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Minimum Data-Rate for Emulating a Linear Feedback System ⋆

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Abstract:
In this work, we ask what the minimum data-rate is for a possibly nonlinear control law acting on a linear system to emulate the closed-loop behavior of a desired linear control law. This result gives an implicit estimate for the information flow in the feedback path of the closed-loop linear system. We formally introduce the notion of emulation and control law, and we present a data-rate theorem. Remarkably, we note that the minimum data-rate varies discontinuously with the parameters of the open and closed-loop systems. This feature is new since the usual data-rate theorem, which only requires stabilization instead of emulation, gives a continuously varying minimum data-rate.

Keywords: Quantized systems, Quantized control, Control under communication constraints, Control emulation, Linear systems

1. INTRODUCTION

Quantization and sampling have become ubiquitous in control systems since the advent of digital circuits. These features, however, constrain the class of problems we can solve because the number of possible distinct control functions we can implement on any given time interval is finite. Moreover, many control tasks have a minimum data-rate, informally understood as the number of distinct possible controls on a given time interval, below which we cannot address them (see, e.g., Delchamps (1990); Wong and Brockett (1999); Nair and Evans (2000); Baillieul (2002); Hespanha et al. (2002); Matveev and Savkin (2004); Tatikonda and Mitter (2004); Matveev and Savkin (2009); Kawan (2013)). Generally, results that present data-rate restrictions are called Data-Rate Theorems (see, e.g., Nair et al. (2007) for an overview).

In this document, we address the following question: given a linear feedback system, what is the minimum asymptotic average data-rate to emulate that system's behavior? This question has an immediate practical consequence since, often, the designer ignores the data-rate constraints and simply designs the controller for a given plant, assuming perfect information. Our paper addresses what is the data-rate we need to use for our control system to behave similarly to the desired designed behavior.

In most of the previous literature, authors addressed the problem of stabilizing the origin in some sense (see, e.g., Delchamps (1990); Wong and Brockett (1999); Brockett and Liberzon (2000); Nair and Evans (2000)). In our case, we require the system's state trajectory to closely mimic a desired behavior. This additional requirement leads us to a data-rate theorem that takes into consideration both the eigenstructure of the open-loop system and that of the desired closed-loop one, which we want to emulate. Surprisingly, unlike the usual data-rate theorem, the minimum data-rate to solve the emulation problem varies discontinuously with the system parameters.

The structure of our work is as follows: in Section 2, we introduce the emulation problem and related notions. Next, we recall the definition of Lyapunov indices and pertinent concepts, which appear in our data-rate theorem statement. Still in Section 2, we present Theorem 7, presenting lower and upper bounds for the minimum data-rate to solve the emulation problem, where the Lyapunov indices of an associated time-varying matrix appear in the upper bound, and a so-called k-dimensional Lyapunov index is used in the lower bound. Then, in Section 3, we give a sufficient condition for the lower and upper bounds to be equal. Also, we show how to compute those Lyapunov indices in terms of the open and closed-loop systems' eigenstructures. We end that section with an example, which illustrates that the minimum data-rate might vary discontinuously with respect to the target system parameters. Finally, we conclude the paper and state some future research directions.

Notations: We denote the sets of complex, real, and integer numbers by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$, respectively. Also, we represent the sets of positive and nonnegative integers by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$, respectively. We define $[n] := \{1, \ldots, n\}$ for each $n \in \mathbb{Z}_{>0}$. For a number $\lambda \in \mathbb{C}$, we denote by $\Re(\lambda) \in \mathbb{R}$ its real part. We denote by $\#\mathcal{S}$ the cardinality of a set $\mathcal{S}$. Further, we denote the usual Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$, and the symbol $\|\cdot\|$ denotes its respective induced norm on $d \times d$ matrices. Given a matrix A, we denote by $(A)_{(i,j)}$ its entry in the i-th row and j-th column. The $d \times d$ identity

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matrix is I_d. Finally, given a $d \times d$ matrix A, we denote by $\Sigma(A) \subset \mathbb{C}$ its set of eigenvalues.

2. THE EMULATION PROBLEM

In this section, we introduce the model and the problem we want to solve. Next, we introduce several notions necessary to make our problem rigorous. After that, we state our main result, which we prove in subsequent works.

2.1 The problem's informal description

Let A and B be real matrices of dimensions $d \times d$ and $d \times m$, respectively. We assume that we want to design a controller for system

$$\dot{z}(t) = Az(t) + Bv(t) \tag{1}$$

with the informal goal of mimicking the behavior of a target closed-loop system

$$\dot{x}(t) = (A + BF)x(t), \tag{2}$$

where F is a $m \times d$ matrix. The idea is the following: suppose we designed a full-state feedback controller $u(x(t), t) = Fx(t)$ for system (1) and obtained the closed-loop system (2). Now, we want to imitate the target closed-loop dynamics (2) over a communication channel, which will force the controller to operate with a finite data-transmission rate. What is the minimum data-rate for us to implement such controller with an error that vanishes asymptotically? We address this problem as the emulation problem since we are interested in emulating the behavior of a system that operates with perfect information by using a controller that is subject to data-rate constraints. This problem statement is only informal; the goal of the remainder of this section is to make it precise and to present a data-rate theorem for it.

We take this opportunity to introduce some notation [1]: $\xi(x, t)$ is the solution of system (2) at time $t \in [0, \infty)$ when the initial condition is $x \in \mathbb{R}^d$. Also, $\phi(x, t, v(\cdot))$ is the solution of (1) at time $t \in [0, \infty)$, when the initial condition is $x \in \mathbb{R}^d$, and the control is $v(\cdot)$. Finally, $\eta(x, t) := \xi(x, t) - \phi(x, t, v(\cdot))$ is the *emulation error* at time t when we start at the state x and apply the control $v(\cdot)$.

2.2 The notion of control law

Now, to talk about a data-rate associated to a controller, we introduce the notion of control law. To get an informal understanding of such idea, let $T \in (0, \infty)$ be a time horizon, $\mathcal{K} \subset \mathbb{R}^d$ be a set of possible initial conditions, and let $u(\cdot, \cdot)$ be a function that associates an initial condition to a function $u(x, \cdot)$ on $[0, \infty)$ with image on $\mathbb{R}^m$, which we interpret as a control. We notice that several initial conditions in $\mathcal{K}$ can have the same function $u(\cdot, \cdot)$ when restricted to the time interval $[0, T]$. If the number of distinct function restrictions on $[0, T]$ is finite, we can encode them on an alphabet and use its growth-rate as a data-rate measure. That is the idea behind the next definition.

[1] The initial time is always equal to zero.

Definition 1. (Control law). A *control law* is a function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^m$ so that $u(x, \cdot)$ is locally bounded and càdlàg for each $x \in \mathbb{R}^d$. Given a set of possible initial conditions $\mathcal{K} \subset \mathbb{R}^d$, which we assume compact, we associate an *asymptotic average data-rate* as

$$b(u) := \limsup_{t \to \infty} \frac{1}{t} \log(\#u(\mathcal{K}, [0, t))). \tag{3}$$

If the asymptotic average data-rate is finite, the control law induces a natural finite partition of $\mathcal{K} \subset \mathbb{R}^d$ for each $T \in (0, \infty)$. We describe the partition using an equivalence relation: two possible initial conditions x and y are equivalent on the time interval $[0, T)$ if, and only if, $u(x, t) = u(y, t)$ for all $t \in [0, T)$. We denote this partition of $\mathcal{K}$ by $\mathscr{P}_u(T)$ and we denote its cardinality by $N_u(T) \in \mathbb{Z}_{>0}$. We further assume that the cells $\mathcal{P} \in \mathscr{P}_u(T)$ are Borel and have nonempty interior. ▲

Remark 2. Our definition of control law is inspired by the definition of control set from Colonius and Kliemann (2000) (Chapter 3). We recall that a control set is a set of admissible control functions. The idea behind the notion of control law is to index the elements of a control set by elements from $\mathcal{K}$. We see that this indexed set is equivalent to a function indexed by its first entry. Thus, both definitions of control law and indexed control set give the same object. Nonetheless, we believe that introducing the control law makes the discussion clearer and more intuitive.

2.3 Formal problem statement

We now can state formally what we mean by emulation.

Definition 3. (Emulation). Let $\varepsilon \in (0, \infty)$ and $\alpha \in [0, \infty)$ be constants. Also, let $\mathcal{K} \subset \mathbb{R}^d$ be a set of possible initial conditions, which is compact and has a nonempty interior. We say that a control law $u(\cdot, \cdot)$ makes system (1) $(\varepsilon, \mathcal{K}, \alpha)$-*emulate* system (2) if

$$|\eta(x, t)| < \varepsilon e^{-\alpha t}$$

for all $x \in \mathcal{K}$ and all $t \in [0, \infty)$. The *minimum data-rate* to solve the emulation problem for any $\varepsilon \in (0, \infty)$ is

$$h(\alpha, \mathcal{K}) := \sup_{\varepsilon > 0} \{\inf\{b(u) : u(\cdot, \cdot) \text{ makes system (1) } (\varepsilon, \alpha, \mathcal{K})\text{-emulate system (2)}\}\}. \tag{4}$$

▲

We note that, although we call it the minimum data-rate, the quantity (4) is actually an infimum in general. The supremum over ε reflects the fact that we are not interested in bounding the initial transients of the emulation error $\eta(\cdot, \cdot)$, so long as it converges to zero faster than $e^{-\alpha t}$.

The goal of the rest of this section is to state Theorem 7, which gives lower and upper bounds for the minimum data-rate to solve the emulation problem. Nonetheless, to do that, we must introduce a geometric notion, which is related to the eigenstructure of systems (1) and (2).

We take this opportunity to introduce some additional notation. We define

$$E(t) := e^{(A+BF)t} - e^{At} \tag{5}$$

for all $t \in [0, \infty)$. Note that we can explicitly solve equations (1) and (2) to compute $\eta(x, t)$ as

$$\eta(x,t) = E(t)x + \int_0^t e^{A(t-s)}Bv(s)ds \tag{6}$$

for all $t \in [0,\infty)$ and any integrable control $v(\cdot)$. Thus, we can interpret $E(t)$ as a "free response" of the emulation error. In this sense, $E(\cdot)$ is similar to semi-flows that arise from differential equations (see, e.g., Chapter 1 from Sell and You (2002)). We note, however, that $E(\cdot)$ does not satisfy the semi-group property since it comprises the difference of two exponentials. This latter fact makes it possible for $E(\cdot)$ to have a nontrivial null space.

2.4 Lyapunov indices and flag

In Liapounoff (1907), the concept of *characteristic number* of a function was introduced (see Chapter 1 Section 6), which is known today as the Lyapunov index of a function[2] (see, e.g., Arnold (1998)) and, in a slightly less general setting, as *Lyapunov exponent*. Lyapunov used that concept to study the asymptotic behavior of solutions of linear time-varying differential equations that appeared in his first method of stability. In our work, we analyse the Lyapunov indices of $E(\cdot)$ since they are related to the solution of the minimum average asymptotic data-rate emulation problem. Explicitly, we use these indices to provide lower and upper bounds to the minimum data-rate for the emulation problem.

The next definition is adapted from Chapter 3 of Arnold (1998). The difference is in the fact that we parametrize the functions by a vector $v \in \mathbb{R}^d$ to make our discussion clearer.

Definition 4. (Lyapunov index). The *Lyapunov index* of a function $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ parametrized by its second argument is the functional

$$\lambda(f,v) \coloneqq \limsup_{t\to\infty} \frac{1}{t} \log\left(|f(t,v)|\right) \in \mathbb{R} \cup \{-\infty,\infty\}, \tag{7}$$

where $\log(0) = -\infty$. ▲

We are interested in studying the Lyapunov indices when $f(t,v) = E(t)v$. Recall that $E(t)$ is a $d \times d$ real matrix for each $t \in [0,\infty)$ (see equation (5)). We denote this Lyapunov index, when evaluated at $v \in \mathbb{R}^d$, by $\lambda(E,v)$, to keep the notation simple. Now, we need the following two properties in our analysis. First, we denote by

$$\boldsymbol{\chi} \coloneqq \lambda(E,\mathbb{R}^d) \cap \mathbb{R} \tag{8}$$

the set of real values the function $\lambda(E,\cdot)$ can take. Further, we denote the cardinality of $\boldsymbol{\chi}$ by

$$q \coloneqq \#\boldsymbol{\chi}. \tag{9}$$

We remark that $q \in [d]$ (see, e.g., Section 2.1 in Chapter 3 from Arnold (1998)). Further, we order the set $\boldsymbol{\chi}$ so that $\boldsymbol{\chi} = \{\chi_1,\ldots,\chi_q\}$ where $\chi_i > \chi_{i+1}$ for $i \in [q-1]$ and define $\chi_{q+1} \coloneqq -\infty$.

The other property of $\lambda(E,\cdot)$ is geometric in nature. More clearly, the set

$$V_\beta \coloneqq \{v \in \mathbb{R}^d : \lambda(E,v) \le \beta\} \tag{10}$$

is a vector subspace of $\mathbb{R}^d$ for each $\beta \in \mathbb{R} \cup \{-\infty,\infty\}$. Notice that the set of vector spaces $\mathcal{F}' \coloneqq \{V_\beta : \beta \in \{\chi_1,\ldots,\chi_q,\chi_{q+1}\}\}$ forms a chain with respect to strict set inclusion, i.e.,

$$V_{\chi_{q+1}} \subsetneq V_{\chi_q} \subsetneq \cdots \subsetneq V_{\chi_1} = \mathbb{R}^d.$$

Recall that a set of subspaces $\tilde{\mathcal{F}} \coloneqq \{F_0,\ldots,F_p\}$ of $\mathbb{R}^d$ is called a *flag* (see, e.g., Chapter 7 from Shafarevich and Remizov (2013)) if $\{0\} \in \tilde{\mathcal{F}}$, $\mathbb{R}^d \in \tilde{\mathcal{F}}$, and if

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_p = \mathbb{R}^d.$$

We see that, if $V_{\chi_{q+1}} = \{0\}$, then $\mathcal{F}'$ is a flag. When $E(t)$ is not invertible, however, we might have $V_{-\infty} \neq \{0\}$. To remedy that, we define $\mathcal{F} \coloneqq \mathcal{F}' \cup \{0\}$ and note that this is always a flag. This is called the *Lyapunov flag* of $E(\cdot)$ in $\mathbb{R}^d$. In some works, this is called the *Oseledets flag*[3].

Another concept that plays a role in our discussion is that of Lyapunov spectrum. We briefly recall that a multiset is a "set" where an element can occur more than once[4]. The next definition is an adaptation from the one given in Chapter 3 from Arnold (1998).

Definition 5. (Lyapunov spectrum). The ordered multiset $\bar{\boldsymbol{\chi}} \coloneqq \{\bar{\chi}_1,\ldots,\bar{\chi}_d\}$ is defined by the properties:

(1) $\bar{\chi}_i \ge \bar{\chi}_{i+1}$ for each $i \in [d]$,
(2) $\bar{\chi}_i \in \boldsymbol{\chi}$ for each $i \in [d]$,
(3) there are exactly $d_j \coloneqq \dim(V_{\chi_j}) - \dim(V_{\chi_{j-1}})$ copies of χ_j in $\bar{\boldsymbol{\chi}}$ for each $j \in [q]$.

We call $\bar{\boldsymbol{\chi}}$ the *Lyapunov spectrum* of E and the quantity d_j is the *multiplicity* of the Lyapunov index χ_j for each $j \in [q]$. Finally, we call the elements from $\bar{\boldsymbol{\chi}}$ the *Lyapunov indices with multiplicity.* ▲

2.5 Compound Matrices and k-dimensional Lyapunov Indices

In this subsection, we introduce the concept of k-dimensional Lyapunov index (see, e.g., Shimada and Nagashima (1979)), which is related to the asymptotic exponential growth rate of k-dimensional volumes and appears in the data-rate lower bound in Theorem 7. To do that, we first recall the notion of multiplicative compound matrix.

We first introduce some notations, which are largely based on Section 3 from Chapter 2 of Kuznetsov and Reitmann (2020) (see also Bar-Shalom et al. (2023)). For $d \in \mathbb{Z}_{>0}$ and $k \in [d]$, define $Q_{k,d} \coloneqq \{(i_1,\ldots,i_k) \in [d]^k : 1 \le i_1 < \cdots < i_k \le d\}$ and consider that its elements are ordered lexicographically, i.e., if $(i_1,\ldots,i_k) < (j_1,\ldots,j_k)$ we must have that there exists some $\ell \in [k]$ so that $j_\ell > i_\ell$ and $i_p = j_p$ for all $p \in [\ell-1]$.[5] Intuitively, we can think of $Q_{k,d}$ as the set of all k-combinations with elements in the set $[d]$. Thus, $\#Q_{k,d} = \binom{k}{d}$. For simplicity, we represent the i-th element from $Q_{k,d}$ as $I \in Q_{k,d}$ for each $i \in [\binom{k}{d}]$. Given a $d \times d$ matrix A, we denote by $[A]_{I,J}$ the $k \times k$ minor with row elements in I and column elements in J, i.e., the $k \times k$ matrix $([A]_{I,J})_{(\ell,p)} = A_{(i_\ell,j_p)}$ with $i_\ell \in I$, $j_p \in J$, $\ell \in [k]$, and $p \in [k]$.

[2] Lyapunov considered the quantity with the sign flipped. We follow the current sign convention in our work.

[3] After V.I. Oseledets who studied such flags in Oseledets (1968). The nomenclature *Oseledets filtration* is also standard in the dynamical systems' literature for such flag.

[4] We believe that this informal description of a multiset is enough for our purposes. We refer to page 694 from Knuth (1997) for a discussion on this terminology and for a rigorous definition.

[5] We adopt the convention that $[0] = \emptyset$.

Definition 6. (Compound matrix). Let A be a $d \times d$ matrix and let $k \in [d]$. We define the *k-th multiplicative compound matrix* of A as the $\binom{k}{d} \times \binom{k}{d}$ matrix

$$(A^{(k)})_{(i,j)} := \det([A]_{I,J}), \tag{11}$$

for each $i \in [\binom{k}{d}]$ and each $j \in [\binom{k}{d}]$. ▲

As mentioned above, our interest in compound matrices comes from the study of the so-called *k-dimensional Lyapunov indices*. For the present results, we only need the maximal index for each dimension $k \in [d]$, which we call the k-dimensional Lyapunov index in this work; we refer to Shimada and Nagashima (1979) and Benettin et al. (1978) for a discussion on the general theory. For $E : [0, \infty) \to \mathbb{R}^{d \times d}$ (equation (5)), we define the k-dimensional Lyapunov index as

$$\chi_1^{(k)} := \limsup_{t \to \infty} \frac{1}{t} \log(\|E^{(k)}(t)\|) \tag{12}$$

for each $k \in [d]$. We briefly note that $\chi_1^{(k)}$ is a Lyapunov index (Definition 4) for the function $f(t, v) = \|E^{(k)}(t)\|$, justifying the nomenclature.

2.6 A data-rate theorem

Finally, we can state our main result.

Theorem 7. Let $\mathcal{K} \subset \mathbb{R}^d$ be a compact set of possible initial conditions with nonempty interior. Also, let $\alpha \in [0, \infty)$ and $\varepsilon \in (0, \infty)$ be constants. Then, any control law $u(\cdot, \cdot)$ (Definition 1) that makes system (1) $(\varepsilon, \mathcal{K}, \alpha)$-emulate (Definition 3) the behavior of system (2) must satisfy

$$b(u) \geq \max_{j \in [d]} \{\max\{\chi_1^{(j)} + j\alpha, 0\}\}. \tag{13}$$

Moreover, for each $\delta \in (0, \infty)$ there exists a control law $u_\delta(\cdot, \cdot)$ so that

$$b(u_\delta) < \sum_{j=1}^{d} \max\{\bar{\chi}_j + \alpha, 0\} + \delta. \tag{14}$$

▲

Although seemingly distinct, the terms on the right-hand sides of inequalities (14) and (13) coincide if we set $\delta = 0$ in some cases. When that happens, we have an expression for the minimum data-rate for the emulation problem. However, even when this occurs, Theorem 7 is not a satisfactory result since we need to know the Lyapunov flag and the Lyapunov indices for $E(\cdot)$. In the next section, we address both of these issues. Explicitly, we state a proposition that gives a sufficient condition for the bounds to coincide when $\delta = 0$ and we characterize the values that $\bar{\chi}_j$ can have for each $j \in [d]$ in terms of the eigenvalues and eigenspaces of the matrices A and $A + BF$ that appear in equations (1) and (2), respectively.

Remark 8. The reader might wonder if we are able to make system (1) mimic more general linear dynamics than those described by (2), i.e., is it possible to design, for each $\varepsilon \in (0, \infty)$, a control law that $(\varepsilon, \alpha, \mathcal{K})$-emulates $\dot{y}(t) = Gy(t)$ for some $d \times d$ real matrix $G \neq A + BF$ for any $d \times m$ matrix F? The answer to that question is negative. Therefore, the feedback form we chose is the most general type of finite-dimensional linear time-invariant system that can be emulated by (1).

3. UNDERSTANDING THE BOUNDS

In this section, we show when the bounds in Theorem 7 are tight by providing a sufficient condition in Proposition 13. Also, we relate the Lyapunov indices of $E(\cdot)$ with the eigenvalues of the open and closed-loop modes. More clearly, we express the Lyapunov indices of $E(\cdot)$ implicitly in terms of the eigenvalues of A and $A + BF$.

3.1 When is our result tight?

First, we recall the notion of root space[6] (see, e.g., Chapter 12 from Gohberg et al. (1986)).

Definition 9. (Root space). Let $\lambda \in \mathbb{C}$ be an eigenvalue of a $d \times d$ real matrix A. Further, let $q_\lambda(x) = x - \lambda$, if $\lambda \in \mathbb{R}$, and $q_\lambda(x) = x^2 - 2\Re(\lambda)x + |\lambda|^2$, otherwise. We define the *root space* $\mathcal{R}(\lambda, A) \subseteq \mathbb{R}^d$ associated with the eigenvalue λ as

$$\mathcal{R}(\lambda, A) := \ker\{(q_\lambda(A))^p\}, \tag{15}$$

where $p \in \mathbb{Z}_{>0}$ is the smallest integer so that $\ker\{(q_\lambda(A))^k\} \subseteq \ker\{(q_\lambda(A))^p\}$, for all $k \in \mathbb{Z}_{>0}$. ▲

Second, let $\mathcal{V} \subseteq \mathbb{R}^d$ be the largest A-invariant subspace that is contained in the null space of BF, i.e., if $\mathcal{W} \subseteq \mathbb{R}^d$ is A-invariant and is contained in the null space of BF, then $\mathcal{W} \subseteq \mathcal{V}$. Informally, the subspace $\mathcal{V}$ corresponds to the eigenstructure that remains unchanged by the feedback F.

To state Proposition 11, we need a technical condition involving the root spaces of the open and closed-loop modes. That statement is clearer if we consider the following sets: for each $\mu \in \Sigma(A)$, we define $\mathcal{J}(\mu) := \{\bar{\mu} \in \Sigma(A + BF) : \Re(\bar{\mu}) = \Re(\mu)\}$, i.e., the set of eigenvalues of $A + BF$ that have the same real part as μ.

Assumption 10. For each $\mu \in \Sigma(A)$ we have either that (i) $\mathcal{J}(\mu) \neq \emptyset$, or that (ii) $\mu \in \Sigma(A + BF)$ and $\mathcal{R}(\mu, A) = \mathcal{R}(\mu, A + BF) \subseteq \mathcal{V}$.

We can interpret the assumption above as follows: the first condition requires the sets of real parts of eigenvalues of A and $A + BF$ to not intersect. The second condition means that if A has an eigenvalue μ and $A + BF$ has some eigenvalue with the same real part as μ, then μ is an eigenvalue of $A + BF$ and the root spaces of A and $A + BF$ corresponding to μ coincide. Furthermore, those root spaces are contained in the "unchanged" eigenstructure. Now, we are ready to state our sufficient condition.

Proposition 11. Suppose that Assumption 10 holds. Then, it is true that

$$\chi_1^{(k)} = \sum_{i=1}^{k} \bar{\chi}_i. \tag{16}$$

Consequently, under Assumption 10, we conclude that

$$h(\alpha, \mathcal{K}) = \sum_{i=1}^{d} \max\{\bar{\chi}_i + \alpha, 0\}. \tag{17}$$

▲

Intuitively, this result states that, under Assumption 10, the k-dimensional Lyapunov index and the sum of the k

[6] When the field of scalars is $\mathbb{C}$, some authors call the root spaces the generalized eigenspaces. See, e.g., Chapter 8 Section B in Axler (2024).

largest Lyapunov indices with multiplicity of $E(\cdot)$ must coincide.

Remark 12. Assumption 10 is technical, and we believe it is unnecessary for the conclusion of Proposition 11 to hold. Explicitly, we conjecture that equation (17) always holds. For example, we can prove this claim for all 2×2 systems.

3.2 How to compute the Lyapunov indices

The next question we need to address is how to express the Lyapunov index of $E(\cdot)$ in terms of the same indices for e^{At} and $e^{(A+BF)t}$. The next proposition gives an exact expression for that.

Proposition 13. For each $x \in \mathbb{R}^d$, we have that

$$\lambda(E(t), x) = \begin{cases} \max\{\lambda(e^{At}, x), \lambda(e^{(A+BF)t}, x)\}, \text{ if } x \in \mathbb{R}^d \setminus \mathcal{V} \\ -\infty, \text{ otherwise.} \end{cases} \tag{18}$$

▲

We take this opportunity to briefly remark that the inequality $\lambda(E, x) \leq \max\{\lambda(e^{At}, x), \lambda(e^{(A+BF)t}, x)\}$ always holds and was proven by Lyapunov in Chapter 7 of Liapounoff (1907) in a more general setting. We also note that the converse inequality does not always hold, however. Clearly, if $BF = 0$, we see that $\lambda(E, x) = -\infty$ for any $x \in \mathbb{R}^d$. This is why we need to split equation (18) into two cases. If $x \in \mathcal{V}$, we have that $E(t)x = 0$ for all $t \in [0, \infty)$. Finally, we note that $\lambda(e^{At}, x)$ equals the real part of some eigenvalue of A (see, e.g., Example 3.2.3 from Arnold (1998)). Thus, equation (18) gives an implicit expression for the $\bar{\chi}_j$'s in terms of the open and closed-loop eigenvalues.

3.3 Illustrative example

The goal of the following example is twofold: to illustrate the results we stated in this paper and to show an example where the minimum data-rate varies discontinuously with the target closed-loop system parameters.

Example 14. For concreteness, let

$$\bar{A} := \begin{pmatrix} -1.5 & 0 \\ 0 & -0.5 \end{pmatrix} \text{ and } \bar{B} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{19}$$

and consider the system

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) \tag{20}$$

for all $t \in [0, \infty)$. We assume that the initial state $x(0)$ belongs to the compact set $\mathcal{K} = [-1, 1]^2$ and let $\alpha = 2$. Further, define the matrices

$$\Lambda := \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, R_\theta := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \text{ and} \tag{21a}$$

$$A_\theta := R_\theta \Lambda R'_\theta. \tag{21b}$$

In this example, our goal is to compute the minimum data-rate $h(\alpha, \mathcal{K})$ for system (20) to $(\varepsilon, \alpha, \mathcal{K})$-emulate the behavior of the closed-loop system

$$\dot{z}(t) = A_\theta z(t), \tag{22}$$

for each fixed choice of $\theta \in [0, 2\pi)$ and an arbitrary $\varepsilon \in (0, \infty)$. Note that $A_\theta = \bar{A} + \bar{B}F$ with $F = A_\theta - \bar{A}$.

Case $\theta = 0$: in this case, we have that

$$A_0 = \Lambda. \tag{23}$$

In this scenario, we can apply Proposition 13 to get that

$$\begin{aligned} \lambda(E, e_1) &= \max\{\lambda(e^{A_0 t}, e_1), \lambda(e^{\bar{A}t}, e_1)\} \\ &= \max\{-2, -1.5\} = -1.5 \text{ and} \\ \lambda(E, e_2) &= \max\{\lambda(e^{A_0 t}, e_2), \lambda(e^{\bar{A}t}, e_2)\} \\ &= \max\{-1, -0.5\} = -0.5. \end{aligned}$$

Since $\lambda(E, \cdot)$ can only have at most 2 distinct values (see the remark after equation (9)), we have found $\bar{\chi}_1$ and $\bar{\chi}_2$. Since the real parts for the eigenvalues of A_0 and $\bar{A}$ are all distinct, Proposition 11 tells us that the minimum data-rate is given by equation (17). Therefore, for $\alpha = 2$, we have that

$$\begin{aligned} h(\alpha, \mathcal{K}) &= \max\{2 - 1.5, 0\} + \max\{2 - 0.5, 0\} \\ &= 2.0 \text{ nats/unit of time.} \end{aligned} \tag{24}$$

Case $\theta \neq 0$: in this case, again by Proposition 13, we have

$$\begin{aligned} \lambda(E, e_1) &= \max\{\lambda(e^{A_\theta t}, e_1), \lambda(e^{\bar{A}t}, e_1)\} \\ &= \max\{-1, -1.5\} = -1 \text{ and} \\ \lambda(E, e_2) &= \max\{\lambda(e^{A_\theta t}, e_2), \lambda(e^{\bar{A}t}, e_2)\} \\ &= \max\{-1, -0.5\} = -0.5, \end{aligned}$$

where we concluded that $\lambda(e^{A_\theta t}, e_1) = -1$ as a consequence of the fact that e_1 belongs only to the largest element of the Lyapunov flag for A_θ for $\theta \neq 0$. As before, we use Proposition 11 with $\alpha = 2$ to write that

$$\begin{aligned} h(\alpha, \mathcal{K}) &= \max\{2 - 1, 0\} + \max\{2 - 0.5, 0\} \\ &= 2.5 \text{ nats/unit of time.} \end{aligned} \tag{25}$$

We note that in both the previous cases, the matrices A_θ, corresponding to the closed-loop system, have the same eigenvalues for each $\theta \in [0, 2\pi)$. However, their eigenspaces differ. The reason why equations (24) and (25) are distinct is because $h(\alpha, \mathcal{K})$ depends not only on the eigenvalues of the open and closed-loop system matrices, but on the relative position between their eigenspaces. Moreover, notice that this example shows us that the minimum data-rate does not vary continuously with the target mode.

4. CONCLUSION

In this work, we introduced the emulation problem. Also, we provided a theorem that characterizes bounds for the minimum data-rate to solve such problem. We further described how to compute that data-rate using information from the open and closed-loop systems' eigenstructure.

In the future, we want to extend this work to deal with time-varying linear systems as well as nonlinear systems. Furthermore, we would like to present an algorithm to compute the minimum data-rate exploiting Proposition 13. Finally, we want to prove or disprove the claim that (17) always holds.

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