Turyn sequences conjecture

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1 NPAF: definition and examples

Definition 1 *For a finite sequence $X = [x_1, \ldots, x_r]$ of length r, the non-periodic autocorrelation function (NPAF) at s, is equal to:*

$$NPAF(X, s) = \sum_{i=1}^{r-s} x_i \cdot x_{i+s},\ \forall s = 1, \ldots, r-1. \tag{1}$$

Remark 1 *For a finite sequence X of length r:*

- *there are exactly $r-1$ non-periodic autocorrelation function coefficients, namely $NPAF(X, s)$, $s = 1, \ldots, r-1$*
- *clearly, $NPAF(X, s)$ is made up of $r-s$ monomials (quadratic terms)*

Example 1 *Let $r = 3$, $X = [x_1, x_2, x_3]$, then we have:*

$$NPAF(X, 1) = \sum_{i=1}^{3-1} x_i \cdot x_{i+1} = x_1 \cdot x_2 + x_2 \cdot x_3$$

$$NPAF(X, 2) = \sum_{i=1}^{3-2} x_i \cdot x_{i+2} = x_1 \cdot x_3$$

Example 2 *Let $r = 5$, $X = [x_1, x_2, x_3, x_4, x_5]$, then we have:*

$$NPAF(X, 1) = \sum_{i=1}^{5-1} x_i \cdot x_{i+1} = x_1 \cdot x_2 + x_2 \cdot x_3 + x_3 \cdot x_4 + x_4 \cdot x_5$$

$$NPAF(X, 2) = \sum_{i=1}^{5-2} x_i \cdot x_{i+2} = x_1 \cdot x_3 + x_2 \cdot x_4 + x_3 \cdot x_5$$

$$NPAF(X, 3) = \sum_{i=1}^{5-3} x_i \cdot x_{i+3} = x_1 \cdot x_4 + x_2 \cdot x_5$$

$$NPAF(X, 4) = \sum_{i=1}^{5-4} x_i \cdot x_{i+4} = x_1 \cdot x_5$$

2 Turyn sequences: definition and examples

Definition 2 *Four sequences X, Y, Z, W of odd length r each, with elements from $\{-1. + 1\}$, are called Turyn sequences if*

$$NPAF(X, s) + NPAF(Y, s) + NPAF(Z, s) + NPAF(W, s) = 0, \forall s = 1, \dots, r-1. \tag{2}$$

Example 3 *Turyn sequences for $r = 3$*

```
X := [-1, -1, -1]
Y := [-1, -1, 1]
Z := [-1, -1, 1]
W := [-1, 1, -1]

[-3, -1, -1, -1] 12 12
```

```
[2, 1]
[0, -1]
[0, -1]
[-2, 1]
```

Example 4 *Turyn sequences for $r = 5$*

```
X := [-1, -1, -1, -1, 1]
Y := [-1, -1, -1, 1, -1]
Z := [-1, -1, 1, -1, 1]
W := [-1, -1, 1, 1, -1]

[-3, -3, -1, -1] 20 20
```

```
[2, 1, 0, -1]
[0, 1, 0, 1]
[-2, 1, 0, -1]
[0, -3, 0, 1]
```

Example 5 *Turyn sequences for $r = 13$*

```
X := [1, 1, -1, 1, 1, 1, -1, 1, -1, 1, -1, -1, -1]
Y := [-1, -1, -1, 1, -1, -1, 1, -1, 1, 1, -1, -1, -1]
Z := [-1, 1, 1, 1, -1, -1, -1, -1, 1, -1, 1, 1, -1]
W := [-1, 1, -1, -1, -1, -1, -1, -1, 1, 1, -1, -1, 1]

[1, -5, -1, -5] 52 52
```

```
[-2, 3, -2, 5, -2, 1, -4, -1, 0, -1, -2, -1]
[0, -1, 2, -3, 2, 1, 0, -1, 0, 3, 2, 1]
[0, -1, -2, -5, 0, -1, 2, 1, 2, -1, -2, 1]
[2, -1, 2, 3, 0, -1, 2, 1, -2, -1, 2, -1]
```

3 Turyn conjecture and Hadamard matrices

Conjecture 1 *For every odd* $r = 3, 5, \ldots,$ *there exist Turyn sequences* X, Y, Z, W *of length* r*.*

Remark 2 *Given Turyn sequences* X, Y, Z, W *of length* r*, denote by* x, y, z, w *the sums of their elements respectively. Then the follwing holds:*

$$x^2 + y^2 + z^2 + w^2 = 4 \cdot r$$

For every odd r*, there is a finite numbers of possible* x, y, z, w *and all such quadruples have been pre-computed. Note that* x, y, z, w *must be all odd, because they are sums of an odd number (namely* r*) of* ± 1*'s.*

Remark 3 *Given Turyn sequences* X, Y, Z, W *of length* r*, we can construct* $HM(4 \cdot r)$

Corollary 1 *Turyn sequences* X, Y, Z, W *of length* $r = 167 \rightsquigarrow HM(4 \cdot 167) = HM(668)$

4 The ISLA algorithm to search for Turyn sequences X, Y, Z, W of length r

We describe the ISLA algorithm to search for Turyn sequences X, Y, Z, W of length r.

ISLA stands for Incremental Substitution Linearization Algorithm.

INPUT An odd number r and four odd integers x, y, z, w, s.t. $x^2 + y^2 + z^2 + w^2 = 4 \cdot r$

OUTPUT Turyn sequences of length r

$$\begin{array}{rcl} X & = & [x_1, \quad x_2, \quad \ldots, \quad x_{r-1}, \quad x_r] \\ Y & = & [y_1, \quad y_2, \quad \ldots, \quad y_{r-1}, \quad y_r] \\ Z & = & [z_1, \quad z_2, \quad \ldots, \quad z_{r-1}, \quad z_r] \\ W & = & [w_1, \quad w_2, \quad \ldots, \quad w_{r-1}, \quad w_r] \end{array} \tag{3}$$

s.t.

$$\begin{array}{rcl} x & = & x_1 + x_2 + \ldots + x_{r-1} + x_r \\ y & = & y_1 + y_2 + \ldots + y_{r-1} + y_r \\ z & = & z_1 + z_2 + \ldots + z_{r-1} + z_r \\ w & = & w_1 + w_2 + \ldots + w_{r-1} + w_r \end{array} \tag{4}$$

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1. We need to solve $r-1$ quadratic equations (2)

2. for every r the $(r-1)-th$ equation, namely:

$$NPAF(X, r-1) + NPAF(Y, r-1) + NPAF(Z, r-1) + NPAF(W, r-1) = 0 \tag{5}$$

has 4 terms and 8 variables and is of the form:

$$x_1x_r + y_1y_r + z_1z_r + w_1w_r = 0 \tag{6}$$

3. We can solve equation (6) by exhaustive search, and obtain its 96 solutions.

4. Note that these 96 solutions will be the same, for every r, i.e. we only need to do this once. This can be a pre-processing (table look-up) step

5. for every r the $(r-2)-th$ equation, namely:

$$NPAF(X, r-2) + NPAF(Y, r-2) + NPAF(Z, r-2) + NPAF(W, r-2) = 0 \tag{7}$$

has 8 terms and 16 variables and is of the form:

$$x_1x_{r-1} + x_2x_r + y_1y_{r-1} + y_2y_r + z_1z_{r-1} + z_2z_r + w_1w_{r-1} + w_2w_r = 0 \tag{8}$$

6. For every of the 96 solutions obtained previously, we substitute the 8 variables $x_1, x_r, y_1, y_r, z_1, z_r, w_1, w_r$ into (8) and obtain 96 linear equations in the 8 variables $x_2, x_{r-1}, y_2, y_{r-1}, z_2, z_{r-1}, w_2, w_{r-1}$

7. Each of the linear equations obtained in the previous step, can be solved by exhaustive search and have a maximum of 70 solutions, depending on the sign pattern they exhibit, i.e. the $\pm$ signs of the 8 variables.

8. Note that there is a limited number of sign patterns, so this can also be a pre-processing (table look-up) step.

9. to determine the exact number of sign patterns, we focus on the number of minus signs:

   (a) 0 minus signs $\rightsquigarrow \binom{8}{0} = 1$ case
   (b) 1 minus signs $\rightsquigarrow \binom{8}{1} = 8$ cases
   (c) 2 minus signs $\rightsquigarrow \binom{8}{2} = 28$ cases
   (d) 3 minus signs $\rightsquigarrow \binom{8}{3} = 56$ cases
   (e) 4 minus signs $\rightsquigarrow \binom{8}{4} = 70$ cases
   (f) 5 minus signs $\rightsquigarrow \binom{8}{5} = 56$ cases
   (g) 6 minus signs $\rightsquigarrow \binom{8}{6} = 28$ cases
   (h) 7 minus signs $\rightsquigarrow \binom{8}{7} = 8$ cases
   (i) 8 minus signs $\rightsquigarrow \binom{8}{8} = 1$ case

   So there are $256 = 2^8$ possible sign patterns.

10. It turns out the number of minus signs must be even. So this leaves us with $1 + 28 + 70 + 28 + 1 = 128$ possible sign patterns.

11. It is of interest to actually count how many and which ones of these 128 possible sign patterns actually appear. i.e. occur, for small lengths, e.g. $r = 3, 5, 7, 9, 11, 13$.

12. **Termination criterion:** After specifying $r - 2$ substitution/linearization steps, the $1 - st$ equation, namely:

$$NPAF(X, 1) + NPAF(Y, 1) + NPAF(Z, 1) + NPAF(W, 1) = 0 \tag{9}$$

   (which has $4(r - 1)$ terms and $4r$ variables)
   will either/or:

   (a) be satisfied automatically, i.e. will be equal to 0
   (b) the 4 "middle" variables $x_{\frac{r+1}{2}}$, $y_{\frac{r+1}{2}}$, $z_{\frac{r+1}{2}}$, $w_{\frac{r+1}{2}}$ will be "free variables", i.e. take any -1 or $+1$ values.

   Experimenting with small lengths, e.g. $r = 3, 5, 7, 9, 11, 13$ will help clarifying the termination criterion a bit better.

Remark 4 *One way to think about the ISLA algorithm is to describe it as some sort of back-tracking in a tree. The root of the tree is the $(r - 1) - th$ equation and will have 96 children nodes. Everyone of these 96 nodes will have a maximum of 70 children nodes. Each layer of the tree corresponds to one of the $r - 2$ NPAF equations that we need to solve. The leaf nodes correspond to the first equation. The solutions, i.e. the Turyn sequences will be located at some of the leaf nodes of the tree.*


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