Finding the number of cells in a volume of the crust

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I'm imagining a volume of the Earth's crust like this:

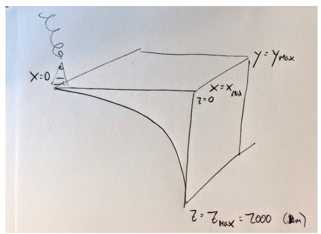


Figure 1: Crust diagram, except that I've drawn the curvature backwards: it should be steepest at the volcano, and nearly flat at the trench.

Note that x is positive to the right, y is positive into the page, and z is positive down.

Magnabosco says that the number of cells per unit volume, as a function of depth, is given by

$$\rho = 10^A z^B$$

Where ρ is the cell concentration along a column and A and B are parameters determined empirically from a linear fit to log-transformed cell data.

We can find the total number of cells in a column by integrating from $z = 0$ to $z = z_{crust}$ (the bottom of the crust),

$$c_{col} = \int_0^{z_{crust}} 10^A z^B dz$$

where c_{col} is the number of cells per unit area in a column. (Magnabosco et al used cells cm^{-3} so this would be the number of cells in a 1 cm^2 column of crust.)

The antiderivative of $\rho(z)$ is given by

$$\int 10^A z^B dz = \frac{10^A z^{B+1}}{b+1} + C$$

Note that $0^{B+1} = 0$ for any B, so

$$\begin{aligned} c_{col}(z_{crust}) &= \int_0^{z_{crust}} 10^A z^B dz \\ &= \frac{10^A z_{crust}^{B+1}}{B+1} \end{aligned}$$

OK, cool. Now we need to plug in the expression for z_{max} as a function of distance from the volcanoes.

Donato likes a logarithmic function to describe depth (I'm agnostic on whether a logarithmic function is better than any other function).

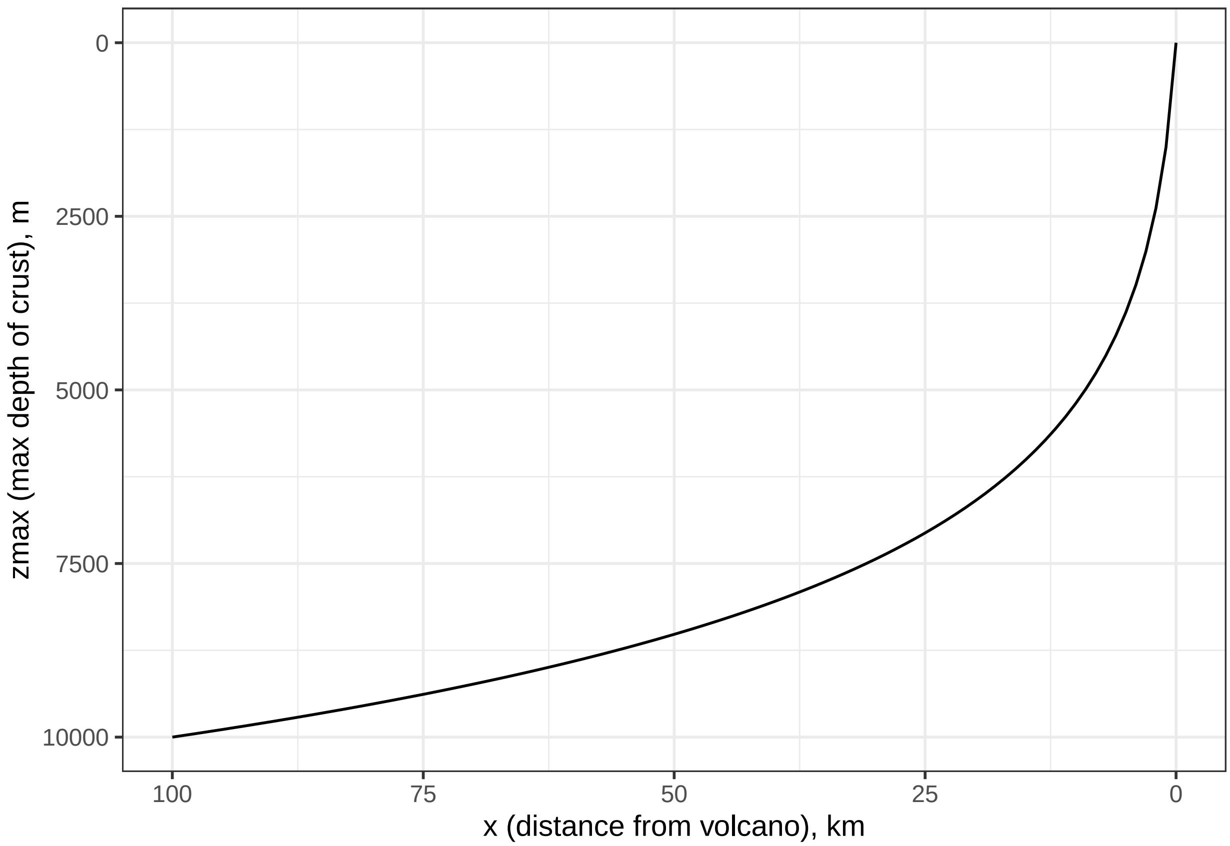
$$z_{crust} = \theta \log (x + 1)$$

θ is a scaling parameter; $\theta = \frac{z_{crust,max}}{\log(x_{max}+1)}$. We have to express z_{crust} as a function of $x + 1$ so that $z_{crust} = 0$ when $x = 0$.

Note that this gives the following profile (assuming 100 km from volcano to the limit of integration):

```
depth.m <- 10000
theta <- depth.m/log10(100+1) # crust is depth.m deep at most
x <- 0:100 # Am using kilometers for x and meters for Z, take note

zmax <- theta*log10(x+1)
d <- data.frame(x, zmax)
ggplot(d, aes(x=x, y=zmax)) +
  geom_line() +
  scale_y_reverse() +
  scale_x_reverse() +
  xlab("x (distance from volcano), km") +
  ylab("zmax (max depth of crust), m")
```



We can integrate over x (left-to-right across the page) to get the number of cells in a unit of cross-sectional area of crust.

$$
\begin{aligned}
c_{area} &= \int_0^{x_{max}} c_{col}(x)dx \\
&= \int_0^{x_{max}} \int_0^{z_{crust}} 10^A z^B dzdx \\
&= \int_0^{x_{max}} \frac{10^A z_{crust}^{B+1}}{B+1} dx \\
&= \frac{10^A z_{crust}^{B+2}}{(B+1)(B+2)}|_{x_{max}} - \frac{10^A z_{crust}^{B+2}}{(B+1)(B+2)}|_0 \\
&= \frac{10^A \theta \log(x_{max}+1)^{B+2}}{B^2+3B+2}
\end{aligned}
$$

This gives a number of cells per cross-sectional area of the volume we want to consider. To get the actual population of cells pop, we multiply by the width of the volume y_{max}:

$$
\text{pop} = y_{max} \frac{10^A \theta \log(x_{max}+1)^{B+2}}{B^2+3B+2}
$$

Magnabosco has $A = 8.16(8.00, 8.45)$ and $B = -0.94(-1.01, -0.81)$ so for a max depth of 10 m, a max volcano-to-outer-arc length of 100,000 m and a width of 100,000 m

```
A <- 8.16
B <- -0.94
calc_theta <- function(z_max, x_max) {
  theta <- z_max / log10(x_max + 1)
  theta
}
#theta <- calc_theta(z_max = depth.max, x_max = 10^5)
calc_cells <- function(x_max, y_max, theta, A = 8.16, B = -0.94) {
  pop <- y_max * (10^A * theta * log10(x_max + 1) ^ (B+2)) / (B^2 + 3*B + 2)
  pop
}
num.cells <- calc_cells(x_max = 1e5, y_max = 1e5, theta = theta, A = 8.16, B = -0.94)
num.cells
```

```
## [1] 6.244305e+18
```

This seems like a plausible number?

There may well be an error or two in these calculations; I'll want to go over them a bit. But I think this is basically the right approach.

Note on error

There is one big problem with the Magnabosco parameters: you can't calculate $\int_0^{z_max} 10^A z^B$ when $B <= -1$. The integral does not converge, since $0^-1 = \frac{1}{0}$. That's not really anyone's fault, it's just that Magnabosco used a model that yields impossible error as depth approaches 0. So... we can't use those data to calculate error. We could recalculate the power law slopes or recalculate the error bounds but we can't use the Magnabosco model as is.

Checking my math

Above, plugging in 10 km for max depth, and a distance of 100 km trench-to-volcano and a width of 100 km (arbitrarily), I get 5e17 cells.

Let's try a simpler calculation to see whether I get