About Shafarevich-Tate groups of algebraic tori and Iwasawa theory for them

Aleksei Samoilenko

June, 2025

1 Introduction

The purpose of this work is to study the arithmetic properties of algebraic tori over number fields and their integral models. Algebraic tori is a multiplicative type k-group schemes which split over a finite Galois extension L/k. Their character modules $\widehat{T}$ form finitely generated $\mathbb{Z}[G]$-modules, where $G = \mathrm{Gal}(L/k)$, allowing the geometry of tori to be studied through the representation theory of finite groups.

In the first part, I explain the basic structure of such $\mathbb{Z}[G]$-modules and recall the notions of permutation representations and similarity between G-modules. These concepts are central in Voskresenskii's theory, where the stable birational classification of algebraic tori is expressed in terms of the Picard class $p(T)$, which measures how far a torus is from being stably rational.

The second part concerns the arithmetic of tori. I introduce the Shafarevich–Tate group $\text{Ш}(T)$ and discuss Voskresenskii's exact sequence connecting $\text{Ш}(T)$, the group of weak approximation $A(T)$, and the Picard class $p(T)$. This exact sequence highlights the cohomological nature of local–global obstructions for tori. Further, I define the class group $Cl(\mathcal{T})$ of an integral model $\mathcal{T}_{\mathcal{O}_k}$ and describe its structure via Nisnevich cohomology, which relates $Cl(\mathcal{T})$ and $\text{Ш}(T)$ through Ono's formula for norm tori.

As a main result, I establish the monotonicity of the Shafarevich–Tate group for norm tori under field extensions, showing that its order does not decrease when the base field is enlarged. In addition, I construct several explicit examples of algebraic tori for which the Galois action on the Shafarevich-Tate group can be described.

2 Integral Representations

Algebraic k-tori admit a simple description:

Theorem 2.1.

$$\begin{pmatrix} \textit{finite-type multiplicative } k\textit{-group schemes} \\ \textit{splitting over } L \end{pmatrix} \simeq^{\mathrm{op}} (\textit{finite-type } \mathrm{Gal}(L/k)\textit{-modules}).$$

$$T \longrightarrow \hat{T} = \mathrm{Hom}_k(T_{\bar{k}}, \mathbb{G}_{m,k})$$

$$T \ s.t. \ T(A) = \mathrm{Hom}(\hat{T}, (A \otimes_k L)) \longleftarrow \hat{T}$$

Definition 2.2. A representation $P \in \mathbb{Z}[G]$-Mod is called *permutation* if there exists a $\mathbb{Z}$-basis of P on which the group G acts by permuting the basis elements.

Remark 2.3. Every permutation representation is isomorphic to a direct sum of modules of the form $\mathbb{Z}[G/H]$ for some (not necessarily normal) subgroups $H \leq G$. Such representations arise as the image of the *linearization functor*

$$G\text{-Sets} \longrightarrow \mathbb{Z}[G]\text{-Mod}.$$

Permutation modules appear in the following criterion of Voskresenskii for stable equivalence of algebraic tori.

Definition 2.4. Two G-modules M and N are called *similar* if there exist permutation modules S_1 and S_2 such that

$$M \oplus S_1 \simeq N \oplus S_2.$$

The *Picard class* is defined as follows. For a G-module $\hat{T}$, consider a short exact sequence

$$0 \longrightarrow \hat{T} \longrightarrow S \longrightarrow N \longrightarrow 0,$$

where S is permutation and N satisfies that its (-1)-st Tate cohomology vanishes for every subgroup of G. The similarity class of N is called the *Picard class* of $\hat{T}$ and is denoted by $p(T)$.

Question 2.5. Can one describe similarity classes of $\mathbb{Z}[G]$-modules as a stable category of modules with respect to the adjunction between the free and forgetful functors

$$G\text{-Sets} \rightleftarrows \mathbb{Z}[G]\text{-Mod}?$$

Does this category admit a structure analogous to that of a triangulated category?

Theorem 2.6. *The following conditions are equivalent:*

- *The torus T is stably rational over k;*
- *The Picard class $p(T)$ is trivial;*
- *The module $\hat{T}$ fits into an exact sequence of G-modules*

  $$0 \longrightarrow \hat{T} \longrightarrow S \longrightarrow S' \longrightarrow 0,$$

  where S and S' are permutation.

3 The Shafarevich–Tate Group

Definition 3.1. The *Shafarevich–Tate group* of a torus T defined over a global field is

$$Ш(T) = \mathrm{Ker}\big(H^1(\mathrm{Gal}(L/k), T(L)) \to H^1(\mathrm{Gal}(L/k), T(A_L))\big),$$

where A_L is the ring of adeles of L.

Class field theory provides the following description of Ш:

When studying arithmetic properties of tori, the property of weak approximation also appears.

Definition 3.2. The *group of weak approximation* is defined as

$$A(T) = \Big(\prod_v G(k_v)\Big)/\overline{G(k)}.$$

These arithmetic invariants of a torus are related by the following short exact sequence of Voskresenskii [1]:

Theorem 3.3. *Let k be a global field and $T \in C(L/k)$. Then there exists an exact sequence*

$$0 \to A(T) \to H^1(\mathrm{Gal}(L/k), p(T)) \to Ш(T) \to 0.$$

The following considerations follow idea of B.Mazur and after that Y. Manin [2], where Iwasawa–Mazur theory is applied to the Shafarevich–Tate group (more precisely, to its ℓ-primary component). In the case of algebraic tori, the situation is simpler: for example, the finiteness of $Ш(T)$ is an immediate consequence of Tate–Nakayama duality.

Let $L_0 = L$ be the splitting field of a $\mathbb{Q}$-torus T, and consider a cyclotomic $\Gamma \simeq \mathbb{Z}_\ell$-extension:

$$\begin{array}{ccccccccc}
L_0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots & \longrightarrow & L_\infty \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
\mathbb{Q} & \longrightarrow & \mathbb{Q}(\zeta_\ell) & \longrightarrow & \mathbb{Q}(\zeta_{\ell^2}) & \longrightarrow & \cdots & \longrightarrow & \mathbb{Q}(\zeta_{\ell^\infty})
\end{array}$$

Let $T_n = T \times_{\mathbb{Q}} K_n$. It splits at least over L_n, but if L and K_n are not linearly disjoint, the minimal splitting field may be smaller.

From Voskresenskii's sequence it follows that for stably rational tori, $Ш(T)$ is necessarily trivial. Examples where this is not the case are given by norm tori.

Definition 3.4. A *norm torus* is the torus $R^1_{L/k}(\mathbb{G}_m)$ corresponding to the module $I(\mathrm{Gal}(L/k))^*$, the dual of the augmentation ideal.

Remark 3.5. Torsors over a norm torus are hypersurfaces

$$\mathrm{N}_{L/k}(x) = a.$$

Then

$$Ш(R^1_{L/k}(\mathbb{G}_m)) = \{a \in k^\times \mid a \text{ is locally but not globally a norm}\}.$$

In this case, the middle term of the Voskresenskii sequence has a simple description.

Proposition 3.6. *Let $T = R^1_{L/k}(\mathbb{G}_m)$, where L/k is a Galois extension. Then*

$$H^1(\mathrm{Gal}(L/k), p(T)) \simeq H^3(\mathrm{Gal}(L/k), \mathbb{Z}).$$

Corollary 3.7. *The groups $Ш(R^1_{L/k}(\mathbb{G}_m)_n)$ and $A(R^1_{L/k}(\mathbb{G}_m)_n)$ are uniformly bounded in order by $H^3(\mathrm{Gal}(L/k), \mathbb{Z})$.*

Moreover, this holds for any torus.

Lemma 3.8. *Consider bi-cartesian square*

$$\begin{tikzcd} L \arrow[r] & L' \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

Then base change commute with norm-one tori $R^1_{L'/k'} \simeq R^1_{L/k} \underset{k}{\times} k'$

Since the restriction functor Res preserves permutation modules, base change does not alter the Picard class, though it can make some summands permutation, meaning the Picard class of the base change contains that of the original torus. Passing to submodules, cohomology cannot increase.

Corollary 3.9. *The groups $Ш(T_n)$ and $A(T_n)$ are uniformly bounded in order by $H^1(\mathrm{Gal}(L/k), p(T))$.*

Next, consider the behavior of $Ш(T_n)$ and $A(T_n)$ in a tower.

Example 3.10. *Let $T = R^1_{\mathbb{Q}(\sqrt{5},\sqrt{13})/\mathbb{Q}}(\mathbb{G}_m)$:*

$$\begin{tikzcd} \mathbb{Q}(\sqrt{5},\sqrt{13}) \arrow[r] & \mathbb{Q}(\sqrt{5},\sqrt{13},\zeta_7) \\ \mathbb{Q} \arrow[u] \arrow[r] & \mathbb{Q}(\zeta_7) \arrow[u] \end{tikzcd}$$

In this case, one can describe the behavior of the weak approximation group: $A(T) = \mathbb{Z}/2\mathbb{Z}$ and $A(T_n) = A(T_1) = 0$, according to the criterion of weak approximation for norm tori (Lemma 6.7 [3]). Indeed, in the first case not all decomposition groups are cyclic, while in the second all are. The middle group of the Voskresenskii sequence remains unchanged; hence the Shafarevich–Tate group stabilizes and equals $\mathbb{Z}/2\mathbb{Z}$. This example, however, does not clarify how $\mathrm{Gal}(K_n/K)$ acts on $Ш(T_n)$. Many similar examples can be constructed, motivating the following question.

Question 3.11. In a cyclotomic Γ-extension, for primes ℓ unramified in the extension corresponding to the torus, the Shafarevich–Tate group does not decrease.

This can be proved for norm tori of abelian extensions.

Theorem 3.12. *Let $L/\mathbb{Q}$ be an abelian Galois extension, and let*

$$T = R^1_{L/\mathbb{Q}}(\mathbb{G}_m)$$

be the norm one torus attached to this extension. For each n, let

$$T_n = R^1_{L_n/\mathbb{Q}(\zeta_{\ell^n})}(\mathbb{G}_m)$$

be the torus obtained by base change along the cyclotomic tower

$$\mathbb{Q} \longrightarrow \mathbb{Q}(\zeta_\ell) \longrightarrow \mathbb{Q}(\zeta_{\ell^2}) \longrightarrow \cdots \longrightarrow \mathbb{Q}(\zeta_{\ell^\infty}).$$

Assume moreover that the fields L and $\mathbb{Q}(\zeta_{\ell^k})$ are linearly disjoint for all k.

Then the norm map induces a surjection

$$Ш(T_n) \twoheadrightarrow Ш(T_m), \qquad m \leq n.$$

Proof. Let $k_n = \mathbb{Q}(\zeta_{\ell^n})$.

In the long Tate cohomology sequence associated to the short exact sequence

$$0 \longrightarrow T(L) \longrightarrow T(A_L) \longrightarrow C_L(T) \longrightarrow 0,$$

consider the segment

$$\hat{H}^0(G, T(A_k)) \xrightarrow{\nu} \hat{H}^0(G, C_L(T)) \xrightarrow{\varphi} \hat{H}^1(G, T(k)) \xrightarrow{\psi} \hat{H}^1(G, T(A_L)),$$

from which one gets

$$Ш(T) = \mathrm{Ker}(\psi) = \mathrm{Coker}(\nu).$$

By the local and global Tate–Nakayama duality [?][pp. 144–145, Ths. 6.22–6.23],

$$\hat{H}^2(G_v, \hat{T}) \simeq \hat{H}^0(G_v, T(k_v))^\vee, \qquad \hat{H}^2(G, \hat{T}) \simeq \hat{H}^0(G, C_L(T))^\vee.$$

Hence

$$Ш(T) = \mathrm{Ker}\Big[H^2(G, \hat{T}) \longrightarrow \prod_v H^2(G_v, \hat{T})\Big]^\vee.$$

For the norm one torus $T = R^1_{L/k}(\mathbb{G}_m)$, by definition $\hat{T} = I_G^* = \mathrm{Hom}_{\mathbb{Z}[G]}(I_G, \mathbb{Z})$, where I_G is the augmentation ideal. Then, from the sequence dual to the one defining I_G, we obtain

$$H^i(G, I_G^*) \simeq H^{i+1}(G, \mathbb{Z}).$$

Substituting into the formula above yields

$$Ш\big(R^1_{L/k}(\mathbb{G}_m)\big) = \mathrm{Ker}\Big[H^3(G, \mathbb{Z}) \longrightarrow \prod_v H^3(G_v, \mathbb{Z})\Big]^\vee.$$

It is known that for a finite abelian group G one has

$$H^3(G, \mathbb{Z}) \simeq \mathrm{Hom}(\wedge^2 G, \mathbb{Q}/\mathbb{Z})$$

[4][p. 123, Th. 6.4]. Therefore,

$$Ш\big(R^1_{L/k}(\mathbb{G}_m)\big) \simeq \mathrm{Coker}\Big(\prod_{v\nmid\infty} \wedge^2 G_v \longrightarrow \wedge^2 G\Big).$$

Under base change, the group G stays the same, while the decomposition groups G_v change as follows:

$$\begin{tikzcd} & L_v \arrow[r] & L'_{v'} \\ k_w \arrow[r] & L_v \cap k'_{w'} \arrow[u, "\mathrm{Gal}\left(L_v/(k'_{w'}\cap L_v)\right)"] \arrow[r] & k'_{w'} \arrow[u, "\mathrm{Gal}(L'_{v'}/k'_{w'})"'] \end{tikzcd}$$

where

$$G_v = \mathrm{Gal}(L_v/k_w) \hookleftarrow \mathrm{Gal}(k'_{w'} \cap L_v) \backsimeq \mathrm{Gal}(L'_{v'}/k'_{w'}) = G_{v'}.$$

Now consider the diagram

$$\begin{tikzcd} Ш_{n+1} \arrow[r] & Ш_n \\ \wedge^2 G_{n+1} \arrow[u] \arrow[r, "\sim"] & \wedge^2 G_n \arrow[u] \\ \prod \wedge^2 G_{n+1,v} \arrow[u] \arrow[r] & \prod \wedge^2 G_{n,v} \arrow[u] \end{tikzcd}$$

Since the bottom square is commutative, one can complete it with a map between the Ш's, and the isomorphism $\wedge^2 G_{n+1} \simeq \wedge^2 G_n$ induces a surjection on cokernels:

$$Ш\big(R^1_{L_{n+1}/k_{n+1}}(\mathbb{G}_m)\big) \twoheadrightarrow Ш\big(R^1_{L_n/k_n}(\mathbb{G}_m)\big).$$

□

To study the action of the Galois group on Ш, consider the simplest case with $\mathrm{Gal}(L/k) = C_3 \times C_3$.

Example 3.13. *Let L be the Galois extension of $\mathbb{Q}$ defined by*

$$x^9 - 26x^7 - 14x^6 + 181x^5 + 154x^4 - 295x^3 - 224x^2 + 84x + 56.$$

Only 7 and 19 ramify, with local degrees 3 and 9 respectively, so weak approximation fails only at 7. Let $T_0 = T = R^1_{L/\mathbb{Q}}$ and its base change $T_1 = R^1_{L_1/\mathbb{Q}(\zeta_\ell)}$. Then $A(T_0) = H^3(\mathrm{Gal}(L/\mathbb{Q}), \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$ and $Ш(T_0) = 0$. Depending on ℓ, these parameters behave differently. For ramified ℓ, the torus splits into a product of norm tori corresponding to cyclic subextensions, so there is no local–global obstruction. For ℓ such that the ideal above (7) splits completely in the tower (e.g. $\ell = 13$), we get $A(T_1) = 0$ and $Ш(T_1) = H^3(\mathrm{Gal}(L/\mathbb{Q}), \mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$. Local norms are easy to describe: by local class field theory, the norm map is surjective on local units for unramified primes, and on principal units in the tamely ramified case. However, verifying that an element is a global norm is more subtle.

Example 3.14. *Consider the same L but with $\ell = 3$. This example shows that Ш stabilizes only on the second level of the tower. Indeed, in $L/\mathbb{Q}$ the decomposition group is non-cyclic only for $p_0 = 19$; since $f(L_1/k_1, p_1) = 3$, it remains non-cyclic on the first level, but on the second it splits into three primes with $f(L_2/k_2, p_2) = 1$, so $Ш(T_2) = \mathbb{Z}/3\mathbb{Z}$.*

Question 3.15. Does the Galois group in a cyclotomic tower act nontrivially on Ш?

4 The Integral Shafarevich–Tate Group

We now consider algebraic tori defined over the rings of integers of number fields. Models of certain tori arise naturally from their definition, for instance: the split torus $\mathbb{G}^n_{m,S}$, the Weil restriction $\mathcal{R}_{S'/S}(\mathcal{T}_{S'})$, and the norm torus $\mathcal{R}^1_{S'/S}$. For tori defined over number fields, there exist several constructions that allow one to define corresponding models, such as the Néron model [5] and the minimal model [6].

Definition 4.1. The *class group* of an algebraic torus $\mathcal{T}_{\mathcal{O}_k}$ is defined as

$$Cl(\mathcal{T}) = \mathcal{T}(\mathbb{A}_k)/\mathcal{T}(k)\,\mathcal{T}(\mathbb{A}_k^{S_\infty}),$$

which, as in the classical number field case, is a finite group. Its order is denoted by h_T.

Remark 4.2. In particular, for $\mathcal{T} = \mathbb{G}_m$, one recovers the ideal class group $Pic(\mathcal{O}_k)$. The ideal class group also admits a description analogous to that of the Shafarevich–Tate group.

Theorem 4.3.

$$Pic(\mathcal{O}_k) = \ker\Big(H^1(k, \mathcal{O}_{\bar{k}}) \longrightarrow \prod_{v \nmid \infty} H^1(k_v, \mathcal{O}_{\bar{k}_v})\Big).$$

Proof. See [7]. □

It turns out that for the class group of an arbitrary torus, there is a similar local–global description.

Theorem 4.4. *Let* Nis *denote the Nisnevich topology on schemes. Then*

$$Cl(\mathcal{T}) = H^1_{\mathrm{Nis}}(\mathcal{O}_k, \mathcal{T}) = \{\mathcal{O}_k\text{-torsors for } \mathcal{T} \text{ that are trivial over } \mathcal{O}_{k_v} \text{ for all } v \nmid \infty\}.$$

Proof. The first equality follows from [8]. For the second, viewing $H^1_{\mathrm{Nis}}(\mathcal{O}_k, \mathcal{T})$ as the set of locally trivial torsors for the Nisnevich topology gives an injection

$$H^1_{\mathrm{Nis}}(\mathcal{O}_k, \mathcal{T}) \hookrightarrow \{\mathcal{O}_k\text{-torsors for } \mathcal{T} \text{ trivial over all } \mathcal{O}_{k_v},\ v \nmid \infty\}.$$

To prove the converse inclusion, one must check that if a torsor admits a point over the completion $\mathcal{O}_{k_v}$, then it also admits a point over the henselization $\mathcal{O}^h_{k_v}$. □

The relation between the classical group Ш and the class group of a norm torus is given by Ono's theorem [9] in the following form. In another form it gives Tamagawa number.

Theorem 4.5.

$$h_T = \frac{|\text{Ш}(T)| \prod e(K_w/k_v)}{[K_{ab} : k]\,[\mathcal{O}_k : N\mathcal{O}_K]} \qquad \tau_T = \frac{|\operatorname{Pic}(T)|}{|\text{Ш}(T)|}$$

This forms connected by analytic class number formula for tori [10].

Theorem 4.6. *Let $L(s, \hat{T})$ be Artin L function corresponding to tori, ρ_T = leading term of $L(1, \hat{T})$, $w_T = [T(A_L)^c \cap T(\mathbb{Q})]$ - quasi roots of unity, D_T - quasi-discriminant and R_T - quasi-regulator, then*

$$h_T = \frac{\tau_T \rho_T w_T \sqrt{D_T}}{R_T}$$

5 Further Directions

In the continuation of this work, we plan to investigate the arithmetic of the class group $Cl(\mathcal{T})$ and the Shafarevich–Tate group $Ш(T)$ using tools from Iwasawa theory.

In particular, I intend to study the L-function of an algebraic torus T, construct its p-adic continuation, and interpret the resulting power series as an element of the completed group algebra. A central question is whether this element can be viewed as a generalization of the classical Stickelberger element, and whether it annihilates the groups $Ш(T)$ and $Cl(\mathcal{T})$.

Furthermore, we aim to explore to what extent analogues of the Herbrand–Ribet type theorems may hold in this context — that is, whether the vanishing of certain components of these p-adic L-functions provides criteria for the triviality of the Shafarevich–Tate group and the class group of the torus.

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