Fundamental math

Calculus

Derivative

Slope and polynomial

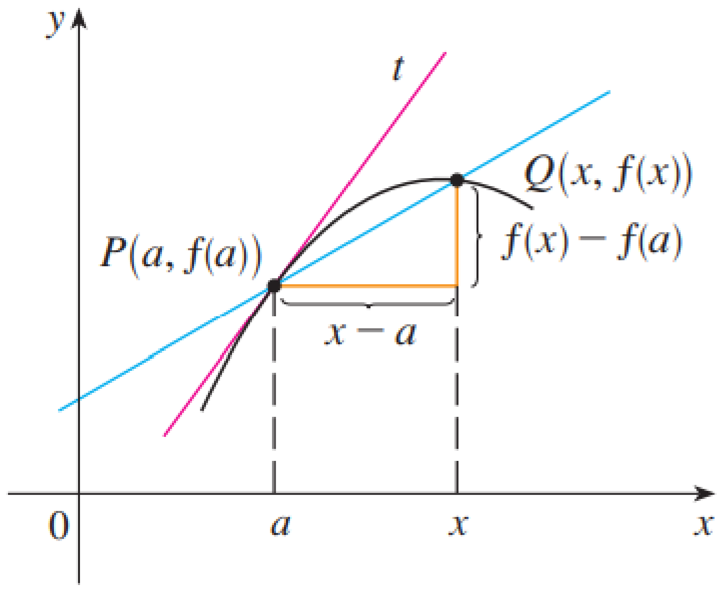
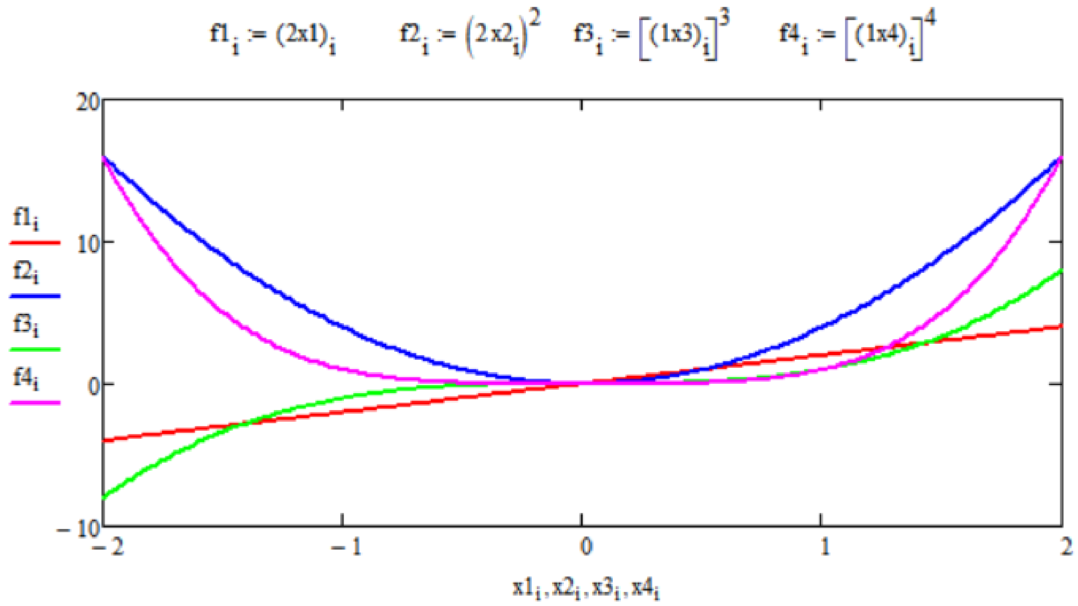


FIGURE 6
The secant line *PQ*

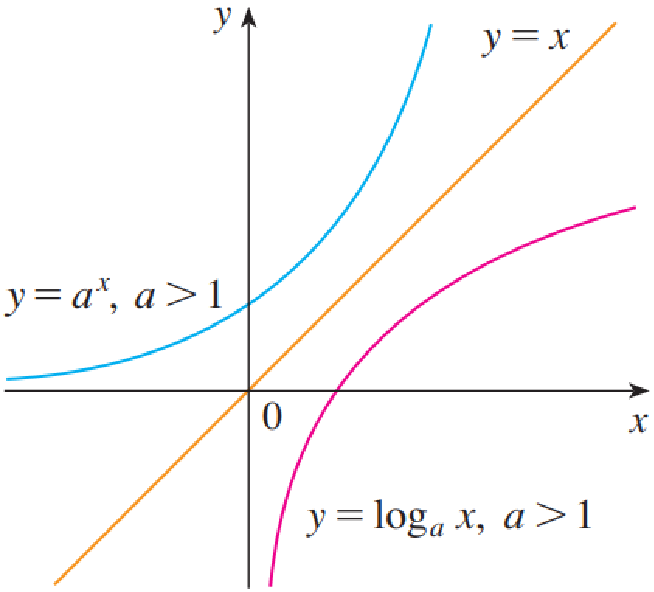
$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

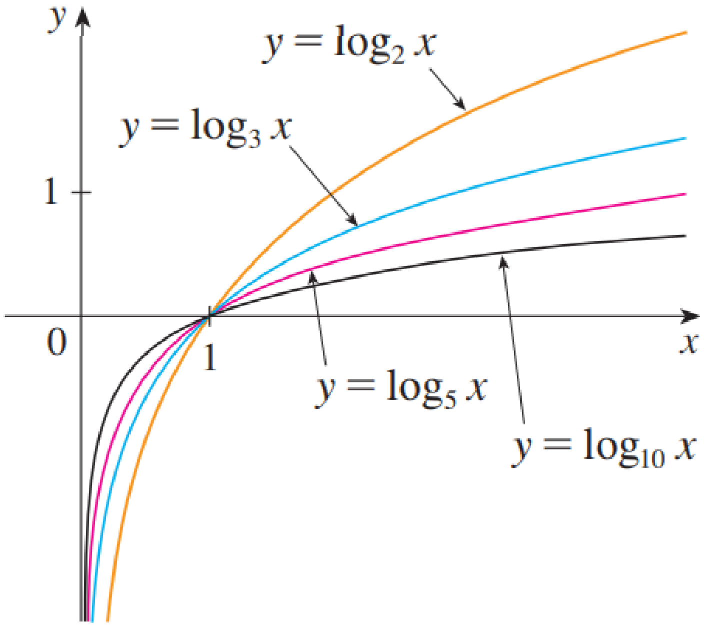
$$m = \lim_{Q \to P} m_{PQ}$$

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$



Logarithms





$$\log_a\left(a^x\right) = x \text{ for every } x \in R$$

$$a^{\log_a x} = x \text{ for every } x > 0$$

Laws of Logarithms

$$\log_a\left(xy\right) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a\left(x^r\right) = r\log_a x$$

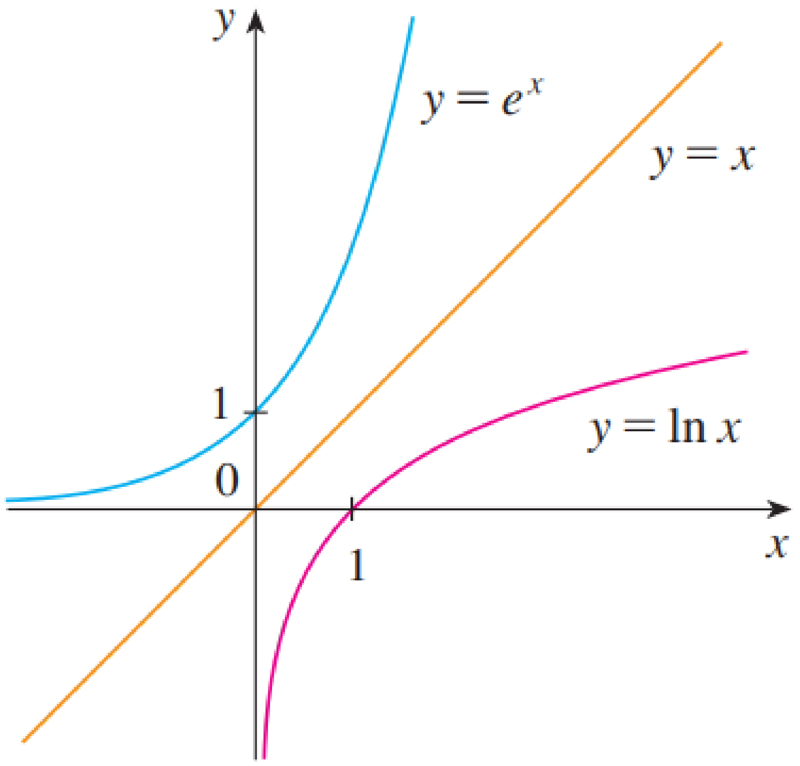
Example

EXAMPLE 6 Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$$

Natural logarithms



$$\log_e x = \ln x$$

$$\ln x = y \iff e^y = x$$

$$\ln\left(e^x\right) = x \quad x \in R$$

$$e^{\ln x} = x \quad x > 0$$

$$\ln e = 1$$

EXAMPLE

Solve the equation $e^{5-3x} = 10$.

SOLUTION

$$\ln(e^{5-3x}) = \ln 10$$

$$5 - 3x = \ln 10$$

$$3x = 5 - \ln 10$$

$$x = \tfrac{1}{3}(5 - \ln 10)$$

EXAMPLE

Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

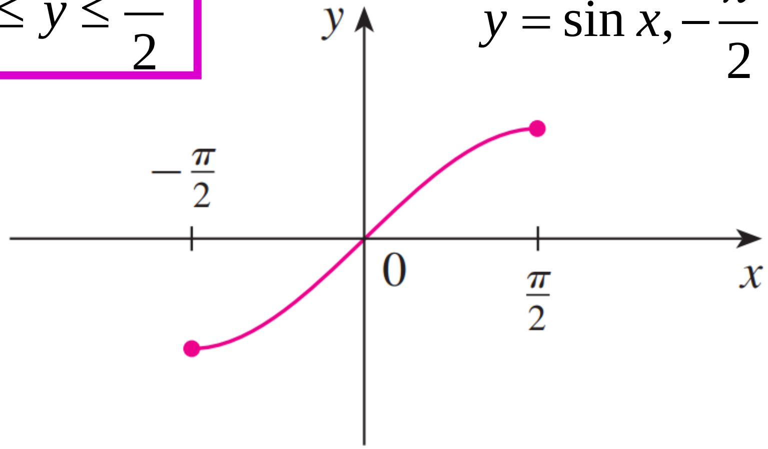
SOLUTION

$$\begin{aligned}\ln a + \tfrac{1}{2}\ln b &= \ln a + \ln b^{1/2}\\ &= \ln a + \ln\sqrt{b}\\ &= \ln\left(a\sqrt{b}\right)\end{aligned}$$

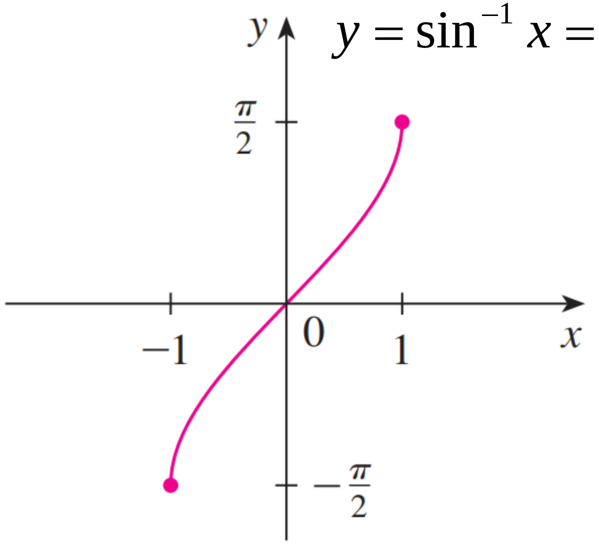
Inverse function

$$\sin^{-1} x = y \Leftrightarrow \sin y = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

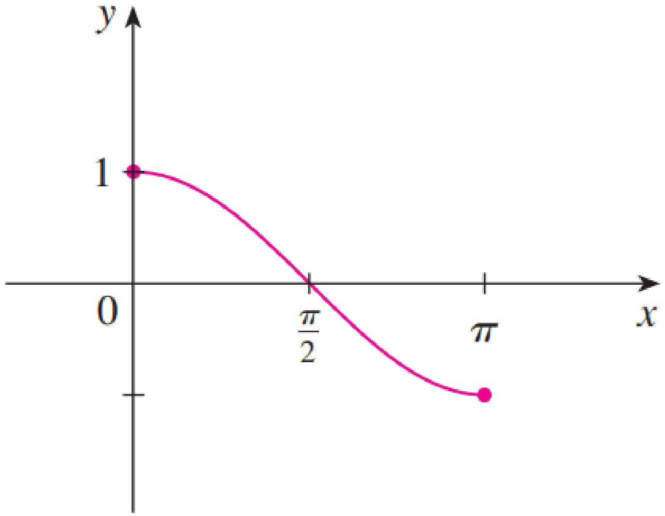


$y = \sin^{-1} x = \arcsin x$

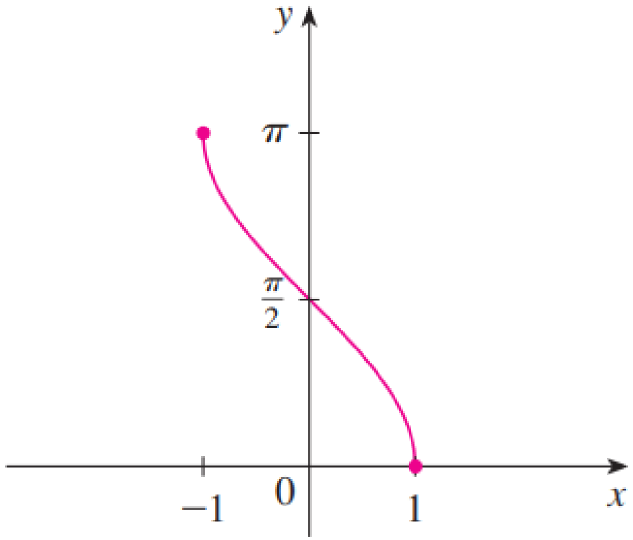


$$\cos^{-1} x = y \Leftrightarrow \cos y = x \text{ and } 0 \le y \le \pi$$

$y = \cos x, 0 \le x \le \pi$



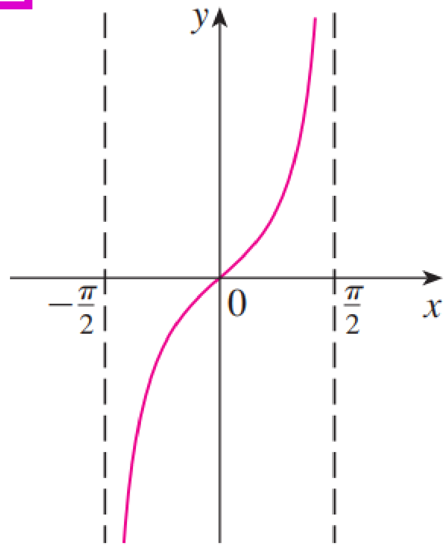
$y = \cos^{-1} x = \arccos x$



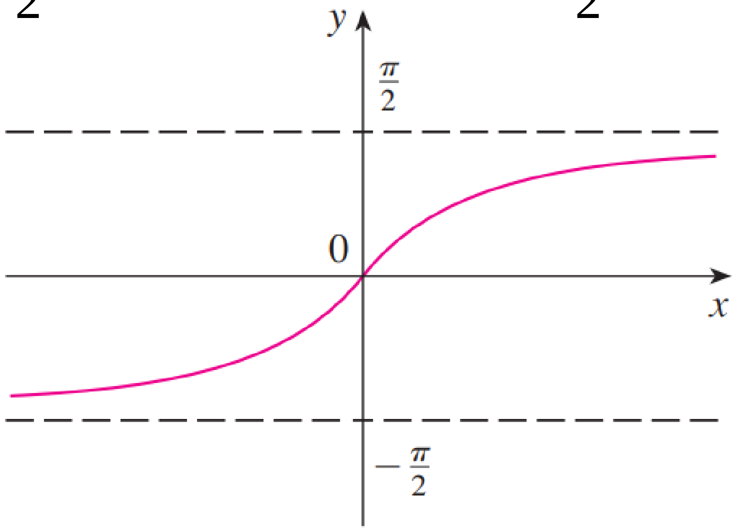
Inverse function

$$\tan^{-1} x = y \Leftrightarrow \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$



$$y = \tan^{-1} x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$



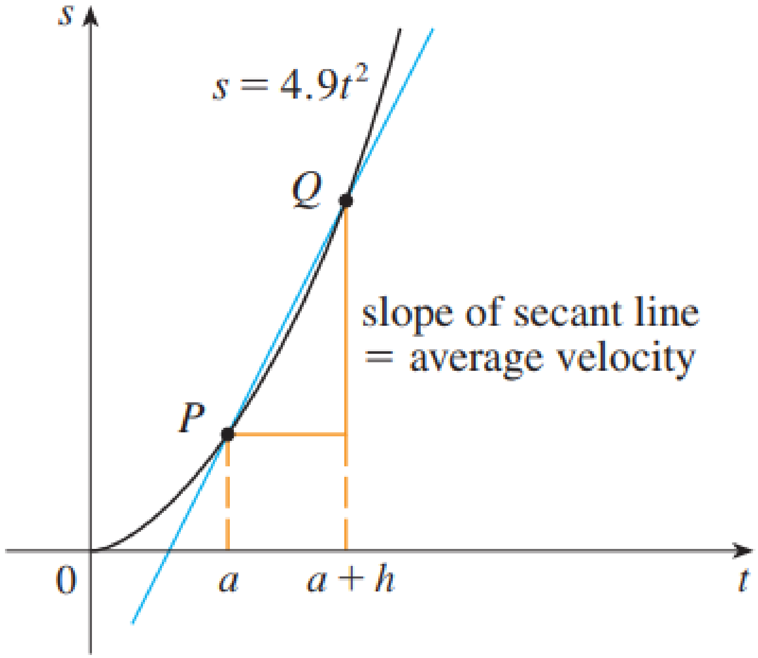
Rarely used
And summary here

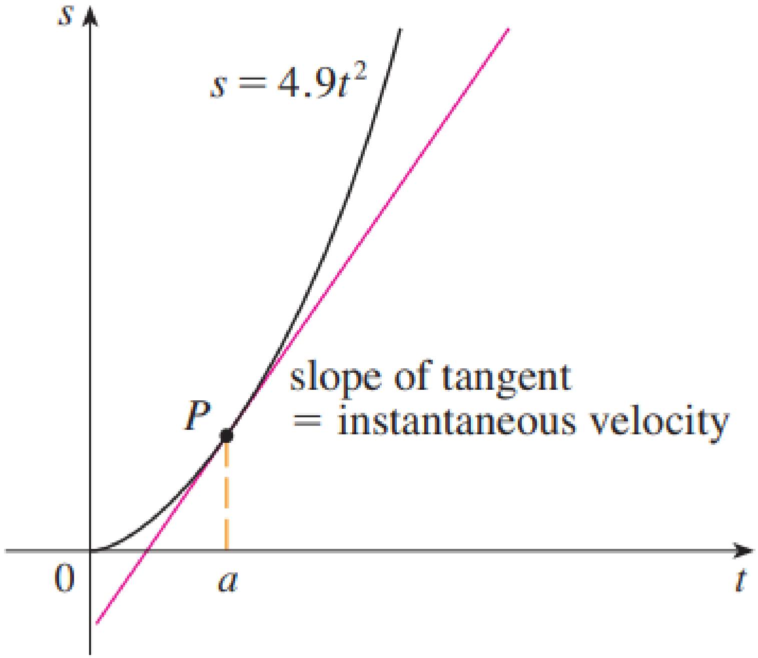
$$y = \csc^{-1} x \left(|x| \geq 1\right) \Leftrightarrow \csc y = x \text{ and } y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$$

$$y = \sec^{-1} x \left(|x| \geq 1\right) \Leftrightarrow \sec y = x \text{ and } y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$$

$$y = \cot^{-1} x \left(x \in R\right) \Leftrightarrow \cot y = x \text{ and } y \in \left(0, \pi\right)$$

Average and instantaneous





EXAMPLE

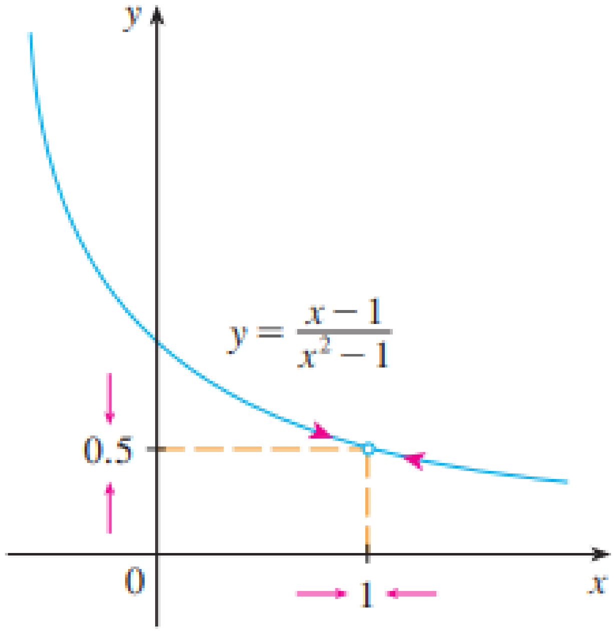
Guess the value of $\lim_{x \to 1} \dfrac{x - 1}{x^2 - 1}$.

SOLUTION Notice that the function $f(x) = (x - 1)/(x^2 - 1)$ is not defined when $x = 1$, but that doesn't matter because the definition of $\lim_{x \to a} f(x)$ says that we consider values of x that are close to a but not equal to a.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

According to Table: $\lim_{x \to 1} \dfrac{x-1}{x^2-1} = 0.5$

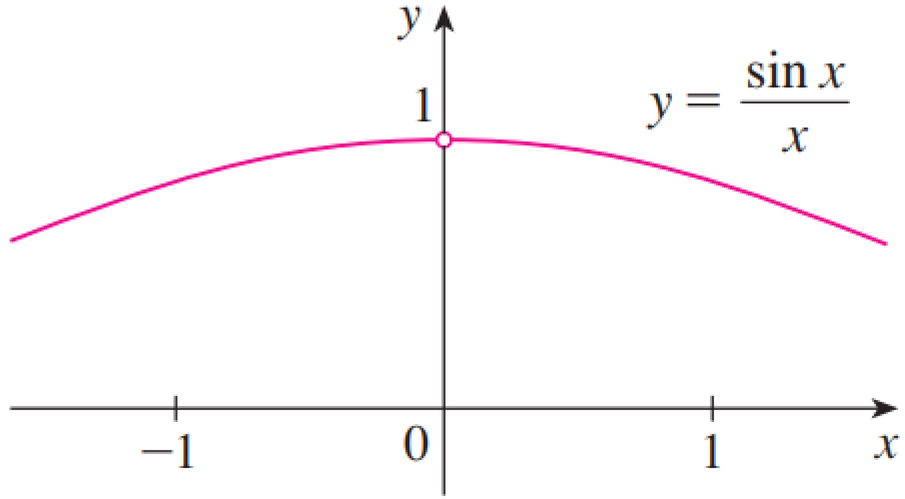


EXAMPLE

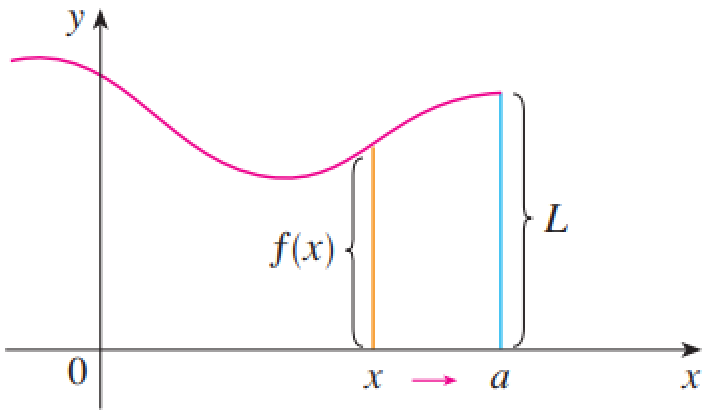
Guess the value of $\lim_{x\to 0}\dfrac{\sin x}{x}$.

x	$\dfrac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

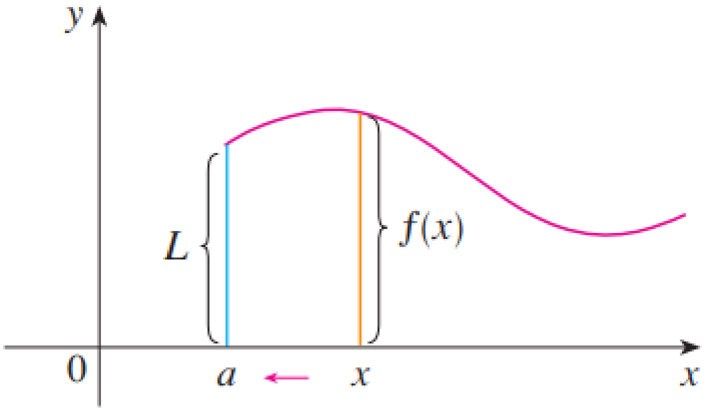
$$\lim_{x\to 0}\frac{\sin x}{x}=1$$



Left hand limit and Right hand limit



(a) $\lim_{x \to a^-} f(x) = L$



(b) $\lim_{x \to a^+} f(x) = L$

$$\lim_{x \to a} f\left(x\right) = L \text{ if and only if } \lim_{x \to a^-} f\left(x\right) = L \text{ and } \lim_{x \to a^+} f\left(x\right) = L$$

EXAMPLE

EXAMPLE 7 The graph of a function g is shown in Figure 10. Use it to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

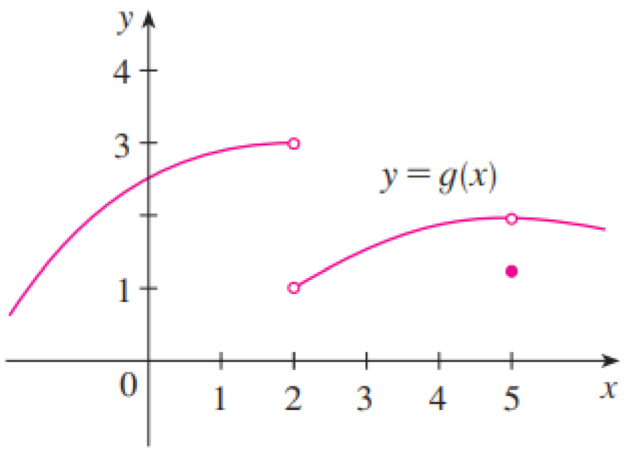


FIGURE 10

SOLUTION

(a) $\lim_{x \to 2^-} g(x) = 3$ and (b) $\lim_{x \to 2^+} g(x) = 1$

(d) $\lim_{x \to 5^-} g(x) = 2$ and (e) $\lim_{x \to 5^+} g(x) = 2$

$$\lim_{x \to 5} g(x) = 2$$

The limit law

Suppose that c is a constant and the limits

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.Then

1. $\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

2. $\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

3. $\lim_{x\to a}\left[cf(x)\right]=c\lim_{x\to a} f(x)$

4. $\lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

5. $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x)\neq 0$

EXAMPLE

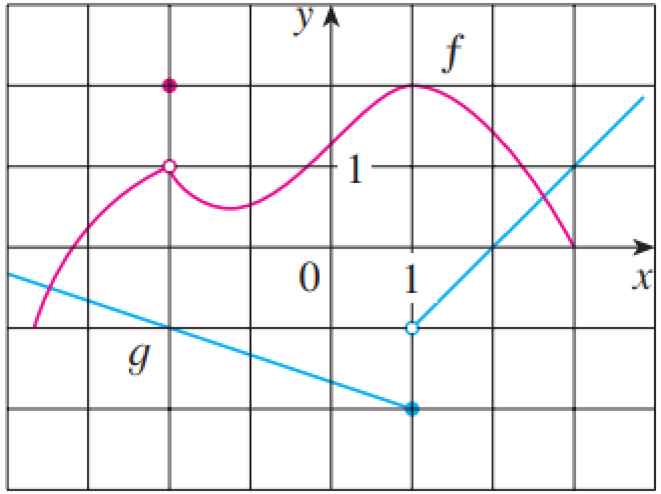


FIGURE 1

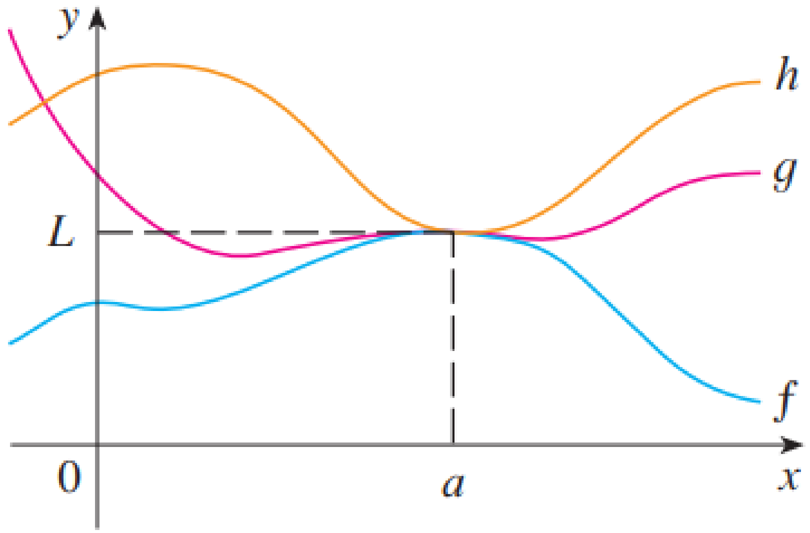
SOLUTION

$$\lim_{x \to -2} [f(x) + 5g(x)]$$

$$\lim_{x \to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to -2} g(x) = -1$$

$$\begin{aligned}\lim_{x \to -2} [f(x) + 5g(x)] &= \lim_{x \to -2} f(x) + \lim_{x \to -2} [5g(x)] && \text{(by Law 1)} \\ &= \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) && \text{(by Law 3)} \\ &= 1 + 5(-1) = -4\end{aligned}$$

The squeeze theorem(Sandwich Theorem)



$$\text{If } f(x) \leq g(x) \leq h(x)$$

when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

$$\lim_{x\to a} g(x) = L$$

EXAMPLE

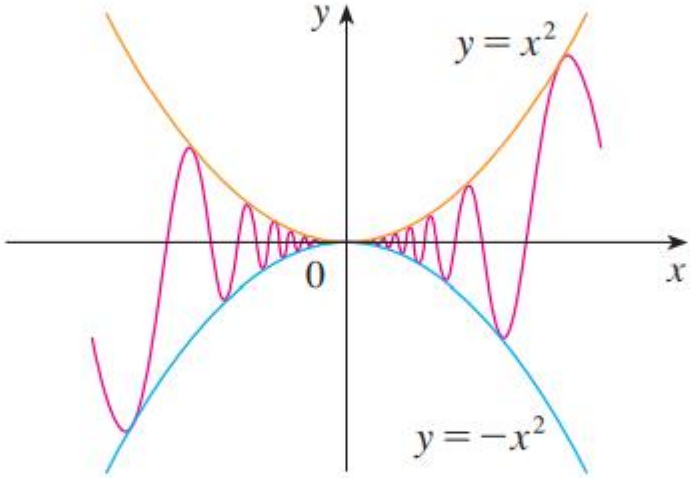


FIGURE 8
$y = x^2 \sin(1/x)$

SOLUTION Show that $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0.$

We cannot use : $\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \dfrac{1}{x}$

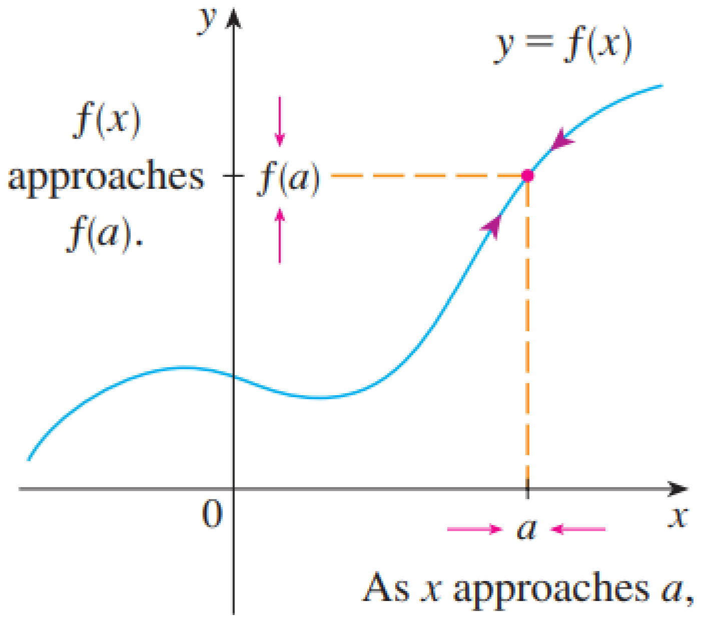
but $-1 \le \sin \dfrac{1}{x} \le 1 \Rightarrow -x^2 \le x^2 \sin \dfrac{1}{x} \le x^2$

We know that $\lim_{x \to 0} x^2 = 0$ and $\lim_{x \to 0} -x^2 = 0$

Taking $f(x) = -x^2, g(x) = x^2 \sin \dfrac{1}{x}, h(x) = x^2$ then

$\lim_{x \to 0} x^2 \sin \dfrac{1}{x} = 0$

Continuity



A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

EXAMPLE

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

SOLUTION

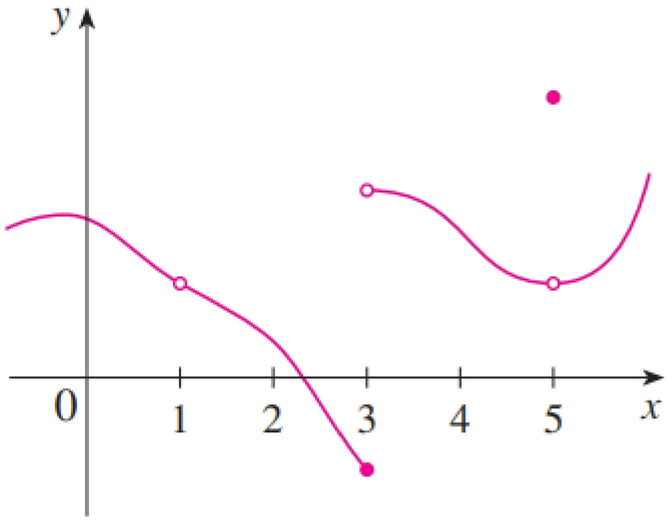


FIGURE 2

$f(1)$ is not defined.

$f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist

$f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists

$$\lim_{x\to 5} f(x) \neq f(5)$$

EXAMPLE

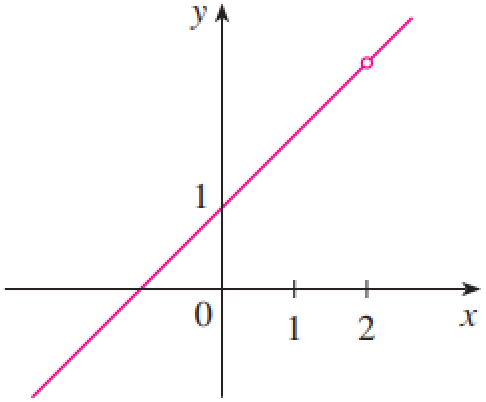
Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

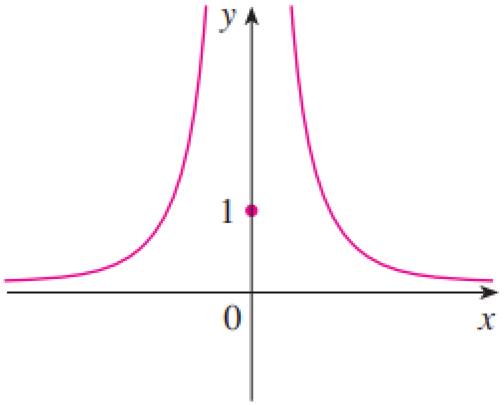
(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

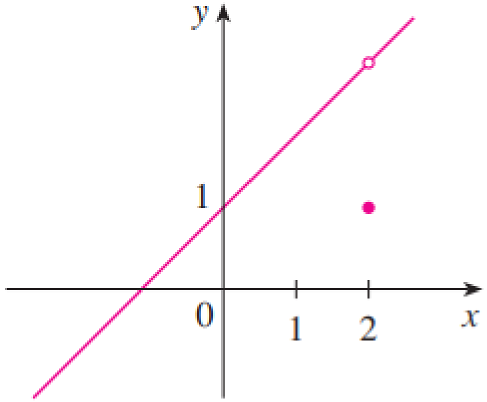
(d) $f(x) = [\![x]\!]$



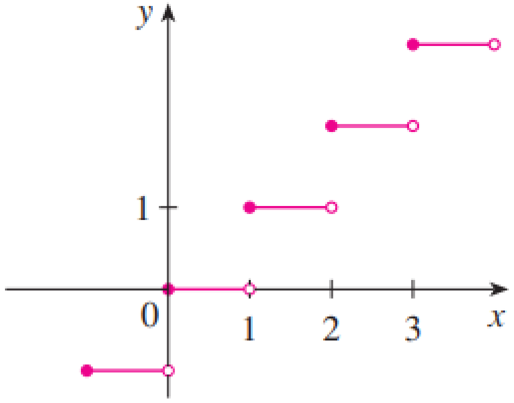
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



(d) $f(x) = [\![x]\!]$

Types of functions are continuous

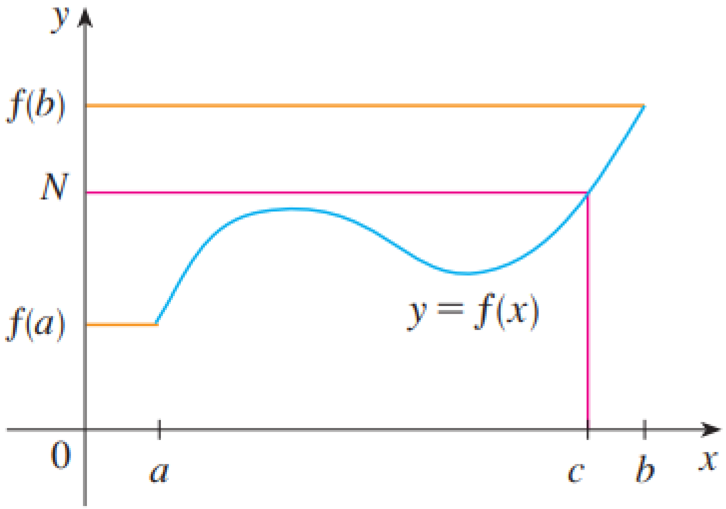
polynomials rational functions root functions

trigonometric functions inverse trigonometric functions

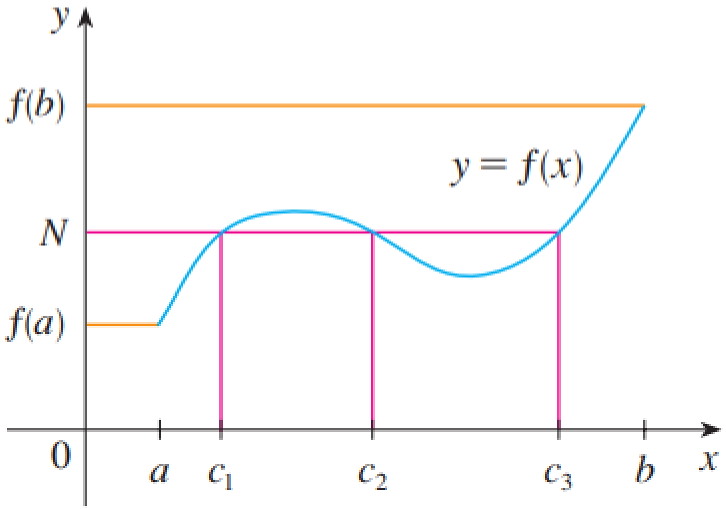
exponential functions logarithmic functions

THE INTERMEDIATE VALUE THEOREM

$\boxed{10}$ **THE INTERMEDIATE VALUE THEOREM** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



(a)



(b)

The intermediate value theorem is not true in general for discontinuous functions

EXAMPLE

EXAMPLE 10 Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

In fact, we can locate a root more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \qquad \text{and} \qquad f(1.3) = 0.548 > 0$$

a root must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \qquad \text{and} \qquad f(1.23) = 0.056068 > 0$$

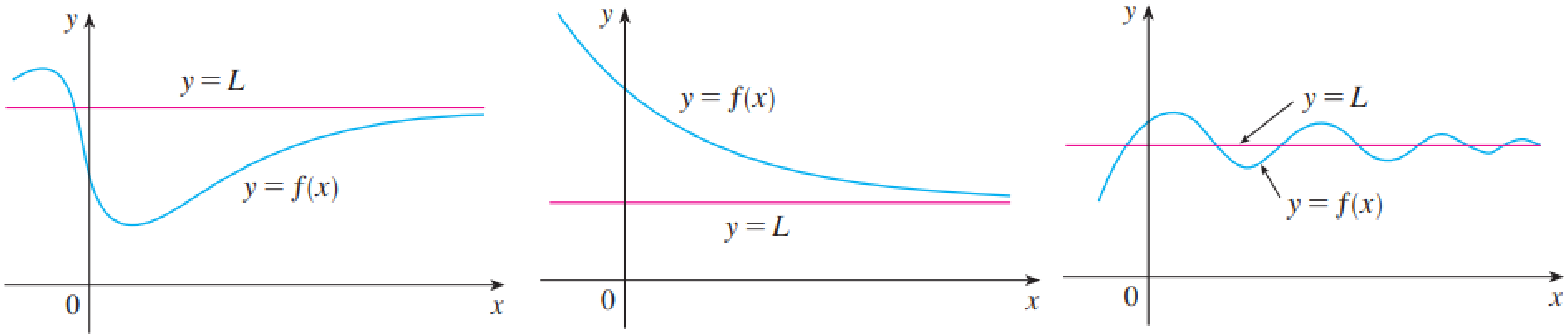
so a root lies in the interval $(1.22, 1.23)$.

LIMITS AT INFINITY

1 DEFINITION Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

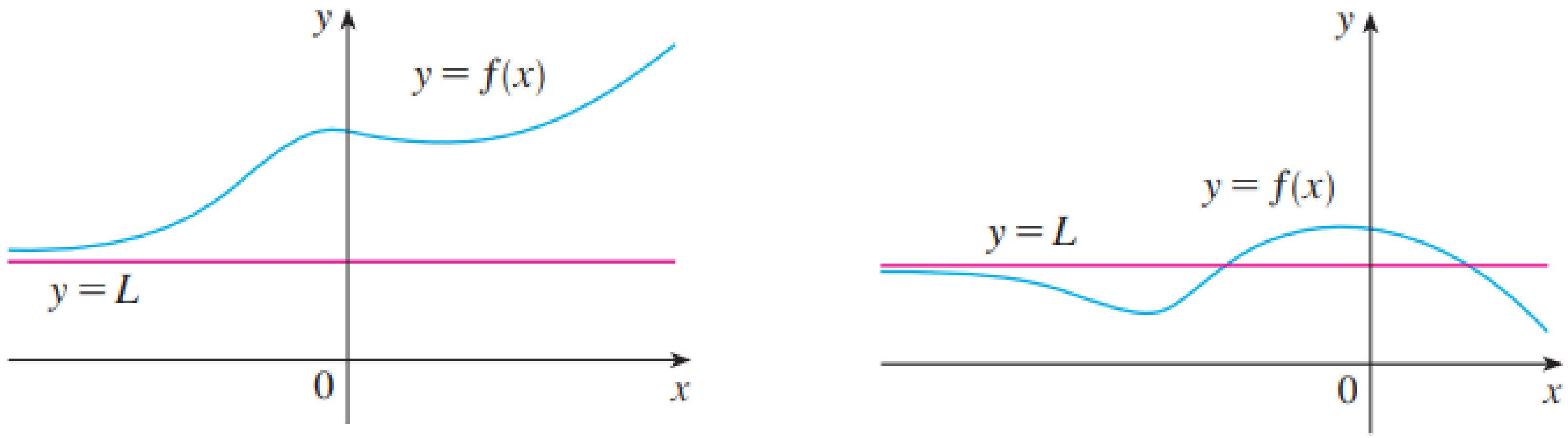


LIMITS AT INFINITY

2 DEFINITION Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.



EXAMPLE

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

SOLUTION

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to \infty} \frac{\sqrt{2 + \dfrac{1}{x^2}}}{3 - \dfrac{5}{x}} \qquad (\text{since } \sqrt{x^2} = x \text{ for } x > 0)$$

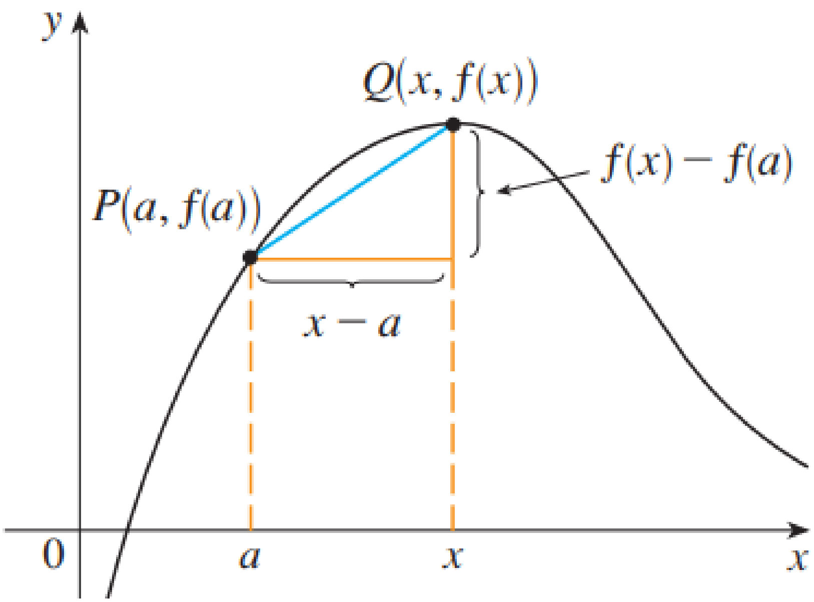
$$= \frac{\lim\limits_{x \to \infty} \sqrt{2 + \dfrac{1}{x^2}}}{\lim\limits_{x \to \infty} \left(3 - \dfrac{5}{x}\right)} = \frac{\sqrt{\lim\limits_{x \to \infty} 2 + \lim\limits_{x \to \infty} \dfrac{1}{x^2}}}{\lim\limits_{x \to \infty} 3 - 5 \lim\limits_{x \to \infty} \dfrac{1}{x}} = \frac{\sqrt{2 + 0}}{3 - 5 \cdot 0} = \frac{\sqrt{2}}{3}$$

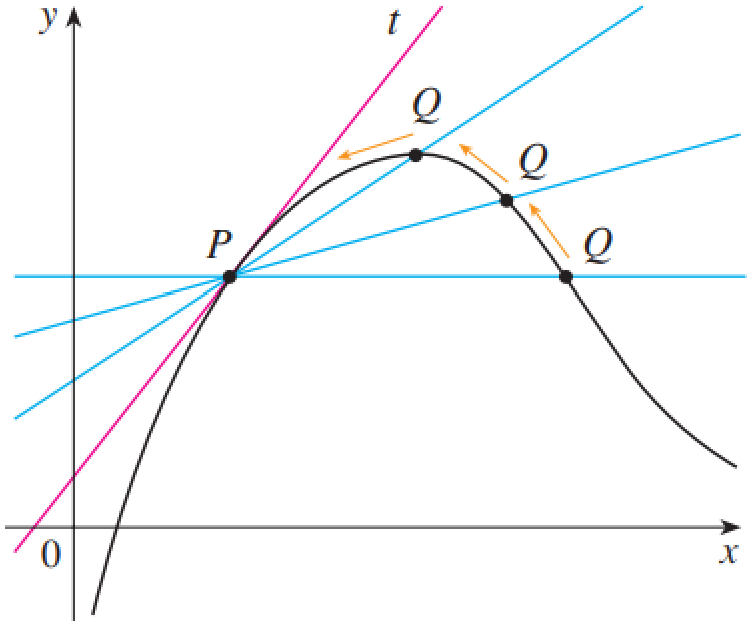
TANGENTS

1 DEFINITION The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.





EXAMPLE

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

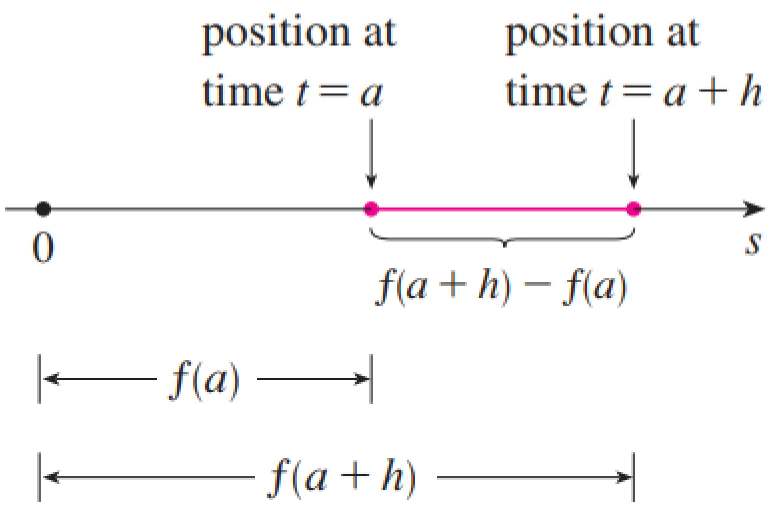
$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

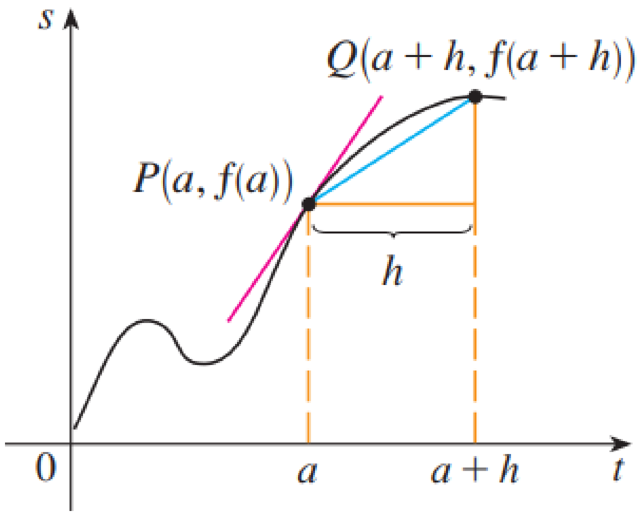
$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

VELOCITIES

$$\text{Average velocity} = \frac{displacement}{time} = \frac{f(a+h) - f(a)}{h}$$

$$\text{Instantaneous velocity:} v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$





$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

$= \text{average velocity}$

EXAMPLE

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

SOLUTION

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}$$

$$= \lim_{h \to 0} 4.9(2a + h) = 9.8a$$

(a) The velocity after 5 s is $v(5) = (9.8)(5) = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t_1 when $s(t_1) = 450$, that is,

$$4.9t_1^2 = 450 \Rightarrow t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

$$v(t_1) = 9.8t_1 = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$

DERIVATIVES

4 DEFINITION The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

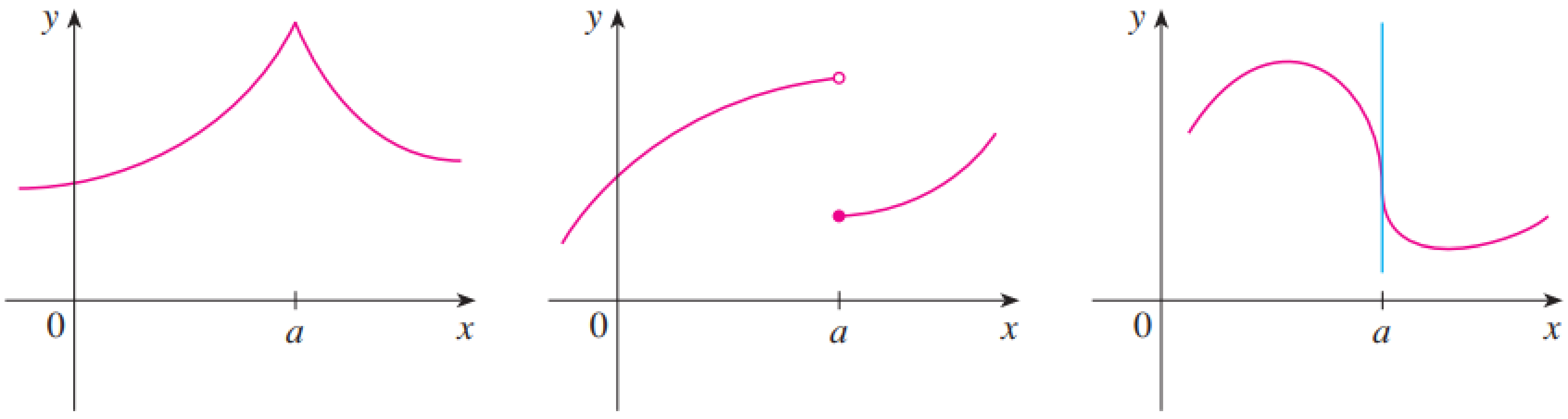
if this limit exists.

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

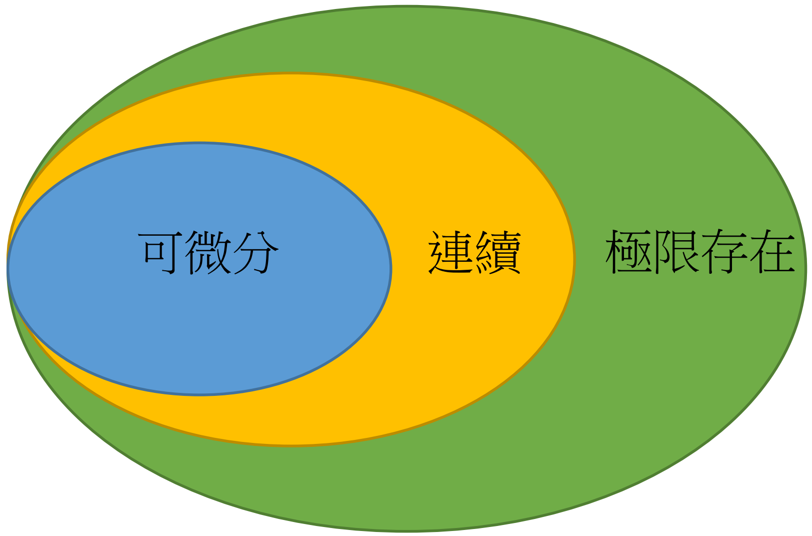
SOLUTION

$$f(x)'\big|_a = \left(x^2 - 8x + 9\right)'\big|_a = 2x - 8\big|_a = 2a - 8$$

HOW CAN A FUNCTION FAIL TO BE DIFFERENTIABLE?



(a) A corner　(b) A discontinuity　(c) A vertical tangent



DIFFERENTIATION RULES

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right] \quad \text{(Product rule)}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2} \quad \text{(Quotient rule)}$$

EXAMPLE

Hint: $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

(a) $f(x) = \frac{1}{x^2}$

(b) $y = \sqrt[3]{x^2}$

Hint:

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$y = ae^v + \frac{b}{v} + \frac{c}{v^2}$$

THE PRODUCT RULE/THE QUOTIENT RULE

THE PRODUCT RULE If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

(a) If $f(x) = xe^x$, find $f'(x)$.

THE QUOTIENT RULE If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Let y= $\dfrac{x^2+x-2}{x^3+6}$ then $y' = ?$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Let's try to confirm our guess that if $f(x) = \sin x$, then $f'(x) = \cos x$. From the definition of a derivative, we have

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

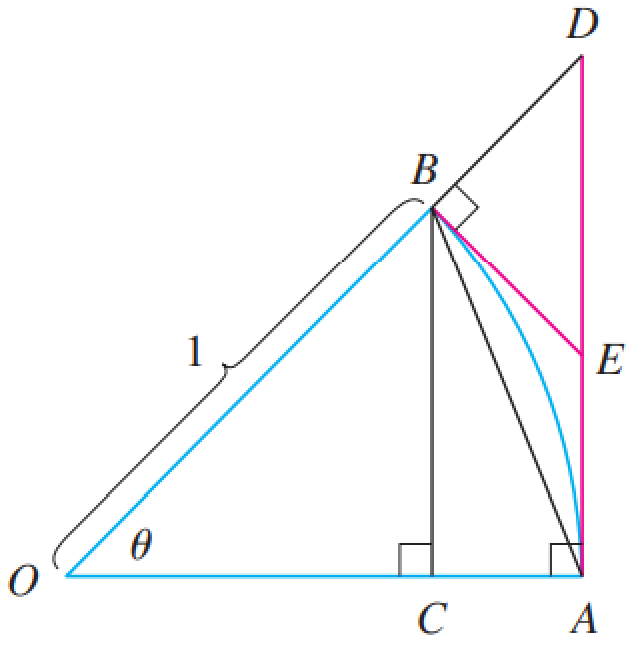
$$= \lim_{h\to 0} \left[\frac{\sin x \cos(\mathrm{h}) - \sin x}{h} + \frac{\cos(x)\sin(h)}{h} \right]$$

$$= \lim_{h\to 0} \left[\sin x \left(\frac{\cos(\mathrm{h}) - 1}{h} \right) + \cos x \left(\frac{\sin(h)}{h} \right) \right]$$

$$= \underbrace{\lim_{h\to 0} \sin x}_{\sin x} \cdot \lim_{h\to 0} \frac{\cos(\mathrm{h}) - 1}{h} + \underbrace{\lim_{h\to 0} \cos x}_{\cos x} \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

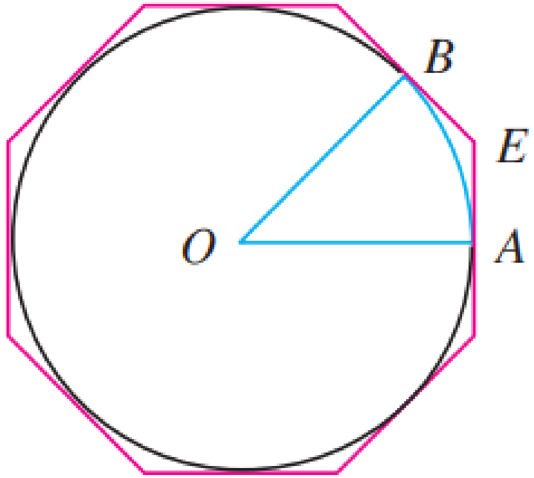
DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\boxed{\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$$



$$|BC| < |AB| < \text{arc } AB$$

$$\sin \theta < \theta \qquad \text{so} \qquad \frac{\sin \theta}{\theta} < 1$$



$$\begin{aligned} \theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan \theta \\ &= \tan \theta \end{aligned}$$

Expression $\quad \theta < \dfrac{\sin \theta}{\cos \theta}$

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos \theta = 1$, so by the Squeeze Theorem

$$\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta} = 1 \quad \xrightarrow{\text{Even function}} \quad \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\boxed{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}}$$

$$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = \lim_{\theta \to 0} \left(\frac{\cos \theta - 1}{\theta} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos \theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos \theta + 1)} = -\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\cos \theta + 1} \right)$$

$$= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\cos \theta + 1}$$

$$= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

EXAMPLE

Differentiate $y = x^2 \sin x$.

Hint: Product Rule

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

EXAMPLE

Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION

$$f'(x) = \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x \dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,(\tan x - 1)}{(1 + \tan x)^2}$$

THE CHAIN RULE

THE CHAIN RULE If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

EXAMPLE

Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$.

THE POWER RULE COMBINED WITH THE CHAIN RULE

$\boxed{4}$ **THE POWER RULE COMBINED WITH THE CHAIN RULE** If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,
$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

EXAMPLE

Differentiate $y = (x^3 - 1)^{100}$.

EXAMPLE

Proof:

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

SOLUTION

$$a^x = (e^{\ln a})^x = e^{(\ln a)x}$$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{(\ln a)x}) = e^{(\ln a)x}\frac{d}{dx}(\ln a)x$$

$$= e^{(\ln a)x} \cdot \ln a = a^x \ln a$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\boxed{\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}}$$

Recall the definition of the arcsine function:

$$y = \sin^{-1}x \qquad \text{means} \qquad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y\,\frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$, since $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

DERIVATIVES OF LOGARITHMIC FUNCTIONS

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Let $y = \log_a x$. Then $a^y = x$

Differentiating this equation implicitly with respect to x,

$$a^y(\ln a)\frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

EXAMPLE

Differentiate $y = \ln(x^3 + 1)$.

THE NUMBER e AS A LIMIT

$$\boxed{e = \lim_{x \to 0} (1 + x)^{1/x}}$$

$$\boxed{n = 1/x}$$

$$\boxed{e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n}$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x}$$

$$e \approx 2.7182818$$

$$= \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$= \lim_{x \to 0} \ln(1+x)^{1/x}$$

RADIOACTIVE DECAY

If m(t) is the mass remaining from an initial mass m_0 of the substance after time t, then the relative decay rate

$$-\frac{1}{m}\frac{dm}{dt}$$

the relative decay rate has been found experimentally to be constant

$$\frac{dm}{dt} = km \quad \Rightarrow \quad m(t) = m_0 e^{kt}$$

EXAMPLE

The half-life of radium-226 is 1590 years.
(a) A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after years.

$$m(t) = m(0)e^{kt} = 100e^{kt}$$

$$100e^{1590k} = 50 \qquad \text{so} \qquad e^{1590k} = \tfrac{1}{2}$$

$$1590k = \ln\tfrac{1}{2} = -\ln 2$$

$$k = -\frac{\ln 2}{1590}$$

$$m(t) = 100e^{-(\ln 2)t/1590}$$

HYPERBOLIC FUNCTIONS

DEFINITION OF THE HYPERBOLIC FUNCTIONS

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

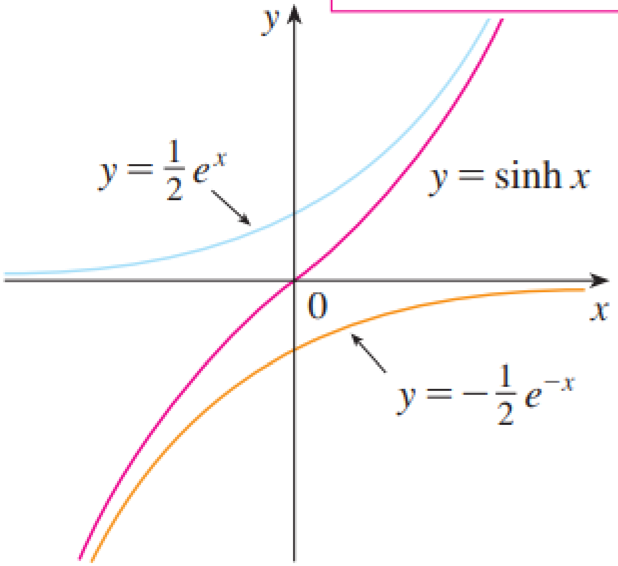


FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

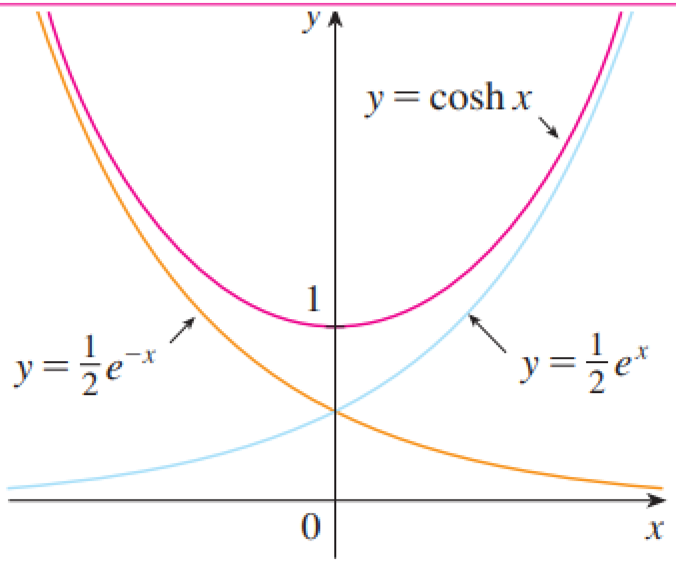


FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

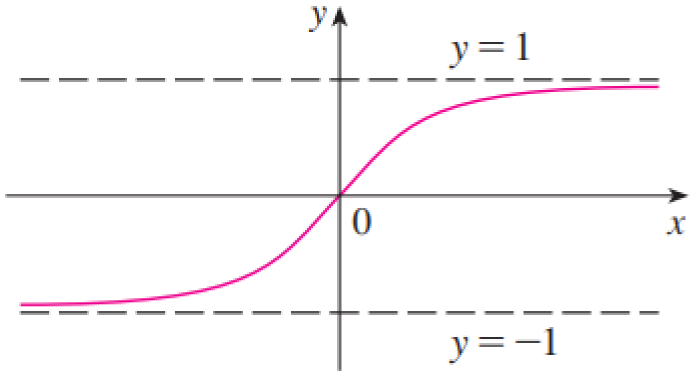


FIGURE 3
$y = \tanh x$

HYPERBOLIC FUNCTIONS

HYPERBOLIC IDENTITIES

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

EXAMPLE

Prove (a) $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1$$

DERIVATIVES OF HYPERBOLIC FUNCTIONS

1 DERIVATIVES OF HYPERBOLIC FUNCTIONS

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

EXAMPLE

$$\frac{d}{dx}\left(\cosh \sqrt{x}\right) \Rightarrow \sinh \sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

INDETERMINATE FORMS AND L'HOSPITAL'S RULE

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

EXAMPLE

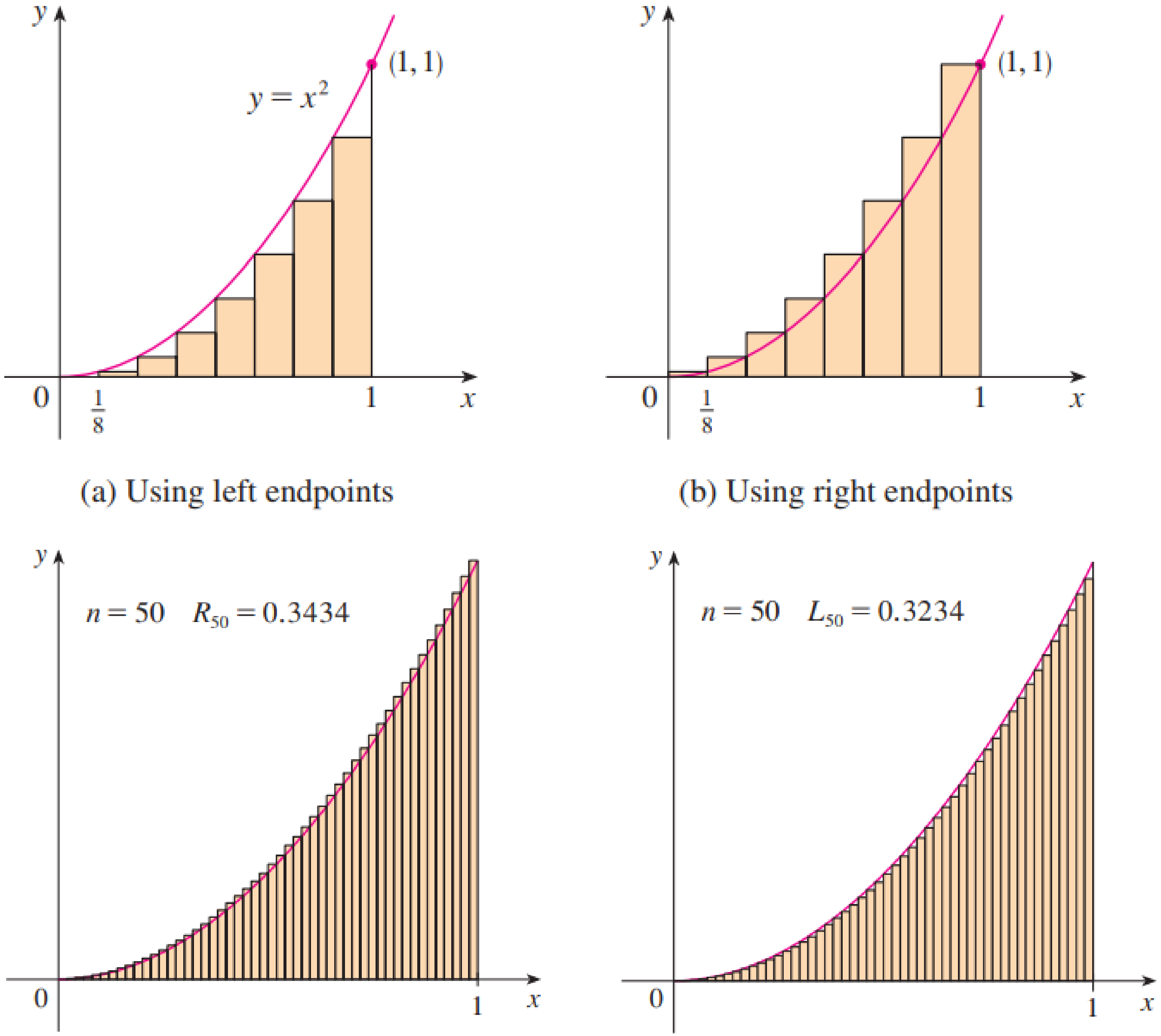
Find $\displaystyle \lim_{x \to 1} \frac{\ln x}{x - 1}$. (indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

SOLUTION

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\dfrac{d}{dx}(\ln x)}{\dfrac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Integral

Riemann sum



(a) Using left endpoints

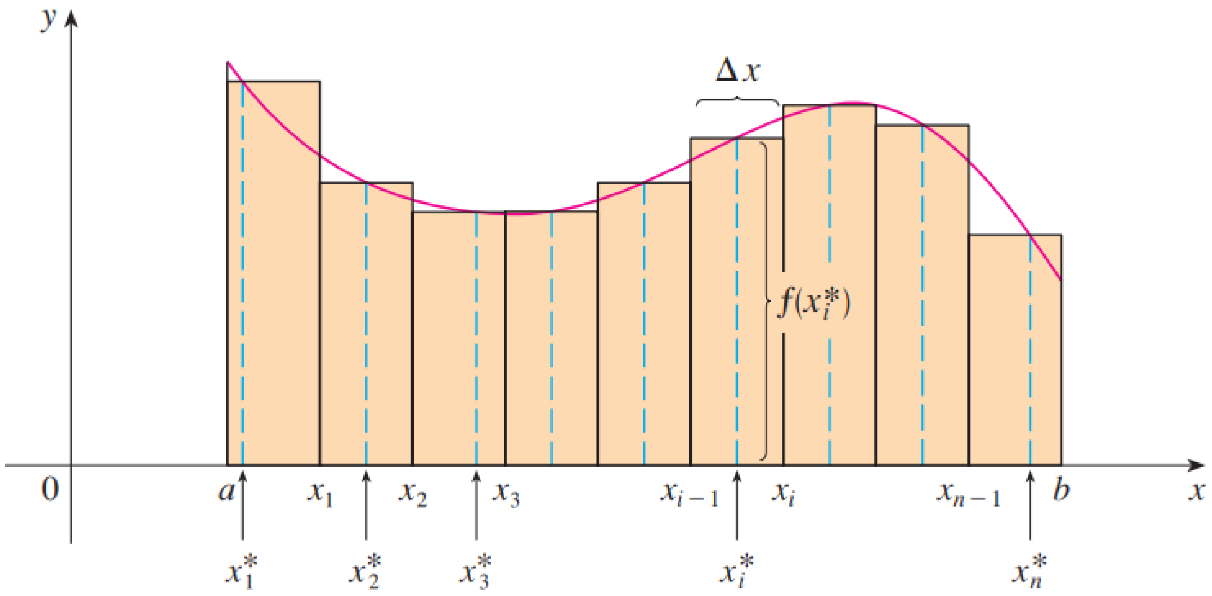
(b) Using right endpoints

THE DEFINITE INTEGRAL

$\boxed{2}$ **DEFINITION OF A DEFINITE INTEGRAL** If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0\ (= a), x_1, x_2, \ldots, x_n\ (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.



EXAMPLE

4 THEOREM If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$

(a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(b) Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

(a)

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\,\Delta x \\ &= f(0.5)\,\Delta x + f(1.0)\,\Delta x + f(1.5)\,\Delta x + f(2.0)\,\Delta x + f(2.5)\,\Delta x + f(3.0)\,\Delta x \\ &= \tfrac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\ &= -3.9375 \end{aligned}$$

PROPERTIES OF THE DEFINITE INTEGRAL

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

PROPERTIES OF THE INTEGRAL

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant

2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

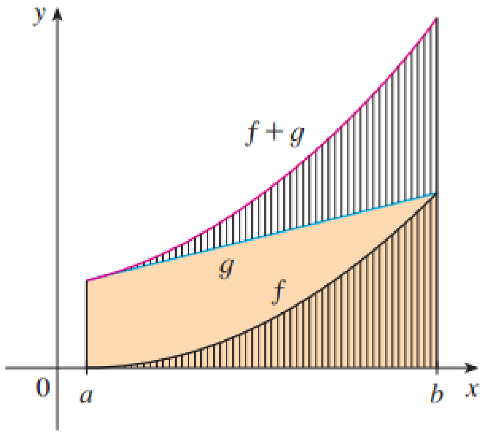
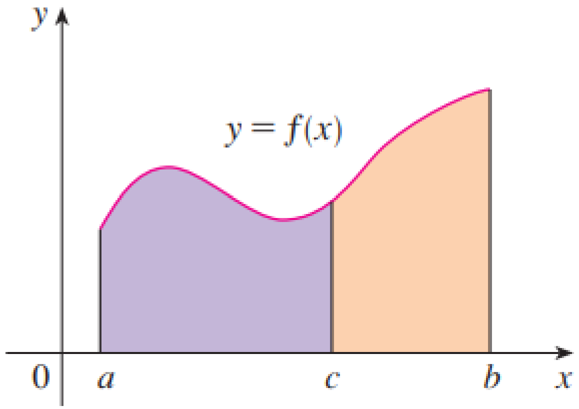


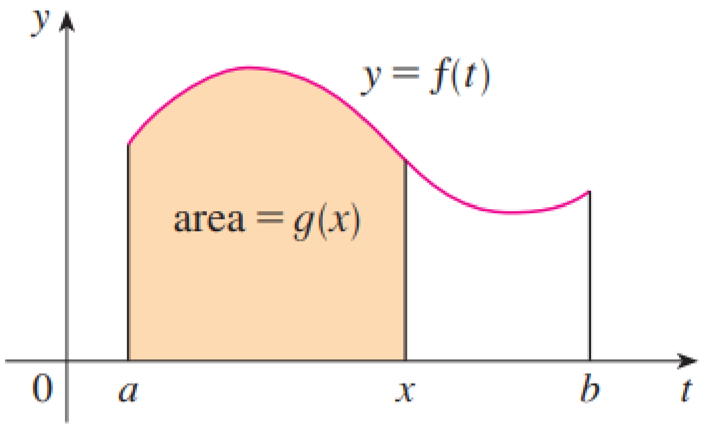
FIGURE 14

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$



THE FUNDAMENTAL THEOREM



THE FUNDAMENTAL THEOREM OF CALCULUS, PART 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

THE FUNDAMENTAL THEOREM OF CALCULUS Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

INDEFINITE INTEGRALS

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function (or family of functions)

EXAMPLE

Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx$$

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\,\frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$$

INTEGRATION BY PARTS

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\int [f(x)g'(x) + g(x)f'(x)]\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx = f(x)g(x)$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

$$\Downarrow$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

EXAMPLE

Find $\int x \sin x \, dx$.

SOLUTION

$$
\begin{aligned}
\int x \sin x \, dx &= f(x)g(x) - \int g(x)f'(x)\, dx \\
&= x(-\cos x) - \int (-\cos x)\, dx \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x + C
\end{aligned}
$$

速解

	微分	積分
+	x	$\sin x$
-	1	$-\cos x$
+	0	$-\sin x$

EXAMPLE

Evaluate $\int e^x \sin x \, dx$.

SOLUTION

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\boxed{\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx}$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

TRIGONOMETRIC INTEGRALS

Strategy for evaluating $\int \sin^m x \cos^m x dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity $\sin x \cos x = \frac{1}{2} \sin 2x$

EXAMPLE

Find $\int \sin^5 x \cos^2 x \, dx$.

SOLUTION

$$\sin^5 x \cos^2 x = (\sin^2 x)^2 \cos^2 x \sin x = (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Substituting $u = \cos x$, we have $du = -\sin x \, dx$ and so

$$\begin{aligned}
\int \sin^5 x \cos^2 x \, dx &= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx \\
&= \int (1 - u^2)^2 u^2 (-du) = -\int (u^2 - 2u^4 + u^6) \, du \\
&= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C \\
&= -\tfrac{1}{3}\cos^3 x + \tfrac{2}{5}\cos^5 x - \tfrac{1}{7}\cos^7 x + C
\end{aligned}$$

INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

CASE I ■ The denominator $Q(x)$ is a product of distinct linear factors.

Evaluate $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

SOLUTION

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \boxed{\frac{x^2+2x-1}{x(2x-1)(x+2)}}dx$$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

$$\Downarrow$$

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

$$=(2A+B+2C)x^2+(3A+2B-C)x-2A$$

$$\begin{cases} 2A+B+2C=1 \\ 3A+B-\ C\ =2 \\ -2A\qquad\quad =-1 \end{cases}$$

$$A=\frac{1}{2};B=\frac{1}{5};C=-\frac{1}{10}$$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int\left(\frac{1}{2}\frac{1}{x}+\frac{1}{5}\frac{1}{2x-1}-\frac{1}{10}\frac{1}{x+2}\right)dx$$

$$=\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+K$$

$$=$$

INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

CASE II ■ $Q(x)$ is a product of linear factors, some of which are repeated.

Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx.$

SOLUTION

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1} = x+1+\boxed{\frac{4x}{(x-1)^2(x+1)}}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$

$$=(A+C)x^2+(B-2C)x+(-A+B+C)$$

$$\begin{cases} A + C = 0 \\ B - 2C = 4 \\ -A+B+C = 0 \end{cases}$$

$$A=1;\ B=2;\ C=-1$$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx = \int\left[x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\right]dx$$

$$=\frac{x^2}{2}+x+\ln\left|\frac{x-1}{x+1}\right|-\frac{2}{x-1}+K$$

INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

CASE III ■ $Q(x)$ **contains irreducible quadratic factors, none of which is repeated.**

Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

SOLUTION

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A\left(x^2 + 4\right) + \left(Bx + C\right)x$$

$$= \left(A + B\right)x^2 + Cx + 4A$$

$$A = 1; B = 1; C = -1$$

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4}\right)dx$$

$$= \int \frac{1}{x}dx + \int \frac{x}{x^2 + 4} - \int \frac{1}{x^2 + 4}dx$$

$$= \ln|x| + \frac{1}{2}\ln\left(x^2 + 4\right) - \frac{1}{2}\tan^{-1}\left(x/2\right) + K$$

Hint:

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

CASE IV ■ $Q(x)$ **contains a repeated irreducible quadratic factor.**

Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

SOLUTION

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

TABLE OF INTEGRATION FORMULAS

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$

3. $\displaystyle\int e^x\,dx = e^x$

4. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a}$

5. $\displaystyle\int \sin x\,dx = -\cos x$

6. $\displaystyle\int \cos x\,dx = \sin x$

7. $\displaystyle\int \sec^2 x\,dx = \tan x$

8. $\displaystyle\int \csc^2 x\,dx = -\cot x$

9. $\displaystyle\int \sec x \tan x\,dx = \sec x$

10. $\displaystyle\int \csc x \cot x\,dx = -\csc x$

11. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\displaystyle\int \csc x\,dx = \ln|\csc x - \cot x|$

TABLE OF INTEGRATION FORMULAS

13. $\int \tan x \, dx = \ln|\sec x|$

14. $\int \cot x \, dx = \ln|\sin x|$

15. $\int \sinh x \, dx = \cosh x$

16. $\int \cosh x \, dx = \sinh x$

17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

***19.** $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x - a}{x + a}\right|$

***20.** $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$