The Euclidean Algorithm

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Outline

Rings and Ring Norms

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$$\begin{aligned} N : \mathbb{Z} &\to \mathbb{Z}^+ \cup \{0\} \\ n &\mapsto \begin{cases} 0 & n = 2k \\ 1 & n = 2k + 1 \end{cases} \end{aligned}$$

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$$\gcd(60, 45) = 15$$

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- $a = q \cdot b + r$ with $r = 0$ or $N(r) < N(b)$

Example Euclidean Domains

$\mathbb{Z}, \mathbb{Z}[\mathrm{i}], \mathbb{Z}[x]$ are examples of rings with a division algorithm

$$\mathbb{Z}[\mathrm{i}] = \{a + b\mathrm{i} : a, b \in \mathbb{Z}\}$$

$$\mathbb{Z}[x] = \{\sum_0^n a_n x^n : a_n \in \mathbb{Z}, n \in \mathbb{Z}^+\}$$

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$$(a, b) = (b, r_1) = (r_1, r_2) = \cdots = (r_{n-1}, r_n)$$

GCD in $\mathbb{Z}$

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$$35 = 2 \cdot 21 - 7$$
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$\mathbb{Z}[\mathrm{i}]$ equipped with the ring norm $N : \mathbb{Z}[\mathrm{i}] \to \mathbb{Z}^{+} \cup \{0\}$ defined as $N(a + b\mathrm{i}) = a^2 + b^2$

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$$\begin{aligned} 8 + 6\mathrm{i} &= (3 - \mathrm{i}) \cdot (2 + 3\mathrm{i}) - 1 - \mathrm{i} \\ 2 + 3\mathrm{i} &= -2 \cdot (-1 - \mathrm{i}) + \mathrm{i} \\ -1 - \mathrm{i} &= (-1 + \mathrm{i}) \cdot \mathrm{i} + 0 \end{aligned}$$

Topological Relationships and Minimal Norms

Let R be a ring and $N : R \to \mathbb{Z}^+ \cup \{0\}$ be a ring norm with the added condition that it defines distance. The ring R is a Euclidean domain if

$$\forall a, b \in R, \exists q, r \in R \text{ such that}$$
$$a = q \cdot b + r \text{ with } r = 0 \text{ or } N(r) < N(b)$$

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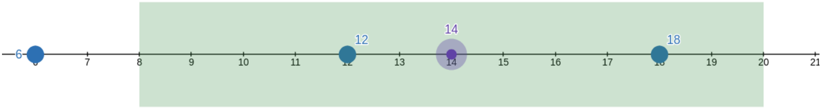
$$\|r\| < \|b\|$$

$$\|q \cdot b - a\| < \|b\|$$

Topological Relationships and Minimal Norms



Integer Worst Case

For any $b < a \in \mathbb{Z}$ there are one or two choices for q, r

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$$\begin{aligned} |r_1| + |r_2| &= |b| \\ \min\{|r_1|, |r_2|\} &\leq \frac{|b|}{2} \end{aligned}$$

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\begin{aligned}
a &= q_1 b + r_1 \quad |r_1| \leq \frac{|b|}{2} \\
b &= q_2 r_1 + r_2 \quad |r_2| \leq \frac{|r_1|}{2} \\
r_1 &= q_3 r_2 + r_3 \quad |r_3| \leq \frac{|r_2|}{2} \\
&\vdots \\
r_{n-1} &= q_{n+1} r_n + r_{n+1} \quad |r_{n+1}| \leq \frac{|r_n|}{2}
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$$f_n = f_{n-1} + f_{n-2}$$

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f_{n+1} &= f_n + f_{n-1} \\
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5 &= 1 \cdot 3 + 2 \\
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$$\begin{aligned} f_{n+1} &= 2f_n - f_{n-2} \\ f_n &= -3 \cdot -f_{n-2} - f_{n-4} \\ -f_{n-2} &= 3 \cdot -f_{n-4} + f_{n-6} \\ &\vdots \\ (-1)^n 5 &= (-1)^n 3 \cdot 2 - 1 \text{ or } (-1)^n 8 = (-1)^n 3 \\ 2 &= -2 \cdot -1 \quad \text{or} \quad 3 = -3 \cdot -1 \end{aligned}$$

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$$\frac{f_n}{f_{n-2}} = \frac{f_n}{f_{n-1}} \frac{f_{n-1}}{f_{n-2}} \to \phi^2 \approx 2.61$$

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$$6 + 7\mathrm{i} = (2 + \mathrm{i}) \cdot (3 + 2\mathrm{i}) + 2 \qquad 6 + 7\mathrm{i} = (3 + \mathrm{i}) \cdot (3 + 2\mathrm{i}) - 1 - 2\mathrm{i}$$

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$$\log_\phi(3-\phi)N$$

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$\mathbb{Z}[x]$ Worst Case has steps and calculations at each step being

$$N = \deg\ p(x) \qquad N^2$$

$$N^3$$

Sources

Abstract Algebra - Dummit and Foote (Book)

The Art of Computer Programming Vol II: Seminumerical Algorithms - Donald Knuth (Book)

The Euclidean Algorithm for Gaussian Integers - Heinrich Rolletschek (Paper)

Prove that the Gaussian Integer's ring is a Euclidean domain (StackExchange)

$\mathbb{Z}[i]$ is a Principal Ideal Domain (StackExchange)

Integer Remainders (Desmos)

Gaussian Remainders (Desmos)