, X_4, X_5, X_6)$.

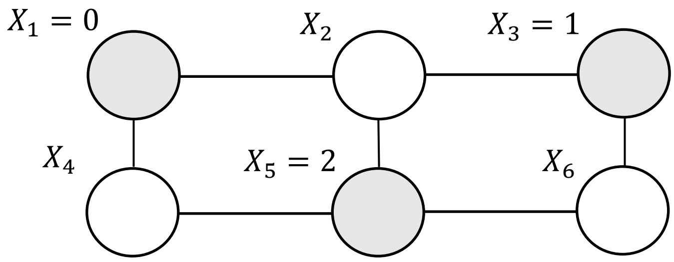


Figure 8.2

X_i	X_j	$\psi(X_i, X_j)$
0	0	1
0	1	5
0	2	7

1	0	2
1	1	4
1	2	8
2	0	3
2	1	6
2	2	9

Table 8.2

Answer:

a.

Joint probability: $p(X,Y) = p(Y_1)\prod_{n=2} p(Y_n \mid Y_{n-1}) \prod_{n=1} p(X_n \mid Y_n)$

$$\min_Y p(X,Y) = \min_Y p(Y_1)\prod_{n=2} p(Y_n \mid Y_{n-1}) \prod_{n=1} p(X_n \mid Y_n)$$

$$= \min_{Y_1}\min_{Y_2}\min_{Y_3} p(Y_1)p(Y_2 \mid Y_1)p(Y_3 \mid Y_2)p(X_1 \mid Y_1)p(X_2 \mid Y_2)p(X_3 \mid Y_3)$$

$$= \min_{Y_3} p(X_3 \mid Y_3)\min_{Y_2} p(X_2 \mid Y_2)p(Y_3 \mid Y_2)\min_{Y_1} p(Y_1)p(X_1 \mid Y_1)p(Y_2 \mid Y_1)$$

Given that the minimum probability equals 0.000216 and occurs at $Y_1 = 0, Y_2 = 1, Y_3 = 0$, this imply:

$$\min_Y p(X,Y) = 0.000216$$

$$p(Y_1 = 0)p(Y_2 = 1 \mid Y_1 = 0)p(Y_3 = 0 \mid Y_2 = 1)p(X_1 \mid Y_1 = 0)p(X_2 \mid Y_2 = 1)p(X_3 \mid Y_3 = 0)$$

$$= (0.2)(\alpha)(0.1)(0.3)(0.4)(0.3) = 0.00072\alpha = 0.000216 \Longrightarrow \alpha = 0.3$$

Since each row of the transition matrix sums to one, we have $0.2 + \alpha + \beta = 1 \Rightarrow \beta = 0.5$

b.

Joint probability:

$$p(X) = \frac{1}{Z}\psi(X_1 = 0, X_2)\psi(X_1 = 0, X_4)\psi(X_2, X_3 = 1)\psi(X_2, X_5 = 2)$$
$$\psi(X_3 = 1, X_6)\psi(X_4, X_5 = 2)\psi(X_5 = 2, X_6)$$

$$\max_{X_2}\max_{X_4}\max_{X_6} p(X) =$$

$\max_{X_2} \psi(X_1 = 0, X_2)\psi(X_2, X_3 = 1)\psi(X_2, X_5 = 2) \max_{X_4} \psi(X_1 = 0, X_4)\psi(X_4, X_5 = 2)$

$\max_{X_6} \psi(X_3 = 1, X_6)\psi(X_5 = 2, X_6)$

Consider

$\max_{X_6} \psi(X_3 = 1, X_6)\psi(X_5 = 2, X_6) = \max[(2)(3), (4)(6), (8)(9)] = \max[6, 24, 72]$

$= 72 \ \ (\boldsymbol{X_6 = 2})$

$\max_{X_4} \psi(X_1 = 0, X_4)\psi(X_4, X_5 = 2) = \max[(1)(7), (5)(8), (7)(9)] = \max[\,7,40,63]$

$= 63 \ (\boldsymbol{X_4 = 2})$

$\max_{X_2} \psi(X_1 = 0, X_2)\psi(X_2, X_3 = 1)\psi(X_2, X_5 = 2)$

$= \max[(1)(5)(7), (5)(4)(8), (7)(6)(9)] = \max[35,160,378] = 378 \ \ (\boldsymbol{X_2 = 2})$

Question 9

Figure 9.1 shows a Bayesian network with both binary and continuous state latent random variables, i.e., $Z \in \{0,1\}$ and $T \in \mathbb{R}$. In addition, $X = 0.5$ is the observed random variable. The maximum log-likelihood of T:

$$\underset{T}{\operatorname{argmax}} \log p(T \mid X),$$

can be obtained from the Expectation-Maximization (EM) algorithm. The EM algorithm iterates between the Expectation step that evaluates the expected complete data log-likelihood with respect to $p(Z \mid X, T^{old})$ and the Maximization step that maximizes T over the expected complete data log-likelihood with respect to $p(Z \mid X, T^{old})$. T^{old} is the value of T from the previous iteration of the EM algorithm. $\{\lambda = 0.1, w_{a0} = 0.5, w_{a1} = 0.5, w_{b0} = 0.8, w_{b1} = 0.2, \tau_a = 1.0, \tau_b = 1.2, U = 0.6\}$ are known hyperparameters of the following distributions:

$p(Z) = \lambda^Z (1 - \lambda)^{(1-Z)}$,

$p(X \mid T, Z) = \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)^Z \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)^{(1-Z)}$,

$\mathcal{N}(X \mid w_0 + w_1 T, \tau) = \sqrt{\frac{\tau}{2\pi}} \exp\{-0.5\tau(X - w_0 - w_1 T)^2\}$,

$p(T) = U.$

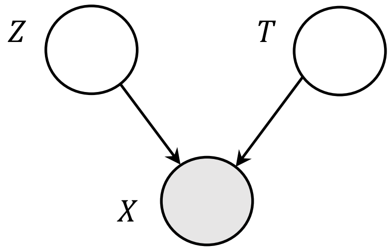


Figure 9.1

a. Derive the expression for the posterior $p(Z \mid X, T^{old})$ from the Bayesian Network.

b. Derive the expression for T that maximizes the expected complete data log-likelihood with respect to $p(Z \mid X, T^{old})$.

c. Given the initial value of $T = 2.0$, find the the value of T in the next EM iteration.

Answer:

Joint distribution:

$$p(X, Z, T) = p(T)\, p(Z) p(X \mid T, Z) = p(Z) p(X \mid T, Z)$$

a. $p(Z \mid X, T^{old}) = \frac{p(Z)p(X|T,Z)}{\sum_Z p(Z)p(X|T,Z)} = \frac{p(Z)p(X|T,Z)}{\sum_Z p(Z)p(X|T,Z)}$

$$= \frac{\lambda^Z (1-\lambda)^{(1-Z)} \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)^Z \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)^{(1-Z)}}{\sum_Z \lambda^Z (1-\lambda)^{(1-Z)} \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)^Z \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)^{(1-Z)}}$$

$$= \frac{\lambda^Z (1-\lambda)^{(1-Z)} \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)^Z \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)^{(1-Z)}}{\lambda \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a) + (1-\lambda) \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)}$$

Let's define:

$$\gamma(Z = 0) = p(Z = 0 \mid X, T^{old})$$

$$= \frac{(1-\lambda) \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)}{\lambda \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a) + (1-\lambda) \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)}$$

$$\gamma(Z = 1) = p(Z = 1 \mid X, T^{old})$$

$$= \frac{\lambda \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)}{\lambda \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a) + (1-\lambda) \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)}$$

b. $Q = \sum_Z p(Z \mid X, T^{old}) \ln p(X, Z \mid T)$

$= \sum_Z \gamma(Z) \ln p(Z) p(X \mid T, Z)$

$= \gamma(Z=0) \ln p(Z=0) p(X \mid T, Z=0) + \gamma(Z=1) \ln p(Z=1) p(X \mid T, Z=1)$

$= \gamma(Z=0) \ln\{(1-\lambda)\mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)\} +$

$\gamma(Z=1) \ln\{\lambda \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)\}$

$$\underset{T}{\operatorname{argmax}}\, \gamma(Z=0) \ln\{(1-\lambda)\mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)\} + \gamma(Z=1) \ln\{\lambda \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)\}$$

$$0 = \gamma(Z=0) \frac{\partial}{\partial T}\{\ln(1-\lambda) + \ln \mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b)\} + \gamma(Z=1) \frac{\partial}{\partial T}\{\ln \lambda + \ln \mathcal{N}(X \mid w_{a0} + w_{a1}T, \tau_a)\}$$

$$0 = \gamma(Z=0) \frac{\partial}{\partial T}\left\{\ln \sqrt{\frac{\tau_b}{2\pi}} - 0.5\tau_b (X - w_{b0} - w_{b1}T)^2\right\} + \gamma(Z=1) \frac{\partial}{\partial T}\left\{\ln \sqrt{\frac{\tau_a}{2\pi}} - 0.5\tau_a (X - w_{a0} - w_{a1}T)^2\right\}$$

$0 = \gamma(Z=0)\{w_{b1}\tau_b(X - w_{b0} - w_{b1}T)\} + \gamma(Z=1)\{w_{a1}\tau_a(X - w_{a0} - w_{a1}T)\}$

$= X(\gamma(Z=0) w_{b1}\tau_b) - w_{b0}(\gamma(Z=0) w_{b1}\tau_b) - w_{b1}T(\gamma(Z=0) w_{b1}\tau_b) +$

$X(\gamma(Z=1) w_{a1}\tau_a) - w_{a0}(\gamma(Z=1) w_{a1}\tau_a) - w_{a1}T(\gamma(Z=1) w_{a1}\tau_a)$

$= X(\gamma(Z=0) w_{b1}\tau_b + \gamma(Z=1) w_{a1}\tau_a) - T(\gamma(Z=0) w_{b1}^2\tau_b + \gamma(Z=0) w_{a1}^2\tau_a) -$

$\gamma(Z=0) w_{b0} w_{b1}\tau_b - \gamma(Z=1) w_{a0} w_{a1}\tau_a$

$T\big(\gamma(Z=0) w_{b1}^2\tau_b + \gamma(Z=0) w_{a1}^2\tau_a\big) =$

$X(\gamma(Z=0) w_{b1}\tau_b + \gamma(Z=1) w_{a1}\tau_a) - \gamma(Z=0) w_{b0} w_{b1}\tau_b - \gamma(Z=1) w_{a0} w_{a1}\tau_a$

$$T = \frac{X(\gamma(Z=0) w_{b1}\tau_b + \gamma(Z=1) w_{a1}\tau_a) - \gamma(Z=0) w_{b0} w_{b1}\tau_b - \gamma(Z=1) w_{a0} w_{a1}\tau_a}{\gamma(Z=0) w_{b1}^2\tau_b + \gamma(Z=0) w_{a1}^2\tau_a}$$

c.

$\mathcal{N}(X \mid w_{b0} + w_{b1}T, \tau_b) = \mathcal{N}(X = 0.5 \mid 0.8 + (0.2)(2.0), 1.2)$

$$= \mathcal{N}(\,X = 0.5 \mid 1.2, 1.2\,)$$

$$= \sqrt{\frac{1.2}{2\pi}}\exp\{-0.5(1.2)(0.5-1.2)^2\} = 0.3257$$

$$\mathcal{N}(\,X \mid w_{a0} + w_{a1}T, \tau_a\,) = \mathcal{N}(\,X = 0.5 \mid 0.5 + (0.5)(2.0), 1.0\,)$$

$$= \mathcal{N}(\,X = 0.5 \mid 1.5, 1.0\,)$$

$$= \sqrt{\frac{1.0}{2\pi}}\exp\{-0.5(1.0)(0.5-1.5)^2\} = 0.242$$

$$\gamma(Z=0) = p(\,Z = 0 \mid X, T^{old}\,)$$

$$= \frac{(1-\lambda)\mathcal{N}(\,X \mid w_{b0} + w_{b1}T, \tau_b\,)}{\lambda\mathcal{N}(\,X \mid w_{a0} + w_{a1}T, \tau_a\,) + \ (1-\lambda)\mathcal{N}(\,X \mid w_{b0} + w_{b1}T, \tau_b\,)}$$

$$= \frac{(1-0.1)0.3257}{(0.1)0.242 + \ (1-0.1)0.3257} = \frac{0.29313}{0.31733} = 0.924$$

$$\gamma(Z=1) = p(\,Z = 1 \mid X, T^{old}\,)$$

$$= \frac{\lambda\mathcal{N}(\,X \mid w_{a0} + w_{a1}T, \tau_a\,)}{\lambda\mathcal{N}(\,X \mid w_{a0} + w_{a1}T, \tau_a\,) + \ (1-\lambda)\mathcal{N}(\,X \mid w_{b0} + w_{b1}T, \tau_b\,)}$$

$$= \frac{(0.1)0.242}{(0.1)0.242 + \ (1-0.1)0.3257} = \frac{0.0242}{0.31733} = 0.076$$

$$X(\gamma(Z=0)w_{b1}\tau_b + \gamma(Z=1)w_{a1}\tau_a)$$

$$= (0.5)\{(0.924)(0.2)(1.2) + (0.076)(0.5)(1.0)\} = 0.130$$

$$-\gamma(Z=0)w_{b0}w_{b1}\tau_b - \gamma(Z=1)w_{a0}w_{a1}\tau_a$$

$$= -(0.924)(0.8)(0.2)(1.2) - (0.076)(0.5)(0.5)(1.0) = -0.196$$

$$\gamma(Z=0)w_{b1}^2\tau_b + \ \gamma(Z=0)w_{a1}^2\tau_a = (0.924)(0.04)(1.2) + (0.076)(0.25)(1.0) = 0.063$$

$$T = \frac{X(\gamma(Z=0)w_{b1}\tau_b + \gamma(Z=1)w_{a1}\tau_a) - \gamma(Z=0)w_{b0}w_{b1}\tau_b - \gamma(Z=1)w_{a0}w_{a1}\tau_a}{\gamma(Z=0)w_{b1}^2\tau_b + \gamma(Z=0)w_{a1}^2\tau_a}$$

$$= \frac{0.130 - 0.196}{0.063} = \mathbf{-1.048}$$

Question 10

a. The objective of image denoising is to recover the clean image (noise-free) from a given noisy image. Figure 10.1 shows a Markov Random Field (MRF) to solve a four-pixel binary image denoising problem. The latent random variable $X_i \in \{-1, +1\}$ represents the pixels of the desired clean image, and the observed random variables $Y_i \in \{-1, +1\}$ represents the pixels of the noisy image. We use the Ising model, i.e., $\psi(X_i, X_j) = \exp(JX_iX_j)$ as the edge potentials, where J is the coupling strength of the smoothness prior between neighboring pixels X_i and X_j. The observation model follows a Gaussian distribution: $p(Y_i \mid X_i) = \mathcal{N}(Y_i \mid X_i, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-0.5 \frac{(Y_i - X_i)^2}{\sigma^2}\right\}$.

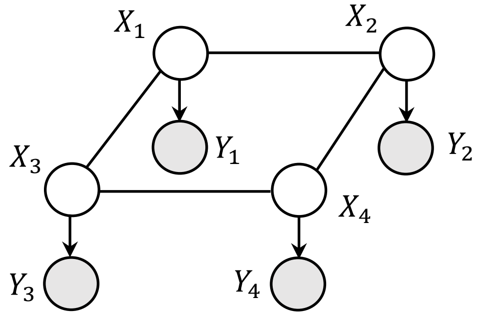


Figure 10.1

Given that we observe $Y_1 = +1, Y_2 = -1, Y_3 = -1, Y_4 = +1$, and the following random numbers are drawn from a uniform distribution $u \sim \mathcal{U}(0,1)$:

$$u = [\ 0.6557 \quad 0.0357 \quad 0.9340 \quad 0.8491\ ],$$

generate one set of samples from the joint distribution $p(X, Y)$ using Gibbs sampling. Use $X_1 = -1, X_2 = -1, X_3 = -1, X_4 = -1$ as the initialization and set $J = 0.01$, $\sigma^2 = 1.0$. Show all workings clearly.

b. Draw the Bayesian Network and write down the factorized joint probability distribution that encodes all the following conditional independences:

1. $X_4 \perp \{X_1, X_2, X_5\} \mid X_3$
2. $X_5 \perp \{X_1, X_3, X_4\} \mid X_2$
3. $X_3 \perp X_5 \mid \{X_1, X_2\}$

4. $X_1 \perp X_2 \mid \emptyset$
5. $X_1 \perp \{X_2, X_5\} \mid \emptyset$

Answers:

Joint probability: $p(X,Y) = \frac{1}{Z}\prod_i p(Y_i \mid X_i) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i, X_j)$

$$= \frac{1}{Z} p(Y_1 \mid X_1) p(Y_2 \mid X_2) p(Y_3 \mid X_3) p(Y_4 \mid X_4) \psi(X_1, X_2)\psi(X_1, X_3)\psi(X_3, X_4)\psi(X_2, X_4)$$

Conditional distribution for each pixel X_i:

$$p(X_i \mid X_{\setminus i}, Y) = \frac{\frac{1}{Z}\prod_i p(Y_i \mid X_i) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i, X_j)}{\sum_{X_i} \frac{1}{Z}\prod_i p(Y_i \mid X_i) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i, X_j)}$$

$$= \frac{p(Y_i \mid X_i) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i, X_j)}{p(Y_i \mid X_i = -1) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i = -1, X_j) + p(Y_i \mid X_i = +1) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i = +1, X_j)}$$

$$p(X_i = +1 \mid X_{\setminus i}, Y)$$

$$= \frac{p(Y_i \mid X_i = +1) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i = +1, X_j)}{p(Y_i \mid X_i = -1) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i = -1, X_j) + p(Y_i \mid X_i = +1) \prod_{j\in \mathrm{nbr}(i)} \psi(X_i = +1, X_j)}$$

$$= \frac{\mathcal{N}(Y_i \mid +1, \sigma^2) \prod_{j\in \mathrm{nbr}(i)} \exp(Jx_j)}{\mathcal{N}(Y_i \mid -1, \sigma^2) \prod_{j\in \mathrm{nbr}(i)} \exp(-Jx_j) + \mathcal{N}(Y_i \mid +1, \sigma^2) \prod_{j\in \mathrm{nbr}(i)} \exp(Jx_j)}$$

$$= \frac{1}{1 + \frac{\mathcal{N}(Y_i \mid -1, \sigma^2)}{\mathcal{N}(Y_i \mid +1, \sigma^2)} \prod_{j\in \mathrm{nbr}(i)} \exp(-2Jx_j)}$$

$$p(X_i = -1 \mid X_{\setminus i}, Y) = 1 - p(X_i = +1 \mid X_{\setminus i}, Y)$$

$$\mathcal{N}(\,Y_i = +1 \mid +1, \sigma^2\,) = \frac{1}{\sqrt{2\pi 1^2}} \exp\left\{-0.5\frac{(1-1)^2}{1^2}\right\} = \frac{1}{\sqrt{2\pi}}$$

$$\mathcal{N}(\,Y_i = +1 \mid -1, \sigma^2\,) = \frac{1}{\sqrt{2\pi 1^2}} \exp\left\{-0.5\frac{(1+1)^2}{1^2}\right\} = \frac{1}{\sqrt{2\pi}} \exp\{-2.0\}$$

$$\mathcal{N}(\,Y_i = -1 \mid +1, \sigma^2\,) = \frac{1}{\sqrt{2\pi 1^2}} \exp\left\{-0.5\frac{(-1-1)^2}{1^2}\right\} = \frac{1}{\sqrt{2\pi}} \exp\{-2.0\}$$

$$\mathcal{N}(\,Y_i = -1 \mid -1, \sigma^2\,) = \frac{1}{\sqrt{2\pi 1^2}} \exp\left\{-0.5\frac{(-1+1)^2}{1^2}\right\} = \frac{1}{\sqrt{2\pi}}$$

Consider $\boldsymbol{X_1}$:

Markov blanket: $X_2 = -1, X_3 = -1, Y_1 = +1,$

$$p\big(\,X_1 = +1 \mid X_{\setminus 1}, Y\,\big) = \frac{1}{1 + \frac{\mathcal{N}\left(\,Y_1 \mid -1, \sigma^2\,\right)}{\mathcal{N}\left(\,Y_1 \mid +1, \sigma^2\,\right)} \prod_{j \in \mathrm{nbr}(1)} \exp(-2Jx_j)}$$

$$= \frac{1}{1 + \frac{\mathcal{N}\left(\,Y_1 = +1 \mid -1, \sigma^2\,\right)}{\mathcal{N}\left(\,Y_1 = +1 \mid +1, \sigma^2\,\right)} \exp(-2J(x_2 + x_3))}$$

$$= \frac{1}{1 + \exp\{-2.0\} \exp\{-(2)(0.01)(-2)\,\}\,)}$$

$$= \frac{1}{1 + \exp\{-2.0 + 0.04\}\,)} = \frac{1}{1 + \exp\{-1.96\}\,)} = \mathbf{0.8765}$$

Since $u_1 = 0.6557 < p\big(\,X_1 = +1 \mid X_{\setminus 1}, Y\,\big) \Longrightarrow \boldsymbol{X_1 = +1}$

Consider $\boldsymbol{X_2}$:

Markov blanket: $X_1 = +1,\; X_4 = -1, Y_2 = -1,$

$$p\big(\,X_2 = +1 \mid X_{\setminus 2}, Y\,\big) = \frac{1}{1 + \frac{\mathcal{N}\left(\,Y_2 \mid -1, \sigma^2\,\right)}{\mathcal{N}\left(\,Y_2 \mid +1, \sigma^2\,\right)} \prod_{j \in \mathrm{nbr}(2)} \exp(-2Jx_j)}$$

$$= \frac{1}{1 + \frac{\mathcal{N}\left(\,Y_2 = -1 \mid -1, \sigma^2\,\right)}{\mathcal{N}\left(\,Y_2 = -1 \mid +1, \sigma^2\,\right)} \exp(-2J(x_1 + x_4))}$$

$$= \frac{1}{1 + \exp\{+2.0\} \exp\{0\}\,)}$$

$$= \frac{1}{1 + \exp\{+2.0\}} = \mathbf{0.1192}$$

Since $u_2 = 0.0357 < p\big(X_2 = +1 \mid X_{\setminus 2}, Y\big) \Rightarrow \boldsymbol{X_2} = +\mathbf{1}$

Consider $\boldsymbol{X_3}$:

Markov blanket: $X_1 = +1,\; X_4 = -1, Y_3 = -1,$

$$p\big(X_3 = +1 \mid X_{\setminus 3}, Y\big) = \frac{1}{1 + \frac{\mathcal{N}(Y_3 \mid -1, \sigma^2)}{\mathcal{N}(Y_3 \mid +1, \sigma^2)} \prod_{j \in \mathrm{nbr}(3)} \exp(-2Jx_j)}$$

$$= \frac{1}{1 + \frac{\mathcal{N}(Y_3 = -1 \mid -1, \sigma^2)}{\mathcal{N}(Y_3 = -1 \mid +1, \sigma^2)} \exp(-2J(x_1 + x_4))}$$

$$= \frac{1}{1 + \exp\{+2.0\} \exp\{0\})}$$

$$= \frac{1}{1 + \exp\{+2.0\}} = \mathbf{0.1192}$$

Since $u_3 = 0.9340 \geq p\big(X_3 = +1 \mid X_{\setminus 3}, Y\big) \Rightarrow \boldsymbol{X_3} = -\mathbf{1}$

Consider $\boldsymbol{X_4}$:

Markov blanket: $X_2 = +1,\; X_3 = -1, Y_4 = +1,$

$$p\big(X_4 = +1 \mid X_{\setminus 4}, Y\big) = \frac{1}{1 + \frac{\mathcal{N}(Y_4 \mid -1, \sigma^2)}{\mathcal{N}(Y_4 \mid +1, \sigma^2)} \prod_{j \in \mathrm{nbr}(4)} \exp(-2Jx_j)}$$

$$= \frac{1}{1 + \frac{\mathcal{N}(Y_4 = +1 \mid -1, \sigma^2)}{\mathcal{N}(Y_4 = +1 \mid +1, \sigma^2)} \exp(-2J(x_2 + x_3))}$$

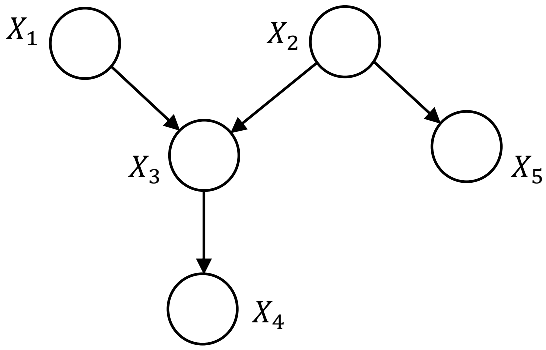
$$= \frac{1}{1 + \exp\{-2.0\} \exp\{0\})}$$

$$= \frac{1}{1 + \exp\{-2.0\}} = \mathbf{0.8808}$$

Since $u_4 = 0.8491 \geq p\big(X_4 = +1 \mid X_{\setminus 4}, Y\big) \Rightarrow \boldsymbol{X_4} = -\mathbf{1}$

b.

$$p(X) = p(x_1)p(x_2)p(x_3|x_1, x_2)p(x_4|x_3)p(x_5|x_2)$$



Question 11:

Table 11.1 shows nine observations $\{\mathbf{x}_1, \dots, \mathbf{x}_9\}$ of 2-dimensional features $[x, y]$, where each observation is generated from an image of 1-out-of-3 handwritten alphabets. We further assume the sampling of each image is fully independent, and the observations given the alphabet follow a bivariate Gaussian distribution (see Equation 1). Figure 11.1 shows a plot of the nine 2-dimensional features in Table 1.1. Given a new observation $\mathbf{x}_{\text{Test}} = [14.65, 11.00]$, find the probability distribution of the alphabet on its corresponding image. **Explain and show all your workings clearly.**

$\mathbf{x}_n$	$[x, y]$
$\mathbf{x}_1$	[3.83, 14.48]
$\mathbf{x}_2$	[0.31, 2.06]
$\mathbf{x}_3$	[13.62, 8.89]
$\mathbf{x}_4$	[5.74, 1.35]
$\mathbf{x}_5$	[4.02, 15.69]
$\mathbf{x}_6$	[11.82, 9.88]
$\mathbf{x}_7$	[12.39, 10.8]
$\mathbf{x}_8$	[1.64, 15.22]
$\mathbf{x}_9$	[1.84, 0.68]

Table 11.1

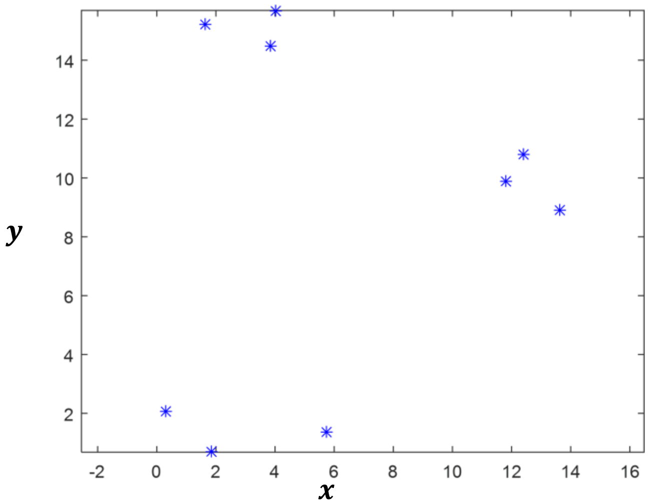


Figure 11.1

Useful Equations:

1. $p(\mathbf{x}) = (2\pi)^{-1} \det(\Sigma)^{-0.5} \exp\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\}$,

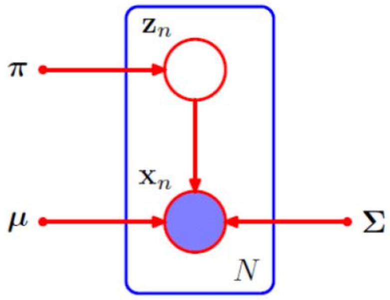
2. $\det\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}$,

3. $\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}^{-1} = \frac{1}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}}\begin{pmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{21} & \sigma_{11}\end{pmatrix}.$

Answer:

The three clusters and their respective observed points can be identified from the plot. Let the latent random variable $Z_n \in \{1,2,3\}$ represent the handwritten alphabet type, and the observed random variable $X_n \in \mathbb{R}^2$ represent the observed 2d features.

Since the sampling of each image is fully independent, and the observations given the alphabet follow a Gaussian distribution, we can model the joint distribution of Z_n and X_n with a Gaussian mixture model.



$$p(X, Z \mid \theta) = \prod_k \pi_k^{Z_k} N(X \mid \mu_k, \sigma_k)^{Z_k}$$

We can directly identify the clusters and the assignments of the observations; we can directly compute μ and σ without EM. Furthermore, we can also see that the three clusters are spaced distinctively apart, and thus the mixing coefficients are equal, i.e $\pi_k = \frac{1}{3}$.

$\pi_1 = \frac{1}{3}$,

$\mu_1 = \dfrac{X_2 + X_4 + X_9}{3} = [2.6300 \quad 1.3633]$,

$\sigma_1 = \frac{1}{3}\sum_{i=\{2,4,9\}}(X_i - \mu_1)^T(X_i - \mu_1) = [\,5.2262 \quad -0.3726;\ -0.3726 \quad 0.3175].$

$\pi_2 = \frac{1}{3}$,

$\mu_2 = \dfrac{X_1 + X_5 + X_8}{3} = [3.1633 \quad 15.1300]$,

$\sigma_2 = \frac{1}{3}\sum_{i=\{1,5,8\}}(X_i - \mu_2)^T(X_i - \mu_2) = [\,1.1663 \quad -0.0302;\ -0.0302 \quad 0.2481].$

$\pi_3 = \frac{1}{3}$,

$$\mu_3 = \frac{X_3 + X_6 + X_7}{3} = [12.6100 \quad 9.8567],$$

$$\sigma_3 = \frac{1}{3}\sum_{i=\{3,6,7\}}(X_i - \mu_3)^T(X_i - \mu_3) = [\,0.5642 \quad -0.4008;\ -0.4008 \quad 0.6083].$$

We can use the model parameters to find the assignment posterior of the test data:

$$\tilde{p}(Z = 1 \mid X_{test}, \theta\,) = \pi_1 N(X_{test} \mid \mu_1, \sigma_1\,) = 0.00$$

$$\tilde{p}(Z = 2 \mid X_{test}, \theta\,) = \pi_2 N(X_{test} \mid \mu_2, \sigma_2\,) = 0.00$$

$$\tilde{p}(Z = 3 \mid X_{test}, \theta\,) = \pi_3 N(X_{test} \mid \mu_3, \sigma_3\,) = 1.00$$

Question 12

Figure 12.1 shows a three time-step Hidden Markov Model (HMM) with binary-state latent $Z_t \in \{0,1\}$ and observed $X_t \in \{0,1\}$ random variables. The local conditional and prior probabilities of the HMM are shown in Table 12.1. Using variational inference, find the approximate posterior distribution of $p(Z_1, Z_2, Z_3 \mid X_1, X_2, X_3)$ using the mean-field approximation, i.e. $q(Z_1, Z_2, Z_3) = \prod_{t=1}^{t=3} q_t(Z_t)$ in one iteration. Assume the initial value of $q_2(Z_2 = 0) = 0.5$, and $X_1 = 0, X_2 = 1, X_3 = 0$. **Explain and show all your workings clearly.**

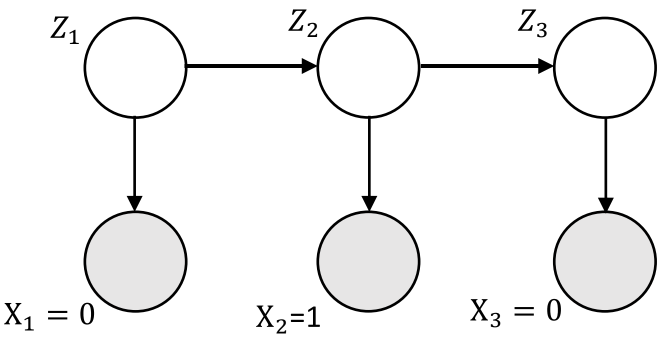


Figure 12.1

z_1	$p(z_1)$
0	0.2
1	0.8

z_2	z_3	$p(z_3 \mid z_2)$
0	0	0.4
0	1	0.6
1	0	0.8
1	1	0.2

z_1	z_2	$p(z_2 \mid z_1)$
0	0	0.3
0	1	0.7
1	0	0.9
1	1	0.1

x_t	z_t	$p(x_t \mid z_t)$
0	0	0.6
0	1	0.7
1	0	0.4
1	1	0.3

Table 12.1

Answer:

Joint probability:

$$p(X,Z) = p(z_1)p(z_2 \mid z_1)p(z_3 \mid z_2)p(x_1 \mid z_1)p(x_2 \mid z_2)p(x_3 \mid z_3)$$

Using mean field approximation: $q(Z) = \prod_{t=1}^{t=3} q_t(z_t)$. The optimal approximation of the posterior $p(Z \mid X, \theta)$is given by:

$$\ln q_1(z_1) = \sum_{z_2}\sum_{z_3} q_2 q_3 \ln p(X \mid Z) + const.$$

$$= \sum_{z_2}\sum_{z_3} q_2 q_3 [\ln p(z_1) + \ln p(z_2 \mid z_1) + \ln p(z_3 \mid z_2) + \ln p(x_1 \mid z_1) + \ln p(x_2 \mid z_2)$$
$$+ \ln p(x_3 \mid z_3) + const.$$

$$= \sum_{z_2} q_2 [\ln p(z_2 \mid z_1) + \ln p(x_2 \mid z_2)] + \sum_{z_3} q_3 [\ln p(z_3 \mid z_2) + \ln p(x_3 \mid z_3)] + \ln p(z_1)$$
$$+ \ln p(x_1 \mid z_1) + const.$$

$$= \sum_{z_2} q_2 \ln p(z_2 \mid z_1) + \ln p(z_1) + \ln p(x_1 \mid z_1) + const.$$

$$\ln q_2(z_2) = \sum_{z_1}\sum_{z_3} q_1 q_3 \ln p(X \mid Z) + const.$$

$$= \sum_{z_1}\sum_{z_3} q_1 q_3 [\ln p(z_1) + \ln p(z_2 \mid z_1) + \ln p(z_3 \mid z_2) + \ln p(x_1 \mid z_1) + \ln p(x_2 \mid z_2) + \ln p(x_3 \mid z_3)] + const.$$

$$= \sum_{z_1} q_1 [\ln p(z_1) + \ln p(x_1 \mid z_1) + \ln p(z_2 \mid z_1)] + \sum_{z_3} q_3 [\ln p(z_3 \mid z_2) + \ln p(x_3 \mid z_3)] + \ln p(x_2 \mid z_2) + const.$$

$$= \sum_{z_1} q_1 \ln p(z_2 \mid z_1) + \sum_{z_3} q_3 \ln p(z_3 \mid z_2) + \ln p(x_2 \mid z_2) + const.$$

$$\ln q_3(z_3) = \sum_{z_1}\sum_{z_2} q_1 q_2 \ln p(X \mid Z) + const.$$

$$= \sum_{z_1}\sum_{z_2} q_1 q_2 [\ln p(z_1) + \ln p(z_2 \mid z_1) + \ln p(z_3 \mid z_2) + \ln p(x_1 \mid z_1) + \ln p(x_2 \mid z_2) + \ln p(x_3 \mid z_3)] + const.$$

$$= \sum_{z_1} q_1 [\ln p(z_1) + \ln p(x_1 \mid z_1)] + \sum_{z_1}\sum_{z_2} q_1 q_2 \ln p(z_2 \mid z_1) + \sum_{z_2} q_2 [\ln p(z_3 \mid z_2) + \ln p(x_2 \mid z_2)] + \ln p(x_3 \mid z_3) + const.$$

$$= \sum_{z_2} q_2 \ln p(z_3 \mid z_2) + \ln p(x_3 \mid z_3) + const.$$

Given $x_1 = 0, x_2 = 1, x_3 = 0$, and initial q_2 follows a uniform distribution, i.e. 0.5, we have:

For $q_1(z_1)$:

$\ln q_1(z_1) = \sum_{z_2} q_2 \ln p(z_2 \mid z_1) + \ln p(z_1) + \ln p(x_1 \mid z_1) + const.$

When $z_1 = 0$,

$$\ln q_1(z_1 = 0) = \sum_{z_2} q_2 \ln p(z_2 \mid z_1 = 0) + \ln p(z_1 = 0) + \ln p(x_1 = 0 \mid z_1 = 0) + const.$$

$$= q_2(z_2 = 0) \ln p(z_2 = 0 \mid z_1 = 0) + q_2(z_2 = 1) \ln p(z_2 = 1 \mid z_1 = 0) + \ln p(z_1 = 0) + \ln p(x_1 = 0 \mid z_1 = 0) + const.$$

$$= (0.5)\ln(0.3) + (0.5)\ln(0.7) + \ln(0.2) + \ln(0.6) + const. = -2.901 + const.$$

$\tilde{q}_1(z_1 = 0) = 0.0550$

When $z_1 = 1$,

$$\ln q_1(z_1=1) = \sum_{z_2} q_2 \ln p(\, z_2 \mid z_1 = 1\,) + \ln p(z_1=1) + \ln p(x_1 = 0 \mid z_1 = 1) + const.$$

$$= \; q_2(z_2=0)\ln p(\, z_2 = 0 \mid z_1 = 1\,) + q_2(z_2=1)\ln p(\, z_2 = 1 \mid z_1 = 1\,) + \ln p(z_1 = 1) + \ln p(\, x_1 = 0 \mid z_1 = 1\,) + const.$$

$$= (0.5)\ln(0.9) + (0.5)\ln(0.1) + \ln(0.8) + \ln(0.7) + const. = -1.784 + const.$$

$\tilde{q}_1(z_1 = 1) = 0.168$

$\boldsymbol{q_1(z_1 = 0) = 0.247}$

$\boldsymbol{q_1(z_1 = 1) = 0.753}$

For $\boldsymbol{q_3(z_3)}$:

$$\ln q_3(z_3) = \sum_{z_2} q_2 \ln p(\, z_3 \mid z_2\,) + \ln p(x_3 \mid z_3) + const.$$

when $\boldsymbol{q_3(z_3 = 0)}$,

$$\ln q_3(z_3=0) = \sum_{z_2} q_2 \ln p(\, z_3 = 0 \mid z_2\,) + \ln p(x_3 = 0 \mid z_3 = 0) + const.$$

$$= q_2(z_2=0)\ln p(\, z_3 = 0 \mid z_2 = 0\,) + q_2(z_2=1)\ln p(\, z_3 = 0 \mid z_2 = 1\,) + \ln p(x_3 = 0 \mid z_3 = 0) + const.$$

$$= (0.5)\ln(0.4) + (0.5)\ln(0.8) + \ln(0.6) + const.$$

$$= -1.081 + const.$$

$\tilde{q}_3(z_3 = 0) = 0.339$

when $\boldsymbol{q_3(z_3 = 1)}$,

$$\ln q_3(z_3=1) = \sum_{z_2} q_2 \ln p(\, z_3 = 1 \mid z_2\,) + \ln p(x_3 = 0 \mid z_3 = 1) + const.$$

$$= q_2(z_2=0)\ln p(\, z_3 = 1 \mid z_2 = 0\,) + q_2(z_2=1)\ln p(\, z_3 = 1 \mid z_2 = 1\,) + \ln p(x_3 = 0 \mid z_3 = 1) + const.$$

$$= (0.5)\ln(0.6) + (0.5)\ln(0.2) + \ln(0.7) + const.$$

$$= -0.360 + const.$$

$\tilde{q}_3(z_3 = 0) = 0.698$

$\boldsymbol{q_3(z_3 = 0) = 0.327}$

$\boldsymbol{q_3(z_3 = 1) = 0.673}$

For $\boldsymbol{q_2(z_2)}$:

$$\ln q_2(z_2) = \sum_{z_1} q_1 \ln p(z_2 \mid z_1) + \sum_{z_3} q_3 \ln p(z_3 \mid z_2) + \ln p(x_2 \mid z_2) + const.$$

when $\boldsymbol{q_2(z_2 = 0)}$,

$$\ln q_2(z_2 = 0) = \sum_{z_1} q_1 \ln p(z_2 = 0 \mid z_1) + \sum_{z_3} q_3 \ln p(z_3 \mid z_2 = 0) + \ln p(x_2 = 1 \mid z_2 = 0) + const.$$

$$= q_1(z_1 = 0) \ln p(z_2 = 0 \mid z_1 = 0) + q_1(z_1 = 1) \ln p(z_2 = 0 \mid z_1 = 1) + q_3(z_3 = 0) \ln p(z_3 = 0 \mid z_2 = 0) + q_3(z_3 = 1)(\ln p(z_3 = 1 \mid z_2 = 0) + \ln p(x_2 = 1 \mid z_2 = 0) + const.$$

$$= (0.247) \ln(0.3) + (0.753) \ln(0.9) + (0.327) \ln(0.4) + (0.673) \ln(0.6) + \ln(0.4) + const.$$

$$= -1.936 + const.$$

$$\tilde{q}_2(z_2 = 0) = 0.144$$

when $\boldsymbol{q_2(z_2 = 1)}$,

$$\ln q_2(z_2 = 1) = \sum_{z_1} q_1 \ln p(z_2 = 1 \mid z_1) + \sum_{z_3} q_3 \ln p(z_3 \mid z_2 = 1) + \ln p(x_2 = 1 \mid z_2 = 1) + const.$$

$$= q_1(z_1 = 0) \ln p(z_2 = 1 \mid z_1 = 0) + q_1(z_1 = 1) \ln p(z_2 = 1 \mid z_1 = 1) + q_3(z_3 = 0) \ln p(z_3 = 0 \mid z_2 = 1) + q_3(z_3 = 1)(\ln p(z_3 = 1 \mid z_2 = 1) + \ln p(x_2 = 1 \mid z_2 = 1) + const.$$

$$= (0.247) \ln(0.7) + (0.753) \ln(0.1) + (0.327) \ln(0.8) + (0.673) \ln(0.2) + \ln(0.3) + const.$$

$$= -4.182 + const.$$

$$\tilde{q}_2(z_2 = 0) = 0.0153$$

$\boldsymbol{q_2(z_2 = 0) = 0.904}$

$\boldsymbol{q_2(z_2 = 1) = 0.096}$

Compute the final approximation to the posterior distribution:

$$q(Z) = \prod_{t=1}^{t=3} q_t(z_t)$$

$$q(z_1 = 0, z_2 = 0, z_3 = 0) = q_1(z_1 = 0)q_2(z_2 = 0)q_2(z_2 = 0) = (0.247)(0.904)(0.327) = \mathbf{0.073}$$

$$q(z_1 = 0, z_2 = 0, z_3 = 1) = q_1(z_1 = 0)q_2(z_2 = 0)q_2(z_2 = 1) = (0.247)(0.904)(0.673) = \mathbf{0.150}$$

$$q(z_1 = 0, z_2 = 1, z_3 = 0) = q_1(z_1 = 0)q_2(z_2 = 1)q_2(z_2 = 0) = (0.247)(0.096)(0.327) = \mathbf{0.00775}$$

$$q(z_1 = 0, z_2 = 1, z_3 = 1) = q_1(z_1 = 0)q_2(z_2 = 1)q_2(z_2 = 1) = (0.247)(0.096)(0.673) = \mathbf{0.0160}$$

$$q(z_1 = 1, z_2 = 0, z_3 = 0) = q_1(z_1 = 0)q_2(z_2 = 0)q_2(z_2 = 0) = (0.753)(0.904)(0.327) = \mathbf{0.223}$$

$$q(z_1 = 1, z_2 = 0, z_3 = 1) = q_1(z_1 = 0)q_2(z_2 = 0)q_2(z_2 = 0) = (0.753)(0.904)(0.673) = \mathbf{0.458}$$

$$q(z_1 = 1, z_2 = 1, z_3 = 0) = q_1(z_1 = 0)q_2(z_2 = 0)q_2(z_2 = 0) = (0.753)(0.096)(0.327) = \mathbf{0.0236}$$

$$q(z_1 = 1, z_2 = 1, z_3 = 1) = q_1(z_1 = 0)q_2(z_2 = 0)q_2(z_2 = 0) = (0.753)(0.096)(0.673) = \mathbf{0.0486}$$

Question 13

Figure 13.1 shows a three-node undirected graphical model, where $X_i \in R_{\geq 0}$, $\psi(X_1, X_2) = \exp\{-\alpha X_1 X_2\}$, $\psi(X_i) = \exp\{-\beta X_i\}$, and $\alpha = 0.5$ and $\beta = 2.5$ are constants.

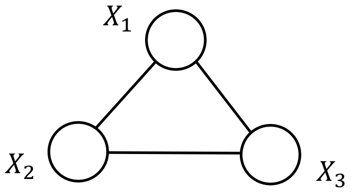


Figure 13.1.

a) Using variational inference, find the expressions of the expectation of X_1, X_2, and X_3 under $q(X_1), q(X_2)$, and $q(X_3)$, respectively, where $q(X_1, X_2, X_3) = q(X_1)q(X_2)q(X_3)$ is the mean-field approximation of the posterior distribution $p(X_1, X_2, X_3)$.

(*15 marks*)

b) Taking the initial expected values of X_2, and X_3 under $q(X_2)$, and $q(X_3)$ to be 2.0, and 1.0, respectively, find the mean-field approximation $q(X_1, X_2, X_3)$ after one iteration.

(*10 marks*)

Show all your workings clearly.

Useful equations:

1. $\int \exp\{kx\}\, dx = \frac{1}{k} \exp kx$ + const, where $k \neq 0$ is a constant.
2. $\int x \exp\{kx\}\, dx = \left(\frac{kx-1}{k^2}\right) \exp\{kx\}$, where $k \neq 0$ is a constant.

<u>Solution</u>

$\int_0^\infty \exp\{-cx\}\ \ dx = \left[-\frac{\exp\{-cx\}}{c}\right]_0^\infty = -\frac{\exp\{-0\}}{c} + \frac{\exp\{-\infty\}}{c} = -\frac{1}{c}$, where $c \neq 0$ is a constant.

$\int_0^\infty x \exp\{-cx\}\ \ dx = \left[\exp\{-cx\}\left(\frac{-cx-1}{c^2}\right)\right]_0^\infty = \exp\{-0\}\left(\frac{-0-1}{c^2}\right) - \exp\{-\infty\}\left(\frac{-\infty-1}{c^2}\right) = -\frac{1}{c^2}$, where $c \neq 0$ is a constant.

(a)

Joint Probability:

$p(X_1, X_2, X_3) = \frac{1}{Z}\psi(X_1, X_2)\psi(X_1, X_3)\psi(X_2, X_3)\psi(X_1)\psi(X_2)\psi(X_3)$

$= \frac{1}{Z}\exp\{-\alpha(X_1X_2 + X_1X_3 + X_2X_3) - \beta(X_1 + X_2 + X_3)\}$

(2 marks)

Mean-field approximation:

$q(X_1, X_2, X_3) = q(X_1)q(X_2)q(X_3)$

Expectation of X_1:

$\ln q^*(X_1)\ = E_{q(X_2)q(X_3)}[\ln\ p(X_1, X_2, X_3)\,] + const$

$= \int\int q(X_2)q(X_3)[-\alpha(X_1X_2 + X_1X_3 + X_2X_3) - \beta(X_1 + X_2 + X_3) - \ln\ Z\,]\, dX_2dX_3 + const$

$= \int\int q(X_2)q(X_3)[-X_1(\alpha X_2 + \alpha X_3) - \beta X_1]dX_2dX_3 + const$

$= -\alpha X_1\int\int X_2q(X_2)q(X_3)dX_2dX_3 - \alpha X_1\int\int X_3q(X_2)q(X_3)dX_2dX_3 - \beta X_1\int\int q(X_2)q(X_3)dX_2dX_3 + const$

$= -\alpha X_1\int X_2q(X_2)dX_2 - \alpha X_1\int X_3q(X_3)dX_3 - \beta X_1 + const$

$= -\alpha X_1\mu_{X_2} - \alpha X_1\mu_{X_3} - \beta X_1 + const$

$q^*(X_1) = \frac{\exp\{-(\alpha\mu_{X_2}+\alpha\mu_{X_3}+\beta)X_1\}}{\int_0^\infty \exp\{-(\alpha\mu_{X_2}+\alpha\mu_{X_3}+\beta)X_1\}\ dX_1} = -(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta)$
$\exp\{-\left(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta\right)X_1\}$

Expectation:

$$\int_0^\infty X_1 q(X_1) dX_1 = -(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta)\int_0^\infty X_1 \exp\{-\left(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta\right)X_1\}\ dX_1$$

$= \frac{(\alpha\mu_{X_2}+\alpha\mu_{X_3}+\beta)}{(\alpha\mu_{X_2}+\alpha\mu_{X_3}+\beta)^2} = \frac{1}{(\alpha\mu_{X_2}+\alpha\mu_{X_3}+\beta)}$.

$\Rightarrow \boldsymbol{\mu_{X_1}} = \frac{\mathbf{1}}{\left(\boldsymbol{\alpha\mu_{X_2}+\alpha\mu_{X_3}+\beta}\right)}$**.**

Expectation of X_2:

$\ln q^*(X_2) \ = E_{q(X_1)q(X_3)}[\ln p(X_1, X_2, X_3)] + const$

$= \int\int q(X_1)q(X_3)[-\alpha(X_1X_2 + X_1X_3 + X_2X_3) - \beta(X_1 + X_2 + X_3) - \ln\ Z\ ]\ dX_1dX_3 + const$

$= \int\int q(X_1)q(X_3)[-X_2(\alpha X_1 + \alpha X_3) - \beta X_2]dX_1dX_3 + const$

$= -\alpha X_2\int\int X_1q(X_1)q(X_3)dX_1dX_3 - \alpha X_2\int\int X_3q(X_2)q(X_3)dX_1dX_3 - \beta X_2\int\int q(X_1)q(X_3)dX_1dX_3 + const$

$= -\alpha X_2\int X_1q(X_1)dX_1 - \alpha X_2\int X_3q(X_3)dX_3 - \beta X_2 + const$

$= -\alpha X_2\mu_{X_1} - \alpha X_2\mu_{X_3} - \beta X_2 + const$

$q^*(X_2) = \frac{\exp\{-(\alpha\mu_{X_1}+\alpha\mu_{X_3}+\beta)X_2\}}{\int_0^\infty \exp\{-(\alpha\mu_{X_1}+\alpha\mu_{X_3}+\beta)X_2\}\, dX_2} = -(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta)$
$\exp\{-\left(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta\right)X_2\}$

Expectation:

$$\int_0^\infty X_2 q(X_2)\, dX_2 = -(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta)\int_0^\infty X_2 \exp\{-\left(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta\right)X_1\}\ dX_2$$

$= \frac{(\alpha\mu_{X_1}+\alpha\mu_{X_3}+\beta)}{(\alpha\mu_{X_1}+\alpha\mu_{X_3}+\beta)^2} = \frac{1}{(\alpha\mu_{X_1}+\alpha\mu_{X_3}+\beta)}$.

$\Rightarrow \boldsymbol{\mu_{X_2}} = \frac{\mathbf{1}}{\left(\boldsymbol{\alpha\mu_{X_1}+\alpha\mu_{X_3}+\beta}\right)}$**.**

Expectation of X_3:

$\ln q^*(X_3) = E_{q(X_1)q(X_2)}[\ln p(X_1, X_2, X_3)\,] + const$

$= \int\int q(X_1)q(X_2)[-\alpha(X_1X_2 + X_1X_3 + X_2X_3) - \beta(X_1 + X_2 + X_3) - \ln Z]\; dX_1dX_2 + const$

$= \int\int q(X_1)q(X_2)[-X_3(\alpha X_1 + \alpha X_2) - \beta X_3]dX_1dX_2 + const$

$= -\alpha X_3 \int\int X_1 q(X_1)q(X_2)dX_1dX_2 - \alpha X_3 \int\int X_2 q(X_1)q(X_2)dX_1dX_2 -$
$\beta X_3 \int\int q(X_1)q(X_2)dX_1dX_2 + const$

$= -\alpha X_3 \int X_1 q(X_1)dX_1 - \alpha X_3 \int X_2 q(X_2)dX_2 - \beta X_3 + const$

$= -\alpha X_3 \mu_{X_1} - \alpha X_3 \mu_{X_2} - \beta X_3 + const$

$q^*(X_3) = \frac{\exp\{-(\alpha\mu_{X_1}+\alpha\mu_{X_2}+\beta)X_3\}}{\int_0^\infty \exp\{-(\alpha\mu_{X_1}+\alpha\mu_{X_2}+\beta)X_3\}\,dX_3} = -\left(\alpha\mu_{X_1} + \alpha\mu_{X_2} + \beta\right)$
$\exp\left\{-\left(\alpha\mu_{X_1} + \alpha\mu_{X_2} + \beta\right)X_3\right\}$

Expectation:

$$\int_0^\infty X_3 q(X_3)\, dX_3 = -(\alpha\mu_{X_1} + \alpha\mu_{X_2} + \beta)\int_0^\infty \quad X_3 \exp\left\{-\left(\alpha\mu_{X_1} + \alpha\mu_{X_2} + \beta\right)X_1\right\} dX_3$$

$= \frac{(\alpha\mu_{X_1}+\alpha\mu_{X_2}+\beta)}{(\alpha\mu_{X_1}+\alpha\mu_{X_2}+\beta)^2} = \frac{1}{(\alpha\mu_{X_1}+\alpha\mu_{X_2}+\beta)}\,.$

$\Rightarrow \boldsymbol{\mu_{X_3}} = \frac{\mathbf{1}}{\left(\boldsymbol{\alpha\mu_{X_1}+\alpha\mu_{X_2}+\beta}\right)}$**.**

(b)

Given the initial values of $\mu_{X_2} = 0.2$ and $\mu_{X_3} = 0.1$, we get

$q^*(X_1) = -(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta)\exp\left\{-\left(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta\right)X_1\right\}$

$= -((0.5)(0.2) + (0.5)(0.1) + 0.25)\exp\{-((0.5)(0.2) + (0.5)(0.1) + 0.25)X_1\}$

$= -0.4\exp(-0.4X_1)$

$$\mu_{X_1} = \frac{1}{\left(\alpha\mu_{X_2} + \alpha\mu_{X_3} + \beta\right)} = \frac{1}{0.4} = 2.5$$

$q^*(X_2) = -(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta)exp\left\{-\left(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta\right)X_2\right\}$

$= -\left((0.5)(2.5) + (0.5)(0.1) + 0.25\right)exp\left\{-\left((0.5)(0.25) + (0.5)(1) + 2.5\right)X_2\right\}$

$= -(1.55)exp\,\{-1.55X_2\}$

$$\mu_{X_2} = \frac{1}{\left(\alpha\mu_{X_1} + \alpha\mu_{X_3} + \beta\right)} = \frac{1}{1.55} = 0.64$$

$q^*(X_3) = -(\alpha\mu_{X_1} + \alpha\mu_{X_2} + \beta)\ exp\ \left\{-\left(\alpha\mu_{X_1} + \alpha\mu_{X_2} + \beta\right)X_3\right\}$

$= -((0.5)(0.25) + (0.5)(0.64) + 0.25)\ exp\ \{-((0.5)(0.25) + (0.5)(0.64) + 0.25)X_3\}$

$= -0.695\ exp\ \{-0.695X_3\}$

$q(X_1, X_2, X_3) = q(X_1)q(X_2)q(X_3)$

$= (-0.4)(-3.125)(-2.875)exp\,\{-0.4X_1 - 1.55X_2 - 0.695X_3\}$

$= -3.59375exp\,\{-0.4X_1 - 1.55X_2 - 0.695X_3\}$

Question 14

Figure 14.1 shows a three-node directed graphical model where $X_i \in \{0,1\}$. The prior and conditional probabilities are given by the following Bernoulli's distributions:

$p(X_1 \mid \lambda) = \lambda^{X_1}(1-\lambda)^{1-X_1}$, where $0 \le \lambda \le 1$;

$p(X_2 \mid X_1, \beta) = \beta_{X_1}^{X_2}\left(1-\beta_{X_1}\right)^{1-X_2}$, where $\beta_{X_1} = \begin{cases} \beta_0,\ 0 \le \beta_0 \le 1, & if\ X_1 = 0 \\ \beta_1,\ 0 \le \beta_1 \le 1, & otherwise \end{cases}$;

$p(X_3 \mid X_1, \gamma) = \gamma_{X_1}^{X_3}\left(1-\gamma_{X_1}\right)^{1-X_3}$, where $\gamma_{X_1} = \begin{cases} \gamma_0,\ 0 \le \gamma_0 \le 1, & if\ X_1 = 0 \\ \gamma_1,\ 0 \le \gamma_1 \le 1, & otherwise \end{cases}$.

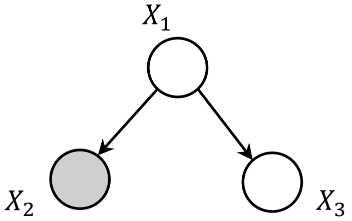


Figure 14.1.

a) Given the observation of $X_2 = 1$ and the initial conditions: $\lambda = 0.5, \beta_0 = 0.2, \beta_1 = 0.6, \gamma_0 = 0.3$, and $\gamma_1 = 0.4$. Draw *one set of samples* for X_1 and X_3 using Gibbs sampling. Assume initial samples of $X_1 = 0, X_2 = 1$ and $X_3 = 1$, and $u = [0.7655, 0.1869]$ are two numbers drawn from a uniform distribution in the range of $[0,1]$, i.e., $u \sim \text{Uniform}(0,1)$.

(*10 marks*)

b) Suppose that the subsequent samples drawn from Gibbs sampling are: $\{[X_1 = 1, X_3 = 0], [X_1 = 0, X_3 = 1], [X_1 = 1, X_3 = 1], [X_1 = 0, X_3 = 0]\}$ under the observations of $X_2 = \{0, 1, 1, 0\}$, respectively. Find the updated optimal parameters $\lambda, \beta_0, \beta_1, \gamma_0$, and γ_1 with all the samples drawn from Gibbs sampling. Ignore the initial set of samples and the burn-in effect.

(*15 marks*)

Show all your workings clearly.

Useful equation: $\frac{d}{dx}\ln\big(f(x)\big) = \frac{1}{f(x)}\frac{df(x)}{dx}$.

Solution

(a)

Joint probability: $p(X_1, X_2, X_3) = p(X_1)p(X_2)p(X_3 \mid X_1)$.

$X_1 \sim p(X_1 \mid X_2 = 1, X_3)$

$= \frac{p(X_1)p(X_2)p(X_3|X_1)}{\sum_{X_1} p(X_1)p(X_2)p(X_3|X_1)} \propto p(X_1)p(X_2)p(X_3 \mid X_1)$

$X_3 \sim p(X_3 \mid X_1, X_2 = 1)$

$= \frac{p(X_1)p(X_2)p(X_3|X_1)}{\sum_{X_3} p(X_1)p(X_2)p(X_3|X_1)} = p(X_3 \mid X_1)$

Given $X_1 = 0$ and $X_3 = 1$, and $\lambda = 0.5, \beta_0 = 0.2, \beta_1 = 0.6, \gamma_0 = 0.3$, and $\gamma_1 = 0.4$,

First iteration:

$p(X_1 = 0 \mid X_2 = 1, X_3 = 1) \propto p(X_1 = 0)p(X_2 = 1)p(X_3 = 1 \mid X_1 = 0)$

$= (1 - \lambda)(\beta_0)(\gamma_0) = (1 - 0.5)(0.2)(0.3) = 0.03$

$p(X_1 = 1 \mid X_2 = 1, X_3 = 1) \propto p(X_1 = 1)p(X_2 = 1)p(X_3 = 1 \mid X_1 = 0)$

$= (\lambda)(\beta_1)(\gamma_1) = (0.5)(0.2)(0.3) = 0.03$

$$p(X_1 = 0 \mid X_2 = 1, X_3 = 1) = \frac{0.03}{0.03 + 0.03} = 0.5$$

$$p(X_1 = 1 \mid X_2 = 1, X_3 = 1) = \frac{0.03}{0.03 + 0.03} = 0.5$$

Since $\boldsymbol{u = 0.7655}$, we have $\boldsymbol{X_1 = 1}$.

$p(X_3 = 0 \mid X_1 = 1, X_2 = 1) \propto p(X_3 = 0 \mid X_1 = 1)$

$= 1 - \gamma_1 = 1 - 0.4 = 0.6$

$p(X_3 = 1 \mid X_1 = 1, X_2 = 1) \propto p(X_3 = 0 \mid X_1 = 1)$

$= \gamma_1 = 0.4$

Since $\boldsymbol{u = 0.1869}$, we have $\boldsymbol{X_3 = 0}$.

(b)

Joint probability: $p(X_1, X_2, X_3) = p(X_1)p(X_1)p(X_3 \mid X_1)$.

$\ln p(X_1, X_2, X_3) = \ln p(X_1) + \ln p(X_1) + \ln p(X_3 \mid X_1)$

$= X_1 \ln \lambda + (1 - X_1) \ln(1 - \lambda) + X_2 \ln \beta_{X_1} + (1 - X_2) \ln\left(1 - \beta_{X_1}\right) + X_3 \ln \gamma_{X_1} + (1 - X_3)$ $\ln\left(1 - \gamma_{X_1}\right)$

$\ln p(X_1, X_2, X_3 \mid \theta) \approx \frac{1}{L}\sum_l \ln p(X_1, X_2, X_3 \mid \theta)$

$$\ln p(X_1, X_2, X_3 \mid \theta) \approx \frac{1}{6}\left\{ \ln p\left(X_1^{(1)}, X_2, X_3^{(1)} \mid \theta\right) + \dots + \ln p\left(X_1^{(6)}, X_2, X_3^{(6)} \mid \theta\right) \right\}$$

$ln\, p(X_1 = 0, X_2 = 0, X_3 = 0) = ln\,(1 - \lambda) + ln\,(1 - \beta_0) + ln\,(1 - \gamma_0)$

$ln\, p(X_1 = 0, X_2 = 0, X_3 = 1) = ln\,(1 - \lambda) + ln\,(1 - \beta_0) + ln\,\gamma_0$

$ln\, p(X_1 = 1, X_2 = 0, X_3 = 0) = ln\,\lambda + ln\,(1 - \beta_1) + ln\,(1 - \gamma_1)$

$$ln\, p(X_1 = 1, X_2 = 0, X_3 = 1) = ln\, \lambda + ln\, (1 - \beta_1) + ln\, \gamma_1$$

$$ln\, p(X_1 = 0, X_2 = 1, X_3 = 0) = ln\, (1 - \lambda) + ln\, \beta_0 + ln\, (1 - \gamma_0)$$

$$ln\, p(X_1 = 0, X_2 = 1, X_3 = 1) = ln\, (1 - \lambda) + ln\, \beta_0 + ln\, \gamma_0$$

$$ln\, p(X_1 = 1, X_2 = 1, X_3 = 0) = ln\, \lambda + ln\, \beta_1 + ln\, (1 - \gamma_1)$$

$$ln\, p(X_1 = 1, X_2 = 1, X_3 = 1) = ln\, \lambda + ln\, ln\, \beta_1 + ln\, \gamma_1$$

Given: $[X_1 = 1, X_2 = 1, X_3 = 0], [X_1 = 1, X_2 = 0, X_3 = 0], [X_1 = 0, X_2 = 1, X_3 = 1], [X_1 = 1, X_2 = 1, X_3 = 1], [X_1 = 0, X_2 = 0, X_3 = 0],$

$$\ln p(X_1, X_2, X_3 \mid \theta) \approx \tfrac{1}{6}[ln\, \lambda + ln\, \beta_1 + ln\, (1 - \gamma_1) + ln\, \lambda + ln\, (1 - \beta_1) + ln\, (1 - \gamma_1) +$$

$$ln\, (1 - \lambda) + ln\, \beta_0 + ln\, \gamma_0 + ln\, \lambda + ln\, \beta_1 + ln\, \gamma_1 +$$

$$ln\, (1 - \lambda) + ln\, (1 - \beta_0) + ln\, (1 - \gamma_0)\,]$$

$$= \frac{1}{6}\big[3ln\, \lambda + 2ln\, \beta_1 + 2ln\, (1 - \gamma_1) + ln\, (1 - \beta_1) + 2ln\, (1 - \lambda) + ln\, \beta_0 + ln\, \gamma_0$$
$$+ ln\, \gamma_1 + ln\, (1 - \beta_0) + ln\, (1 - \gamma_0)\big]$$

$$\max_\theta \ln p(X_1, X_2, X_3 \mid \theta) \approx \max_\theta \frac{1}{L} \sum_l \ln p(X_1, X_2, X_3 \mid \theta)$$

$$\frac{\delta Q}{\delta \lambda} = \frac{3}{\lambda} - \frac{2}{1-\lambda} \Rightarrow 2\lambda = 3(1 - \lambda) \Rightarrow \boldsymbol{\lambda} = \frac{\mathbf{3}}{\mathbf{5}}.$$

$$\frac{\delta Q}{\delta \beta_0} = \frac{1}{\beta_0} - \frac{1}{1-\beta_0} \Rightarrow 1 - \beta_0 = \beta_0 \Rightarrow \boldsymbol{\beta_0} = \frac{\mathbf{1}}{\mathbf{2}}.$$

$$\frac{\delta Q}{\delta \beta_1} = \frac{2}{\beta_1} - \frac{1}{1-\beta_1} \Rightarrow 2 - 2\beta_1 = \beta_1 \Rightarrow \boldsymbol{\beta_1} = \frac{\mathbf{2}}{\mathbf{3}}.$$

$$\frac{\delta Q}{\delta \gamma_0} = \frac{1}{\gamma_0} - \frac{1}{1-\gamma_0} \Rightarrow 1 - \gamma_0 = \gamma_0 \Rightarrow \boldsymbol{\gamma_0} = \frac{\mathbf{1}}{\mathbf{2}}.$$

$$\frac{\delta Q}{\delta \gamma_1} = \frac{1}{\gamma_1} - \frac{2}{1-\gamma_1} \Rightarrow 1 - \gamma_1 = 2\gamma_1 \Rightarrow \boldsymbol{\gamma_1} = \frac{\mathbf{1}}{\mathbf{3}}.$$

Question 15

Figure 15.1 shows a hidden Markov model with binary latent and observed random variables, i.e. $Z_i \in \{0,1\}$ and $X_i \in \{0,1\}$. The prior, transition and emission probabilities are given in Table 3.1, 3.2 and 3.3, respectively.

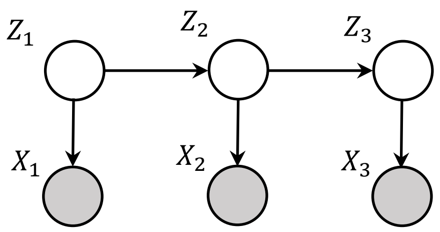


Figure. 15.1

Table. 15.1

Z_1	$p(Z_1)$
0	0.2
1	0.8

Table. 15.2

Z_j	Z_i	$p(Z_j \mid Z_i)$
0	0	0.2
1	0	0.8
0	1	0.7
1	1	0.3

Table. 15.3

X_i	Z_i	$p(X_i \mid Z_i)$
0	0	0.4
1	0	0.6
0	1	0.1
1	1	0.9

a) Find the values of a, b and c that give the conditional probability of $p(Z_2 = 0 \mid X_1 = a, X_2 = b, X_3 = c) = 0.579$.

(*15 marks*)

b) Given $X_1 = 0, X_2 = 1, X_3 = 0$, find Z_1, Z_2 and Z_3 that give the maximum joint probability of $p(X_1, X_2, X_3, Z_1, Z_2, Z_3)$.

(*10 marks*)

Show all your workings clearly.

<u>Solution</u>

Joint probability:

$$p(X, Z) = p(Z_1)p(Z_2 \mid Z_1)p(Z_3 \mid Z_2)p(X_1 \mid Z_1)p(X_2 \mid Z_2)p(X_3 \mid Z_3)$$

(a)

$p(X_1, X_2, X_3) = \frac{p(Z_2, X_1, X_2, X_3)}{p(X_1, X_2, X_3)}$, where

$p(Z_2, X_1, X_2, X_3) = p(X_2 \mid Z_2) \sum_{Z_1} p(Z_1) p(Z_2 \mid Z_1) p(X_1 \mid Z_1) \sum_{Z_3} p(Z_3 \mid Z_2)\, p(X_3 \mid Z_3)$

$p(X_1, X_2, X_3) = \sum_{Z_2} p(Z_2, X_1, X_2, X_3)$

Z_2	X_3	$m_{Z_3}(X_3) = \sum_{Z_3} p(Z_3 \mid Z_2)\, p(X_3 \mid Z_3)$
0	0	p(Z3=0 | Z2=0)p(X3=0 | Z3=0) + p(Z3=1 | Z2=0)p(X3=0 | Z3=1) = (0.200000)(0.400000) + (0.800000)(0.100000) = 0.160000
0	1	p(Z3=0 | Z2=0)p(X3=1 | Z3=0) + p(Z3=1 | Z2=0)p(X3=1 | Z3=1) = (0.200000)(0.600000) + (0.800000)(0.900000) = 0.840000
1	0	p(Z3=0 | Z2=1)p(X3=0 | Z3=0) + p(Z3=1 | Z2=1)p(X3=0 | Z3=1) = (0.700000)(0.400000) + (0.300000)(0.100000) = 0.310000
1	1	p(Z3=0 | Z2=1)p(X3=1 | Z3=0) + p(Z3=1 | Z2=1)p(X3=1 | Z3=1) = (0.700000)(0.600000) + (0.300000)(0.900000) = 0.690000

Z_2	X_1	$m_{Z_1}(Z_2, X_1) = \sum_{Z_1} p(Z_1) p(Z_2 \mid Z_1) p(X_1 \mid Z_1)$
0	0	p(Z1=0)p(Z2=0.000000 | Z1=0)p(X1=0.000000 | Z1=0) + p(Z1=1)p(Z2=0.000000 | Z1=1)p(X1=0.000000 | Z1=1) = (0.200000)(0.200000)(0.400000) + (0.800000)(0.700000)(0.100000) = 0.072000
0	1	p(Z1=0)p(Z2=0.000000 | Z1=0)p(X1=1.000000 | Z1=0) + p(Z1=1)p(Z2=0.000000 | Z1=1)p(X1=1.000000 | Z1=1) = (0.200000)(0.200000)(0.600000) + (0.800000)(0.700000)(0.900000) = 0.528000
1	0	p(Z1=0)p(Z2=1.000000 | Z1=0)p(X1=0.000000 | Z1=0) + p(Z1=1)p(Z2=1.000000 | Z1=1)p(X1=0.000000 | Z1=1) = (0.200000)(0.800000)(0.400000) + (0.800000)(0.300000)(0.100000) = 0.088000
1	1	p(Z1=0)p(Z2=1.000000 | Z1=0)p(X1=1.000000 | Z1=0) + p(Z1=1)p(Z2=1.000000 | Z1=1)p(X1=1.000000 | Z1=1) = (0.200000)(0.800000)(0.600000) + (0.800000)(0.300000)(0.900000) = 0.312000

Z_2	X_1	X_2	X_3	$p(Z_2, X_1, X_2, X_3) = p(X_2 \mid Z_2) m_{Z_1}(Z_2, X_1) m_{Z_3}(Z_2, X_3)$
0	0	0	0	p(X1=0 | Z2=0)mZ1(Z2=0,X1=0)mZ3(Z2=0,X3=0) = (0.400000)(0.072000)(0.160000) = 0.004608
0	0	0	1	p(X1=0 | Z2=0)mZ1(Z2=0,X1=0)mZ3(Z2=0,X3=1) = (0.400000)(0.072000)(0.840000) = 0.024192

0	0	1	0	p(X1=1 | Z2=0)mZ1(Z2=0,X1=0)mZ3(Z2=0,X3=0) = (0.600000)(0.072000)(0.160000) = 0.006912
0	0	1	1	p(X1=1 | Z2=0)mZ1(Z2=0,X1=0)mZ3(Z2=0,X3=1) = (0.600000)(0.072000)(0.840000) = 0.036288
0	1	0	0	p(X1=0 | Z2=0)mZ1(Z2=0,X1=1)mZ3(Z2=0,X3=0) = (0.400000)(0.528000)(0.160000) = 0.033792
0	1	0	1	p(X1=0 | Z2=0)mZ1(Z2=0,X1=1)mZ3(Z2=0,X3=1) = (0.400000)(0.528000)(0.840000) = 0.177408
0	1	1	0	p(X1=1 | Z2=0)mZ1(Z2=0,X1=1)mZ3(Z2=0,X3=0) = (0.600000)(0.528000)(0.160000) = 0.050688
0	1	1	1	p(X1=1 | Z2=0)mZ1(Z2=0,X1=1)mZ3(Z2=0,X3=1) = (0.600000)(0.528000)(0.840000) = 0.266112
1	0	0	0	p(X1=0 | Z2=1)mZ1(Z2=1,X1=0)mZ3(Z2=1,X3=0) = (0.100000)(0.088000)(0.310000) = 0.002728
1	0	0	1	p(X1=0 | Z2=1)mZ1(Z2=1,X1=0)mZ3(Z2=1,X3=1) = (0.100000)(0.088000)(0.690000) = 0.006072
1	0	1	0	p(X1=1 | Z2=1)mZ1(Z2=1,X1=0)mZ3(Z2=1,X3=0) = (0.900000)(0.088000)(0.310000) = 0.024552
1	0	1	1	p(X1=1 | Z2=1)mZ1(Z2=1,X1=0)mZ3(Z2=1,X3=1) = (0.900000)(0.088000)(0.690000) = 0.054648
1	1	0	0	p(X1=0 | Z2=1)mZ1(Z2=1,X1=1)mZ3(Z2=1,X3=0) = (0.100000)(0.312000)(0.310000) = 0.009672
1	1	0	1	p(X1=0 | Z2=1)mZ1(Z2=1,X1=1)mZ3(Z2=1,X3=1) = (0.100000)(0.312000)(0.690000) = 0.021528
1	1	1	0	p(X1=1 | Z2=1)mZ1(Z2=1,X1=1)mZ3(Z2=1,X3=0) = (0.900000)(0.312000)(0.310000) = 0.087048
1	1	1	1	p(X1=1 | Z2=1)mZ1(Z2=1,X1=1)mZ3(Z2=1,X3=1) = (0.900000)(0.312000)(0.690000) = 0.193752

X_1	X_2	X_3	$p(X_1, X_2, X_3) = p \sum_{Z_2} p(Z_2, X_1, X_2, X_3)$
0	0	0	p(Z2=0,X1=0,X2=0,X3=0) + p(Z2=1,X1=0,X2=0,X3=0) = (0.004608)+(0.002728) = 0.007336
0	0	1	p(Z2=0,X1=0,X2=0,X3=1) + p(Z2=1,X1=0,X2=0,X3=1) = (0.024192)+(0.006072) = 0.030264
0	1	0	p(Z2=0,X1=0,X2=1,X3=0) + p(Z2=1,X1=0,X2=1,X3=0) = (0.006912)+(0.024552) = 0.031464
0	1	1	p(Z2=0,X1=0,X2=1,X3=1) + p(Z2=1,X1=0,X2=1,X3=1) = (0.036288)+(0.054648) = 0.090936
1	0	0	p(Z2=0,X1=1,X2=0,X3=0) + p(Z2=1,X1=1,X2=0,X3=0) = (0.033792)+(0.009672) = 0.043464
1	0	1	p(Z2=0,X1=1,X2=0,X3=1) + p(Z2=1,X1=1,X2=0,X3=1) = (0.177408)+(0.021528) = 0.198936
1	1	0	p(Z2=0,X1=1,X2=1,X3=0) + p(Z2=1,X1=1,X2=1,X3=0) = (0.050688)+(0.087048) = 0.137736
1	1	1	p(Z2=0,X1=1,X2=1,X3=1) + p(Z2=1,X1=1,X2=1,X3=1) = (0.266112)+(0.193752) = 0.459864

X_1	X_2	X_3	$p(Z_2 = 0 \mid X_1 = a, X_2 = b, X_3 = c)$
0	0	0	0.004608/0.007336 = 0.6281
0	0	1	0.024192/0.030264 = 0.7994
0	1	0	0.006912/0.031464 = 0.2196
0	1	1	0.036288/0.090936 = 0.3990
1	0	0	0.033792/0.043464 = 0.7775
1	0	1	0.177408/0.198936 = 0.8918
1	1	0	0.050688/0.137736 = 0.3679
1	1	1	0.266112/0.459864 = 0.5787

(b)

Z_1	Z_2	Z_3	$p(Z_1)p(Z_2 \mid Z_1)p(Z_3 \mid Z_2)p(X_1 = 0 \mid Z_1)p(X_2 = 1 \mid Z_2)p(X_3 = 0 \mid Z_3)$
0	0	0	p(Z1=0)p(Z3=0 | Z2=0)p(Z2=0 | Z1=0)p(X1=0 | Z1=0)p(X2=1 | Z2=0)p(X3=0 | Z3=0) = (0.200000)(0.200000)(0.200000)(0.600000)(0.600000)(0.600000) = 0.000768
0	0	1	p(Z1=0)p(Z3=1 | Z2=0)p(Z2=0 | Z1=0)p(X1=0 | Z1=0)p(X2=1 | Z2=0)p(X3=0 | Z3=1) = (0.200000)(0.800000)(0.200000)(0.600000)(0.600000)(0.900000) = 0.000768
0	**1**	**0**	p(Z1=0)p(Z3=0 | Z2=1)p(Z2=1 | Z1=0)p(X1=0 | Z1=0)p(X2=1 | Z2=1)p(X3=0 | Z3=0) = (0.200000)(0.700000)(0.800000)(0.600000)(0.900000)(0.600000) = 0.016128
0	1	1	p(Z1=0)p(Z3=1 | Z2=1)p(Z2=1 | Z1=0)p(X1=0 | Z1=0)p(X2=1 | Z2=1)p(X3=0 | Z3=1) = (0.200000)(0.300000)(0.800000)(0.600000)(0.900000)(0.900000) = 0.001728
1	0	0	p(Z1=1)p(Z3=0 | Z2=0)p(Z2=0 | Z1=1)p(X1=0 | Z1=1)p(X2=1 | Z2=0)p(X3=0 | Z3=0) = (0.800000)(0.200000)(0.700000)(0.900000)(0.600000)(0.600000) = 0.002688
1	0	1	p(Z1=1)p(Z3=1 | Z2=0)p(Z2=0 | Z1=1)p(X1=0 | Z1=1)p(X2=1 | Z2=0)p(X3=0 | Z3=1) = (0.800000)(0.800000)(0.700000)(0.900000)(0.600000)(0.900000) = 0.002688
1	1	0	p(Z1=1)p(Z3=0 | Z2=1)p(Z2=1 | Z1=1)p(X1=0 | Z1=1)p(X2=1 | Z2=1)p(X3=0 | Z3=0) = (0.800000)(0.700000)(0.300000)(0.900000)(0.900000)(0.600000) = 0.006048
1	1	1	p(Z1=1)p(Z3=1 | Z2=1)p(Z2=1 | Z1=1)p(X1=0 | Z1=1)p(X2=1 | Z2=1)p(X3=0 | Z3=1) = (0.800000)(0.300000)(0.300000)(0.900000)(0.900000)(0.900000) = 0.000648

--End--